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THE 1-SKELETON OF POLYTOPES, ORIENTED MATROIDS AND SOME OTHER LATTICES

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ABSTRACT

The 1-skeleton of a lattice is the graph whose vertices are the atoms, two being adjacent if they are covered by their join. An H-lattice is a ranked lattice in which every length 2 interval has 4 elements and the 1-skeleton of each interval is connected. With $d = r(\hat{1})-r(\hat{0})-1$, we show that the 1-skeleton of an H-lattice is d-connected and has a subdivision of K_{d+1}. This extends known results about polytopes to oriented matroids and certain cell complexes.

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The 1-skeleton of a convex d-polytope is d-connected [1] and contains a subdivision of the complete graph K_{d+1} [3]. Those results have been generalized in several ways ([5], [8]); see also [3]). One would expect these results to hold for oriented matroid lattices; indeed, the desire of proving such an extension has led us to proofs of the results above in a purely combinatorial setting, and this is the content of this paper. Some conversations with Komei Fukuda were very helpful in the course of finding these proofs.

We shall work with an object which we call an <u>H-lattice</u>: it is a ranked finite lattice, whith rank normalized by $r(\bar{0}) = -1$, in which the following two conditions hold:

a) (<u>The rhombus condition</u>) Every length 2 interval contains precisely 4 elements.

a) The 1-skeleton of each interval is connected. The 1-skeleton of a lattice has as vertices the atoms, two being adjacent if they are covered by their join.

<u>Theorem 1</u>: <u>The 1-skeleton of a rank d H-lattice is d-connected</u>

<u>Theorem 2</u>: <u>The 1-skeleton of a rank d H-lattice contains a subdivision of K_{d+1}; indeed, one may choose one atom v so that the principal vertices</u>

of the subdivided K_{d+1} are v and neighbours of v, and one of these neighbours may also be specified arbitrarily.

It is well known and easily proved that the face-lattice of a d-polytope is a rank d H-lattice. The same is true of the Las Vergnas lattice of an acyclic oriented matroid [6], and its dual, the tope-lattice of [7]. More generally, suppose that P is a regular finite CW-complex in which the intersection of closed cells is a closed cell or empty, and whose space is a homology d-manifold. Order P by inclusion, adjoin a maximum and a minimum element; there results a rank d+1 H-lattice. Thus Theorems 1 and 2 apply to a lot more than polytope lattices

Heretofore P denotes a rank d H-lattice. Note that each interval of P is also an H-lattice. The 1-skeleton will be denoted as skel P. For simplicity, the atoms and rank 1 elements of P will be called respectively vertices and edges. The rhombus condition implies that each edge of P covers exactly two vertices, whence an alternative description of skel P follows: it has the vertices and edges of P in their natural roles, the ends of an edge being the vertices it covers.

The main idea of the proofs is to transform paths in skel[v, T], v a vertex, into paths in skel P. This is done in the following Lemmas.

Lemma 1: If d=2, Skel P is a polygon, with at least 3 vertices.

Proof: The rhombus condition for an interval [v,T], v a vertex, implies that each vertex has degree 2 in skel P; by hypothesis this graph is connected, therefore it is a polygon. As P is a lattice, skel P is simple, hence there are at last 3 vertices.

Suppose now that $d \geq 3$, and fix a vertex v. If α is a vertex of $[v,\hat{1}]$, it is an edge of P, incident to v; let α denote its other end. If $\beta = \alpha_1\alpha_2$ is an edge of $[v,\hat{1}]$, in skel $[\hat{0},\beta]$ there exists a path from $\bar{\alpha}_1$ to $\bar{\alpha}_2$ which does not contain v. Denote this path by $\bar{\beta}$.

<u>Lemma 2:</u> Let β_1,β_2 be distinct edges of $[v,\hat{1}]$. If β_1 and β_2 have a common vertex u, then $u = \bar{\alpha}$ for a common end α of β_1 and β_2 in skel$[v,\hat{1}]$.

<u>Proof:</u> As both v and u are $\leq \beta_1$ and β_2, we have that $v \vee u \leq \beta_1 \wedge \beta_2$. Since $v \neq u$ and $\beta_1 \neq \beta_2$, $1 \leq r(v \vee u) \leq r(\beta_1 \wedge \beta_2) \leq 1$. Hence equality holds troughout, $v \vee u = \beta_1 \wedge \beta_2 = \alpha$, where α is an edge of P. Since $\alpha = v \vee u$, $u = \bar{\alpha}$. Since $v < \alpha < \beta_1$ and $v < \alpha < \beta_2$, α is a common end of β_1 and β_2 in skel$[v,\hat{1}]$. □

Now, given a trail $p = \alpha_0\beta_1\alpha_1 \ldots \beta_k\beta_k$ in skel$[v,\hat{1}]$, let $\bar{p} = \bar{\alpha}_0\bar{\beta}_1\bar{\alpha}_1 \ldots \bar{\beta}_k\bar{\alpha}_k$. Clearly $\bar{p}$ is a trail in skel P, and v is not a vertex of $\bar{p}$. It follows directly from Lemma 2 that:

<u>Lemma 3:</u> a) If p is a path in skel$[v,\hat{1}]$, then $\bar{p}$ is also a path

b) b)]If $p_1,p_2,\ldots,p_n$ are (internally) disjoint paths in skel$[v,\hat{1}]$, then likewise are $\bar{p}_1,\bar{p}_2,\ldots,\bar{p}_n$.

The scheme of the two proofs below closely resembles those by Barnette [2].

Proof of Theorem 1: The result is true if $d \leqq 2$ by virtue of Lemma 1. We proceed by induction for $d \geqq 3$. Choose a minimum separating set X in skel P and single out $v \in X$. By the minimality of X, there are neighbours u_1 and u_2 of v, not in X, which are separated by X. Let $\alpha_1 = v \vee u_1$, $\alpha_2 = v \vee u_2$. Since $[v, \hat{1}]$ is a rank $d-1$ H-lattice, its 1-skeleton is $(d-1)$-connected, hence by Whitney's or Menger's Theorem there exist $d-1$ internally disjoint paths $p_1, \ldots, p_{d-1}$ in skel$[v, \hat{1}]$ from α_1 to α_2. Hence

$\bar{p}_1, \ldots, \bar{p}_{d-1}, p_d = (u_1 \alpha_1, v, \alpha_2, u_2)$ are d internally disjoint paths in skel P from u_1 to u_2. It follows that $|X| \geqq d$. □

Proof of Theorem 2: We again proceed by induction, for $d \geqq 3$.

Choose a vertex v and one neighbour u; let $\alpha = v \vee u$. Applying the theorem to $[v, \hat{1}]$, we obtain a subdivision G of K_d, of which v is one of the principal vertices. This yields in the obvious way a subgraph $\bar{G}$ of skel P, which by Lemma 3 is still a subdivision of K_d. The principal vertices of $\bar{G}$ are images of principal vertices of G, hence adjacent to v, and one of them is $\bar{\alpha} = u$. Adjoining v and the edges linking it to the principal vertices of $\bar{G}$ we obtain the required subdivision of K_{d+1}.

Theorems 1 and 2 also extend to the dual 1-skeleton, by virtue of the next result.

Theorem 3: The dual of an H-lattice is an H-lattice.

Proof: It is enough to show that the dual 1-skeleton of each interval is connected. Since each interval is an H-lattice, everything reduces to proving that skel P^*, the dual 1-skeleton of P is connected . As before, we use Lemma 1 when $d \leq 2$ and induction if $d \geq 3$.

Note that if $\hat{0} \neq x \in P$, skel$[x,\hat{1}]^*$, which is a subgraph of skel P^*, is connected, by induction. Suppose now that skel P^* is disconnected and let (X,Y) be a partition of its vertices such that no member of X is joined to a member of Y by a path. Let $V = \{v \in P \mid r(v) = 0,\ v \leq x \text{ for some } x \in X\}$, and let U be defined similarly with respect to Y; clearly every vertex of P is in V or in U. Suppose that $v \in V$ and $u \in U$ are adjacent in skel P. Choose an $x > v \vee u$ such that $r(x) = d-1$; whithout loss, assume that $x \in X$, and choose a $y \in Y$ such that $y > u$. By connectivity of skel$[u,\hat{1}]^*$, a path links x to y in skel P^*, contradicting the choice of the partition (X,Y). It follows that no member of V is adjacent to a member of U. But this contradicts the hypothesis that skel P is connected. This contradiction proves the Theorem. $\square$

Remarks: The ideas in these proofs can be generalized further:

a) The connectivity of skeleta of all intervals of P is never used in Theorems 1 and 2; it suffices to assume it for intervals of type $[x,\hat{1}]$ and length 3 intervals. It is not clear whether this will admit new examples.

b) One may substitute the rhombus condition in the definition of H-lattice by the requirement that the 1-skeleton of each length 3 interval be 2-connected. This would admit, for instance, matroid lattices. The paths $\bar{p}$ would then not be uniquely defined, but would still satisfy Lemma 3, entailing all the theorems.

c) Barnette's [2] proofs of Theorems 1 and 2 for polytopes could be completely repeated in the present setting, using the combinatorial version of "pulling" given in [7].

d) One notices that the set $v_0, v_1, \ldots, v_d$ of principal vertices of the subdivided K_{d+1} detected by the proof of Theorem 2 satisfies, upon appropriate indexing: for all $0 \leq i \leq j \leq d$, $r(v_0 \vee v_1 \vee \ldots \vee v_{i-1} \vee v_i \vee v_j) = i+1$. Conversely, any such sequence may be specified in advance to be the principal vertices of a subdivided K_{d+1} (this is slightly weaker than the result in [4]). □

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