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# Construction of Invariant Relations of *n* Symmetric Second-Order Tensors

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*Dedicated to the 85th birthday of Professor Roger Fosdick*

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**Abstract**
A methodology is presented to find either implicit or explicit relations, called syzygies, between invariants in a minimal integrity basis for $n$ symmetric second-order tensors defined on a three-dimensional euclidean space. The methodology i) yields explicit non-polynomial expressions for certain invariants in terms of the remaining invariants in the integrity basis and ii) allows the construction of the implicit relations. The results of this investigation are important in modeling biological structures, which, in general, are non-homogeneous and made of anisotropic viscoelastic materials that are subjected to large deformations and are modeled through constitutive relations that depend on symmetric tensors.

**Keywords** Mechanics of materials · Biological structure · Response function · Second-order tensor · Syzygy

**Mathematics Subject Classification (2020)** 74B20 · 74A20

## 1 Introduction

The invariant theory has a long history that dates back to the XIX century (Popov [17]). The application of this theory in continuum mechanics is due largely to the works of Reiner [18, 19], Rivlin [21, 22], and Rivlin and Ericksen [23], who have used this theory to obtain constitutive relations of isotropic solid and liquid materials. Since then, the invariant theory has been used to obtain constitutive relations of a wide class of materials, such as bones (Kichenko *et al.* [11]), ligaments (Pioletti *et al.* [16]), arteries (Balzani *et al.* [2]), myocardium (Holzapfel and Ogden [8]), wood (Mackenzie-Helnwein *et al.* [12]), soil (Zhao

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and Guo [31]), ceramics (Schröder and Romanowski [24]), polymers (Richards and Odegard [20]), viscoelastic materials (Wineman [30]), electroelastic materials (Bustamante and Rajagopal [3]), etc.

Recent mathematical developments of the theory are reported by Kemper [9] and [10], who have worked on optimal homogeneous systems of parameters and separating sets, Olive and Auffray [13] and Chen *et al.* [4], on isotropic invariants of third-order tensors, Olive *et al.* [15], Desmorat *et al.* [6], Desmorat *et al.* [7], on minimal integrity basis and separating set for the fourth-order elasticity tensor, and Olive and Desmorat [14], on effectivce rationality of second-order symmetric tensors. Our approach is based on the invariant theory found in classical texts of mechanics, such as Spencer [29], Smith [28], and Zheng [32].

The application of invariance principles in continuum mechanics leads to the proposition of constitutive relations that depend on a list of invariants of physical variables, such as vectors and second-order tensors. Given a group of transformations acting on these variables, the central problem of the associated theory of invariants is to find a list of invariants from which all the other invariants can be generated without having redundant members. In this work, the invariants are real-valued polynomial functions of their arguments and this list is called an integrity basis if any invariant can be expressed as a polynomial of the members in the list. The integrity basis is minimal if it contains the smallest possible number of members.

The construction of minimal integrity bases in continuum mechanics has been the subject of intense investigation since the 1950s, an account of which can be found in Spencer [29], and is, by now, well established. Concerning symmetric second-order tensors defined on the three-dimensional euclidean space, Spencer [29] presents a thorough analysis about the construction of the minimum integrity basis for a finite number $n$ of these tensors. As noted in Sect. 2, the number of elements in this basis increases rapidly with $n$ (see also Olive and Desmorat [14]), which represents a serious limitation for the application of the corresponding results in practice.

The members of a minimal integrity basis may satisfy polynomial relations between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder. These relations may be explicit, in which case an invariant may be expressed as a rational function of the remainder, or, implicit. If the invariants do not depend on the other invariants through any type of relation, implicit or explicit, they are called independent. The number of independent variables in a minimal integrity basis is $6n-3$ (see, for instance, da Rocha and Aguiar [5], Aguiar and da Rocha [1], Shariff [25, 26], and Shariff *et al.* [27]).

The determination of the number of syzygies in a minimal integrity basis together with the construction of these syzygies continues to be an active area of research. In terms of applications in continuum mechanics, syzygies provide additional expressions which the invariants of the constitutive relations must satisfy. Thus, in case the values of the invariants are obtained experimentally, the syzygies can be used to verify the accuracy between the experimental values and the corresponding theoretical values of the invariants.

In this work, we propose a methodology, based on the works of Smith [28] and Zheng [32], that i) yields explicit non-polynomial expressions for certain invariants in terms of the remaining invariants in the integrity basis and ii) allows the construction of implicit relations between the invariants. To obtain these results, the first step in the methodology is to construct the set of $6n-3$ independent invariants. In Sect. 2 we present some preliminary results, which include the total number of invariants in a minimal integrity basis for $n$ symmetric tensors. In Sect. 3 we investigate the cases $n=1,\dots,5$, and then generalize the results for $n>5$. In Sect. 4 we present some concluding remarks.

## 2 Preliminaries

We are concerned with symmetric second-order tensors in three-dimensional euclidean space. The summation convention is not used and indices of tensor components take the values 1, 2, 3.

Let us consider that all the symmetric tensors in the set $\mathcal{A}^{(n)} \stackrel{\text{def}}{=} \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}\}$, where $n \geq 1$, have three distinct eigenvalues and that no two tensors in this set do not have parallel eigenvectors. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the set of eigenvectors of $\mathbf{A}^{(1)}$ with associated eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, we write

$$\mathbf{A}^{(1)} = \sum_{i=1}^{3} \lambda_i \, \mathbf{e}_i \otimes \mathbf{e}_i \,, \qquad \mathbf{A}^{(r)} = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_{ij}^{(r)} \mathbf{e}_i \otimes \mathbf{e}_j \,, \qquad r = 2, \ldots, n \,, \tag{1}$$

where

$$\alpha_{ij}^{(r)} \stackrel{\text{def}}{=} \mathbf{e}_i \cdot \mathbf{A}^{(r)} \, \mathbf{e}_j \,, \quad i, j = 1, 2, 3 \,, \tag{2}$$

are the six components of the tensor $\mathbf{A}^{(r)}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. It is then clear from (1) that the maximum number of distinct components of all the tensors in the set $\mathcal{A}^{(n)}$ is $6n - 3$. Also, the trace of $\mathbf{A}^{(r)}$ is given by

$$\operatorname{tr} \mathbf{A}^{(1)} = \lambda_1 + \lambda_2 + \lambda_3 \,, \qquad \operatorname{tr} \mathbf{A}^{(r)} = \sum_{i=1}^{3} \alpha_{ii}^{(r)} \,, \quad r = 2, \ldots, n \,. \tag{3}$$

Now, let $\mathbf{Q}$ be a second-order orthogonal tensor, such that $\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$, where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$ and $\mathbf{1}$ is the second-order identity tensor. We say that a real-valued function $f : \mathcal{A}^{(n)} \to \mathbb{R}$ is an invariant of the tensors in $\mathcal{A}^{(n)}$ under the group of second-order orthogonal tensors if

$$f\big(\bar{\mathbf{A}}^{(1)}, \bar{\mathbf{A}}^{(2)}, \ldots, \bar{\mathbf{A}}^{(n)}\big) = f\big(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(n)}\big) \,, \quad \bar{\mathbf{A}}^{(r)} \stackrel{\text{def}}{=} \mathbf{Q}\mathbf{A}^{(r)}\mathbf{Q}^T \,, \tag{4}$$

for every $\mathbf{Q}$ of the orthogonal group. From classical invariant theory,[1] all the invariants of second-order tensors only can be expressed in terms of traces of products of the tensors in $\mathcal{A}^{(n)}$. We consider invariants that are real-valued polynomial functions in their arguments.

The central problem of the theory of invariants, as it applies to this work, is to determine a set of polynomial invariants from which all the other polynomial invariants can be generated and which contains the smallest possible number of members. In our work, this set is the minimal integrity basis for the set $\mathcal{A}^{(n)}$ under orthogonal transformations and is completely characterized in Spencer [1971]. For completeness, in Table 2 of Appendix A we present the matrix products whose traces generate the minimal integrity basis for $\mathcal{A}^{(n)}$. Observe from this table that the total number of invariants of $\mathcal{A}^{(n)}$, $n > 0$, is given by

$$N(n) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \beta_i \binom{n}{i} \,, \tag{5}$$

where $\beta_i$ is an integer given in the first column of Table 2 and $\binom{n}{i} \stackrel{\text{def}}{=} \frac{n!}{i!(n-i)!}$ is the binomial coefficient. In particular, if $n \geq 6$, we then have that $N(n) = 3n + 4\binom{n}{2} + 7\binom{n}{3} + 20\binom{n}{4} +$

[1] See Spencer [1971] for an application of this theory in continuum mechanics.

$26\binom{n}{5}+10\binom{n}{6}$, which shows that $N(n)$ increases rapidly with $n$. This expression is presented in Table 1 of Olive and Desmorat [14].[2]

Based on the works of Smith [28] and Zheng [32], we present below a methodology to find either implicit or explicit relations between invariants in the minimal integrity basis for $\mathcal{A}^{(n)}$. The methodology i) yields explicit non-polynomial expressions for certain invariants in terms of the remaining invariants in the integrity basis, and ii) allows the construction of implicit relations between the invariants. To obtain these results, the first step is to construct the set of $6n-3$ independent invariants, which is included here for completeness of presentation.

## 3 Methodology

We investigate the cases $n=1,\dots,5$, and then generalize for $n>5$. Similarly to the works of Smith [28] and Zheng [32], the basic idea is to construct a bijection between subsets of invariants and sets of combinations of products of the components $\alpha_{ij}^{(r)}$ defined in (2). Based on this bijection, we clearly identify the independent invariants and syzygies between elements of the subsets of invariants.

### 3.1 The Case $n=1$

**Claim** *All the* 3 *classical invariants are independent.*

***Proof*** The three invariants

$$I_1^1 \overset{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}, \quad I_2^1 \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2, \quad I_3^1 \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^3 \tag{6}$$

depend on the three eigenvalues $\lambda_i$, $i=1,2,3$, which are independent variables. In addition, these variables are roots of the characteristic equation $\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$, where $J_1 \overset{\text{def}}{=} I_1^1$, $J_2 \overset{\text{def}}{=} [(I_1^1)^2 - I_2^1]/2$, $J_3 \overset{\text{def}}{=} \det\mathbf{A}^{(1)} = (I_3^1 - I_1^1\, I_2^1 + I_1^1\, J_2)/3$. It is well known that this characteristic equation yields three real-valued expressions for $\lambda$ in terms of the invariants $J_i$, $i=1,2,3$, and, in view of (6), in terms of the invariants $I_i^1$, $i=1,2,3$. Thus, the three invariants in (6) are independent. □

### 3.2 The Case $n=2$

**Claim** *From* $N(2)=10$ *classical invariants*, 9 *invariants are independent and the remaining invariant satisfies a syzygy.*

***Proof*** In addition to the 3 invariants in (6), the minimal integrity basis for 2 symmetric tensors contains the invariants (Appendix A)

$$\begin{gathered} I_1^2 \overset{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(2)}, \quad I_2^2 \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(2)}\big)^2, \quad I_3^2 \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(2)}\big)^3, \quad I_1^{12} \overset{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\mathbf{A}^{(2)}, \\ I_2^{12} \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2\mathbf{A}^{(2)}, \quad I_3^{12} \overset{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\big(\mathbf{A}^{(2)}\big)^2, \quad I_4^{12} \overset{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2\big(\mathbf{A}^{(2)}\big)^2. \end{gathered} \tag{7}$$

Thus, in accordance to (5), the total number of invariants is $N(2)=10$.

[2] In this table, for $n=5$, instead of 261, it should be 251. See Table 1 of our work.

It follows from (1) and (2) that the tensors $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are uniquely determined by the nine components $\lambda_i$, $\alpha_{ij}^{(2)}$, $i, j = 1, 2, 3$, $j > i$, and, therefore, the 10 invariants in both (6) and (7) are given in terms of these components. In fact, each invariant in the set $\Psi \stackrel{\text{def}}{=} \{I_1^1, I_2^1, I_3^1, I_1^2, I_2^2, I_3^2, I_1^{12}, I_2^{12}, I_3^{12}, I_4^{12}\}$ can be expressed as a polynomial of elements in the set $\Omega \stackrel{\text{def}}{=} \{\lambda_1, \lambda_2, \lambda_3, \alpha_{11}^{(2)}, \alpha_{22}^{(2)}, \alpha_{33}^{(2)}, (\alpha_{12}^{(2)})^2, (\alpha_{23}^{(2)})^2, (\alpha_{13}^{(2)})^2, \alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\}$ (see Appendix B). In addition, there is a bijection between the sets $\Psi$ and $\Omega$. To show this, we only need to express the elements of $\Omega$ in terms of the elements of $\Psi$, which we accomplish by following the steps below.

- **a)** In view of Sect. 3.1, the components $\lambda_1$, $\lambda_2$, $\lambda_3$ are given in terms of the invariants $I_1^1$, $I_2^1$, $I_3^1$.
- **b)** The components $\alpha_{11}^{(2)}, \alpha_{22}^{(2)}, \alpha_{33}^{(2)}$ are obtained from the solution of a system of linear equations obtained from the expressions of $I_1^2$, $I_1^{12}$, $I_2^{12}$, where $\lambda_1$, $\lambda_2$, $\lambda_3$ were determined in Step **a)**.
- **c)** The terms $(\alpha_{12}^{(2)})^2$, $(\alpha_{23}^{(2)})^2$, $(\alpha_{13}^{(2)})^2$ are obtained from the solution of a system of linear equations obtained from the expressions of $I_2^2$, $I_3^{12}$, $I_4^{12}$, where $\lambda_i$, $\alpha_{ii}$, $i = 1, 2, 3$, were determined in steps **a)** and **b)**.
- **d)** The term $\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}$ is obtained from the expression of $I_3^2$, the other terms in this expression having been determined in the previous steps.

In this way, we have shown that there is a bijection between the sets $\Psi$ and $\Omega$.

Since

$$\left(\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\right)^2 = \left(\alpha_{12}^{(2)}\right)^2\left(\alpha_{23}^{(2)}\right)^2\left(\alpha_{13}^{(2)}\right)^2 \tag{8}$$

and since the integrity basis is minimal and contains the 10 invariants given by both (6) and (7), it follows from steps **a)–d)** above that relation (8) yields a syzygy between $I_3^2$ and the other invariants. Since the elements of the set $\Omega_I \stackrel{\text{def}}{=} \Omega \backslash \{\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\}$ are independent, the claim is proved. □

### 3.3 The Case $n = 3$

**Claim** *From* $N(3) = 28$ *classical invariants*, 15 *invariants are independent*, 11 *invariants satisfy, at least*, 19 *syzygies, and the remaining* 2 *invariants are rational functions of the other invariants.*

***Proof*** In addition to the 10 invariants defined in the expressions (6) and (7), the integrity basis for 3 symmetric tensors also have the 18 invariants given by $I_i^3$, $i = 1, 2, 3$, defined similarly to (6), $I_j^q$, $j = 1, \dots, 4$, $q = 13, 23$, defined similarly to (7), and (Appendix A)

$$\begin{aligned}
&I_1^{123} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\mathbf{A}^{(2)}\mathbf{A}^{(3)}, && I_2^{123} \stackrel{\text{def}}{=} \operatorname{tr}\left(\mathbf{A}^{(1)}\right)^2\mathbf{A}^{(2)}\mathbf{A}^{(3)},\\
&I_3^{123} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\left(\mathbf{A}^{(2)}\right)^2\mathbf{A}^{(3)}, && I_4^{123} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\mathbf{A}^{(2)}\left(\mathbf{A}^{(3)}\right)^2,\\
&I_5^{123} \stackrel{\text{def}}{=} \operatorname{tr}\left(\mathbf{A}^{(1)}\right)^2\left(\mathbf{A}^{(2)}\right)^2\mathbf{A}^{(3)}, && I_6^{123} \stackrel{\text{def}}{=} \operatorname{tr}\left(\mathbf{A}^{(1)}\right)^2\mathbf{A}^{(2)}\left(\mathbf{A}^{(3)}\right)^2,\\
&I_7^{123} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\left(\mathbf{A}^{(2)}\right)^2\left(\mathbf{A}^{(3)}\right)^2.
\end{aligned} \tag{9}$$

Thus, in accordance to (5), the total number of invariants is given by $N(3) = 28$.

As in Sect. 3.2, it follows from (1) and (2) that the tensors $\mathbf{A}^{(i)}$, $i=1,2,3$, are uniquely determined by the 15 components $\lambda_i$, $\alpha_{ij}^{(2)}$, $\alpha_{ij}^{(3)}$, $i,j=1,2,3,\ j>i$, and, therefore, the 28 invariants defined through (6), (7), and (9) are given in terms of these components.

Let the set of 28 invariants be given by $\Psi \overset{\text{def}}{=} \{I_i^p, I_j^q, I_k^{123}\}$, $i,p=1,2,3,\ j=1,\dots,4,$ $q=12,23,13,\ k=1,\dots,7$, and, analogous to Section (3.2), let $\Omega$ be a set of products between the components of the symmetric tensors in $\mathcal{A}^{(3)}$, which will be determined from these invariants by following the steps below.

**a)** Following the steps **a)** thru **d)** in Sect. 3.2, we find that there is a bijection between the set of 17 invariants given by $\Psi_a \overset{\text{def}}{=} \{I_i^1\} \cup \{I_i^p, I_j^{1p}\}$, $i=1,2,3,\ j=1,\dots,4,\ p=2,3$, and the set of 17 terms given by $\Omega_a \overset{\text{def}}{=} \{\lambda_k, \alpha_{kk}^{(p)}, (\alpha_{kl}^{(p)})^2, \alpha_{12}^{(p)}\alpha_{23}^{(p)}\alpha_{13}^{(p)}\}$, $k,l=1,2,3,\ l>k$, $p=2,3$. Not all the terms in the set $\Omega_a$ are independent, since the terms $\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}$, $\alpha_{12}^{(3)}\alpha_{23}^{(3)}\alpha_{13}^{(3)}$ satisfy 2 relations, given by, respectively, the relation (8) and

$$\big(\alpha_{12}^{(3)}\alpha_{23}^{(3)}\alpha_{13}^{(3)}\big)^2 = \big(\alpha_{12}^{(3)}\big)^2\big(\alpha_{23}^{(3)}\big)^2\big(\alpha_{13}^{(3)}\big)^2. \tag{10}$$

**b)** The 3 invariants in the set $\Psi_b \overset{\text{def}}{=} \{I_1^{23}, I_1^{123}, I_2^{123}\}$ depend linearly on the 3 elements in the set $\Omega_b \overset{\text{def}}{=} \{\alpha_{12}^{(2)}\alpha_{12}^{(3)}, \alpha_{13}^{(2)}\alpha_{13}^{(3)}, \alpha_{23}^{(2)}\alpha_{23}^{(3)}\}$ and, therefore, each element in the set $\Omega_b$ can be obtained from the solution of a system of linear equations obtained from the expressions of the invariants in $\Psi_b$. Thus, there is a bijection between the sets $\Psi_b$ and $\Omega_b$. The terms in $\Omega_b$ are not independent. They must satisfy the 4 relations

$$\begin{aligned}&\big(\alpha_q^{(2)}\alpha_q^{(3)}\big)^2 = \big(\alpha_q^{(2)}\big)^2\big(\alpha_q^{(3)}\big)^2, \quad q=12,23,13,\\ &\big(\alpha_{12}^{(2)}\alpha_{12}^{(3)}\big)\big(\alpha_{23}^{(2)}\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\alpha_{13}^{(3)}\big) = \big(\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\big)\big(\alpha_{12}^{(3)}\alpha_{23}^{(3)}\alpha_{13}^{(3)}\big),\end{aligned} \tag{11}$$

where $(\alpha_{ij}^{(2)})^2$, $(\alpha_{ij}^{(3)})^2$, $i=1,2,3,\ j>i$, $\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}$, and $\alpha_{12}^{(3)}\alpha_{23}^{(3)}\alpha_{13}^{(3)}$ were obtained in Step **a)**. The bijection between the sets $\Psi_b$ and $\Omega_b$ then implies that the elements of $\Psi_b$ are not independent and must satisfy syzygies obtained from the expressions in (11).

**c)** Analogously, each invariant in the set $\Psi_c \overset{\text{def}}{=} \{I_2^{23}, I_3^{123}, I_5^{123}\} \cup \{I_3^{23}, I_4^{123}, I_6^{123}\}$ depends linearly on the elements in the set $\Omega_c \overset{\text{def}}{=} \{\alpha_{12}^{(2)}\alpha_{13}^{(2)}\alpha_{23}^{(3)}, \alpha_{13}^{(2)}\alpha_{23}^{(2)}\alpha_{12}^{(3)}, \alpha_{23}^{(2)}\alpha_{12}^{(2)}\alpha_{13}^{(3)}\} \cup \{\alpha_{12}^{(2)}\alpha_{13}^{(3)}\alpha_{23}^{(3)}, \alpha_{13}^{(2)}\alpha_{23}^{(3)}\alpha_{12}^{(3)}, \alpha_{23}^{(2)}\alpha_{12}^{(3)}\alpha_{13}^{(3)}\}$ and, therefore, the elements in $\Omega_c$ can be given in terms of the invariants in $\Psi_c$ by solving a system of linear equations. Thus, there is a bijection between the sets $\Psi_c$ and $\Omega_c$. The terms in $\Omega_c$ are not independent and must satisfy the 13 relations

$$\begin{aligned}&\big(\alpha_{12}^{(2)}\alpha_{13}^{(2)}\alpha_{23}^{(3)}\big)^2 = \big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{13}^{(2)}\big)^2\big(\alpha_{23}^{(3)}\big)^2\ (*),\\ &\big(\alpha_{12}^{(2)}\alpha_{13}^{(3)}\alpha_{23}^{(3)}\big)^2 = \big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{13}^{(3)}\big)^2\big(\alpha_{23}^{(3)}\big)^2\ (*),\\ &\big(\alpha_{12}^{(2)}\alpha_{13}^{(2)}\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\alpha_{23}^{(2)}\alpha_{12}^{(3)}\big)\big(\alpha_{23}^{(2)}\alpha_{12}^{(2)}\alpha_{13}^{(3)}\big) =\\ &\qquad \big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{13}^{(2)}\big)^2\big(\alpha_{23}^{(2)}\big)^2\big(\alpha_{12}^{(3)}\alpha_{23}^{(3)}\alpha_{13}^{(3)}\big) =\\ &\qquad \big(\alpha_{12}^{(2)}\alpha_{12}^{(3)}\big)\big(\alpha_{23}^{(2)}\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\alpha_{13}^{(3)}\big)\big(\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\big),\\ &\big(\alpha_{12}^{(2)}\alpha_{13}^{(3)}\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\alpha_{23}^{(3)}\alpha_{12}^{(3)}\big)\big(\alpha_{23}^{(2)}\alpha_{12}^{(3)}\alpha_{13}^{(3)}\big) =\\ &\qquad \big(\alpha_{12}^{(3)}\big)^2\big(\alpha_{13}^{(3)}\big)^2\big(\alpha_{23}^{(3)}\big)^2\big(\alpha_{12}^{(2)}\alpha_{23}^{(2)}\alpha_{13}^{(2)}\big) =\end{aligned} \tag{12}$$

$$\big(\alpha_{12}^{(2)}\,\alpha_{12}^{(3)}\big)\big(\alpha_{23}^{(2)}\,\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\,\alpha_{13}^{(3)}\big)\big(\alpha_{12}^{(3)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(3)}\big),$$
$$\big(\alpha_{12}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{23}^{(3)}\big)\big(\alpha_{13}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{12}^{(3)}\big) = \big(\alpha_{12}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{13}^{(2)}\big)\big(\alpha_{12}^{(3)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(3)}\big)\ (*)\,,$$

where $(*)$ means cyclic permutation of the indexes in the expression, i.e., $12 \to 23 \to 31$ (or, $13) \to 12$. Also, $(\alpha_{ij}^{(2)})^2$, $(\alpha_{ij}^{(3)})^2$, $i, j = 1, 2, 3$, $j > i$, and $\alpha_{12}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{13}^{(2)}$, $\alpha_{12}^{(3)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(3)}$ were obtained in Step **a)** and $\alpha_{12}^{(2)}\,\alpha_{12}^{(3)}$, $\alpha_{13}^{(2)}\,\alpha_{13}^{(3)}$, $\alpha_{23}^{(2)}\,\alpha_{23}^{(3)}$ were obtained in Step **b)**. Other relations are also possible by combining terms in $\Omega_c$ with terms in $\Omega_a$ and $\Omega_b$. For instance, we have that

$$\begin{aligned}\big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{12}^{(3)}\,\alpha_{13}^{(2)}\,\alpha_{23}^{(2)}\big) &= \big(\alpha_{12}^{(2)}\,\alpha_{12}^{(3)}\big)\big(\alpha_{12}^{(2)}\,\alpha_{13}^{(2)}\,\alpha_{23}^{(2)}\big)\ (*)\,,\\ \big(\alpha_{12}^{(3)}\big)^2\big(\alpha_{12}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{23}^{(3)}\big) &= \big(\alpha_{12}^{(2)}\,\alpha_{12}^{(3)}\big)\big(\alpha_{12}^{(3)}\,\alpha_{13}^{(3)}\,\alpha_{23}^{(3)}\big)\ (*)\,.\end{aligned} \tag{13}$$

The determination of the complete set of relations for the case $n = 3$ is beyond the scope of this work.
The bijection between the sets $\Psi_c$ and $\Omega_c$ then implies that the elements of $\Psi_c$ are not independent and must satisfy syzygies obtained from the expressions in (12).

**d)** Observe from the sets $\Omega_a$, $\Omega_b$, and $\Omega_c$ that all the possible combinations between the components of the tensors were considered. Thus, the two remaining invariants, $I_4^{23}$ and $I_7^{123}$, are given in terms of these combinations, which, in turn, are given in terms of the invariants in the set $\Psi \stackrel{\text{def}}{=} \Psi_a \cup \Psi_b \cup \Psi_c$.

In summary, we observe from steps **a)–d)** that only 15 elements in the set $\Omega \stackrel{\text{def}}{=} \Omega_a \cup \Omega_b \cup \Omega_c$ are independent. They are the elements in the set $\Omega_I \stackrel{\text{def}}{=} \Omega_a \backslash \{\alpha_{12}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{13}^{(2)}, \alpha_{12}^{(3)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(3)}\}$. Since, by Step **a)**, there is a bijection between $\Omega_I$ and the set $\Psi_I \stackrel{\text{def}}{=} \Psi_a \backslash \{I_3^2, I_3^3\}$, we see that $\Psi_I$ contains 15 independent invariants. All the other invariants depend on these invariants through both explicit expressions discussed in Step **d)** and implicit relations obtained from (8) and (10) thru (12). Thus, the claim is proved. □

### 3.4 The Case $n = 4$

**Claim** *From* $N(4) = 84$ *classical invariants*, 21 *invariants are independent*, 36 *invariants satisfy*, *at least*, 60 *syzygies*, *and the remaining* 27 *invariants are rational functions of the other invariants*.

***Proof*** In addition to the 28 invariants defined in the expressions (6), (7), and (9), the integrity basis for 4 symmetric tensors also have the 56 invariants given by $I_i^4$, $i = 1, 2, 3$, defined similarly to (6), $I_j^q$, $j = 1, \ldots, 4$, $q = 14, 24, 34$, defined similarly to (7), $I_k^r$, $k = 1, \ldots, 7$, $r = 124, 134, 234$, defined similarly to (9), and (Appendix A)

$$\begin{aligned}
&I_1^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}, && I_2^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)},\\
&I_3^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\ (\dagger)\,, && I_7^{1234} \stackrel{\text{def}}{=} \big(\operatorname{tr}\mathbf{A}^{(1)}\big)^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}\ (\dagger)\,,\\
&I_{11}^{1234} \stackrel{\text{def}}{=} \big(\operatorname{tr}\mathbf{A}^{(1)}\big)^2\big(\mathbf{A}^{(2)}\big)^2\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}, && I_{12}^{1234} \stackrel{\text{def}}{=} \big(\operatorname{tr}\mathbf{A}^{(1)}\big)^2\big(\mathbf{A}^{(3)}\big)^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)},\\
&I_{13}^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2\big(\mathbf{A}^{(4)}\big)^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}, && I_{14}^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(2)}\big)^2\big(\mathbf{A}^{(3)}\big)^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(4)},\\
&I_{15}^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(2)}\big)^2\big(\mathbf{A}^{(4)}\big)^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}, && I_{16}^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(3)}\big)^2\big(\mathbf{A}^{(4)}\big)^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(2)},\\
&I_{17}^{1234} \stackrel{\text{def}}{=} \operatorname{tr}\big(\mathbf{A}^{(1)}\big)^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\ (\dagger)\,,
\end{aligned} \tag{14}$$

where (†) means cyclic permutation of the superscripts. Thus, $I_{l_1}^{1234}, l_1 = 4, 5, 6$, $I_{l_2}^{1234}, l_2 = 8, 9, 10$, and $I_{l_3}^{1234}, l_3 = 18, 19, 20$, are obtained by cyclic permutations of the superscripts on the right-hand side of the expressions of $I_3^{1234}$, $I_7^{1234}$, and $I_{17}^{1234}$, respectively. In accordance with (5), the total number of invariants is then given by $N(4) = 84$.

As in the previous two sections, the tensors $\mathbf{A}^{(i)}$, $i = 1, 2, 3, 4$, are uniquely determined by the 21 components $\lambda_i, \alpha_{ij}^{(k)}, i, j = 1, 2, 3, j > i, k = 1, \ldots, 4$. Therefore, the 84 invariants defined through (6), (7), (9), and (14) are given in terms of these components.

Let the set of 84 invariants be given by $\Psi \overset{\text{def}}{=} \{I_i^p, I_j^q, I_k^r, I_l^{1234}\}$, $i = 1, 2, 3$, $j, p = 1, \ldots, 4$, $q = 12, 13, 14, 23, 24, 34$, $k = 1, \ldots, 7$, $r = 123, 124, 134, 234$, $l = 1, \ldots, 20$, and, analogous to the previous sections, let $\Omega$ be a set of products between the components of the symmetric tensors in $\mathcal{A}^{(4)}$, which will be determined from these invariants by following the steps below.

**a)** Following the steps **a)** thru **d)** in Sect. 3.3, we have that there is a bijection between the set of 24 invariants given by $\Psi_a \overset{\text{def}}{=} \{I_i^1\} \cup \{I_i^p, I_j^{1p}\}$, $i = 1, 2, 3$, $j = 1, \ldots, 4$, $p = 2, 3, 4$, and the set of 24 terms given by $\Omega_a \overset{\text{def}}{=} \{\lambda_k, \alpha_{kk}^{(p)}, (\alpha_{kl}^{(p)})^2, \alpha_{12}^{(p)}\alpha_{23}^{(p)}\alpha_{13}^{(p)}\}$, $k, l = 1, 2, 3$, $l > k$, $p = 2, 3, 4$. We also have that the last 3 terms of $\Omega_a$ yield 3 relations, given by (8), (10), and

$$\left(\alpha_{12}^{(4)}\alpha_{23}^{(4)}\alpha_{13}^{(4)}\right)^2 = \left(\alpha_{12}^{(4)}\right)^2\left(\alpha_{23}^{(4)}\right)^2\left(\alpha_{13}^{(4)}\right)^2. \tag{15}$$

**b)** The 9 invariants in the set $\Psi_b \overset{\text{def}}{=} \{I_1^{pq}, I_1^{1pq}, I_2^{1pq}\}$, $p, q = 2, 3, 4$, $q > p$, depend linearly on the 9 elements in the set $\Omega_b \overset{\text{def}}{=} \{\alpha_{12}^{(p)}\alpha_{12}^{(q)}, \alpha_{13}^{(p)}\alpha_{13}^{(q)}, \alpha_{23}^{(p)}\alpha_{23}^{(q)}\}$ for $p, q = 2, 3, 4$, $q > p$, and, therefore, each element in the set $\Omega_b$ can be obtained from the solution of a system of linear equations obtained from the expressions of the invariants in $\Psi_b$. Thus, there is a bijection between the sets $\Psi_b$ and $\Omega_b$. The terms in $\Omega_b$ are not independent. They must satisfy (11) together with the 8 relations

$$\begin{aligned}
&\left(\alpha_r^{(i)}\alpha_r^{(4)}\right)^2 = \left(\alpha_r^{(i)}\right)^2\left(\alpha_r^{(4)}\right)^2, \ r = 12, 23, 13, \\
&\left(\alpha_{12}^{(i)}\alpha_{12}^{(4)}\right)\left(\alpha_{23}^{(i)}\alpha_{23}^{(4)}\right)\left(\alpha_{13}^{(i)}\alpha_{13}^{(4)}\right) = \left(\alpha_{12}^{(i)}\alpha_{23}^{(i)}\alpha_{13}^{(i)}\right)\left(\alpha_{12}^{(4)}\alpha_{23}^{(4)}\alpha_{13}^{(4)}\right),
\end{aligned} \tag{16}$$

where $i = 2, 3$, and the terms $(\alpha_r^{(j)})^2$ and $\alpha_{12}^{(j)}\alpha_{23}^{(j)}\alpha_{13}^{(j)}$, $j = 2, 3, 4$, were obtained in Step **a)**. The bijection between $\Psi_b$ and $\Omega_b$ then implies that the elements of $\Psi_b$ are not independent and must satisfy syzygies obtained from the expressions in (16).

**c)** Analogously, the 18 invariants in the set $\Psi_c \overset{\text{def}}{=} \{I_2^{pq}, I_3^{1pq}, I_5^{1pq}\} \cup \{I_3^{pq}, I_4^{1pq}, I_6^{1pq}\}$, $p, q = 2, 3, 4$, $q > p$, depend linearly on the 18 elements in the set $\Omega_c \overset{\text{def}}{=} \{\alpha_{12}^{(p)}\alpha_{13}^{(p)}\alpha_{23}^{(q)}, \alpha_{13}^{(p)}\alpha_{23}^{(p)}\alpha_{12}^{(q)}, \alpha_{23}^{(p)}\alpha_{12}^{(p)}\alpha_{13}^{(q)}\} \cup \{\alpha_{12}^{(p)}\alpha_{13}^{(q)}\alpha_{23}^{(q)}, \alpha_{13}^{(p)}\alpha_{23}^{(q)}\alpha_{12}^{(q)}, \alpha_{23}^{(p)}\alpha_{12}^{(q)}\alpha_{13}^{(q)}\}$, $p, q = 2, 3, 4$, $q > p$, and, therefore, the elements of $\Omega_c$ can be given in terms of the invariants in the set $\Psi_c$ by solving a system of linear equations. Thus, there is a bijection between the sets $\Psi_c$ and $\Omega_c$. The terms in $\Omega_c$ are not independent and must satisfy (12) and the 26 relations

$$\begin{aligned}
&\left(\alpha_{12}^{(i)}\alpha_{13}^{(i)}\alpha_{23}^{(4)}\right)^2 = \left(\alpha_{12}^{(i)}\right)^2\left(\alpha_{13}^{(i)}\right)^2\left(\alpha_{23}^{(4)}\right)^2 \ (*), \\
&\left(\alpha_{12}^{(i)}\alpha_{13}^{(4)}\alpha_{23}^{(4)}\right)^2 = \left(\alpha_{12}^{(i)}\right)^2\left(\alpha_{13}^{(4)}\right)^2\left(\alpha_{23}^{(4)}\right)^2 \ (*), \\
&\left(\alpha_{12}^{(i)}\alpha_{13}^{(i)}\alpha_{23}^{(4)}\right)\left(\alpha_{13}^{(i)}\alpha_{23}^{(i)}\alpha_{12}^{(4)}\right)\left(\alpha_{23}^{(i)}\alpha_{12}^{(i)}\alpha_{13}^{(4)}\right) =
\end{aligned}$$

$$
\begin{aligned}
&\big(\alpha_{12}^{(i)}\big)^2\big(\alpha_{13}^{(i)}\big)^2\big(\alpha_{23}^{(i)}\big)^2\big(\alpha_{12}^{(4)}\,\alpha_{23}^{(4)}\,\alpha_{13}^{(4)}\big) = \\
&\big(\alpha_{12}^{(i)}\,\alpha_{12}^{(4)}\big)\big(\alpha_{23}^{(i)}\,\alpha_{23}^{(4)}\big)\big(\alpha_{13}^{(i)}\,\alpha_{13}^{(4)}\big)\big(\alpha_{12}^{(i)}\,\alpha_{23}^{(i)}\,\alpha_{13}^{(i)}\big), \\
&\big(\alpha_{12}^{(i)}\,\alpha_{13}^{(4)}\,\alpha_{23}^{(4)}\big)\big(\alpha_{13}^{(i)}\,\alpha_{23}^{(4)}\,\alpha_{12}^{(4)}\big)\big(\alpha_{23}^{(i)}\,\alpha_{12}^{(4)}\,\alpha_{13}^{(4)}\big) = \\
&\big(\alpha_{12}^{(4)}\big)^2\big(\alpha_{13}^{(4)}\big)^2\big(\alpha_{23}^{(4)}\big)^2\big(\alpha_{12}^{(i)}\,\alpha_{23}^{(i)}\,\alpha_{13}^{(i)}\big) = \\
&\big(\alpha_{12}^{(i)}\,\alpha_{12}^{(4)}\big)\big(\alpha_{23}^{(i)}\,\alpha_{23}^{(4)}\big)\big(\alpha_{13}^{(i)}\,\alpha_{13}^{(4)}\big)\big(\alpha_{12}^{(4)}\,\alpha_{23}^{(4)}\,\alpha_{13}^{(4)}\big), \\
&\big(\alpha_{12}^{(i)}\,\alpha_{13}^{(4)}\,\alpha_{23}^{(4)}\big)\big(\alpha_{13}^{(i)}\,\alpha_{23}^{(i)}\,\alpha_{12}^{(4)}\big) = \big(\alpha_{12}^{(i)}\,\alpha_{23}^{(i)}\,\alpha_{13}^{(i)}\big)\big(\alpha_{12}^{(4)}\,\alpha_{23}^{(4)}\,\alpha_{13}^{(4)}\big)\ (*)\,,
\end{aligned}
\tag{17}
$$

where $i = 2, 3$, and we recall from Sect. 3.3 that $(*)$ means cyclic permutation of the indexes. Also, $(\alpha_r^{(j)})^2$, $\alpha_{12}^{(j)}\,\alpha_{23}^{(j)}\,\alpha_{13}^{(j)}$, $j = 2, 3, 4$, $r = 12, 23, 13$ were obtained in Step **a)** and $\alpha_{12}^{(i)}\,\alpha_{12}^{(4)}$, $\alpha_{13}^{(i)}\,\alpha_{13}^{(4)}$, $\alpha_{23}^{(i)}\,\alpha_{23}^{(4)}$ were obtained in Step **b)**. Again, other relations are also possible by combining terms in $\Omega_c$ with terms in $\Omega_a$ and $\Omega_b$, but will not be pursued in this work. The bijection between the sets $\Psi_c$ and $\Omega_c$ then implies that the elements of $\Psi_c$ are not independent and must satisfy syzygies obtained from the expressions in (17).

**d)** The 6 invariants in the set $\Psi_d \overset{\text{def}}{=} \{I_1^{234}, I_1^{1234}, I_2^{1234}\} \cup \{I_3^{1234}, I_7^{1234}, I_{17}^{1234}\}$ depend linearly on the elements in the set $\Omega_d \overset{\text{def}}{=} \{\alpha_{12}^{(2)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(4)}, \alpha_{23}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{12}^{(4)}, \alpha_{13}^{(2)}\,\alpha_{12}^{(3)}\,\alpha_{23}^{(4)}\} \cup \{\alpha_{12}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{23}^{(4)}, \alpha_{23}^{(2)}\,\alpha_{12}^{(3)}\,\alpha_{13}^{(4)}, \alpha_{13}^{(2)}\,\alpha_{23}^{(3)}\,\alpha_{12}^{(4)}\}$. Analogously as before, the elements of $\Omega_d$ can be solved in terms of the invariants of $\Psi_d$. These elements are not independent and must satisfy the 6 relations

$$
\begin{aligned}
\big(\alpha_{12}^{(2)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(4)}\big)^2 &= \big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{23}^{(3)}\big)^2\big(\alpha_{13}^{(4)}\big)^2\ (*)\,, \\
\big(\alpha_{12}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{23}^{(4)}\big)^2 &= \big(\alpha_{12}^{(2)}\big)^2\big(\alpha_{13}^{(3)}\big)^2\big(\alpha_{23}^{(4)}\big)^2\ (*)\,,
\end{aligned}
\tag{18}
$$

where, again, $(*)$ means cyclic permutation of indexes.

**e)** The set $\Omega \overset{\text{def}}{=} \Omega_a \cup \Omega_b \cup \Omega_c \cup \Omega_d$ has 57 elements, which determine all the possible combinations between the components of the tensors and yield all the 84 invariants. Since there is a bijection between the sets $\Omega$ and $\Psi \overset{\text{def}}{=} \Psi_a \cup \Psi_b \cup \Psi_c \cup \Psi_d$, the difference between the 84 invariants and the number of invariants in $\Psi$ yields the 27 invariants given by $I_4^p, p = 23, 24, 34$, $I_7^q, q = 123, 124, 134, 234$, $I_i^{234}, i = 2, \ldots, 6$, $I_j^{1234}, j = 4, 5, 6, 8, 9, \ldots, 16, 18, 19, 20$, which, in turn, are given in terms of the invariants in the set $\Psi$.

In summary, observe from steps **a)** – **d)** that only 21 elements in the set $\Omega$ are independent. They are the elements in the set $\Omega_I \overset{\text{def}}{=} \Omega_a \backslash \{\alpha_{12}^{(r)}\,\alpha_{23}^{(r)}\,\alpha_{13}^{(r)}\}$, $r = 2, 3, 4$. Since, by Step **a)**, there is a bijection between $\Omega_I$ and the set $\Psi_I \overset{\text{def}}{=} \Psi_a \backslash \{I_3^r\}$, $r = 2, 3, 4$, we see that $\Psi_I$ contains 21 independent invariants. All the other invariants depend on these invariants through both explicit expressions discussed in Step **e)** and implicit relations obtained from (8), (10) thru (12), and (15) thru (18). Thus, the claim is proved. □

## 3.5 The Case $n = 5$

**Claim** *From* $N(5) = 251$ *classical invariants*, 27 *invariants are independent*, 82 *invariants satisfy*, *at least*, 130 *syzygies*, *and the remaining* 142 *invariants are rational functions of the other invariants*.

***Proof*** In addition to the 84 invariants defined in the expressions (6), (7), (9), and (14), the integrity basis for 5 symmetric tensors also have the 167 invariants given by $I_i^5, i = 1, 2, 3$, defined similarly to (6), $I_j^q, j = 1, \ldots, 4, q = 15, 25, 35, 45$, defined similarly to (7), $I_k^r, k = 1, \ldots, 7, r = 125, 135, 145, 235, 245, 345$, defined similarly to (9), $I_l^s, l = 1, \ldots, 20, s = 1235, 1245, 1345, 2345$, defined similarly to (14), and

$$
\begin{aligned}
&I_1^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(2)} \mathbf{A}^{(3)} \mathbf{A}^{(4)} \mathbf{A}^{(5)}, \qquad I_2^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(2)} \mathbf{A}^{(4)} \mathbf{A}^{(5)} \mathbf{A}^{(3)}, \\
&I_3^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(2)} \mathbf{A}^{(5)} \mathbf{A}^{(3)} \mathbf{A}^{(4)}, \qquad I_4^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(3)} \mathbf{A}^{(2)} \mathbf{A}^{(5)} \mathbf{A}^{(4)}, \\
&I_5^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(3)} \mathbf{A}^{(4)} \mathbf{A}^{(2)} \mathbf{A}^{(5)}, \qquad I_6^{12345} \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{A}^{(1)} \mathbf{A}^{(4)} \mathbf{A}^{(2)} \mathbf{A}^{(3)} \mathbf{A}^{(5)}, \\
&I_7^{12345} \stackrel{\text{def}}{=} \left(\operatorname{tr} \mathbf{A}^{(1)}\right)^2 \mathbf{A}^{(2)} \mathbf{A}^{(3)} \mathbf{A}^{(4)} \mathbf{A}^{(5)} \ (\dagger), \\
&I_{12}^{12345} \stackrel{\text{def}}{=} \left(\operatorname{tr} \mathbf{A}^{(1)}\right)^2 \mathbf{A}^{(2)} \mathbf{A}^{(3)} \mathbf{A}^{(5)} \mathbf{A}^{(4)} \ (\dagger), \\
&I_{17}^{12345} \stackrel{\text{def}}{=} \left(\operatorname{tr} \mathbf{A}^{(1)}\right)^2 \mathbf{A}^{(3)} \mathbf{A}^{(2)} \mathbf{A}^{(4)} \mathbf{A}^{(5)} \ (\dagger), \\
&I_{22}^{12345} \stackrel{\text{def}}{=} \operatorname{tr}\left(\mathbf{A}^{(1)}\right)^2 \mathbf{A}^{(2)} \mathbf{A}^{(5)} \mathbf{A}^{(4)} \mathbf{A}^{(3)} \ (\dagger),
\end{aligned}
\tag{19}
$$

where we recall from Sect. 3.4 that (†) means cyclic permutation of the superscripts. Thus, $I_{l_1}^{12345}, l_1 = 8, \ldots, 11$, $I_{l_2}^{12345}, l_2 = 13, \ldots, 16$, $I_{l_3}^{12345}, l_3 = 18, \ldots, 21$, and $I_{l_4}^{12345}, l_4 = 23, \ldots, 26$, are obtained by cyclic permutations on the right-hand side of the expressions of $I_7^{12345}$, $I_{12}^{12345}$, $I_{17}^{12345}$, and $I_{22}^{12345}$, respectively. In accordance to (5), the total number of invariants is then given by $N(5) = 251$.

Similarly as before, the tensors $\mathbf{A}^{(i)}$, $i = 1, \ldots, 5$, are uniquely determined by the 27 components $\lambda_i, \alpha_{ij}^{(k)}, i, j = 1, 2, 3, j > i, k = 1, \ldots, 5$. Therefore, the 251 invariants defined through (6), (7), (9), (14), and (19) are given in terms of these components.

Let the set of 251 invariants be given by $\Psi \stackrel{\text{def}}{=} \{I_i^p, I_j^q, I_k^r, I_l^s, I_m^{12345}\}$, $i = 1, 2, 3$, $p = 1, \ldots, 5$, $j = 1, \ldots, 4$, $q = 12, 13, 14, 15, 23, 24, 25, 34, 35, 45$, $k = 1, \ldots, 7$, $r = 123, 124, 125, 134, 135, 145, 234, 235, 245, 345$, $l = 1, \ldots, 20$, $s = 1234, 1235, 1245, 1345, 2345$, $m = 1, \ldots, 26$, and, analogous to the previous sections, let $\Omega$ be a set of products between the components of the symmetric tensors in $\mathcal{A}^{(5)}$, which will be determined from these invariants by following the steps below.

- **a)** There is a bijection between the set of 31 invariants given by $\Psi_a \stackrel{\text{def}}{=} \{I_i^1\} \cup \{I_i^p, I_j^{1p}\}$, $i = 1, 2, 3$, $j = 1, \ldots, 4$, $p = 2, \ldots, 5$, and the set of 31 terms given by $\Omega_a \stackrel{\text{def}}{=} \{\lambda_k, \alpha_{kk}^{(p)}, (\alpha_{kl}^{(p)})^2, \alpha_{12}^{(p)} \alpha_{23}^{(p)} \alpha_{13}^{(p)}\}$, $k, l = 1, 2, 3$, $l > k$, $p = 2, \ldots, 5$. We also have that the last 4 terms of $\Omega_a$ satisfy 4 relations, given by (8), (10), (15), and a relation that can be obtained from (15) by replacing the superscript (4) with the superscript (5).
- **b)** The 18 invariants in the set $\Psi_b \stackrel{\text{def}}{=} \{I_1^{pq}, I_1^{1pq}, I_2^{1pq}\}$, $p, q = 2, \ldots, 5$, $q > p$, depend linearly on the 18 terms in the set $\Omega_b \stackrel{\text{def}}{=} \{\alpha_{12}^{(p)} \alpha_{12}^{(q)}, \alpha_{13}^{(p)} \alpha_{13}^{(q)}, \alpha_{23}^{(p)} \alpha_{23}^{(q)}\}$ for $p, q = 2, \ldots, 5$, $q > p$, and, therefore, there is a bijection between the sets $\Psi_b$ and $\Omega_b$. The terms in $\Omega_b$ are not independent. They must satisfy (11), (16), and 12 relations that can be obtained from (16) by replacing the superscript (4) with the superscript (5), where, here, $i = 2, 3, 4$, and the terms $(\alpha_r^{(j)})^2$ and $\alpha_{12}^{(j)} \alpha_{23}^{(j)} \alpha_{13}^{(j)}$, $j = 2, 3, 4, 5$, were obtained in Step **a**). The bijection between the sets $\Psi_b$ and $\Omega_b$ then implies that the elements of $\Psi_b$ are not independent and must satisfy syzygies obtained from expressions analogous to (16).
- **c)** The 36 invariants in the set $\Psi_c \stackrel{\text{def}}{=} \{I_2^{pq}, I_3^{1pq}, I_5^{1pq}\} \cup \{I_3^{pq}, I_4^{1pq}, I_6^{1pq}\}$, $p, q = 2, \ldots, 5$, $q > p$, depend linearly on the 36 elements in the set $\Omega_c \stackrel{\text{def}}{=} \{\alpha_{12}^{(p)} \alpha_{13}^{(p)} \alpha_{23}^{(q)}, \alpha_{13}^{(p)} \alpha_{23}^{(p)} \alpha_{12}^{(q)},$

$\alpha_{23}^{(p)}\,\alpha_{12}^{(p)}\,\alpha_{13}^{(q)}\}\cup\{\alpha_{12}^{(p)}\,\alpha_{13}^{(q)}\,\alpha_{23}^{(q)},\alpha_{13}^{(p)}\,\alpha_{23}^{(q)}\,\alpha_{12}^{(q)},\alpha_{23}^{(p)}\,\alpha_{12}^{(q)}\,\alpha_{13}^{(q)}\}$, $p,q=2,3,4,5$, $q>p$, and, as before, there is a bijection between the sets $\Psi_c$ and $\Omega_c$. The terms in $\Omega_c$ are not independent and must satisfy (12), (17), and 39 relations that can be obtained from (17) by replacing the superscript (4) with the superscript (5), where, here, $i=2,3,4$, $(\alpha_r^{(j)})^2$ and $\alpha_{12}^{(j)}\,\alpha_{23}^{(j)}\,\alpha_{13}^{(j)}$, $j=2,3,4,5$, $r=12,23,13$ were obtained in Step **a)** and $\alpha_{12}^{(i)}\,\alpha_{12}^{(5)}$, $\alpha_{13}^{(i)}\,\alpha_{13}^{(5)}$, $\alpha_{23}^{(i)}\,\alpha_{23}^{(5)}$ were obtained in Step **b)**. Again, other relations are also possible by combining terms in $\Omega_c$ with terms in $\Omega_a$ and $\Omega_b$. The bijection between the sets $\Psi_c$ and $\Omega_c$ then implies that the elements of $\Psi_c$ are not independent and must satisfy syzygies obtained from expressions analogous to (17).

**d)** The 24 invariants in the set $\Psi_d \stackrel{\text{def}}{=} \{I_1^{pqr}, I_1^{1pqr}, I_2^{1pqr}\}\cup\{I_3^{pqr}, I_7^{1pqr}, I_{17}^{1pqr}\}$, $p,q,r=2,3,4,5$, $r>q>p$, depend linearly on the 24 terms in the set $\Omega_d \stackrel{\text{def}}{=} \{\alpha_{12}^{(p)}\,\alpha_{23}^{(q)}\,\alpha_{13}^{(r)}, \alpha_{23}^{(p)}\,\alpha_{13}^{(q)}\,\alpha_{12}^{(r)}, \alpha_{13}^{(p)}\,\alpha_{12}^{(q)}\,\alpha_{23}^{(r)}\}\cup\{\alpha_{12}^{(p)}\,\alpha_{13}^{(q)}\,\alpha_{23}^{(r)}, \alpha_{23}^{(p)}\,\alpha_{12}^{(q)}\,\alpha_{13}^{(r)}, \alpha_{13}^{(p)}\,\alpha_{23}^{(q)}\,\alpha_{12}^{(r)}\}$, where $p,q,r=2,3,4,5$, $r>q>p$. As before, the elements of $\Omega_d$ can be solved in terms of the invariants of $\Psi_d$. These elements are not independent and, in addition to (18), must satisfy the 18 relations

$$\begin{aligned}\big(\alpha_{12}^{(i)}\,\alpha_{23}^{(j)}\,\alpha_{13}^{(5)}\big)^2 &= \big(\alpha_{12}^{(i)}\big)^2\big(\alpha_{23}^{(j)}\big)^2\big(\alpha_{13}^{(5)}\big)^2\ (*)\,,\\ \big(\alpha_{12}^{(i)}\,\alpha_{13}^{(j)}\,\alpha_{23}^{(5)}\big)^2 &= \big(\alpha_{12}^{(i)}\big)^2\big(\alpha_{13}^{(j)}\big)^2\big(\alpha_{23}^{(5)}\big)^2\ (*)\,,\end{aligned}\tag{20}$$

where $i,j=2,3,4$, $j>i$, and, as before, $(*)$ means cyclic permutation of indexes.

**e)** Unlike the steps c) in Case $n=3$ and d) in Case $n=4$, here, no new set of terms are possible. To see this, first, consider the invariant $I_1^{12345}$ defined in (19.a). In view of (1), it is given by $I_1^{12345}=\sum_{i,j,k,l,m=1}^{3}\lambda_i\,\alpha_{ij}^{(2)}\,\alpha_{jk}^{(3)}\,\alpha_{kl}^{(4)}\,\alpha_{li}^{(5)}$. We see that if $i=1$, $j=2$, $k=3$, $l=1$, then the corresponding term would be a combination of the terms $\lambda_1$ and $\alpha_{11}^{(5)}$, obtained in step **a)**, and the term $\alpha_{12}^{(2)}\,\alpha_{23}^{(3)}\,\alpha_{13}^{(4)}$, obtained in step **d)**. This example is representative of all the invariants in (19), except the ones containing squares of $\mathbf{A}^{(i)}$, $i=2,\dots,5$.

Next, we repeat the analysis for the invariant $I_8^{12345}=\operatorname{tr}\mathbf{A}^{(1)}\,(\mathbf{A}^{(2)})^2\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(5)}$. In view of (1), it is given by $I_8^{12345}=\sum_{i,j,k,l,m=1}^{3}\lambda_i\,\alpha_{ij}^{(2)}\,\alpha_{jk}^{(2)}\,\alpha_{kl}^{(3)}\,\alpha_{lm}^{(4)}\,\alpha_{mi}^{(5)}$. We see that if $i=1$, $j=2$, $k=3$, $l=1$, $m=2$, then the corresponding term would be a combination of $\lambda_1$, obtained in step **a)**, $\alpha_{12}^{(4)}\,\alpha_{12}^{(5)}$, obtained in step **b)**, and the term $\alpha_{12}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{13}^{(3)}$, obtained in step **c)**. This combination is not unique since it could also be given by $\lambda_1$, obtained in step **a)**, $\alpha_{12}^{(2)}\,\alpha_{12}^{(5)}$, obtained in step **b)**, and $\alpha_{23}^{(2)}\,\alpha_{13}^{(3)}\,\alpha_{12}^{(4)}$, obtained in step **d)**.

Thus, the set $\Omega \stackrel{\text{def}}{=} \Omega_a\cup\Omega_b\cup\Omega_c\cup\Omega_d$ has 109 elements, which determine all the possible combinations between the components of the tensors and yield all the 251 invariants. Since there is a bijection between the sets $\Omega$ and $\Psi_\cup \stackrel{\text{def}}{=} \Psi_a\cup\Psi_b\cup\Psi_c\cup\Psi_d$, the difference between the 251 invariants and the number of invariants in $\Psi_\cup$ yields the 142 invariants given by $I_4^p$, $p=23,24,25,34,35,45$, $I_7^q$, $q=123,124,125,134,135,145,234,235,245,345$, $I_i^r$, $i=2,\dots,6$, $r=234,235,245,345$, $I_j^s$, $j=4,5,6,8,9,\dots,16,18,19,20$, $s=1234,1235,1245,1345$, $I_k^{2345}$, $k=1,\dots,20$, $I_l^{12345}$, $l=1,\dots,26$, which, in turn, are given explicity in terms of the invariants in the set $\Psi_\cup$.

In summary, observe from steps **a)–d)** that only 27 elements in the set $\Omega$ are independent. They are the elements in the set $\Omega_I \stackrel{\text{def}}{=} \Omega_a\backslash\{\alpha_{12}^{(r)}\,\alpha_{23}^{(r)}\,\alpha_{13}^{(r)}\}$, $r=2,\dots,5$. Since, by Step **a)**, there is a bijection between $\Omega_I$ and the set $\Psi_I \stackrel{\text{def}}{=} \Psi_a\backslash\{I_3^r\}$, $r=2,\dots,5$, we see that $\Psi_I$ contains 27 independent invariants. All the other invariants depend on these invariants through both explicit expressions discussed in Step **e)** and implicit relations obtained from (8), (10) thru (12), (15) thru (18), (20), and expressions analogous to (15) thru (17) with the superscript (4) replaced by the superscript (5). Thus, the claim is proved. □

### 3.6 The Case $n > 5$

**Claim** *Recall from* (5) *that the total number of polynomial invariants in the integrity basis for* $\mathcal{A}^{(n)}$ *is given by* $N(n) = 3n + 4\binom{n}{2} + 7\binom{n}{3} + 20\binom{n}{4} + 26\binom{n}{5} + 10\binom{n}{6}$, *from which* $N_I(n) \stackrel{\text{def}}{=} 6n - 3$ *invariants are independent*, $N_D(n) \stackrel{\text{def}}{=} (n-1)(n^2 - n/2 - 2)$ *invariants satisfy, at least*, $(n-1)(n^2 + 7n/2 - 10)$ *syzygies, and the remaining invariants, which add up to* $N(n) - (N_I(n) + N_D(n)) = (n-1)(n-2)(n+3)/6 + 20\binom{n}{4} + 26\binom{n}{5} + 10\binom{n}{6}$, *are rational functions of the other invariants.*

***Proof*** The results of the previous claims are summarized in Table 1, where $N(n)$ is the total number of classical invariants, $N_I(n)$ is the number of independent invariants, $N_D(n)$ is the number of invariants that satisfy $N_S(n)$ syzygies, and $N_R(n) = N(n) - (N_I(n) + N_D(n))$ is the number of remaining invariants, which are rational functions of the other invariants. Details about generalizations of these results for $n > 5$ are discussed below.

The components of the $n$ tensors in $\mathcal{A}^{(n)}$ yield the set $\Omega_I \stackrel{\text{def}}{=} \{\lambda_i, \alpha_{ii}^{(r)}, (\alpha_{ij}^{(r)})^2\}$, $i, j = 1, 2, 3$, $j > i$, $r = 2, \ldots, n$. The elements of this set are clearly independent and yield the $6n - 3$ invariants in the set $\Psi_I \stackrel{\text{def}}{=} \{I_1^1, I_2^1, I_3^1, I_1^r, I_2^r, I_i^{1r}\}$, $i, = 1, \ldots, 4$, $r = 2, \ldots, n$, where these invariants are defined similarly to (7). Following the arguments of Step **a)** in Case $n = 3$, it can be seen that there is a bijection between the sets $\Omega_I$ and $\Psi_I$ and, therefore, that the $6n - 3$ invariants of $\Psi_I$ are independent.

The $n - 1$ invariants $I_3^r$, $r = 2, \ldots, n$, are determined from the terms in $\Omega_I$ together with $\alpha_{12}^{(r)}\alpha_{23}^{(r)}\alpha_{13}^{(r)}$, which satisfy the relations

$$\left(\alpha_{12}^{(r)}\alpha_{23}^{(r)}\alpha_{13}^{(r)}\right)^2 = \left(\alpha_{12}^{(r)}\right)^2\left(\alpha_{23}^{(r)}\right)^2\left(\alpha_{13}^{(r)}\right)^2, \qquad r = 2, \ldots, n. \tag{21}$$

The bijection between the sets $\Omega_I$ and $\Psi_I$ and the expressions in (21) yield $n - 1$ syzygies relating $I_3^r$ to the invariants of $\Psi_I$. Similarly to the previous sections, we have that $\Omega_a \stackrel{\text{def}}{=} \Omega_I \cup \{\alpha_{12}^{(r)}\alpha_{23}^{(r)}\alpha_{13}^{(r)}\}$.

Next, following the arguments of Step **b)** in Case $n = 4$, we find that there is a bijection between the sets $\Psi_b \stackrel{\text{def}}{=} \{I_1^{rs}, I_1^{1rs}, I_2^{1rs}\}$ and $\Omega_b \stackrel{\text{def}}{=} \{\alpha_{12}^{(r)}\alpha_{12}^{(s)}, \alpha_{13}^{(r)}\alpha_{13}^{(s)}, \alpha_{23}^{(r)}\alpha_{23}^{(s)}\}$ for $r, s = 2, \ldots, n$, $s > r$. Based on the previous claims, we find that both sets have $3(n-1)(n-2)/2$ elements. We also find that the terms in $\Omega_b$ are not independent and satisfy $2(n-1)(n-2)$ relations having the forms given in (16) with the superscript (4) replaced by $(n)$.

Similarly, following the arguments of Step **c)** in Case $n = 4$, we find that there is a bijection between the sets $\Psi_c \stackrel{\text{def}}{=} \{I_2^{rs}, I_3^{1rs}, I_5^{1rs}\} \cup \{I_3^{rs}, I_4^{1rs}, I_6^{1rs}\}$ and $\Omega_c \stackrel{\text{def}}{=} \{\alpha_{12}^{(r)}\alpha_{13}^{(r)}\alpha_{23}^{(s)}, \alpha_{13}^{(r)}\alpha_{23}^{(r)}\alpha_{12}^{(s)}, \alpha_{23}^{(r)}\alpha_{12}^{(r)}\alpha_{13}^{(s)}\} \cup \{\alpha_{12}^{(r)}\alpha_{13}^{(s)}\alpha_{23}^{(s)}, \alpha_{13}^{(r)}\alpha_{23}^{(s)}\alpha_{12}^{(s)}, \alpha_{23}^{(r)}\alpha_{12}^{(s)}\alpha_{13}^{(s)}\}$, $r, s = 2, \ldots, n$, $s > r$. Based on the previous claims, we find that both sets have $3(n-1)(n-2)$ elements. We also find that the terms in $\Omega_c$ are not independent and satisfy $13(n-1)(n-2)/2$ relations having the forms given in (17) with the superscript (4) replaced by $(n)$.

Finally, following the arguments of Step **d)** in Case $n = 5$, we find that there is a bijection between the sets $\Psi_d \stackrel{\text{def}}{=} \{I_1^{rst}, I_1^{1rst}, I_2^{1rst}\} \cup \{I_3^{1rst}, I_7^{1rst}, I_{17}^{1rst}\}$ and $\Omega_d \stackrel{\text{def}}{=} \{\alpha_{12}^{(r)}\alpha_{23}^{(s)}\alpha_{13}^{(t)}, \alpha_{23}^{(r)}\alpha_{13}^{(s)}\alpha_{12}^{(t)}, \alpha_{13}^{(r)}\alpha_{12}^{(s)}\alpha_{23}^{(t)}\} \cup \{\alpha_{12}^{(r)}\alpha_{13}^{(s)}\alpha_{23}^{(t)}, \alpha_{23}^{(r)}\alpha_{12}^{(s)}\alpha_{13}^{(t)}, \alpha_{13}^{(r)}\alpha_{23}^{(s)}\alpha_{12}^{(t)}\}$, $r, s, t = 2, \ldots, n$, $t > s > r$. Based on the previous claims, we find that both sets have $(n-1)(n-$

**Table 1** Number of invariants and syzygies

| $n$ | $N(n)$ | $N_I(n)$ | $N_D(n)$ | $N_S(n)$ | $N_R(n)$ |
|---|---|---|---|---|---|
| 1 | 3 | 3 | 0 | 0 | 0 |
| 2 | 10 | 9 | 1 | 1 | 0 |
| 3 | 28 | 15 | 11 | 19 | 2 |
| 4 | 84 | 21 | 36 | 60 | 27 |
| 5 | 251 | 27 | 82 | 130 | 142 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | § | $6n-3$ | $(n-1)(n^2-n/2-2)$ | $(n-1)(n^2+7n/2-10)$ | §§ |

§: See (5). §§: $(n-1)\,(n-2)\,(n+3)/6+20\binom{n}{4}+26\binom{n}{5}+10\binom{n}{6}$.

2) $(n-3)$ elements. We also find that the elements of $\Omega_d$ are not independent and satisfy $(n-1)\,(n-2)(n-3)$ relations having the forms given in (20) with the superscript (5) replaced by $(n)$.

It follows from above that the set $\Omega \overset{\text{def}}{=} \Omega_a \cup \Omega_b \cup \Omega_c \cup \Omega_d$ has $n^3+3(-n^2+3n)/2-1$ elements, which correspond to $N_I(n)+N_D(n)$ from Table 1. These elements determine all the possible combinations between the components of the tensors and yield all the $N(n)$ invariants for $\mathcal{A}^{(n)}$. Since there is a bijection between the sets $\Omega$ and $\Psi_\cup \overset{\text{def}}{=} \Psi_a \cup \Psi_b \cup \Psi_c \cup \Psi_d$, the difference between $N(n)$ and the number of invariants in $\Psi_\cup$ yields the number of invariants that can be obtained explicity in terms of the invariants in the set $\Psi_\cup$, which is given by $N_R(n)$ in Table 1. It also follows from above that $(n-1)(n^2-n/2-2)$ invariants in $\Psi_\cup$ satisfy, at least, $(n-1)(n^2+7n/2-10)$ implicit relations, and these numbers correspond to, respectively, $N_D(n)$ and $N_S(n)$ in Table 1. □

## 4 Conclusions

The minimal integrity basis for $n$ symmetric second-order tensors in the set $\mathcal{A}^{(n)}$ has $N(n)$ invariants given by (5), from which $N_I(n)=6n-3$ invariants are independent, $N_D(n)=(n-1)(n^2-n/2-2)$ invariants satisfy, at least, $N_S(n)=(n-1)(n^2+7n/2-10)$ implicit relations between the invariants, and the remaining $N_R(n)=N(n)-(N_I(n)+N_D(n))$ invariants are rational functions of the other invariants and are, therefore, determined explicitly.

## Appendix A: Minimal Integrity Basis for $\mathcal{A}^{(n)}$

Let $\mathbf{A}^{(r)}$, $r=1,\ldots,n$, be given by (1). For $n \le 6$, the minimal integrity basis is formed by the traces of the tensor products in Table 2, where $\beta_i, i=1,\ldots,6$, is the number of these products in the $i$-th row and we recall from Sect. 3.4 that (†) means cyclic permutation of the superscripts, together with the traces of tensor products of all the subsets of tensors that can be chosen from the given set. For example, the minimal integrity basis for the case $n=3$ has $\beta_1 n+\beta_2 n+\beta_3=28$ elements and its elements are given by the expressions in (6), (7), and (9). For $n>6$, the integrity bases are obtained by forming the sum of the integrity bases taken six at a time in all the possible combinations.

**Table 2** Products of tensors in $\mathcal{A}^{(n)}$ for $n \leq 6$. Adapted from Spencer [29]

| $\beta_i$ | Tensor products of $\mathbf{A}^{(r)}$, $r = 1, \ldots, n$ |
|---|---|
| 3 | $\mathbf{A}^{(1)}$; $(\mathbf{A}^{(1)})^2$; $(\mathbf{A}^{(1)})^3$ |
| 4 | $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}$; $\mathbf{A}^{(1)}\,(\mathbf{A}^{(2)})^2(\dagger)$; $(\mathbf{A}^{(1)})^2(\mathbf{A}^{(2)})^2$ |
| 7 | $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}(\dagger)$; $(\mathbf{A}^{(1)})^2(\mathbf{A}^{(2)})^2\,\mathbf{A}^{(3)}(\dagger)$ |
| 20 | $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}(\dagger)$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}(\dagger)$; $(\mathbf{A}^{(1)})^2(\mathbf{A}^{(2)})^2\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}$; $(\mathbf{A}^{(1)})^2(\mathbf{A}^{(3)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}$; $(\mathbf{A}^{(1)})^2(\mathbf{A}^{(4)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}$; $(\mathbf{A}^{(2)})^2(\mathbf{A}^{(3)})^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}$; $(\mathbf{A}^{(2)})^2(\mathbf{A}^{(4)})^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}$; $(\mathbf{A}^{(3)})^2(\mathbf{A}^{(4)})^2\,\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}(\dagger)$ |
| 26 | $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(3)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(4)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(5)}$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(5)}(\dagger)$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(4)}(\dagger)$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(5)}(\dagger)$; $(\mathbf{A}^{(1)})^2\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}$ |
| 10 | $\mathbf{A}^{(1)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(6)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(6)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(6)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(6)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(6)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(5)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(6)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(6)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(6)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(6)}$; $\mathbf{A}^{(1)}\,\mathbf{A}^{(5)}\,\mathbf{A}^{(4)}\,\mathbf{A}^{(3)}\,\mathbf{A}^{(2)}\,\mathbf{A}^{(6)}$ |

## Appendix B: Invariants in $\Psi$ Expressed in Terms of Elements in $\Omega$

The case $n = 1$:

$$I_1^1 = \lambda_1 + \lambda_2 + \lambda_3\,, \quad I_2^1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \quad I_3^1 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3\,. \tag{22}$$

The case $n = 2$: In addition to the list of invariants given by (22), we have the list of invariants presented below.

$$\begin{aligned}
I_1^2 &= \alpha_{11}^{(2)} + \alpha_{22}^{(2)} + \alpha_{33}^{(2)}\,,\\
I_2^2 &= \big(\alpha_{11}^{(2)}\big)^2 + \big(\alpha_{22}^{(2)}\big)^2 + \big(\alpha_{33}^{(2)}\big)^2 + 2\big(\big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{13}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2\big),\\
I_3^2 &= \big(\alpha_{11}^{(2)}\big)^3 + \big(\alpha_{22}^{(2)}\big)^3 + \big(\alpha_{33}^{(2)}\big)^3 + 3\big[\alpha_{11}^{(2)}\big(\big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{13}^{(2)}\big)^2\big) +\\
&\qquad \alpha_{22}^{(2)}\big(\big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2\big) + \alpha_{33}^{(2)}\big(\big(\alpha_{13}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2\big) + 2\,\alpha_{12}^{(2)}\,\alpha_{23}^{(2)}\,\alpha_{13}^{(2)}\big],\\
I_1^{12} &= \lambda_1\,\alpha_{11}^{(2)} + \lambda_2\,\alpha_{22}^{(2)} + \lambda_3\,\alpha_{33}^{(2)}, \quad I_2^{12} = \lambda_1^2\,\alpha_{11}^{(2)} + \lambda_2^2\,\alpha_{22}^{(2)} + \lambda_3^2\,\alpha_{33}^{(2)}\,,\\
I_3^{12} &= \lambda_1\big(\big(\alpha_{11}^{(2)}\big)^2 + \big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{13}^{(2)}\big)^2\big) + \lambda_2\big(\big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{22}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2\big) +\\
&\qquad \lambda_3\big(\big(\alpha_{13}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2 + \big(\alpha_{33}^{(2)}\big)^2\big),\\
I_4^{12} &= \lambda_1^2\big(\big(\alpha_{11}^{(2)}\big)^2 + \big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{13}^{(2)}\big)^2\big) + \lambda_2^2\big(\big(\alpha_{12}^{(2)}\big)^2 + \big(\alpha_{22}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2\big) +\\
&\qquad \lambda_3^2\big(\big(\alpha_{13}^{(2)}\big)^2 + \big(\alpha_{23}^{(2)}\big)^2 + \big(\alpha_{33}^{(2)}\big)^2\big).
\end{aligned} \tag{23}$$

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## Declarations

**Competing interests** The authors declare no competing interests.

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