



# Noncommutative Noether's problem vs classic Noether's problem

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## Abstract

We prove that if two complex affine irreducible varieties are birational (that is their coordinate rings have isomorphic fields of fractions) then their rings of differential operators are birationally equivalent. It allows to address the Noncommutative Noether's Problem on the invariants of Weyl fields for linear actions of finite groups. In fact, we show for any field  $k$  of characteristic 0 that rationality of the quotient variety  $\mathbb{A}^n(k)/G$  implies that the Noncommutative Noether's Problem is positively solved for  $G$ . In particular, this gives affirmative answer for all pseudo-reflections groups, for the alternating groups ( $n = 3, 4, 5$ ) and for any finite group when  $n = 3$  and  $k$  is algebraically closed (covering and generalizing all previously known cases). Alternative proofs are given for the complex field and for all pseudo-reflections groups. In the later case an effective algorithm of finding the Weyl generators is described.

## 1 Introduction

We will assume that all algebras are considered over the field  $k$  of characteristic zero.

Let  $G$  be a finite group acting linearly on the ring of polynomials  $k[x_1, \dots, x_n]$ . Extend the linear action of  $G$  to the action on the field of rational functions  $K_n = k(x_1, \dots, x_n)$ . The Classical Noether's Problem (CNP for short), related to the 14-th Hilbert's problem, asks whether  $K_n^G$  is a purely transcendental extension of  $k$  or, equivalently, whether  $\mathbb{A}^n(k)/G$  is *birational* to  $\mathbb{A}^m(k)$  for some  $m$ . In the latter case we say that the quotient variety  $\mathbb{A}^n(k)/G$  is *rational* (cf. [29]). More generally, for an irreducible affine variety  $X$  with an action of the group  $G$  we say that the quotient variety  $X/G$  is birational to  $X$  if the field of functions on  $X$  is isomorphic to the subfield of  $G$ -invariants.

Well known cases with positive solution for the CNP include  $n = 1$  (which is the classic Luroth theorem for algebraic curves),  $n = 2$  (due to Miyata [24]) or  $n = 3$  and  $k$  is algebraically closed (due to Burnside [3]) and any  $n$  when the representation of  $G$  is isomorphic to a direct sum of one dimensional representations (due to Fischer). It also holds for alter-

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nating groups  $G = A_n$  for  $n = 3, 4, 5$  (due to Maeda [21]). Detailed references including the counter-examples of Swan, Voskresenskii and Saltman can be found in [6, 14].

We remark that the question of rationality of invariants of the rational function field makes perfect sense for linear algebraic groups. For such generality see [27], II.2. In particular, if the action is triangular then the invariants of the rational function field define again a rational function field. This is the case, for instance, when the group is connected solvable and the field is algebraically closed (by the Lie–Kolchin Theorem).

Passing to a noncommutative case consider the  $\mathbf{k}$ -algebra of differential operators on the polynomial ring  $\mathbf{k}[x_1, \dots, x_n]$ , which is the  $n$ -th Weyl algebra  $A_n(\mathbf{k})$ , and extend the action of  $G$  to a linear action on  $A_n(\mathbf{k})$ . The algebra  $A_n(\mathbf{k})$  is a simple Noetherian Ore domain which admits the skew field of fractions which we will denote by  $F_n(\mathbf{k})$ . The action of  $G$  extends naturally to  $F_n(\mathbf{k})$ . For convenience we set  $F_0(\mathbf{k}) := \mathbf{k}$ .

An analog of the Noether’s Problem for the Weyl algebra  $A_n$  was first considered by Alev and Dumas in [1], where they asked whether  $F_n(\mathbf{k})^G$  is isomorphic to  $F_m(L)$  for some  $m \geq 1$  and some purely transcendental extension  $L$  of  $\mathbf{k}$  of transcendence degree  $t \geq 0$ .

In fact, if such isomorphism holds, then  $m = n$  and  $L = \mathbf{k}$  [1]. This is the case, for instance, for  $n = 1$  and  $n = 2$  and an arbitrary finite group  $G$  [1], for any  $n$  and any  $G$  whose natural representation decomposes into a direct sum of one dimensional representations [1]. In particular, it holds for all  $n \geq 1$  if  $G$  is abelian and  $\mathbf{k}$  is algebraically closed. It was shown in [7] that it also holds for any  $n$  and any complex reflection group.

The case of infinite groups  $G$  was also considered previously. In this case  $m + \text{trdeg}_{\mathbf{k}} L \leq n$  and if the action is triangular then one can not guarantee that the invariants form a Weyl field [1, Remark 1.2.3, 1.3.2]. However, if the action of  $G$  decomposes as a direct sum of one dimensional  $G$ -modules, then indeed  $F_n(\mathbf{k})^G \simeq F_m(L)$ , and  $m + \text{trdeg}_{\mathbf{k}} L = n$ . Also, all values  $1, \dots, n$  can appear as  $\text{trdeg}_{\mathbf{k}} L$  for actions of subgroups of the torus  $\mathbb{T}^n$  [1], including the extreme case  $n = 0$ , where the skew field of invariants is commutative.

We will only be interested in the case of finite  $G$ . Due to the importance of Weyl algebras and to the fact that they are the simplest noncommutative deformations of polynomial algebras, we call the analog of the Noether’s Problem for  $A_n$ , following [1], the *Noncommutative Noether’s Problem* (NNP for short).

The cases when the NNP has a positive solution are of special interest in view of the rigidity of the Weyl algebras proven by Alev and Polo [2]:  $A_n(\mathbf{k})^G$  is not isomorphic to  $A_n(\mathbf{k})$  when  $\mathbf{k}$  is algebraically closed, for any non trivial linear action of  $G$ . Hence, positive solutions of the NNP give examples for the question posed by Kirkman et al. [16], asking for which rigid algebras the skew field of fractions and its skew subfield of invariants are isomorphic (two algebras with isomorphic skew field of fractions are called *birationally equivalent*). The Noncommutative Noether’s Problem is also connected to the Gelfand–Kirillov Conjecture on the birational equivalence between the universal enveloping algebras and Weyl algebras. It can be used to reprove the Gelfand–Kirillov Conjecture for  $gl_n$  and  $sl_n$  [9] and show it for all finite  $W$ -algebras of type  $A$  [11].

We will say that the NNP holds for a group  $G$  if it has positive solution for  $G$ . The first goal of our paper is to prove the following statement which was first conjectured in a less general form in [28].

**Theorem 1.1** *For any field  $\mathbf{k}$  of zero characteristic and any linear action of a finite group  $G$  if the quotient variety  $\mathbb{A}^n(\mathbf{k})/G$  is rational then the NNP holds for  $G$ , that is the CNP implies the NNP.*



**Proof** Call  $K := \text{Frac } R$ . We can realize  $D(R)$  as a subset of  $D(K)$  in the following way [22, 15.5.5(iii)]:

$$D(R) = \{d \in D(K) \mid d(R) \subset R\}.$$

Now, since  $K$  is finite field extension of  $\mathbf{k}$ ,  $D(K)$  is a non-commutative domain with finite Gelfand–Kirillov dimension. Since  $D(R)$  is a subring of  $D(K)$ , the same properties hold for it. Hence,  $D(R)$  does not contain a subalgebra isomorphic to the free associative algebra in two variables. It follows, then, by a result of Jategaonkar [18, Proposition 4.13], that  $D(R)$  is an Ore domain.  $\square$

It was shown by Faith [8] that if  $R$  is an Ore domain (both left and right) then any invariant subring  $R^G$  corresponding to a finite group  $G$  is also an Ore domain. In this case,  $R$  admits the skew field of fractions which we denote by  $\mathcal{F}(R)$ . Then the action of  $G$  on  $R$  extends uniquely to an action on  $\mathcal{F}(R)$  and  $(\mathcal{F}(R))^G \simeq \mathcal{F}(R^G)$ . Hence,  $\mathcal{F} : R \mapsto \mathcal{F}(R)$  defines a functor from the category of Ore domains with injective homomorphisms to the category of skew fields. If  $\phi : R_1 \rightarrow R_2$  is a morphism of domains such that  $\mathcal{F}(\phi)$  is an isomorphism of skew fields then we say that  $R_1$  and  $R_2$  are birationally equivalent.

Since  $A_n(\mathbf{k})$  is a Noetherian Ore domain, it admits the skew field of fractions  $F_n(\mathbf{k}) := \mathcal{F}(A_n(\mathbf{k}))$ , called the *Weyl field*.

### 3 CNP implies NNP

In this section we prove our main result: the CNP implies the NNP for an arbitrary field of characteristic zero and arbitrary linear action of a finite group.

Consider an arbitrary finite group  $G$  acting linearly on an  $n$ -dimensional  $\mathbf{k}$ -vector space  $V$  and its naturally extended action on  $\mathcal{O}(V^*) = \mathbf{k}[x_1, \dots, x_n]$ . This action extends to the Weyl algebra  $A_n(\mathbf{k}) \simeq D(\mathcal{O}(V^*))$ . Recall that a positive solution for the Classical Noether's Problem for this action means that  $\text{Frac}(\mathcal{O}(V^*)^G) \cong \mathbf{k}(x_1, \dots, x_n)$ .

Denote  $\mathcal{B}$  the subalgebra of  $A_n(\mathbf{k})^G$  generated by  $\mathcal{O}(V^*)^G = \mathbf{k}[x_1, \dots, x_n]^G$  and  $\mathcal{O}(V)^G = \mathbf{k}[\partial_1, \dots, \partial_n]^G$ . Then  $\mathcal{B} = A_n(\mathbf{k})^G$  by [19], Theorem 5. We will closely follow the argument in the proof of this fact.

Set  $S = \mathcal{O}(V^*)^G \setminus \{0\}$ . Since  $S$  is ad-nilpotent on  $A_n(\mathbf{k})$ , and hence on  $\mathcal{B} = A_n(\mathbf{k})^G$ , it is an Ore set in both algebras [18, Theorem 4.9].

Denote  $F := \text{Frac}(\mathcal{O}(V^*)^G)$ .

Recall the following lemma from [19]:

**Lemma 3.1** [19, Lemma 8] *Let  $L$  be a finite field extension of  $\mathbf{k}$ , with  $\text{trdeg}_{\mathbf{k}} L = l$ , and consider the ring of differential operators on  $L$ ,  $D(L)$ . Let  $A$  be a subalgebra of  $D(L)$  containing  $L$ , with a filtration induced from that of  $D(L)$  (by order of differential operator). If the associated graded algebra of  $A$  contains as a subalgebra a finitely generated graded  $L$ -algebra  $B$  such that  $K\dim B = l$ , then  $A = D(L)$ .*

**Lemma 3.2**  $\text{Frac}(A_n(\mathbf{k})^G) \cong \text{Frac}(D(F))$ .

**Proof** We have  $\mathcal{B} \subset A_n(\mathbf{k})^G$  by definition. On the other hand,  $A_n(\mathbf{k})^G \subset D(\mathcal{O}(V^*)^G)$  by restriction of domain. We have

$$D(\mathcal{O}(V^*)^G)_S = D(\mathcal{O}(V^*)_S^G) = D(F)$$

by [26], Proposition 1.8. After localization by  $S$  we obtain:

$$\mathcal{B}_S \subset A_n(\mathbf{k})_S^G \subset D(\mathcal{O}(V^*)^G)_S = D(F).$$

Consider the filtration on  $\mathcal{B}_S$  induced from  $D(F)$ . Since  $\mathcal{O}(V^*)^G \subset \mathcal{B}$ , we have that  $gr \mathcal{B}_S$  contains  $F \otimes \mathcal{O}(V^*)^G$  as a graded  $F$ -subalgebra. Since  $\mathcal{O}(V^*)$  is finite over  $\mathcal{O}(V^*)^G$  then it has the Krull dimension  $n$  [22]. Applying Lemma 3.1 we have  $\mathcal{B}_S = D(F)$ . We conclude that  $\text{Frac}(D(F)) \subset \text{Frac}(A_n(\mathbf{k})^G) \subset \text{Frac}(D(F))$  which implies the desired equality.  $\square$

**Remark 3.3** Note that in fact in the above proof we do not need the equality  $\mathcal{B} = A_n(\mathbf{k})^G$  but only the embedding  $\mathcal{B} \subset A_n(\mathbf{k})^G$  by [19, Lemma 9].

As a consequence of Lemma 3.2 we immediately obtain our main result.

**Theorem 3.4** *The CNP implies the NNP for any linear representation of a finite group over any field of characteristic zero.*

**Proof** Under the condition of the theorem we have that  $F \cong \mathbf{k}(x_1, \dots, x_n)$ . Then  $D(F)$  is isomorphic to the localization of  $A_n$  by  $\mathbf{k}[x_1, \dots, x_n] \setminus \{0\}$  and the statement follows.  $\square$

We immediately have the following application of the main result

**Corollary 3.5** *The Noncommutative Noether's Problem holds in the following cases for any field of characteristic zero:*

- for all linear representations of all pseudo-reflection groups;
- for alternating groups  $A_n$  with usual permutation action for  $n = 3, 4, 5$ ;
- for any group when  $n = 3$  and  $\mathbf{k}$  is algebraically closed.

We note that Theorem 3.4 allows to recover all results for finite groups from [1] and all results from [7]. Also it allows us to give a shorter proof of the following fact shown in [1].

**Theorem 3.6** [1] *If  $F_n(\mathbf{k})^G$  is isomorphic to  $F_m(L)$  for some  $m$  and some purely transcendental extension  $L$  of  $\mathbf{k}$  of transcendence degree  $t$ , then  $m = n$  and  $t = 0$ .*

**Proof** We have that  $F_m(L) \simeq \text{Frac}(D(F))$ . Now use [6, Lemma 3.2.2]. The center of  $F_m(L)$  has the transcendence degree  $t$  over  $\mathbf{k}$ . By the primitive element theorem,  $F = \mathbf{k}(y_1, \dots, y_n)(f)$ , for certain algebraically independent  $y_1, \dots, y_n$ , and by [22, 15.2.4], the second skew-field has center of transcendence degree 0. So  $t = 0$  and  $m = n$ .  $\square$

## 4 Proof of Theorem 1.2

Let  $X$  be an affine variety over  $\mathbf{k}$  with the coordinate ring  $\mathcal{O}(X)$ . The ring of differential operators  $D(X)$  on  $X$  is defined as  $D(\mathcal{O}(X))$ . If  $X$  is irreducible then  $D(X)$  is an Ore domain by Proposition 2.1. If  $G$  is a finite group action of the variety  $X$ , then this action extends to an action on the ring of differential operators. Throughout this section, the field  $\mathbf{k}$  will be the field of complex numbers.

The following result was established by Cannings and Holland [5]:

**Theorem 4.1** *Let  $X$  be an affine irreducible algebraic variety over  $\mathbb{C}$  with an action of a finite group  $G$  on it.*

- (1) *There exists a maximal open dense  $G$ -invariant subset  $V \subset X$ , on which the induced action of  $G$  is free. Let  $\pi : X \rightarrow X/G$  be the canonical projection and  $V' = \pi(V)$ . Then  $V'$  is open dense in  $X/G$  and, since  $V$  is a complete pre-image,  $V = \pi^{-1}(V')$ , the map  $\pi$  restricts to the quotient map:*

$$\pi|_V : V \rightarrow V'.$$

- (2) *If  $G$  acts freely, we have the following isomorphism  $D(X)^G \simeq D(X/G)$ .*

**Lemma 4.2** *Let  $X$  be a complex irreducible affine variety. Suppose that a finite group  $G$  is acting by automorphisms on  $X$  and that the action is free. If  $X/G$  is birationally equivalent to an affine irreducible variety  $Y$  then  $\text{Frac}(D(X)^G) \cong \text{Frac}(D(Y))$ .*

**Proof** Let  $S$  be the set of regular elements in  $\mathcal{O}(X)^G$ . Since  $X/G$  is birational to  $Y$  we have

$$\text{Frac}(\mathcal{O}(X)^G) = \text{Frac}(\mathcal{O}(X)_S^G) \cong \text{Frac}(\mathcal{O}(Y)).$$

Since  $\text{Frac}(D(X)^G) \cong \text{Frac}(D(X/G))$  by Theorem 4.1, then applying [26, Proposition 1.8], we have

$$\text{Frac}(D(X)^G) \cong \text{Frac}(D(X/G)_S) \cong \text{Frac}(D(\mathcal{O}(X)_S^G)) \cong \text{Frac}D(\text{Frac}(\mathcal{O}(Y))),$$

and hence  $\text{Frac}(D(X)^G) \cong \text{Frac}(D(Y))$ . Note that unlike in a similar statement in [22] we do not assume the variety  $X$  to be smooth.  $\square$

We proceed to the proof of Theorem 1.2.

Suppose  $X$  is a complex irreducible affine variety. Fix a finite group  $G$  acting on  $X$  and satisfying the hypotheses of the theorem. By Theorem 4.1, (1), there exists an open dense subset  $V$  on which the action of  $G$  is free and such that the quotient map  $\pi : X \rightarrow X/G$  restricts to the quotient  $\pi : V \rightarrow V'$ , where  $V'$  is open dense in  $X/G$ . By the Hilbert–Noether theorem, the map  $\pi$  is finite, hence affine [12, Exercise 5.17]. Let  $W'$  be a principal open subset of  $V'$ . Since  $\pi$  is affine, then  $W = \pi^{-1}(W')$  is affine. Since  $W$  is a union of orbits,  $G$  restricts to a free action on it. We now have a quotient map  $\pi : W \rightarrow W'$  with  $W$  affine and smooth (hence normal). Also, since  $W' \subset X/G$ , then  $W'$  is birational to  $Y$ . Applying Lemma 4.2 we obtain

**Lemma 4.3**  $\text{Frac}(D(W)^G) \cong \text{Frac}(D(Y))$ .

For  $f \in \mathcal{O}(X)$  denote  $\text{Spec } \mathcal{O}(X)_f$  the principal open subset. These sets constitute a basis of the Zariski topology, and hence there exists a principal open subset  $\text{Spec } \mathcal{O}(X)_h \subset W$  for  $h \in \mathcal{O}(X)$ . Set  $f = \prod_{g \in G} g.h$ . Then  $f$  is  $G$ -invariant. Thus we have

**Lemma 4.4** *There exists a principal open set  $\text{Spec } \mathcal{O}(X)_f \subset W$  with  $G$ -invariant  $f$ .*

Now we generalize the argument given in the proof for unitary reflection groups in [7]. By Lemma 4.4 there exists a principal open set  $\text{Spec } \mathcal{O}(X)_f \subset W$  with  $G$ -invariant  $f$ . Then we have the following inclusions of varieties:  $\text{Spec } \mathcal{O}(X)_f \subset W \subset X$ . Let  $D(\cdot)$  be the sheaf of differential operators functor which associates the ring of differential operators to a given variety. Functor  $D(\cdot)$  is contravariant and we have chain of inclusions

$$D(\mathcal{O}(X)) \subset D(W) \subset D(\mathcal{O}(X)_f) = D(\mathcal{O}(X))_f$$

(cf. [20, Proposition 2.4.18]). Taking the field of fractions, and then the  $G$ -invariants, we have the following chain:

$$\begin{aligned} \text{Frac } D(\mathcal{O}(X))^G &\subset \text{Frac } (D(W))^G \subset \text{Frac } (D(\mathcal{O}(X))_f)^G \\ &= \text{Frac } (D(\mathcal{O}(X))_f^G) = \text{Frac } (D(\mathcal{O}(X))^G). \end{aligned}$$

Then applying Lemma 4.3 we have

$$\text{Frac } (D(X))^G \simeq \text{Frac } (D(W))^G \cong \text{Frac } (D(Y)),$$

which implies the statement of Theorem 1.2. We obtain Corollary 1.3 when  $Y = X$  or  $Y = \mathbb{A}^n(\mathbb{C})$ ,  $n = \dim X$ , respectively.

**Example 4.5** The following gives an example of the situation in Corollary 1.3(a). Namely, consider an elliptic curve  $E$  and define the map  $\tau : E \rightarrow E$  which sends  $P \mapsto P + P$  (multiplication by 2). The map  $\tau$  is an isogeny, and hence it is surjective with finite kernel,  $E[2]$  ([23] I.7). Since  $E$  is an abelian variety, we can view  $E[2]$  as a finite group of automorphisms of  $E$  with the action given by translations:  $Q \in E[2]$  maps  $P \in E$  to  $Q + P$ . With this we have  $E/E[2] \simeq E$ . Removing a finite number of points from  $E$  and from the corresponding inverse image, we obtain desired birational affine varieties.

## 5 NNP for pseudo-reflection groups

By the Chevalley–Shephard–Todd theorem the CNP holds for all pseudo-reflection groups over any field of zero characteristic. In this section we give an alternative proof that the Noncommutative Noether's Problem has a positive solution for all pseudo-reflection groups over any field of zero characteristic (for complex reflection groups this was shown in [7, Theorem 2]). Such approach allows us to exhibit explicitly the Weyl generators of the invariant skew subfield of  $F_n(\mathbf{k})$  using a simple algorithm.

### 5.1 Localizations and invariants

As before  $\Lambda$  denote the polynomial algebra over  $\mathbf{k}$  with  $n$  variables. Let  $W$  be an arbitrary pseudo-reflection group acting by linear automorphisms on  $\Lambda$ . Recall the following statement independent on the field  $\mathbf{k}$  [7, Proposition 1]:

**Proposition 5.1** *Let  $\Delta$  be a  $W$ -invariant element of  $\Lambda$ ,  $S$  a multiplicatively closed set in  $\Lambda$ . Then*

- (1)  $(\Lambda_\Delta)^W = (\Lambda^W)_\Delta$ ;
- (2)  $D(\Lambda_S) = D(\Lambda)_S$ ;
- (3)  $(D(\Lambda)_\Delta)^W \cong (D(\Lambda^W)_\Delta)$ .

Consider a  $W$ -invariant element  $\Delta \in \Lambda$ , localization  $\Lambda_\Delta$  with the induced action of  $W$  and the  $W$ -invariants  $\Lambda_\Delta^W$  in  $\Lambda_\Delta$ . We have an embedding  $\Lambda_\Delta^W \rightarrow \Lambda_\Delta$ . By the restriction of domain we have an induced map

$$\phi_\Delta : D(\Lambda_\Delta)^W \rightarrow D(\Lambda_\Delta^W).$$

**Proposition 5.2** *Let  $\Delta$  be a  $W$ -invariant element in  $\Lambda$ . Then the map  $\phi_\Delta$  is injective.*

**Proof** Note that  $D(\Lambda_\Delta)$  is a simple ring and  $W$  acts by outer automorphisms. Then  $D(\Lambda_\Delta)^W$  is a simple ring, by [25, Corollary 2.6]. Since  $\phi_\Delta$  is not trivial, it is injective.  $\square$

Our goal now is to find an adequate  $\Delta$  such that the  $\phi_\Delta$  is surjective. The case  $W = S_n$  was considered in [10].

## 5.2 Proof of the NNP for irreducible pseudo-reflection groups

We proceed by considering first irreducible pseudo-reflection groups. Recall that a pseudo-reflection group  $W$  is called *irreducible* if its natural representation is irreducible.

We will make use of the following notion of the *field of definition* of  $G$  - the smallest subfield where a representation of the group  $G$  is defined. More precisely,

**Definition 5.3** Let  $\rho : G \rightarrow GL_n(\mathbb{k})$  be a linear representation of a finite group  $G$ . Let  $\mathbb{k}' \subset \mathbb{k}$  be a subfield. Suppose there exists a homomorphism  $\rho' : G \rightarrow GL_n(\mathbb{k}')$  such that  $\rho$  can be obtained from  $\rho'$  by the extension of scalars. We say that  $\rho$  has  $\mathbb{k}'$  as the field of definition if  $\mathbb{k}'$  is the smallest subfield with this property.

Given a linear representation  $\rho : G \rightarrow GL_n(\mathbb{k})$  denote by  $\chi$  the corresponding character function. Let  $\mathbb{Q}(\chi)$  be the field extension of  $\mathbb{Q}$  by  $Im \chi$ .

By [15, Appendix B], we have

**Proposition 5.4** Let  $W$  be an irreducible pseudo-reflection group and  $\rho : W \rightarrow GL_n(\mathbb{k})$  a representation of  $W$ . Then  $\rho$  has  $\mathbb{Q}(\chi)$  as the field of definition.

We shall also need the following fact from the invariant theory of pseudo-reflection groups. Let  $M$  be the  $n \times n$  matrix whose  $ij$ 's entry is  $\partial_{x_j} e_i$ , where  $\Lambda^W = \mathbb{k}[x_1, \dots, x_n]^W = \mathbb{k}[e_1, \dots, e_n]$ . Let  $J'$  be the determinant of  $M$ .

Let  $\mathcal{S}$  be the set of all pseudo-reflections in  $W$ . Each  $s \in \mathcal{S}$  fixes a hyperplane  $H_s$ . Let  $L_s$  be a linear form whose kernel is  $H_s$  for each  $s \in \mathcal{S}$ . Set  $J = \prod_{s \in \mathcal{S}} L_s$ . It has the following properties:

**Proposition 5.5** [15, 20-2, Proposition A and B, 21-1, Proposition A and B]  $J \neq 0$  and  $w.J = \det(w)J$  for every  $w \in W$ . Moreover,  $J$  is a multiple of  $J'$ .

As in the case of complex reflection groups set  $\Delta = J^{|W|}$  [7, Section 3].

Let  $E_i, i = 1, \dots, n$  be the column vector, where we have 1 in the  $i$ th position and 0 in all others. Let

$$F_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in} \end{pmatrix}$$

be a solution of the linear system

$$(*) \quad M F_i = E_i.$$

By the Kramer's rule,  $f_{ij} \in \Lambda_J$ ,  $1 \leq i, j \leq n$ , where  $\Lambda_J$  is the localization of  $\Lambda$  by  $J$ .

For each  $i = 1, \dots, n$  set  $d_i = \sum_{k=1}^n f_{ik} \partial_k$ . Then  $d_i \in D(\Lambda_\Delta) = D(\Lambda)_\Delta$  and we have  $d_i(e_j) = \delta_{ij}$ ,  $i, j = 1, \dots, n$ .

We will show that all differential operators  $d_i$ ,  $i = 1, \dots, n$  are  $W$ -invariant. By Theorem 5.4 we can assume that  $e_i$ 's, and hence  $d_i$ 's, have coefficients in  $\mathbb{Q}(\chi)$ . Observe the following: let  $\mathbb{k}' \subset \mathbb{k}$  be a subfield fixed by  $W$  and  $d$  a differential operator with coefficients in  $\mathbb{k}'$ , then the question of  $W$ -invariance of  $d$  is the same, whether we consider the base field  $\mathbb{k}$  or  $\mathbb{k}'$ . As  $\mathbb{Q}(\chi)$  is fixed by  $W$ , to show that the  $d_i$ 's are invariant differential operators on  $\Lambda_\Delta^W$ , we

can replace  $\mathbf{k}$  by  $\mathbb{Q}(\chi)$ . Now our field of definition is a subfield of  $\mathbb{C}$ . Repeating the above argument we can assume that  $\mathbf{k} = \mathbb{C}$ .

Recall the following result of Knop:

**Theorem 5.6** [17, Theorem 3.1]. *Let  $X$  be a complex affine irreducible normal variety. Then  $D(X)^W = \{d \in D(X) | d(\mathcal{O}(X)^W) \subseteq \mathcal{O}(X)^W\}$ .*

Denote by  $\Delta'$  the result of expressing  $\Delta$  in a polynomial on the  $e_i$ ,  $i = 1, \dots, n$ . By the Chevalley-Shephard-Todd theorem,  $\Lambda_{\Delta}^W \simeq \mathbf{k}[e_1, \dots, e_n]_{\Delta'}$ . Taking into account the action of operators  $d_i$ 's and Theorem 5.6 we obtain the desired invariance of  $d_i$ 's under the action of  $W$ . Indeed, for each  $i = 1, \dots, n$  the operator  $d_i$  sends every  $e_j$  to an element of  $\mathbf{k}[e_1, \dots, e_n]_{\Delta'}$  and the same holds for  $\Delta'$ . Since these elements generate  $\mathbf{k}[e_1, \dots, e_n]_{\Delta'}$  the statement follows.

**Proposition 5.7** *The map  $\phi_{\Delta} : D(\Lambda_{\Delta})^W \rightarrow D(\Lambda_{\Delta}^W)$  is surjective.*

**Proof** It is sufficient to show that the images of  $d_i$ ,  $e_i$ ,  $i = 1, \dots, n$  under  $\phi_{\Delta}$  are the Weyl generators of  $D(\Lambda_{\Delta}^W)$ . Let  $A := \Lambda_{\Delta}^W$ . The  $A$ -module of Kähler differentials  $\Omega_{\mathbf{k}}(A)$  is freely generated over  $A$  with basis  $d_{e_1}, \dots, d_{e_n}$ . Then, by [22, 15.1.12], the  $A$ -module of derivations  $\text{Der}_{\mathbf{k}}(A)$  is freely generated by the unique extensions of  $\partial_{e_i}$ ,  $i = 1, \dots, n$  from  $\mathbf{k}[e_1, \dots, e_n]$  to  $A$ . Clearly,  $\phi_{\Delta}(d_i) = \partial_{e_i}$ ,  $i = 1, \dots, n$ .  $\square$

Combining Proposition 5.2 and Proposition 5.7 we conclude

$$D(\Lambda_{\Delta})^W \simeq D(\Lambda_{\Delta}^W).$$

Applying Proposition 5.1 we finally have

**Corollary 5.8** *Let  $\mathbf{k}$  be an arbitrary field of zero characteristic and  $W$  an irreducible pseudo-reflection group. Then the NNP holds for  $W$ .*

### 5.3 Proof of the NNP for general pseudo-reflection groups

In this subsection we consider general pseudo-reflection groups.

Let  $V$  be a finite dimensional vector space. If  $g$  is a linear automorphism of  $V$  then we set  $\text{Fix } g = \{v \in V | gv = v\} = \text{Ker}(Id - g)$ , and  $[V, g] = \text{Im}(Id - g)$ .

If  $g$  is a pseudo-reflection,  $g \neq id$  then  $\text{Fix } g$  is a hyperplane and  $[V, g]$  is one dimensional. If  $a \in V$  generates  $[V, g]$  then for every  $v \in V$  there exists  $\psi(v) \in \mathbf{k}$  such that  $v - gv = \psi(v)a$ . Then  $\psi$  is a linear functional on  $V$  and  $\text{Ker } \psi = \text{Fix } g$ .

In the following we collect basic properties of pseudo-reflections.

- Lemma 5.9** (1) *Let  $g$  be a pseudo-reflection of order  $m > 1$ ,  $H = \text{Fix } g$ ,  $L_H$  any linear functional such that  $H = \text{Ker } L_H$ . Let  $a$  be a generator of  $[V, g]$ . Then there exists an  $m$ -th primitive root of unity  $\mu$  such that  $gv = v - (1 - \mu) \frac{L_H(v)}{L_H(a)} a$ , for all  $v \in V$ .*
- (2) *Let  $r, s \neq id$  be pseudo-reflections,  $H = \text{Fix } r$ ,  $J = \text{Fix } s$ ,  $x$  a generator of  $[V, r]$ , and  $y$  a generator of  $[V, s]$ . If  $x \in J$  and  $y \in H$  then  $rs = sr$ .*
- (3) *A subspace  $V' \subset V$  is invariant by a pseudo-reflection  $g \neq id$  if and only if  $V' \subseteq \text{Fix } g$  or  $[V, g] \subseteq V'$ .*

**Proof** Given a pseudo-reflection  $g$  consider a linear functional  $\psi$  such that  $gv = v - \psi(v)a$  and  $H = \text{Ker } \psi$ , as above. Hence  $gx = \mu x$  for a primitive  $m$ -th root of unity  $\mu$ , and hence

$\psi(a) = 1 - \mu$ . We have  $\psi = \lambda L_H$ , where  $0 \neq \lambda \in \mathbf{k}$ . This gives  $\lambda = (1 - \mu)/L_H(a)$  and implies statement (1). Applying (1), we have  $\mu, \nu \in \mathbf{k}$  such that  $\forall v \in V$

$$rs(v) = v - (1 - \mu) \frac{L_H(v)}{L_H(a)} x - (1 - \nu) \frac{L_J(v)}{L_J(y)} y + (1 - \mu)(1 - \nu) \frac{L_H(y)L_J(v)}{(L_H(x)L_J(y))} x.$$

If  $y \in H$  then  $L_H(y) = 0$  and the last term is 0. Analogously, the last term in the expression of  $sr(v)$  is 0 and other terms in both expressions are equal. Therefore  $rs = sr$ .

Finally, if  $V' \subseteq \text{Fix } g$  or  $[V, g] \subseteq V'$ , then  $V'$  is invariant by the statement (1). Conversely, if  $V'$  is  $g$ -invariant and is not contained in  $\text{Fix } g$ , then  $[V', g] \neq 0$ , and hence  $[V, g] = [V', g] \subseteq V'$ .  $\square$

The following is probably well known but we include the proof for the sake of completeness.

**Proposition 5.10** *Let  $W$  be a finite group of pseudo-reflections on  $V$ . Consider a decomposition  $V_1 \oplus \cdots \oplus V_m$  of the  $\mathbf{k}W$ -module  $V$  into irreducible submodules and set  $W_i$  to be the restriction of  $W$  to  $V_i$ ,  $i = 1, \dots, m$ . Then  $W_i$  is either a pseudo-reflection group or trivial, and  $W \cong W_1 \times \cdots \times W_m$ .*

**Proof** By Lemma 5.9, (3), if  $g$  is a non-identity pseudo-reflection then  $[V, g] \subseteq V_i$  for some  $i$ . Let  $W_i$  be the subgroup of  $W$  generated by the pseudo-reflections  $g$  such that  $[V, g] \subset V_i$  (if there is no such pseudo-reflections then  $W_i = \text{Id}$ ). The subgroup  $W_i$  acts trivially on all  $V_j$ ,  $j \neq i$ , and by By Lemma 5.9, (2),  $W_i$  and  $W_j$  commute. Therefore,  $W$  is the direct product of the subgroups  $W_i$ 's, and each  $W_i$  is irreducible pseudo-reflection group on  $V_i$ , or trivial.  $\square$

Consider now the Weyl algebra  $A_n(\mathbf{k})$  with a linear action of a pseudo-reflection group  $W$  extended from a linear action on  $n$ -dimensional vector space  $V$ . By Proposition 5.10 we have  $W \cong W_1 \times \cdots \times W_m$ . Suppose that  $n = n_1 + \cdots + n_m + k$  and  $A_n(\mathbf{k}) = A_{n_1}(\mathbf{k}) \otimes \cdots \otimes A_{n_m}(\mathbf{k}) \otimes A_k(\mathbf{k})$ . Then for each  $i = 1, \dots, m$ ,  $W_i$  acts on  $A_{n_i}(\mathbf{k})$  and fixes all  $A_{n_j}(\mathbf{k})$  with  $j \neq i$ . The whole group  $W$  fixes  $A_k(\mathbf{k})$ . Then we have

$$A_n(\mathbf{k})^W \cong A_{n_1}(\mathbf{k})^{W_1} \otimes \cdots \otimes A_{n_m}(\mathbf{k})^{W_m} \otimes A_k(\mathbf{k}).$$

Applying Corollary 5.8 we immediately obtain

**Theorem 5.11** *The NNP holds for all pseudo-reflection groups over fields of zero characteristic.*

## 5.4 Algorithmic aspects

We now present an algorithmic procedure how to exhibit explicitly the Weyl generators in  $F_n(\mathbf{k})^W$  which realize its isomorphism with  $F_n(\mathbf{k})$  for an irreducible pseudo-reflection group  $W$ .

Given an irreducible pseudo-reflection group  $W$  acting on  $\mathbf{k}[x_1, \dots, x_n]$ , the Weyl generators in  $F_n(\mathbf{k})^W$  are, as we saw, the algebraically independent generators  $e_1, \dots, e_n$  of  $\mathbf{k}[x_1, \dots, x_n]^W$ , and the operators  $d_i$ ,  $i = 1, \dots, n$  introduced immediately after the linear equation (\*) (cf. Proposition 5.7). So the procedure of finding the Weyl generators reduces to essentially 3 steps. The first step is the classical problem of finding generators of the ring of invariant polynomials under a finite group action. In our case, we are looking for a minimal generating set and suitable algorithms are known (see, [4]). We also note that for

finite Coxeter groups explicit invariant bases are well known ([13] and references for 3.12). The second step is to obtain the matrix  $M$  in (\*), which is done by  $n^2$  operations of formal partial derivations. Finally, the third step is the solution of the system (\*). We illustrate this algorithm with the following examples.

**Example 5.12** Assume  $n = 3$  and  $W = S_3$ . Set  $J = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ . The following elements are the Weyl generators of  $F_3(\mathbf{k})^{S_3}$ , where  $S_3$  acts by permutations:

$$\begin{aligned} x_1 + x_2 + x_3 &\rightarrow X_1, \quad x_1x_2 + x_2x_3 + x_1x_3 \rightarrow X_2, \quad x_1x_2x_3 \rightarrow X_3; \\ \frac{x_1^2(x_2 - x_3)}{J} \partial_1 + \frac{x_2^2(x_3 - x_1)}{J} \partial_2 + \frac{x_3^2(x_1 - x_2)}{J} \partial_3 &\rightarrow Y_1; \\ \frac{x_1(x_3 - x_2)}{J} \partial_1 + \frac{x_2(x_1 - x_3)}{J} \partial_2 + \frac{x_3(x_2 - x_1)}{J} \partial_3 &\rightarrow Y_2; \\ \frac{(x_2 - x_3)}{J} \partial_1 + \frac{(x_3 - x_1)}{J} \partial_2 + \frac{(x_1 - x_2)}{J} \partial_3 &\rightarrow Y_3. \end{aligned}$$

Here,  $Y_i X_j - X_j Y_i = \delta_{ij}$  for  $i, j = 1, 2, 3$ .

**Example 5.13** Assume  $n = 2$  and  $W = B_2$ . Setting  $J = 8x_1x_2(x_2^2 - x_1^2)$ , we have the following Weyl generators of  $F_2(\mathbf{k})^{B_2}$ :

$$\begin{aligned} x_1^2 + x_2^2 &\rightarrow X_1, \quad x_1^4 + x_2^4 \rightarrow X_2; \\ \frac{4x_2^3 \partial_1 - 4x_1^3 \partial_2}{J} &\rightarrow Y_1; \\ \frac{-2x_2 \partial_1 + 2x_1 \partial_2}{J} &\rightarrow Y_2. \end{aligned}$$

**Example 5.14** Assume  $n = 2$  and  $W = I_2(8)$ , the dihedral group of order 16. Setting  $J = x_1^7x_2 - 7x_1^5x_2^3 + 7x_1^3x_2^5 - x_1x_2^7$ , we have the following Weyl generators of  $F_2(\mathbf{k})^{I_2(8)}$ :

$$\begin{aligned} x_1^2 + x_2^2 &\rightarrow X_1, \quad (1/4)x_1^6x_2^2 - (1/2)x_1^4x_2^4 + (1/4)x_1^2x_2^6 \rightarrow X_2; \\ \frac{((1/2)x_1^6x_2 - 2x_1^4x_2^3 + (3/2)x_1^2x_2^5)\partial_1 - ((1/2)x_1x_2^6 - 2x_1^3x_2^4 + (3/2)x_1^5x_2^2)\partial_2}{J} &\rightarrow Y_1; \\ \frac{-2x_2 \partial_1 + 2x_1 \partial_2}{J} &\rightarrow Y_2. \end{aligned}$$

## 6 Invariant cross products

In this section we apply the result above to the subalgebras of invariants of cross products.

Let  $G$  be a finite group of automorphisms of field  $L$ ,  $\mathcal{M}$  a monoid of automorphisms of  $L$  on which  $G$  acts by conjugations. Denote by  $L * \mathcal{M}$  the cross product, where  $(lm)(l'm') = (lm')(mm')$  for  $l, l' \in L$  and  $m, m' \in \mathcal{M}$ . We have a well defined action of  $G$  on the cross product  $L * \mathcal{M}$ :  $g(lm) = g(l)g(m)$ ,  $g \in G$ ,  $l \in L$ ,  $m \in \mathcal{M}$ . Consider the ring of invariants  $(L * \mathcal{M})^G$  by the action of  $G$ .

Suppose  $L * \mathcal{M}$  is an Ore domain. Then  $(L * \mathcal{M})^G$  is an Ore domain and the skew field of fractions  $\mathcal{F}((L * \mathcal{M})^G)$  is isomorphic to  $(\mathcal{F}(L * \mathcal{M}))^G$  with induced actions of  $G$  on the skew field of fractions.

Assume  $L \simeq \mathbf{k}(t_1, \dots, t_n)$  to be the field of fractions of the symmetric algebra  $S(V)$  for some  $n$ -dimensional  $\mathbf{k}$ -vector space  $V$ . If  $G < GL_n$  is a finite group then it acts linearly on  $L$ . If  $G$  normalizes  $\mathcal{M}$  then  $(L * \mathcal{M})^G$  is a *linear Galois order* [7]. Suppose

$L = \mathbf{k}(t_1, \dots, t_n; z_1, \dots, z_m)$ , for some integers  $n, m$ ,  $\mathcal{M} \simeq \mathbb{Z}^n$  is generated by  $\varepsilon_1, \dots, \varepsilon_n$ , where  $\varepsilon_i(t_j) = t_j + \delta_{ij}$ ,  $\varepsilon_i(z_k) = z_k$   $i, j = 1, \dots, n$ ,  $k = 1, \dots, m$  (in this case we say that  $\mathcal{M}$  acts by shifts on  $L$ ). It was shown in [7, Theorem 6], that for such cross products and for any complex reflection group  $G$ ,  $(L * \mathcal{M})^G$  is birationally equivalent to  $A_n(\mathbf{k}) \otimes \mathbf{k}[z_1, \dots, z_m]$ . Applying Theorem 3.4 we can extend this result to other groups.

**Theorem 6.1** *Let  $\mathbf{k}$  has characteristic zero,  $L = \mathbf{k}(t_1, \dots, t_n; z_1, \dots, z_m)$ , for some integers  $n, m$ , and  $\mathcal{M} \simeq \mathbb{Z}^n$  acting by shifts on  $L$ . Then*

- (1)  *$(L * \mathcal{M})^G$  is birationally equivalent to  $A_n(\mathbf{k}) \otimes \mathbf{k}[z_1, \dots, z_m]$  for any pseudo-reflection group  $G$ ;*
- (2) *If  $L^G \simeq L$  for a given group  $G$  then  $(L * \mathcal{M})^G$  is birationally equivalent to  $A_n(\mathbf{k}) \otimes \mathbf{k}[z_1, \dots, z_m]$ .*

**Proof** Indeed, we have an embedding of the Weyl algebra  $A_n(\mathbf{k})$  to  $\mathbf{k}[t_1, \dots, t_n] * \mathbb{Z}^n$  and their skew fields of fractions are isomorphic [9]. Hence, item (1) follows from Theorem 5.11. If  $L^G \simeq L$  then the CNP holds and (2) follows from Theorem 3.4.  $\square$

We finish with the following problem:

**Problem:** Find an example of a linear action of a finite group such that the CNP does not hold but the NNP holds.

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