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Units in some group rings over the ring of p-cyclotomic integers

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Describing the group of units of a group ring is a classical problem. Let p be a rational prime number. We set θ_p a primitive root of unity of order p, $\mathbb{Z}[\theta_p]$ the ring of p-cyclotomic integers, G a finite abelian p-group and $U_1(\mathbb{Z}[\theta_p]G)$ the group of the units u of $\mathbb{Z}[\theta_p]G$ such that $\varepsilon(u) \equiv 1 \bmod (\theta_p - 1)$, where ε is the augmentation map. We will prove that all the elements of the group $U_1(\mathbb{Z}[\theta_p]G)$ arise from the units of the group ring $\mathbb{Z}(C_p \times G)$, where C_p is the cyclic group of order p. As an application, we describe explicitly the group of units of the group ring $\mathbb{Z}[\theta_p]G$ when G is an elementary abelian p-group and $p < 67$ is a regular prime number.

Keywords: Units in group rings; p-groups; p-cyclotomic integers; elementary abelian p-groups.

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1. Introduction

Let us fix some notations. In this paper, p will denote a prime number, θ_p will denote a p-primitive root of unit and G will denote a finite abelian p-group. The symbol ε will denote the augmentation map and $U_1(\mathbb{Z}[\theta_p]G)$ will denote the following group of units:

$$U_1(\mathbb{Z}[\theta_p]G) = \{u \in U(\mathbb{Z}[\theta_p]G) \mid \varepsilon(u) \equiv 1 \bmod (\theta_p - 1)\}.$$

As usual, C_p will denote the group of order p and g will denote a generator of C_p ($C_p = \langle g \rangle$). If R is any ring, then $U(R)$ will denote the group of units of R and if

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H is any group then $U_1(\mathbb{Z}H)$ will denote the group of units v of the integral group ring $\mathbb{Z}H$ such that $\varepsilon(v) = 1$.

Describing the group of units of a group ring is a classical open problem, and a lot of work has been done in this direction (see, for example [6, 7, 10, 14]). A particular problem of interest is to describe a set of generators of the group of units of a group ring or a set of generators of a subgroup of finite index. The problem of describing the group of central units is also of interest in the subject (see, for example [1, 4, 8, 12]).

Some work has also been done in describing the group of units of group rings of type $\mathbb{Z}[\theta_p]G$ (see [3, 13, 15]), but it is not known yet a way to evaluate explicitly a set of generators for the group of units of such group rings.

We state the following theorem.

Theorem 1.1 ([3, Theorem 2.2]). *Let G be a finite abelian group, and let U be the group of units of RG, where R is the ring of integers in an algebraic number field K. If $n \,|\, |G|$, let c_n be the number of elements in G of order n and define $v_n = [K(\theta_n) : K]$, $a_n = c_n v_n^{-1}$ and d_n the rank of the unit group of the ring of algebraic integers of $K(\theta_n)$. Then*

$$U \cong G \times H \times \mathbb{Z}^t,$$

where H is the (cyclic) group of units of finite order in R and $t = \sum a_n d_n$, the sum being taken over all divisors n of $|G|$.

In the case where $R = \mathbb{Z}[\theta_p]$, we have that the group H defined in the theorem above is simply $\langle -1, \theta_p \rangle$. One of our goals in this paper is to provide a complement for $G \times \langle -1, \theta_p \rangle$ in the group of units of the group ring $\mathbb{Z}[\theta_p]G$ for some groups G.

The following units in $\mathbb{Z}[\theta_p]$, defined for $0 \leq i \leq p-2$, are well known:

$$\mu_i = \sum_{j=0}^{i} \theta_p^j$$

and we have that $\mu_i \equiv i + 1 \bmod (\theta_p - 1)$. It might also be useful to the reader to remember that $\mathbb{Z}[\theta_p]/(\theta_p - 1) \cong \mathbb{F}_p$. So, in order to find the whole group of units in $\mathbb{Z}[\theta_p]G$ it is enough to find a complement for $G \times \langle \theta_p \rangle$ in $U_1(\mathbb{Z}[\theta_p]G)$, since every other unit would be a product between a unit $u \in U_1(\mathbb{Z}[\theta_p]G)$ and some μ_i.

In Sec. 2, we will prove a general result that provides a relation between $U_1(\mathbb{Z}C_p \times G)$ and $U_1(\mathbb{Z}[\theta_p]G)$ whenever G is a finite abelian p-group.

In Sec. 3, we will present an application of the result presented in Sec. 2, when G is a finite elementary abelian p-group with $p < 67$ a regular prime, by finding a complete set of independent generators of a complement for $G \times \langle \theta_p \rangle$ in $U_1(\mathbb{Z}[\theta_p]G)$. By "independent set of generators" we mean in this case a finite set of generators $\{g_1, \ldots, g_k\}$, such that $g_1^{r_1} \cdots g_k^{r_k} = 1$ if, and only if $r_j = 0$ for all j.

In Sec. 4, we finish by providing a concrete example and evaluating explicitly a complement for $\langle \theta_5 \rangle \times C_5$ in $U_1(\mathbb{Z}[\theta_5]C_5)$.

2. Main Result

Before we state the next lemma, we need to define the following morphisms, for $a_{i,h}, a_h \in \mathbb{Z}$, $b_h \in \mathbb{Z}[\theta_p]$:

$\sigma : U(\mathbb{Z}(C_p \times G)) \to U(\mathbb{Z}[\theta_p]G)$, given by

$$\sum_{\substack{i=0,\dots,p-1\\ h\in G}} a_{i,h} g^i h \mapsto \sum_{\substack{i=0,\dots,p-1\\ h\in G}} a_{i,h} \theta_p^i h.$$

$\pi : U(\mathbb{Z}(C_p \times G)) \to U(\mathbb{Z}G)$, given by

$$\sum_{\substack{i=0,\dots,p-1\\ h\in G}} a_{i,h} g^i h \mapsto \sum_{\substack{i=0,\dots,p-1\\ h\in G}} a_{i,h} h.$$

$\alpha : U(\mathbb{Z}[\theta_p]G) \to U(\mathbb{F}_pG)$, which takes the coefficients to their classes modulo $\theta_p - 1$:

$$\sum_{h\in G} b_h h \mapsto \sum_{h\in G} \overline{b_h} h.$$

$\rho : U(\mathbb{Z}G) \to U(\mathbb{F}_pG)$, which takes the coefficients to their classes modulo p:

$$\sum_{h\in G} a_h h \mapsto \sum_{h\in G} \overline{a_h} h.$$

Lemma 2.1 ([13, Lemma 2.7]). *In the commutative diagram below, the morphism σ restricted to* $\ker(\pi)$ *is an isomorphism over* $\ker(\alpha)$.

$$\begin{tikzcd} U(\mathbb{Z}C_p \times G) \arrow[r, "\pi"] \arrow[d, "\sigma"'] & U(\mathbb{Z}G) \arrow[d, "\rho"] \\ U(\mathbb{Z}[\theta_p]G) \arrow[r, "\alpha"] & U(\mathbb{F}_pG) \end{tikzcd}$$

Remark 2.2. As a consequence, the same result holds if we consider the same restricted maps in the following diagram:

$$\begin{tikzcd} U_1(\mathbb{Z}C_p \times G) \arrow[r, "\pi"] \arrow[d, "\sigma"'] & U(\mathbb{Z}G) \arrow[d, "\rho"] \\ U_1(\mathbb{Z}[\theta_p]G) \arrow[r, "\alpha"] & U(\mathbb{F}_pG) \end{tikzcd}$$

We are now able to state our main result.

Theorem 2.3. *With the same definitions above, if G is a finite abelian p-group and $u \in U_1(\mathbb{Z}[\theta_p]G)$, then there is $\tilde{u} \in U_1(\mathbb{Z}C_p \times G)$ such that $u = \sigma(\tilde{u})$.*

Proof. By Remark 2.2, we consider from now on that the domain of σ is $U_1(\mathbb{Z}C_p \times G)$. Therefore, we must prove that σ is surjective.

Let $u \in U_1(\mathbb{Z}[\theta_p]G)$. We shall prove that $u \in \mathrm{Im}(\sigma)$.

Consider $\eta : \mathbb{Z}[\theta_p]G \to \mathbb{Z}[\theta_p]G$ the $\mathbb{Z}[\theta_p]$-linear morphism induced by $h \mapsto h^p$, for all $h \in G$.

First, we prove that $\eta(u)u^{-p} \in \mathrm{Im}(\sigma)$: in fact, we can write

$$u = \sum_{h \in G} a_h h,$$

where $a_h \in \mathbb{Z}[\theta_p]$, for all h. Thus,

$$\alpha(u^p) = \alpha(u)^p = \left(\sum_{h \in G} \overline{a_h} h \right)^p,$$

as $\mathbb{F}_p G$ is a commutative ring of characteristic p, we have

$$\alpha(u^p) = \sum_{h \in G} \overline{a_h^p} h^p = \sum_{h \in G} \overline{a_h} h^p = \alpha(\eta(u)).$$

Therefore, $\eta(u)u^{-p} \in \ker(\alpha)$ and by Lemma 2.1, we have that $\eta(u)u^{-p} \in \mathrm{Im}(\sigma)$.

Now we define, for $1 \leq s \leq p-1$, the $\mathbb{Z}$-linear morphisms $\psi_s : \mathbb{Z}[\theta_p]G \to \mathbb{Z}[\theta_p]G$ given by $\theta_p \mapsto \theta_p^s$ that fix the elements of G - that is, the maps are induced by $\mathrm{Gal}(\mathbb{Q}(\theta_p)/\mathbb{Q})$ on $\mathbb{Z}[\theta_p]G$.

Thus, we have that $u\psi_s(u)^{-1} \in \ker(\alpha)$, since $\alpha(u) = \alpha(\psi_s(u))$, therefore, by Lemma 2.1 we have that $u\psi_s(u)^{-1} \in \mathrm{Im}(\sigma)$, for all $1 \leq s \leq p-1$.

Now we define the following unit:

$$v = \prod_{s=1}^{p-1} \psi_s(u) \in \mathbb{Z}[\theta_p]G.$$

We have that $\psi_s(v) = v$, for all $1 \leq s \leq p-1$. Then, if we write v as

$$v = \sum_{i=0}^{p-2} z_i \theta_p^i,$$

where $z_i \in \mathbb{Z}G$, we have:

$$v = \psi_s(v) = \sum_{i=0}^{p-2} z_i \theta_p^{si}.$$

We can write $z_j = \sum_{h \in G} a_{j,h} h$, with $a_{j,h} \in \mathbb{Z}$, and we have, for all $h \in G$, $1 \leq s \leq p-1$:

$$\sum_{i=0}^{p-2} a_{i,h} \theta_p^i = \sum_{i=0}^{p-2} a_{i,h} \theta_p^{si} = \psi_s \left(\sum_{i=0}^{p-2} a_{i,h} \theta_p^i \right),$$

that is, $\sum_{i=0}^{p-2} a_{i,h} \theta_p^i$ (the coefficient in $\mathbb{Z}[\theta_p]$ of h of the unit v) is fixed by the action of the group $\mathrm{Gal}(\mathbb{Q}(\theta_p)/\mathbb{Q})$, and then we have, by the correspondence given in the Fundamental Theorem of Galois Theory, that

$$\sum_{i=0}^{p-2} a_{i,h} \theta_p^i \in \mathbb{Q},$$

then $v \in \mathbb{Z}G \subset \mathbb{Z}(C_p \times G)$, and we obtain $v \in \mathrm{Im}(\sigma)$ (since $v = \sigma(v)$ — in this equation, in the left-hand side we are considering $v \in \mathbb{Z}[\theta_p]G$, and in the right-hand side we are considering $v \in \mathbb{Z}(C_p \times G)$).

Then we have $v \in \mathrm{Im}(\sigma)$, $\eta(u)u^{-p} \in \mathrm{Im}(\sigma)$ and for all $1 \leq s \leq p-1$, we have $u\psi_s(u)^{-1} \in \mathrm{Im}(\sigma)$. With this information in mind, we obtain

$$\prod_{s=1}^{p-1} u\psi_s(u)^{-1} = u^{p-1}v^{-1} \in \mathrm{Im}(\sigma),$$

as $v, v^{-1} \in \mathrm{Im}(\sigma)$, we have $u^{p-1} \in \mathrm{Im}(\sigma)$, and since $\eta(u)u^{-p} \in \mathrm{Im}(\sigma)$, then we obtain

$$u^{-1}\eta(u) \in \mathrm{Im}(\sigma).$$

Applying the above for $\eta^{n-1}(u)$, we obtain, for every positive integer n:

$$(\eta^{n-1}(u))^{-1}(\eta^n(u)) \in \mathrm{Im}(\sigma). \tag{1}$$

Let e be an integer such that p^e is the exponent of G. We have $\eta^e(u) \in \mathbb{Z}[\theta_p]$ and $\eta^e(u) \equiv 1 \bmod (\theta_p - 1)$, then $\eta^e(u) \in \ker(\alpha)$ and by Lemma 2.1, we have that $\eta^e(u) \in \mathrm{Im}(\sigma)$, and we conclude by (1) that $\eta^{e-1}(u) \in \mathrm{Im}(\sigma)$.

Using this inclusion and (1) inductively, we conclude that $\eta(u) \in \mathrm{Im}(\sigma)$. Since we know that $u^{-1}\eta(u) \in \mathrm{Im}(\sigma)$, we conclude that $u \in \mathrm{Im}(\sigma)$, as desired. $\square$

With this result, when provided a set of generators of $U_1(\mathbb{Z}(C_p \times G))$, we are able to describe explicitly a set of generators of the group $U_1(\mathbb{Z}[\theta_p]G)$.

3. Application to Finite Elementary Abelian p-Groups

First we note that in [5], Ferraz constructed an independent set of units that complements $\langle g \rangle = C_p$ in the group $U_1(\mathbb{Z}C_p)$ for all primes $p < 67$.

We denote $(C_p)^n$ the direct product of n copies of C_p. If p is a regular prime number (for example, all prime numbers $p < 37$ are regular — see [2]), then the following result is known.

Theorem 3.1 ([9]). *Let p be a regular prime number. Then, for every natural n, the elements of $U_1(\mathbb{Z}(C_p)^n)$ are all generated by the units of subrings $\mathbb{Z}C$, where C runs over the set of cyclic subgroups of $(C_p)^n$ of order p.*

From now on we will consider p a regular prime number and $G = (C_p)^n$.

Let us fix $C_p = \langle g \rangle$ and $\{u_i(g)\}_{1 \leq i \leq \frac{p-3}{2}}$ an independent set of units in $\mathbb{Z}C_p$ that generates a complement for $\langle g \rangle$ in $U_1(\mathbb{Z}C_p)$.

If $u_i(g) = \sum_{i=0,\ldots,p-1} a_i g^i$, with $a_i \in \mathbb{Z}$ and x is any element of any group ring RG such that $\mathbb{Z} \subset R$ and $x^p = 1$, we define the unit: $u_i(x) = \sum_{i=0,\ldots,p-1} a_i x^i \in RG$.

Also, let us write $G = \langle g \rangle \times G_2$, where $G_2 \cong (C_p)^{n-1}$. By Theorem 4.5 of [11], we know that $U_1(\mathbb{Z}G) \cong G \times F$, where F is a free abelian group of rank $\frac{1}{2}(p-3)l$, with $l = \frac{p^n-1}{p-1}$ being the number of nontrivial cyclic subgroups of G. Let us denote $l_2 = \frac{p^{n-1}-1}{p-1}$ the number of nontrivial cyclic subgroups of G_2 and for $1 \leq i \leq l_2$, we fix $\langle c_i \rangle$ the nontrivial cyclic subgroups of G_2 (each one generated by a certain c_i).

Thus, by Theorems 3.1 and 1.1, we have that the following set generates such F and is independent (by comparing the rank given in Theorem 1.1):

$$\mathcal{B} = \left(\bigcup_{i,j} \{u_1(g^j c_i), \ldots, u_{\frac{p-3}{2}}(g^j c_i)\} \right) \cup \{u_1(g), \ldots, u_{\frac{p-3}{2}}(g)\}, \tag{2}$$

with $0 \leq j \leq p-1, 1 \leq i \leq l_2$.

Now we consider the map $\sigma : U_1(\mathbb{Z}G) = U_1(\mathbb{Z}(\langle g \rangle \times G_2)) \to U_1(\mathbb{Z}[\theta_p]G_2)$ as before ($g \mapsto \theta_p$).

We will find the kernel of σ. First, we need the following lemma.

Lemma 3.2. *If $c \in G_2$ is a nontrivial element and $u(c) = \sum_{i=0,\ldots,p-1} a_i c^i \in U_1(\mathbb{Z}\langle c \rangle)$, with $a_i \in \mathbb{Z}$, then*

$$u(c)u(\theta_p c) \ldots u(\theta_p^{p-1} c) = 1.$$

Proof. First, let $\psi : \mathbb{Q}[\theta_p]G_2 \to R = \bigoplus_i (R_i)$ be a $\mathbb{Q}(\theta_p)$-linear isomorphism given by the Wedderburn–Artin theorem, where $R_i \cong \mathbb{Q}[\theta_p]$ are the simple components, for all i. We have that $\theta_p^j c \in \mathbb{Z}[\theta_p]G$ is mapped to a cyclotomic integer x on each simple component of R, and we get $x^p = 1$. Then, on each component, we have that there exists an integer $0 \leq k \leq p-1$ such that $\theta_p^j c \mapsto \theta_p^k$ on that component.

We fix any simple component R_i. If for all j we have that $\theta_p^j c$ is mapped to 1 in the fixed component, then it is clear that in this component we have that $u(c)u(\theta_p c) \ldots u(\theta_p^{p-1} c)$ is mapped to 1, since $u(c)$ has augmentation 1. Now suppose there is j such that $\theta_p^j c$ is mapped to θ_p^k, for a certain $0 < k \leq p-1$, in the fixed component. In this case, we have that $u(c)u(\theta_p c) \ldots u(\theta_p^{p-1} c)$ is mapped to $u(\theta_p^{k-j})u(\theta_p^{k-j+1}) \ldots u(\theta_p^{k-j+p-1}) = N_{\mathbb{Q}(\theta_p)|\mathbb{Q}}(u(\theta_p^k)) = 1$, where $N_{\mathbb{Q}(\theta_p)|\mathbb{Q}} : \mathbb{Q}(\theta_p) \to \mathbb{Q}$ is the field norm.

Therefore, on each simple component the product $u(c_i)u(\theta_p c_i) \ldots u(\theta_p^{p-1} c_i)$ is mapped to 1, and this proves the lemma. □

Corollary 3.3. *Let $\sigma : U_1(\mathbb{Z}G) = U_1(\mathbb{Z}(\langle g \rangle \times G_2)) \to U_1(\mathbb{Z}[\theta_p]G_2)$ as defined above, then*

$$\ker(\sigma) = \left\langle u_j(c_i)u_j(gc_i) \ldots u_j(g^{p-1}c_i) | 1 \leq i \leq (l_2 - 1),\ 1 \leq j \leq \frac{p-3}{2} \right\rangle.$$

Proof. By Lemma 3.2, we have that $u_j(c_i)u_j(gc_i) \ldots u_j(g^{p-1}c_i) \in \ker(\sigma)$, and as we noted above, these units form an independent set of units. Then we conclude that these units generate a group of rank $\frac{(l_2-1)(p-3)}{2}$. We also have that $\operatorname{rank}(U_1(\mathbb{Z}G)) = \frac{1}{2}(p-3)l$ and $\operatorname{rank}(U_1(\mathbb{Z}[\theta_p]G_2)) = \frac{1}{2}(p-3)p^{n-1}$, by Theorem 1.1.

Now, as σ is surjective, we have that $\operatorname{rank}(U_1(\mathbb{Z}G)) - \operatorname{rank}(U_1(\mathbb{Z}[\theta_p]G_2)) = \operatorname{rank}(\ker(\sigma))$, and we can easily verify that this applies to the set we presented.

So we need only to note that $\ker(\sigma)$ is torsion-free, and this is true, since the only torsion-elements of $U_1(\mathbb{Z}G)$ are the elements of G, none of which is in the kernel of σ. □

Now we are able to describe an independent set of units of $U_1(\mathbb{Z}[\theta_p]G_2)$ that complements $G_2 \times \langle\theta_p\rangle$.

Corollary 3.4. *With the notations we are using, the set S below is independent and generates a complement for $G_2 \times \langle\theta_p\rangle$ in $U_1(\mathbb{Z}[\theta_p]G_2)$.*

$$S := \left(\bigcup_{1\le k\le \frac{p-3}{2}} \{u_k(\theta_p)\}\right) \cup \left(\bigcup_{\substack{1\le k\le \frac{p-3}{2}\\ 1\le i\le l_2\\ 1\le j\le p-1}} \{u_k(\theta_p^j c_i)\}\right)$$

Proof. The proof of the above corollary is immediate from Corollary 3.3 and by considering the set $\mathcal{B}$ that we defined in (2), since the elements of S are simply σ applied to elements of $\mathcal{B}$ that are not in $\ker(\sigma)$, the elements of $\ker(\sigma)$ are generated by elements of $\mathcal{B}$ and σ is surjective. Finally, we can conclude independence by comparing with the rank given in Theorem 1.1. □

The units described in [5] for any regular prime number $p < 67$ and Corollary 3.4 describe fully the set S for such primes.

4. A Concrete Example: $\mathbb{Z}[\theta_5]C_5$

We finish by illustrating the results with a concrete example: we will describe an independent set of units that complements $C_5 \times \langle\theta_5\rangle$ in $U_1(\mathbb{Z}[\theta_5]C_5)$. Let us consider the unit $u(g) = -1 + g + g^4$ that generates a complement for C_5 in $U_1(\mathbb{Z}C_5)$ (see [5]). Then, by Corollary 3.4 we have that the following set is independent and complements $C_5 \times \langle\theta_5\rangle$ in $U_1(\mathbb{Z}[\theta_5]C_5)$:

$$\begin{aligned}&\{u(\theta_p), u(\theta_p g), u(\theta_p^2 g), u(\theta_p^3 g), u(\theta_p^4 g)\}\\ &\quad = \{-1 + \theta_p + \theta_p^4, -1 + \theta_p g + \theta_p^4 g^4, -1 + \theta_p^2 g + \theta_p^3 g^4,\\ &\qquad -1 + \theta_p^3 g + \theta_p^2 g^4, -1 + \theta_p^4 g + \theta_p g^4\}.\end{aligned}$$

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