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Generalized Verma Modules over $U_q(\mathfrak{sl}_n(\mathbb{C}))$

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Abstract
We construct realizations of quantum generalized Verma modules for $U_q(\mathfrak{sl}_n(\mathbb{C}))$ by quantum differential operators. Taking the classical limit $q \to 1$ provides a realization of classical generalized Verma modules for $\mathfrak{sl}_n(\mathbb{C})$ by differential operators.

Keywords Quantum group · Quantum Weyl algebra · Generalized Verma module

1 Introduction

Generalized Verma modules for complex simple finite-dimensional Lie algebras play an important role in representation theory of Lie algebras. They were first introduced by Garland and Lepowsky in [9]. The theory was further developed by many authors, see [1, 2, 4–8, 11, 15, 16, 18, 20, 21] and references therein. The generalized Verma modules are a natural generalization of the Verma modules defined in [24], they are obtained by the *parabolic* induction for a given choice of a parabolic subalgebra. When a parabolic subalgebra coincides with a Borel subalgebra we obtain the corresponding Verma module. The importance of generalized Verma modules was shown in [2, 4, 6, 8] by proving that any *weight* (with respect to a fixed Cartan subalgebra) simple module over a complex simple finite-dimensional Lie algebra $\mathfrak{g}$ is either *cuspidal* or a quotient of a certain generalized Verma module, which in turn is obtained by a parabolic induction from the simple weight module over the Levi factor of the parabolic subalgebra. Let us note that the concept of cuspidality depends whether the weight subspaces have finite or infinite dimension [6, 8]. Also, the structure theory of generalized Verma modules differs significantly depending on

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whether the inducing module over the Levi subalgebra is cuspidal or not. The case of cuspidal inducing modules with finite-dimensional weight spaces was fully settled in [20] where it was shown that the block of the category of such modules is equivalent to certain blocks of the category $\mathcal{O}$. On the other hand, the classical construction of generalized Verma modules in [9] uses finite-dimensional inducing modules over the Levi subalgebra. Such induced modules have certain universal properties but at the same time they are quotients of the corresponding Verma modules.

It is always useful and important to have a concrete realization of simple modules in terms of differential operators. Such realizations for different representations of $\mathfrak{sl}_n(\mathbb{C})$ can be obtained, for instance, via the embedding into the Witt algebra W_{n-1} [23]. The purpose of the present paper is to study quantum deformations of the generalized Verma modules and construct realizations of these modules (which are simple generically) by quantum differential operators (Theorems 4.1 and 4.2). We note that our construction holds for finite and infinite-dimensional inducing modules over parabolic subalgebras. Similar realizations can be be constructed for quantum groups of all types. Taking the classical limit $q \to 1$ provides a realization of classical generalized Verma modules by differential operators.

Throughout the article we use the standard notation $\mathbb{N}$ and $\mathbb{N}_0$ for the set of positive integers and the set of nonnegative integers numbers, respectively.

2 Quantum Weyl Algebras

For $q \in \mathbb{C}^\times$ satisfying $q \neq \pm 1$ and $v \in \mathbb{C}$, the q-number $[v]_q$ is defined by

$$[v]_q = \frac{q^v - q^{-v}}{q - q^{-1}}. \tag{2.1}$$

If $n \in \mathbb{N}_0$, then we introduce the q-factorial $[n]_q!$ by

$$[n]_q! = \prod_{k=1}^{n} [k]_q. \tag{2.2}$$

The q-binomial coefficients are defined by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{2.3}$$

where $n, k \in \mathbb{N}_0$ and $n \geq k$.

Let us consider an associative $\mathbb{C}$-algebra A. Let $\sigma \colon A \to A$ be a $\mathbb{C}$-algebra automorphism. Then a twisted derivation of A relative to σ is a linear mapping $D \colon A \to A$ satisfying

$$D(ab) = D(a)\sigma(b) + \sigma^{-1}(a)D(b) \tag{2.4}$$

for all $a, b \in A$. An element $a \in A$ induces an inner twisted derivation $\mathrm{ad}_\sigma a$ relative to σ defined by the formula

$$(\mathrm{ad}_\sigma a)(b) = a\sigma(b) - \sigma^{-1}(b)a \tag{2.5}$$

for all $a, b \in A$. Let us note that also $D_\sigma = \sigma - \sigma^{-1}$ is a twisted derivation of A relative to σ.

Lemma 2.1 *Let D be a twisted derivation of A relative to σ. Then we have*

$$\begin{aligned} \sigma \circ \lambda_a &= \lambda_{\sigma(a)} \circ \sigma, & D \circ \lambda_a - \lambda_{\sigma^{-1}(a)} \circ D &= \lambda_{D(a)} \circ \sigma, \\ \sigma \circ \rho_a &= \rho_{\sigma(a)} \circ \sigma, & D \circ \rho_a - \rho_{\sigma(a)} \circ D &= \rho_{D(a)} \circ \sigma^{-1} \end{aligned} \tag{2.6}$$

for all $a \in A$, where λ_a and ρ_a denote the left and the right multiplications by $a \in A$, respectively.

Proof We have

$$(\sigma \circ \lambda_a)(b) = \sigma(ab) = \sigma(a)\sigma(b) = (\lambda_{\sigma(a)} \circ \sigma)(b)$$
$$(D \circ \lambda_a)(b) = D(ab) = D(a)\sigma(b) + \sigma^{-1}(a)D(b) = (\lambda_{D(a)} \circ \sigma + \lambda_{\sigma^{-1}(a)} \circ D)(b)$$

for all $a, b \in A$. □

Let V be a finite-dimensional complex vector space and let $\mathbb{C}[V]$ be the $\mathbb{C}$-algebra of polynomial functions on V. Further, let $\{x_1, x_2, \dots, x_n\}$ be the linear coordinate functions on V with respect to a basis $\{e_1, e_2, \dots, e_n\}$ of V. Then there exists a canonical isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[V]$ and $\mathbb{C}[x_1, x_2, \dots, x_n]$.

Let $q \in \mathbb{C}^\times$ satisfy $q \neq \pm 1$. We define a $\mathbb{C}$-algebra automorphism γ_{q,x_i} of $\mathbb{C}[V]$ by

$$\gamma_{q,x_i} = q^{x_i \partial_{x_i}} \tag{2.7}$$

and a twisted derivation ∂_{q,x_i} of $\mathbb{C}[V]$ relative to γ_{q,x_i} through

$$\partial_{q,x_i} = \frac{1}{x_i} \frac{q^{x_i \partial_{x_i}} - q^{-x_i \partial_{x_i}}}{q - q^{-1}} \tag{2.8}$$

for $i = 1, 2, \dots, n$.

Lemma 2.2 *Let $q \in \mathbb{C}^\times$ satisfy $q \neq \pm 1$. Further, let D be a twisted derivation of $\mathbb{C}[V]$ relative to γ_{q,x_i} for some $i = 1, 2, \dots, n$. Then we have*

$$D = f_i \partial_{q,x_i}, \tag{2.9}$$

where $f_i \in \mathbb{C}[V]$.

Proof For $j = 1, 2, \dots, n$ satisfying $j \neq i$, we have

$$D(x_i x_j) = D(x_i)\gamma_{q,x_i}(x_j) + \gamma_{q,x_i}^{-1}(x_i)D(x_j) = x_j D(x_i) + q^{-1} x_i D(x_j),$$
$$D(x_j x_i) = D(x_j)\gamma_{q,x_i}(x_i) + \gamma_{q,x_i}^{-1}(x_j)D(x_i) = q x_i D(x_j) + x_j D(x_i),$$

which implies that $D(x_j) = 0$ for all $j = 1, 2, \dots, n$ such that $j \neq i$. If we set $f_i = D(x_i)$, then we get

$$(D - f_i \partial_{q,x_i})(x_j) = 0$$

for all $j = 1, 2, \dots, n$, which gives us $D = f_i \partial_{q,x_i}$. □

Let $q \in \mathbb{C}^\times$ satisfy $q \neq \pm 1$. Then based on the previous lemma, we define the quantum Weyl algebra $\mathcal{A}_V^q$ of the complex vector space V as an associative $\mathbb{C}$-subalgebra of $\operatorname{End} \mathbb{C}[V]$ generated by x_i, ∂_{q,x_i} and $\gamma_{q,x_i}^{\pm 1}$ for $i = 1, 2, \dots, n$. Let us note that the definition of $\mathcal{A}_V^q$ depends on the choice of a basis $\{e_1, e_2, \dots, e_n\}$ of V. Moreover, we have the following nontrivial relations

$$\gamma_{q,x_i} x_i = q x_i \gamma_{q,x_i}, \qquad \gamma_{q,x_i} \partial_{q,x_i} = q^{-1} \partial_{q,x_i} \gamma_{q,x_i} \tag{2.10}$$

and

$$\partial_{q,x_i} x_i - q x_i \partial_{q,x_i} = \gamma_{q,x_i}^{-1}, \qquad \partial_{q,x_i} x_i - q^{-1} x_i \partial_{q,x_i} = \gamma_{q,x_i} \tag{2.11}$$

for $i = 1, 2, \dots, n$.

3 Generalized Verma Modules

3.1 Generalized Verma Modules over Lie Algebras

Let us consider a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, by Δ^+ a positive root system in Δ, and by $\Pi \subset \Delta$ the set of simple roots.

Let $\operatorname{rank} \mathfrak{g} = r$ and $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. Then we denote by $\alpha_i^\vee \in \mathfrak{h}$ the coroot corresponding to the root α_i and by $\omega_i \in \mathfrak{h}^*$ the fundamental weight defined through $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all $j = 1, 2, \dots, r$. We also set

$$Q = \sum_{\alpha \in \Delta^+} \mathbb{Z}\alpha = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i \qquad \text{and} \qquad Q_+ = \sum_{\alpha \in \Delta^+} \mathbb{N}_0\alpha = \bigoplus_{i=1}^{r} \mathbb{N}_0\alpha_i \tag{3.1}$$

and call Q the root lattice and Q_+ the positive root lattice. The Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq r}$ of $\mathfrak{g}$ is given by $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$.

Further, we denote by $s_i \in \mathrm{GL}(\mathfrak{h}^*)$ the reflection about the hyperplane perpendicular to the root α_i. Then we obtain $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Let $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$ generated by s_i for $i = 1, 2, \dots, r$. Then $W_\mathfrak{g}$ is a finite Coxeter group with generators $\{s_1, s_2, \dots, s_r\}$ and the relations

$$(s_i s_j)^{m_{ij}} = 1, \tag{3.2}$$

where $m_{ii} = 1$ and $m_{ij} = 2, 3, 4$ or 6 for $a_{ij}a_{ji} = 0, 1, 2$ or 3, respectively, provided $i \neq j$. Together with the Weyl group $W_\mathfrak{g}$ it is useful to introduce the (generalized) braid group $B_\mathfrak{g}$ of $\mathfrak{g}$. It is an infinite group with generators $\{T_1, T_2, \dots, T_r\}$ and the braid relations

$$\underbrace{T_i T_j T_j \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ji}} \tag{3.3}$$

for $i \neq j$, where $m_{ij} = m_{ji}$. Let us note that the Weyl group $W_\mathfrak{g}$ is the quotient of $B_\mathfrak{g}$ under the further relations $T_i^2 = 1$ for $i = 1, 2, \dots, r$. For an element $w \in W_\mathfrak{g}$ we introduce the length $\ell(w)$ by

$$\ell(w) = |\Delta^+ \cap w(-\Delta^+)|. \tag{3.4}$$

Let us note that the length $\ell(w)$ of $w \in W_\mathfrak{g}$ is the smallest nonnegative integer $k \in \mathbb{N}_0$ required for an expression of w into the form

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}, \tag{3.5}$$

where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, r\}$. Such an expression is called a *reduced expression* of w if $k = \ell(w)$. It is well known that there exists a unique element $w_0 \in W_\mathfrak{g}$ of the maximal length $\ell(w_0) = |\Delta^+|$ called the *longest element*.

The standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is defined through $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with the nilradical $\mathfrak{n}$ and the opposite nilradical $\overline{\mathfrak{n}}$ given by

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}. \tag{3.6}$$

Moreover, we have a triangular decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n} \tag{3.7}$$

of the Lie algebra $\mathfrak{g}$.

Further, let us consider a subset Σ of Π and denote by Δ_Σ the root subsystem in $\mathfrak{h}^*$ generated by Σ. Then the standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\Sigma$ of $\mathfrak{g}$ associated to Σ

is defined through $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ with the nilradical $\mathfrak{u}$ and the opposite nilradical $\overline{\mathfrak{u}}$ given by

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma} \mathfrak{g}_\alpha \qquad \text{and} \qquad \overline{\mathfrak{u}} = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_\Sigma} \mathfrak{g}_{-\alpha} \tag{3.8}$$

and with the reductive Levi subalgebra $\mathfrak{l}$ defined by

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha. \tag{3.9}$$

Moreover, we have a triangular decomposition

$$\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u} \tag{3.10}$$

of the Lie algebra $\mathfrak{g}$. Furthermore, we define the Σ-height $\mathrm{ht}_\Sigma(\alpha)$ of $\alpha \in \Delta$ by

$$\mathrm{ht}_\Sigma\left(\sum_{\alpha \in \Pi} m_\alpha \alpha\right) = \sum_{\alpha \in \Pi \setminus \Sigma} m_\alpha. \tag{3.11}$$

This gives us a structure of a $|k|$-graded Lie algebra on $\mathfrak{g}$ for some $k \in \mathbb{N}_0$. Let us note that if $\Sigma = \emptyset$ then $\mathfrak{p} = \mathfrak{b}$ and if $\Sigma = \Pi$ then $\mathfrak{p} = \mathfrak{g}$.

Definition 3.1 Let V be a simple $\mathfrak{p}$-module satisfying $\mathfrak{u}V = 0$. Then the *generalized Verma module* $M_\mathfrak{p}^\mathfrak{g}(V)$ is the induced module

$$M_\mathfrak{p}^\mathfrak{g}(V) = \mathrm{Ind}_{U(\mathfrak{p})}^{U(\mathfrak{g})}(V) \equiv U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V \simeq U(\overline{\mathfrak{u}}) \otimes_\mathbb{C} V, \tag{3.12}$$

where the last isomorphism of $U(\overline{\mathfrak{u}})$-modules follows from Poincaré–Birkhoff–Witt theorem.

If $\mathfrak{l}$ is the Cartan subalgebra $\mathfrak{h}$, then $\mathfrak{p}$ is the Borel subalgebra $\mathfrak{b}$. In that case, any simple $\mathfrak{p}$-module V is 1-dimensional and $M_\mathfrak{p}^\mathfrak{g}(V)$ is the corresponding Verma module. Moreover, if V is a finite-dimensional $\mathfrak{p}$-module, then $M_\mathfrak{p}^\mathfrak{g}(V)$ is a homomorphic image of a certain Verma module $M_\mathfrak{b}^\mathfrak{g}(V')$, where V' is a 1-dimensional $\mathfrak{b}$-module, see e.g. [10]. Let us note that $M_\mathfrak{p}^\mathfrak{g}(V)$ has a unique simple quotient $L_\mathfrak{p}^\mathfrak{g}(V)$ and generically $M_\mathfrak{p}^\mathfrak{g}(V) \simeq L_\mathfrak{p}^\mathfrak{g}(V)$.

3.2 Generalized Verma Modules over Quantum Groups

In this section we describe generalized Verma modules for quantum groups. For more detailed information concerning quantum groups see e.g. [3, 13, 14]. We use the notation introduced in the previous section.

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank r together with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$, the Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq r}$ and $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$ for $i = 1, 2, \dots, r$, where $(\cdot\,, \cdot)$ is the inner product on $\mathfrak{h}^*$ induced by the Cartan–Killing form on $\mathfrak{g}$ and normalized so that $(\alpha, \alpha) = 2$ for short roots $\alpha \in \Delta^+$.

Let $q \in \mathbb{C}^\times$ satisfy $q^{d_i} \neq \pm 1$ for $i = 1, 2, \dots, r$. Then the quantum group $U_q(\mathfrak{g})$ is a unital associative $\mathbb{C}$-algebra generated by e_i, f_i, k_i, k_i^{-1} for $i = 1, 2, \dots, r$ subject to the relations

$$\begin{gathered} k_i k_i^{-1} = 1, \qquad k_i k_j = k_j k_i, \qquad k_i^{-1} k_i = 1, \\ k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \qquad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \qquad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j \end{gathered} \tag{3.13}$$

for $i, j = 1, 2, \dots, r$ and the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \qquad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \tag{3.14}$$

for $i, j = 1, 2, \dots, r$ satisfying $i \neq j$, where $q_i = q^{d_i}$ for $i = 1, 2, \dots, r$.

There is a unique Hopf algebra structure on the quantum group $U_q(\mathfrak{g})$ with the coproduct $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, the counit $\varepsilon\colon U_q(\mathfrak{g}) \to \mathbb{C}$ and the antipode $S\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ given by

$$\begin{aligned} &\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i, && \Delta(k_i) = k_i \otimes k_i, && \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i, \\ &\varepsilon(e_i) = 0, && \varepsilon(k_i) = 1, && \varepsilon(f_i) = 0, \\ &S(e_i) = -e_i k_i^{-1}, && S(k_i) = k_i^{-1}, && S(f_i) = -k_i f_i \end{aligned} \tag{3.15}$$

for $i = 1, 2, \dots, r$.

Moreover, there exists a homomorphism of the braid group $B_\mathfrak{g}$ into the group of $\mathbb{C}$-algebra automorphisms of $U_q(\mathfrak{g})$ determined by

$$T_i(e_i) = -k_i^{-1} f_i, \qquad T_i(k_j) = k_j k_i^{-a_{ij}}, \qquad T_i(f_i) = -e_i k_i \tag{3.16}$$

for $i, j = 1, 2, \dots, r$ and

$$\begin{aligned} T_i(e_j) &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} \frac{e_i^s}{[s]_{q_i}!} e_j \frac{e_i^{-a_{ij}-s}}{[-a_{ij}-s]_{q_i}!}, \\ T_i(f_j) &= \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} \frac{f_i^{-a_{ij}-s}}{[-a_{ij}-s]_{q_i}!} f_j \frac{f_i^s}{[s]_{q_i}!} \end{aligned} \tag{3.17}$$

for $i, j = 1, 2, \dots, r$ satisfying $i \neq j$.

Let $w_0 \in W_\mathfrak{g}$ be the longest element in the Weyl group $W_\mathfrak{g}$ with a reduced expression

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_n}, \tag{3.18}$$

where $n = |\Delta^+|$. If we set

$$\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \tag{3.19}$$

for $k = 1, 2, \dots, n$, then the sequence $\beta_1, \beta_2, \dots, \beta_n$ exhausts all positive roots Δ^+ of $\mathfrak{g}$. Hence, we define

$$e_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(e_{i_k}) \qquad \text{and} \qquad f_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(f_{i_k}) \tag{3.20}$$

and get elements of $U_q(\mathfrak{g})$ called root vectors of $U_q(\mathfrak{g})$ corresponding to the roots β_k and $-\beta_k$ for $k = 1, 2, \dots, n$, respectively.

Let $U_q(\mathfrak{n})$ and $U_q(\overline{\mathfrak{n}})$ be the $\mathbb{C}$-subalgebras of $U_q(\mathfrak{g})$ generated by the root vectors e_i for $i = 1, 2, \dots, r$ and f_i for $i = 1, 2, \dots, r$, respectively. For the quantum group $U_q(\mathfrak{g})$ we have a direct sum decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q^\alpha(\mathfrak{g}), \tag{3.21}$$

where

$$U_q^\alpha(\mathfrak{g}) = \{u \in U_q(\mathfrak{g});\ k_i u k_i^{-1} = q^{\langle \alpha, \alpha_i^\vee \rangle} u \text{ for } i = 1, 2, \dots, r\}. \tag{3.22}$$

Since $U_q^\alpha(\mathfrak{g})U_q^\beta(\mathfrak{g}) \subset U_q^{\alpha+\beta}(\mathfrak{g})$, the preceding shows that $U_q(\mathfrak{g})$ is a Q-graded $\mathbb{C}$-algebra. Moreover, this grading induces Q-grading on the $\mathbb{C}$-subalgebras $U_q(\mathfrak{n})$ and $U_q(\overline{\mathfrak{n}})$ as well. In particular, we have

$$U_q(\mathfrak{n}) = \bigoplus_{\alpha \in Q_+} U_q^\alpha(\mathfrak{n}) \quad \text{and} \quad U_q(\overline{\mathfrak{n}}) = \bigoplus_{\alpha \in Q_+} U_q^{-\alpha}(\overline{\mathfrak{n}}), \tag{3.23}$$

where $U_q^\alpha(\mathfrak{n}) = U_q^\alpha(\mathfrak{g}) \cap U_q(\mathfrak{n})$ and $U_q^\alpha(\overline{\mathfrak{n}}) = U_q^\alpha(\mathfrak{g}) \cap U_q(\overline{\mathfrak{n}})$ for $\alpha \in Q$.

Further, we denote by $U_q(\mathfrak{h})$ and $U_q(\mathfrak{b})$ the $\mathbb{C}$-subalgebras of $U_q(\mathfrak{g})$ generated by the elements k_i, k_i^{-1} for $i = 1, 2, \ldots, r$ and e_i, k_i, k_i^{-1} for $i = 1, 2, \ldots, r$, respectively. Then we have

$$U_q(\mathfrak{b}) \simeq U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}). \tag{3.24}$$

Moreover, we have a triangular decomposition

$$U_q(\mathfrak{g}) \simeq U_q(\overline{\mathfrak{n}}) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}) \tag{3.25}$$

of the quantum group $U_q(\mathfrak{g})$. Let us note that $U_q(\mathfrak{h})$ and $U_q(\mathfrak{b})$ are Hopf subalgebras of $U_q(\mathfrak{g})$ unlike $U_q(\mathfrak{n})$ and $U_q(\overline{\mathfrak{n}})$.

Let Σ be a subset of Π. Then we have the standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ associated to Σ with the nilradical $\mathfrak{u}$, the opposite nilradical $\overline{\mathfrak{u}}$ and the Levi subalgebra $\mathfrak{l}$.

Let $U_q(\mathfrak{u})$ and $U_q(\overline{\mathfrak{u}})$ be the $\mathbb{C}$-subalgebras of $U_q(\mathfrak{g})$ generated by the root vectors e_α for $\alpha \in \Delta^+$ satisfying $\mathrm{ht}_\Sigma(\alpha) \neq 0$ and f_α for $\alpha \in \Delta^+$ satisfying $\mathrm{ht}_\Sigma(\alpha) \neq 0$, respectively. Further, we denote by $U_q(\mathfrak{l})$ the Levi quantum subgroup of $U_q(\mathfrak{g})$ generated by the elements k_i, k_i^{-1} for $i = 1, 2, \ldots, r$ and the root vectors e_i, f_i for $i = 1, 2, \ldots, r$ such that $\alpha_i \in \Sigma$. Finally, we define the parabolic quantum subgroup $U_q(\mathfrak{p})$ of $U_q(\mathfrak{g})$ as the $\mathbb{C}$-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, k_i for $i = 1, 2, \ldots, r$ and f_i for $i = 1, 2, \ldots, r$ such that $\alpha_i \in \Sigma$. Then we have

$$U_q(\mathfrak{p}) \simeq U_q(\mathfrak{l}) \otimes U_q(\mathfrak{u}). \tag{3.26}$$

Moreover, we have a triangular decomposition

$$U_q(\mathfrak{g}) \simeq U_q(\overline{\mathfrak{u}}) \otimes U_q(\mathfrak{l}) \otimes U_q(\mathfrak{u}) \tag{3.27}$$

of the quantum group $U_q(\mathfrak{g})$. Let us note that $U_q(\mathfrak{l})$ and $U_q(\mathfrak{p})$ are Hopf subalgebras of $U_q(\mathfrak{g})$ unlike $U_q(\mathfrak{u})$ and $U_q(\overline{\mathfrak{u}})$.

Definition 3.2 Let V be a simple $U_q(\mathfrak{p})$-module satisfying $U_q(\mathfrak{u})V = 0$. Then the generalized Verma module $M_{\mathfrak{p},q}^{\mathfrak{g}}(V)$ is the induced module

$$M_{\mathfrak{p},q}^{\mathfrak{g}}(V) = \mathrm{Ind}_{U_q(\mathfrak{p})}^{U_q(\mathfrak{g})}(V) \equiv U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V \simeq U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V, \tag{3.28}$$

where the last isomorphism of $U_q(\overline{\mathfrak{u}})$-modules follows from Poincaré–Birkhoff–Witt theorem.

It is well known that a class of simple highest weight modules for $U_q(\mathfrak{g})$ can be obtained as flat deformations of simple highest weight modules for $\mathfrak{g}$ in the sense of Lusztig [17], that is these modules have the same character formula and the latter can be obtained by the classical limit via the $\mathbb{A}$-forms of $U_q(\mathfrak{g})$. We refer to the paper [19] where the $\mathbb{A}$-forms technique in quantum deformation was described in details. Using this method one can easily show that some generalized Verma modules for $U_q(\mathfrak{g})$ are flat deformations of generalized Verma modules for $\mathfrak{g}$.

4 Representations of the Quantum Group $U_q(\mathfrak{sl}_n(\mathbb{C}))$

4.1 The Quantum Group $U_q(\mathfrak{sl}_n(\mathbb{C}))$

Let us consider the finite-dimensional complex simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of rank $n-1$ together with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ and the Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n-1}$ given by $a_{ii} = 2$, $a_{ij} = -1$ if $|i-j| = 1$ and $a_{ij} = 0$ if $|i-j| > 1$.

Let $q \in \mathbb{C}^\times$ satisfy $q \neq \pm 1$. Then the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$ is a unital associative $\mathbb{C}$-algebra generated by e_i, f_i, k_i, k_i^{-1} for $i = 1, 2, \dots, n-1$ subject to the relations

$$
\begin{gathered}
k_i k_i^{-1} = 1, \qquad k_i k_j = k_j k_i, \qquad k_i^{-1} k_i = 1, \\
k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \qquad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \qquad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j
\end{gathered}
\tag{4.1}
$$

for $i, j = 1, 2, \dots, n-1$ and the quantum Serre relations

$$
\begin{aligned}
e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0, \quad f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i-j| = 1), \\
e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i, \quad (|i-j| > 1).
\end{aligned}
\tag{4.2}
$$

Moreover, there exists a unique Hopf algebra structure on the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$ with the coproduct $\Delta_1 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to U_q(\mathfrak{sl}_n(\mathbb{C})) \otimes U_q(\mathfrak{sl}_n(\mathbb{C}))$, the counit $\varepsilon_1 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to \mathbb{C}$ and the antipode $S_1 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to U_q(\mathfrak{sl}_n(\mathbb{C}))$ given by

$$
\begin{gathered}
\Delta_1(e_i) = e_i \otimes k_i + 1 \otimes e_i, \qquad \Delta_1(k_i) = k_i \otimes k_i, \qquad \Delta_1(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i, \\
\varepsilon_1(e_i) = 0, \qquad \varepsilon_1(k_i) = 1, \qquad \varepsilon_1(f_i) = 0, \\
S_1(e_i) = -e_i k_i^{-1}, \qquad S_1(k_i) = k_i^{-1}, \qquad S_1(f_i) = -k_i f_i
\end{gathered}
\tag{4.3}
$$

for $i = 1, 2, \dots, n-1$. Let us note that we can introduce a different unique Hopf algebra structure on $U_q(\mathfrak{sl}_n(\mathbb{C}))$ with the coproduct $\Delta_2 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to U_q(\mathfrak{sl}_n(\mathbb{C})) \otimes U_q(\mathfrak{sl}_n(\mathbb{C}))$, the counit $\varepsilon_2 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to \mathbb{C}$ and the antipode $S_2 \colon U_q(\mathfrak{sl}_n(\mathbb{C})) \to U_q(\mathfrak{sl}_n(\mathbb{C}))$ given by

$$
\begin{gathered}
\Delta_2(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i, \qquad \Delta_2(k_i) = k_i \otimes k_i, \qquad \Delta_2(f_i) = f_i \otimes 1 + k_i \otimes f_i, \\
\varepsilon_2(e_i) = 0, \qquad \varepsilon_2(k_i) = 1, \qquad \varepsilon_2(f_i) = 0, \\
S_2(e_i) = -e_i k_i, \qquad S_2(k_i) = k_i^{-1}, \qquad S_2(f_i) = -k_i^{-1} f_i
\end{gathered}
\tag{4.4}
$$

for $i = 1, 2, \dots, n-1$.

Furthermore, there is a homomorphism of the braid group $B_{\mathfrak{sl}_n(\mathbb{C})}$ into the group of $\mathbb{C}$-algebra automorphisms of $U_q(\mathfrak{sl}_n(\mathbb{C}))$ determined by

$$
T_i(e_i) = -f_i k_i^{-1}, \qquad T_i(k_i) = k_i^{-1}, \qquad T_i(f_i) = -k_i e_i \tag{4.5}
$$

for $i = 1, 2, \dots, n-1$ and

$$
\begin{aligned}
T_i(e_j) &= e_i e_j - q e_j e_i, \quad T_i(k_j) = k_i k_j, \quad T_i(f_j) = f_j f_i - q^{-1} f_i f_j \quad (|i-j| = 1), \\
T_i(e_j) &= e_j, \quad T_i(k_j) = k_j, \quad T_i(f_j) = f_j \quad (|i-j| > 1).
\end{aligned}
\tag{4.6}
$$

Let us note that a simple computation shows that

$$
T_i T_j(e_i) = e_j \qquad \text{and} \qquad T_i T_j(f_i) = f_j \tag{4.7}
$$

for $i, j = 1, 2, \dots, n-1$ such that $|i-j| = 1$.

Now, we construct root basis of $U_q(\mathfrak{n})$ and $U_q(\overline{\mathfrak{n}})$ by the approach described in the previous section. The longest element w_0 in the Weyl group $W_{\mathfrak{sl}_n(\mathbb{C})}$ has a reduced expression

$$w_0 = s_1 \cdots s_{n-1} s_1 \cdots s_{n-2} \cdots s_1 s_2 s_1. \tag{4.8}$$

If we set

$$w_{i,j} = s_1 \cdots s_{n-1} \cdots s_1 \cdots s_{n-i+1} s_1 \cdots s_{j-i-1}, \tag{4.9}$$

we obtain

$$w_{i,j}(\alpha_{j-i}) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \tag{4.10}$$

for $1 \leq i < j \leq n$. Hence, we denote by

$$E_{i,j} = T_{w_{i,j}}(e_{j-i}) \qquad \text{and} \qquad E_{j,i} = T_{w_{i,j}}(f_{j-i}) \tag{4.11}$$

elements of $U_q(\mathfrak{n})$ and $U_q(\overline{\mathfrak{n}})$ for $1 \leq i < j \leq n$, respectively, where $T_{w_{i,j}}$ stands for

$$T_{w_{i,j}} = T_1 \cdots T_{n-1} \cdots T_1 \cdots T_{n-i+1} T_1 \cdots T_{j-i-1}. \tag{4.12}$$

Furthermore, we define by

$$K_{i,j} = k_i k_{i+1} \cdots k_{j-1} \tag{4.13}$$

elements of $U_q(\mathfrak{h})$ for $1 \leq i < j \leq n$.

Proposition 4.1 *We have*

$$E_{i,i+1} = e_i, \qquad E_{i+1,i} = f_i \tag{4.14}$$

for $i = 1, 2, \ldots, n-1$ *and*

$$\begin{aligned} E_{i,j} &= E_{i,k} E_{k,j} - q E_{k,j} E_{i,k} && \text{for } 1 \leq i < k < j \leq n, \\ E_{i,j} &= E_{i,k} E_{k,j} - q^{-1} E_{k,j} E_{i,k} && \text{for } n \geq i > k > j \geq 1. \end{aligned} \tag{4.15}$$

Proof Let us assume that $i < j$. For $1 \leq i < k \leq n$ we have

$$T_1 \cdots T_k(e_i) = T_1 \cdots T_i T_{i+1}(e_i) = T_1 \cdots T_{i-1}(e_{i+1}) = e_{i+1},$$

which implies $E_{i,i+1} = e_i$ for $i = 1, 2, \ldots, n-1$. Further, we may write

$$\begin{aligned} E_{i,j} &= T_{w_{i,j}}(e_{j-i}) = T_{w_{i,j-1}} T_{j-i-1}(e_{j-i}) = T_{w_{i,j-1}}(e_{j-i-1} e_{j-i} - q e_{j-i} e_{j-i-1}) \\ &= T_{w_{i,j-1}}(e_{j-i-1}) T_{w_{i,j-1}}(e_{j-i}) - q T_{w_{i,j-1}}(e_{j-i}) T_{w_{i,j-1}}(e_{j-i-1}) \\ &= E_{i,j-1} E_{j-1,j} - q E_{j-1,j} E_{i,j-1} \end{aligned}$$

for $j - i > 1$. Hence, we proved the statement for $j - i = 1$ and $j - i = 2$. The rest of the proof is by induction on $j-i$. For $j-i > 2$ we have $E_{i,j} = E_{i,j-1} E_{j-1,j} - q E_{j-1,j} E_{i,j-1}$, which together with the induction assumption $E_{i,j-1} = E_{i,k} E_{k,j-1} - q E_{k,j-1} E_{i,k}$ for $1 \leq i < k < j-1 < n$ gives us

$$\begin{aligned} E_{i,j} &= (E_{i,k} E_{k,j-1} - q E_{k,j-1} E_{i,k}) E_{j-1,j} - q E_{j-1,j} (E_{i,k} E_{k,j-1} - q E_{k,j-1} E_{i,k}) \\ &= E_{i,k} (E_{k,j-1} E_{j-1,j} - q E_{j-1,j} E_{k,j-1}) - q (E_{k,j-1} E_{j-1,j} - q E_{j-1,j} E_{k,j-1}) E_{i,k} \\ &= E_{i,k} E_{k,j} - q E_{k,j} E_{i,k}, \end{aligned}$$

where we used $E_{i,k}E_{j-1,j} = E_{j-1,j}E_{i,k}$ in the second equality. For $i > j$ the proof goes along the same lines. This finishes the proof. □

Let us note that the root vectors $E_{i,j}$ of $U_q(\mathfrak{sl}_n(\mathbb{C}))$ coincide with the elements introduced by Jimbo in [12]. Moreover, these vectors are linearly independent in $U_q(\mathfrak{sl}_n(\mathbb{C}))$ and they have analogous properties as the corresponding elements $E_{i,j}$, $i, j = 1, 2, \ldots, n$ and $i \neq j$, in the matrix realization of $\mathfrak{sl}_n(\mathbb{C})$.

Lemma 4.1 *We have*

$$\begin{aligned}
E_{i,k}E_{k,j}^m &= q^{-m}E_{k,j}^m E_{i,k} + [m]_q E_{k,j}^{m-1}E_{i,j} \quad \text{for } i > k > j,\\
E_{i,k}^m E_{k,j} &= q^{-m}E_{k,j}E_{i,k}^m + [m]_q E_{i,j}E_{i,k}^{m-1} \quad \text{for } i > k > j,\\
E_{i,k}^m E_{i,j} &= q^m E_{i,j}E_{i,k}^m \quad \text{for } i > k > j,\\
E_{i,j}E_{k,j}^m &= q^m E_{k,j}^m E_{i,j} \quad \text{for } i > k > j,\\
E_{i,i+1}E_{i+1,i}^m &= E_{i+1,i}^m E_{i,i+1} + [m]_q E_{i+1,i}^{m-1}\frac{q^{-m+1}K_{i,i+1} - q^{m-1}K_{i,i+1}^{-1}}{q - q^{-1}},\\
E_{i,j}E_{k,i}^m &= E_{k,i}^m E_{i,j} - q^{m-2}[m]_q E_{k,i}^{m-1}E_{k,j}K_{i,j}^{-1} \quad \text{for } i < j < k,\\
E_{j,k}E_{k,i}^m &= E_{k,i}^m E_{j,k} + [m]_q E_{j,i}E_{k,i}^{m-1}K_{j,k} \quad \text{for } i < j < k,\\
E_{\ell,i}E_{k,j} &= E_{k,j}E_{\ell,i} \quad \text{for } i < j < k < \ell,\\
E_{\ell,j}E_{k,i} &- E_{k,i}E_{\ell,j} = (q - q^{-1})E_{k,j}E_{\ell,i} \quad \text{for } i < j < k < \ell,\\
E_{\ell,i}E_{j,k} &= E_{j,k}E_{\ell,i} \quad \text{for } i < j < k < \ell
\end{aligned}$$

in the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$.

Proof All formulas are easy to be verified by induction. □

Lemma 4.2 *We have*

$$\Delta_2(E_{j,i}) = K_{i,j} \otimes E_{j,i} + E_{j,i} \otimes 1 + (q - q^{-1}) \sum_{i<k<j} E_{k,i}K_{k,j} \otimes E_{j,k} \tag{4.16}$$

for $1 \leq i < j \leq n$.

Proof We prove the statement by induction on $j - i$. The case $j - i = 1$ follows immediately from (4.4). Further, for $j - i > 1$ we have $E_{j,i} = E_{j,i+1}E_{i+1,i} - q^{-1}E_{i+1,i}E_{j,i+1}$. Therefore, we may write $\Delta_2(E_{j,i}) = \Delta_2(E_{j,i+1})\Delta_2(E_{i+1,i}) - q^{-1}\Delta_2(E_{i+1,i})\Delta_2(E_{j,i+1})$. By induction assumption we have

$$\Delta_2(E_{j,i+1}) = K_{i+1,j} \otimes E_{j,i+1} + E_{j,i+1} \otimes 1 + (q - q^{-1}) \sum_{i+1<k<j} E_{k,i+1}K_{k,j} \otimes E_{j,k}$$

and also

$$\Delta_2(E_{i+1,i}) = K_{i,i+1} \otimes E_{i+1,i} + E_{i+1,i} \otimes 1,$$

which gives us

$$\begin{aligned}\Delta_2(E_{j,i}) = {} & E_{j,i+1}E_{i+1,i} \otimes 1 - q^{-1}E_{i+1,i}E_{j,i+1} \otimes 1 \\ & + K_{i+1,j}E_{i+1,i} \otimes E_{j,i+1} - q^{-1}E_{i+1,i}K_{i+1,j} \otimes E_{j,i+1} \\ & + E_{j,i+1}K_{i,i+1} \otimes E_{i+1,i} - q^{-1}K_{i,i+1}E_{j,i+1} \otimes E_{i+1,i} \\ & + K_{i+1,j}K_{i,i+1} \otimes E_{j,i+1}E_{i+1,i} - q^{-1}K_{i,i+1}K_{i+1,j} \otimes E_{i+1,i}E_{j,i+1} \\ & + (q-q^{-1}) \sum_{i+1<k<j} E_{k,i+1}K_{k,j}E_{i+1,i} \otimes E_{j,k} \\ & - q^{-1}(q-q^{-1}) \sum_{i+1<k<j} E_{i+1,i}E_{k,i+1}K_{k,j} \otimes E_{j,k} \\ & + (q-q^{-1}) \sum_{i+1<k<j} E_{k,i+1}K_{k,j}K_{i,i+1} \otimes E_{j,k}E_{i+1,i} \\ & - q^{-1}(q-q^{-1}) \sum_{i+1<k<j} K_{i,i+1}E_{k,i+1}K_{k,j} \otimes E_{i+1,i}E_{j,k}.\end{aligned}$$

Further, using the relations $K_{i+1,j}E_{i+1,i} = qE_{i+1,i}K_{i+1,j}$, $K_{i,i+1}E_{j,i+1} = qE_{j,i+1}K_{i,i+1}$ and $K_{i,i+1}E_{k,i+1} = qE_{k,i+1}K_{i,i+1}$ we may simplified $\Delta_2(E_{j,i})$ into the form

$$\begin{aligned}\Delta_2(E_{j,i}) = {} & (E_{j,i+1}E_{i+1,i} - q^{-1}E_{i+1,i}E_{j,i+1}) \otimes 1 + K_{i,j} \otimes (E_{j,i+1}E_{i+1,i} - q^{-1}E_{i+1,i}E_{j,i+1}) \\ & + (q-q^{-1})E_{i+1,i}K_{i+1,j} \otimes E_{j,i+1} \\ & + (q-q^{-1}) \sum_{i+1<k<j} (E_{k,i+1}E_{i+1,i} - q^{-1}E_{i+1,i}E_{k,i+1})K_{k,j} \otimes E_{j,k}.\end{aligned}$$

Therefore, we have

$$\Delta_2(E_{j,i}) = K_{i,j} \otimes E_{j,i} + E_{j,i} \otimes 1 + (q-q^{-1}) \sum_{i<k<j} E_{k,i}K_{k,j} \otimes E_{j,k},$$

which finishes the proof. □

4.2 The Parabolic Induction for $U_q(\mathfrak{sl}_{n+m}(\mathbb{C}))$

For simplicity we concentrate now on one particular choice of a parabolic quantum subgroup of $U_q(\mathfrak{sl}_{n+m}(\mathbb{C}))$. This offers a good insight into the construction for a general case.

Let $\Sigma = \{\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}, \dots, \alpha_{n+m-1}\}$ be a subset of $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n+m-1}\}$ and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the corresponding parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}_{n+m}(\mathbb{C})$ with the nilradical $\mathfrak{u}$, the opposite nilradical $\overline{\mathfrak{u}}$ and the Levi subalgebra $\mathfrak{l}$. We have a triangular decomposition

$$\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u} \qquad (4.17)$$

of the Lie algebra $\mathfrak{g}$, where $\mathfrak{l} \simeq \mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_m(\mathbb{C}) \oplus \mathbb{C}$, $\overline{\mathfrak{u}} \simeq \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ and $\mathfrak{u} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. Furthermore, we have the corresponding quantum parabolic subgroup $U_q(\mathfrak{p})$ of $U_q(\mathfrak{g})$ with the $\mathbb{C}$-subalgebras $U_q(\mathfrak{u})$, $U_q(\overline{\mathfrak{u}})$ and the Levi quantum subgroup $U_q(\mathfrak{l})$. Moreover, we have a triangular decomposition

$$U_q(\mathfrak{g}) \simeq U_q(\overline{\mathfrak{u}}) \otimes U_q(\mathfrak{l}) \otimes U_q(\mathfrak{u}) \qquad (4.18)$$

of the quantum group $U_q(\mathfrak{g})$.

Let V be a $U_q(\mathfrak{p})$-module. Then for the induced module $M^{\mathfrak{g}}_{\mathfrak{p},q}(V)$ we have

$$U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V \simeq U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V, \tag{4.19}$$

where the isomorphism of vector spaces is in fact an isomorphism of $U_q(\overline{\mathfrak{u}})$-modules. Hence, the action of $U_q(\overline{\mathfrak{u}})$ on $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ is just the left multiplication, like in the classical case. Our next step is to describe the action of the Levi quantum subgroup $U_q(\mathfrak{l})$ on $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$, since in the classical case the action of the Levi subalgebra $\mathfrak{l}$ on $U(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ is equal to the tensor product of the adjoint action on $U(\overline{\mathfrak{u}})$ and the action on V.

Let us recall that the Levi quantum subgroup $U_q(\mathfrak{l})$ of $U_q(\mathfrak{g})$ has a Hopf algebra structure determined either by (4.3) or by (4.4). However, we introduce a different (mixed) Hopf algebra structure on $U_q(\mathfrak{l})$ with the coproduct $\Delta\colon U_q(\mathfrak{l}) \to U_q(\mathfrak{l}) \otimes U_q(\mathfrak{l})$, the counit $\varepsilon\colon U_q(\mathfrak{l}) \to U_q(\mathfrak{l})$ and the antipode $S\colon U_q(\mathfrak{l}) \to U_q(\mathfrak{l})$ given by

$$\begin{aligned} \Delta(e_i) &= e_i \otimes k_i^{-1} + 1 \otimes e_i, & \Delta(k_i) &= k_i \otimes k_i, & \Delta(f_i) &= f_i \otimes 1 + k_i \otimes f_i,\\ \varepsilon(e_i) &= 0, & \varepsilon(k_i) &= 1, & \varepsilon(f_i) &= 0,\\ S(e_i) &= -e_i k_i, & S(k_i) &= k_i^{-1}, & S(f_i) &= -k_i^{-1} f_i \end{aligned} \tag{4.20}$$

for $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} \Delta(k_n) &= k_n \otimes k_n,\\ \varepsilon(k_n) &= 1,\\ S(k_n) &= k_n^{-1}, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i, & \Delta(k_i) &= k_i \otimes k_i, & \Delta(f_i) &= f_i \otimes 1 + k_i^{-1} \otimes f_i,\\ \varepsilon(e_i) &= 0, & \varepsilon(k_i) &= 1, & \varepsilon(f_i) &= 0,\\ S(e_i) &= -e_i k_i^{-1}, & S(k_i) &= k_i^{-1}, & S(f_i) &= -k_i f_i \end{aligned} \tag{4.22}$$

for $i = n+1, n+2, \ldots, n+m-1$.

The Hopf algebra structure on $U_q(\mathfrak{l})$ ensures that we can define the (left) adjoint action of $U_q(\mathfrak{l})$ on $U_q(\mathfrak{g})$ by

$$\operatorname{ad}(a)b = \sum a_{(1)} b S(a_{(2)}), \tag{4.23}$$

where

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \tag{4.24}$$

for all $a \in U_q(\mathfrak{l})$ and $b \in U_q(\mathfrak{g})$. Let us note that we also have

$$\operatorname{ad}(a)bc = \sum (\operatorname{ad}(a_{(1)})b)(\operatorname{ad}(a_{(2)})c) \tag{4.25}$$

for all $a \in U_q(\mathfrak{l})$ and $b, c \in U_q(\mathfrak{g})$.

Proposition 4.2 *The $\mathbb{C}$-subalgebra $U_q(\overline{\mathfrak{u}})$ of $U_q(\mathfrak{g})$ is a $U_q(\mathfrak{l})$-module with respect to the adjoint action. Moreover, we have*

$$\begin{gathered} ad(e_i)E_{n+j,k} = -q^{-1}\delta_{i,k}E_{n+j,k+1}, \qquad ad(f_i)E_{j+n,k} = -q\delta_{i+1,k}E_{n+j,k-1},\\ ad(k_i)E_{n+j,k} = q^{-\delta_{i,k}+\delta_{i+1,k}}E_{n+j,k} \end{gathered} \tag{4.26}$$

for $i = 1, 2, \ldots, n-1$,

$$ad(k_n)E_{n+j,k} = q^{-\delta_{1,j}-\delta_{n,k}}E_{n+j,k} \tag{4.27}$$

and

$$ad(e_{n+i})E_{n+j,k} = \delta_{i+1,j}E_{n+j-1,k}, \qquad ad(f_{n+i})E_{n+j,k} = \delta_{i,j}E_{n+j+1,k}, \\ ad(k_{n+i})E_{n+j,k} = q^{\delta_{i,j}-\delta_{i+1,j}}E_{n+j,k} \tag{4.28}$$

for $i = 1, 2, \ldots, m-1$, *where* $1 \leq j \leq m$ *and* $1 \leq k \leq n$.

Proof Due to the formula (4.25), it is enough to verify that the set of generators $\{E_{n+j,k};\ 1 \leq j \leq m, 1 \leq k \leq n\}$ of the $\mathbb{C}$-subalgebra $U_q(\overline{\mathfrak{u}})$ of $U_q(\mathfrak{g})$ is preserved by $U_q(\mathfrak{l})$ with respect to the adjoint action. The formulas (4.26), (4.28) and (4.27) are easy consequence of Lemma 4.1. □

Proposition 4.3 *Let* V *be a* $U_q(\mathfrak{p})$*-module. Then the* $U_q(\mathfrak{l})$*-module structure on* $M^{\mathfrak{g}}_{\mathfrak{p},q}(V)$ *is given by*

$$a(u \otimes v) = \sum (ad\, a_{(1)})u \otimes a_{(2)}v, \tag{4.29}$$

where

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \tag{4.30}$$

for $a \in U_q(\mathfrak{l})$, $u \in U_q(\overline{\mathfrak{u}})$ *and* $v \in V$. *In particular, we get that* $M^{\mathfrak{g}}_{\mathfrak{p},q}(V)$ *is isomorphic to* $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ *as* $U_q(\mathfrak{l})$*-module, where the* $U_q(\mathfrak{l})$*-module structure on* $U_q(\overline{\mathfrak{u}})$ *is given through the adjoint action.*

Proof For an element $a \in U_q(\mathfrak{l})$ we have $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, $\Delta(a_{(1)}) = \sum a_{(11)} \otimes a_{(12)}$ and $\Delta(a_{(2)}) = \sum a_{(21)} \otimes a_{(22)}$. Then for $u \in U_q(\overline{\mathfrak{u}})$ and $v \in V$ we may write

$$\begin{aligned} \sum (\mathrm{ad}\, a_{(1)})u \otimes a_{(2)}v &= \sum a_{(11)}uS(a_{(12)}) \otimes a_{(2)}v = \sum a_{(11)}uS(a_{(12)})a_{(2)} \otimes v \\ &= \sum a_{(1)}uS(a_{(21)})a_{(22)} \otimes v = \sum a_{(1)}u\varepsilon(a_{(2)}) \otimes v \\ &= \sum a_{(1)}\varepsilon(a_{(2)})u \otimes v = au \otimes v, \end{aligned}$$

where we used $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ in the third equality, $m \circ (S \otimes \mathrm{id}) \circ \Delta = i \circ \varepsilon$ in the fourth equality, and $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ in the last equality. Since $U_q(\overline{\mathfrak{u}})$ is a $U_q(\mathfrak{l})$-module by Proposition 4.2, we immediately obtain that $M^{\mathfrak{g}}_{\mathfrak{p},q}(V)$ is isomorphic to $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ as $U_q(\mathfrak{l})$-module. □

Let us note that the formula (4.29) holds for an arbitrary Hopf algebra structure on $U_q(\mathfrak{l})$. However, the main difficulty is to find such a Hopf algebra structure that $U_q(\overline{\mathfrak{u}})$ is a $U_q(\mathfrak{l})$-module with respect to the adjoint action (4.23).

As a consequence of Proposition 4.2 we have that the vector space

$$\overline{\mathfrak{u}}_q = \langle\{E_{n+j,k};\ 1 \leq j \leq m, 1 \leq k \leq n\}\rangle \tag{4.31}$$

is a $U_q(\mathfrak{l})$-submodule of $U_q(\overline{\mathfrak{u}})$. By the specialization $q \to 1$ of the root vectors $E_{n+j,k}$, we obtain the canonical root vectors $x_{j,k}$ of $\overline{\mathfrak{u}}$ for $1 \leq j \leq m$ and $1 \leq k \leq n$. Hence, we define an isomorphism $\psi_q : \overline{\mathfrak{u}} \to \overline{\mathfrak{u}}_q$ of vector spaces by

$$\psi_q(x_{j,k}) = E_{n+j,k} \tag{4.32}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$. Let us note that $x = (x_{j,k})_{1 \leq j \leq m, 1 \leq k \leq n}$ gives us linear coordinate functions on $\overline{\mathfrak{u}}^*$. Further, we introduce a $U_q(\mathfrak{l})$-module structure on $\overline{\mathfrak{u}}$ through $\tau_q \colon U_q(\mathfrak{l}) \to \operatorname{End} \overline{\mathfrak{u}}$ defined by

$$\tau_q(a) = \psi_q^{-1} \circ \operatorname{ad}(a) \circ \psi_q \tag{4.33}$$

for all $a \in U_q(\mathfrak{l})$. Moreover, when q is specialized to 1, we get the original $\mathfrak{l}$-module structure on $\overline{\mathfrak{u}}$.

For now, let us assume that q is not a root of unity. Then we have $\overline{\mathfrak{u}} \simeq L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(\omega_{n-1} - 2\omega_n + \omega_{n+1})$ as $U_q(\mathfrak{l})$-modules, where $L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(\lambda)$ is the simple highest weight $U_q(\mathfrak{l})$-module with highest weight q^λ for $\lambda \in \mathfrak{h}^*$. Further, since we have

$$\begin{aligned} \overline{\mathfrak{u}} \otimes_{\mathbb{C}} \overline{\mathfrak{u}} \simeq L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(2\omega_{n-1} - 4\omega_n + 2\omega_{n+1}) \oplus L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(\omega_{n-2} - 2\omega_n + \omega_{n+2}) \\ \oplus L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(2\omega_{n-1} - 3\omega_n + \omega_{n+2}) \oplus L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(\omega_{n-2} - 3\omega_n + 2\omega_{n+1}) \end{aligned} \tag{4.34}$$

as $U_q(\mathfrak{l})$-modules, we define

$$S_q(\overline{\mathfrak{u}}) = T(\overline{\mathfrak{u}})/I_q, \tag{4.35}$$

where I_q is the two-sided ideal of the tensor algebra $T(\overline{\mathfrak{u}})$ generated by

$$L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(\omega_{n-2} - 3\omega_n + 2\omega_{n+1}) = \langle v^+_{i,k,\ell}, w^+_{i,j,k,\ell};\ 1 \leq i < j \leq m, 1 \leq k < \ell \leq n \rangle, \tag{4.36}$$

where

$$\begin{aligned} v^+_{i,k,\ell} &= x_{i,\ell} \otimes x_{i,k} - q x_{i,k} \otimes x_{i,\ell}, \\ w^+_{i,j,k\ell} &= x_{j,\ell} \otimes x_{i,k} - x_{i,k} \otimes x_{j,\ell} - q x_{j,k} \otimes x_{i,\ell} + q^{-1} x_{i,\ell} \otimes x_{j,k}, \end{aligned} \tag{4.37}$$

and by

$$L^{\mathfrak{l}}_{\mathfrak{b} \cap \mathfrak{l}, q}(2\omega_{n-1} - 3\omega_n + \omega_{n+2}) = \langle v^-_{i,j,k}, w^-_{i,j,k,\ell};\ 1 \leq i < j \leq m, 1 \leq k < \ell \leq n \rangle, \tag{4.38}$$

where

$$\begin{aligned} v^-_{i,j,k} &= x_{j,k} \otimes x_{i,k} - q x_{i,k} \otimes x_{j,k}, \\ w^-_{i,j,k,\ell} &= x_{j,\ell} \otimes x_{i,k} - x_{i,k} \otimes x_{j,\ell} + q^{-1} x_{j,k} \otimes x_{i,\ell} - q x_{i,\ell} \otimes x_{j,k}, \end{aligned} \tag{4.39}$$

which gives us

$$S_q(\overline{\mathfrak{u}}) \simeq \mathbb{C}_q[\overline{\mathfrak{u}}^*] \tag{4.40}$$

with

$$\begin{aligned} \mathbb{C}_q[\overline{\mathfrak{u}}^*] = \mathbb{C}\langle x \rangle / (&x_{i,\ell} x_{i,k} - q x_{i,k} x_{i,\ell}, x_{j,k} x_{i,k} - q x_{i,k} x_{j,k}, x_{j,k} x_{i,\ell} - x_{i,\ell} x_{j,k}, \\ &x_{j,\ell} x_{i,k} - x_{i,k} x_{j,\ell} - (q - q^{-1}) x_{i,\ell} x_{j,k};\ 1 \leq i < j \leq m, 1 \leq k < \ell \leq n). \end{aligned} \tag{4.41}$$

In the previous discussion, we assumed that q is not a root of unity. However, the definition of $S_q(\overline{\mathfrak{u}})$ makes sense for all $q \in \mathbb{C}^\times$ satisfying $q \neq \pm 1$. Moreover, since the two-sided ideal I_q is a $U_q(\mathfrak{l})$-submodule of $T(\overline{\mathfrak{u}})$, we obtain that also $S_q(\overline{\mathfrak{u}})$ is a $U_q(\mathfrak{l})$-module for all $q \in \mathbb{C}^\times$ satisfying $q \neq \pm 1$. The specialization $q \to 1$ gives us $I_q \to I$, hence $\mathbb{C}_q[\overline{\mathfrak{u}}^*] \to \mathbb{C}[\overline{\mathfrak{u}}^*]$. Let us note that the $\mathbb{C}$-algebra $\mathbb{C}_q[\overline{\mathfrak{u}}^*]$ is usually called the coordinate algebra of the quantum vector space $\overline{\mathfrak{u}}^*$ introduced in [22].

It follows immediately from Lemma 4.1 that the mapping (4.32) may be uniquely extended to a $\mathbb{C}$-algebra homomorphism

$$\psi_q \colon \mathbb{C}_q[\overline{\mathfrak{u}}^*] \to U_q(\overline{\mathfrak{u}}). \tag{4.42}$$

Moreover, since the set

$$\{E^{r_{1,1}}_{n+1,1} E^{r_{1,2}}_{n+1,2} \cdots E^{r_{1,n}}_{n+1,n} \cdots E^{r_{m,1}}_{n+m,1} \cdots E^{r_{m,n}}_{n+m,n};\ r \in M_{m,n}(\mathbb{N}_0)\} \tag{4.43}$$

forms a basis of $U_q(\overline{\mathfrak{u}})$, we obtain that ψ_q is an isomorphism of $\mathbb{C}$-algebras. Further, by the formula (4.25) and the fact that $\psi_q \colon \mathbb{C}_q[\overline{\mathfrak{u}}^*] \to U_q(\overline{\mathfrak{u}})$ is an isomorphism of $\mathbb{C}$-algebras, we get that ψ_q is an isomorphism of $U_q(\mathfrak{l})$-modules.

For an $(m \times n)$-matrix $r = (r_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ with nonnegative integer entries we denote by x^r an element of $\mathbb{C}_q[\overline{\mathfrak{u}}^*]$ defined by

$$x^r = x_{1,1}^{r_{1,1}} x_{1,2}^{r_{1,2}} \cdots x_{1,n}^{r_{1,n}} \cdots x_{m,1}^{r_{m,1}} \cdots x_{m,n}^{r_{m,n}} \tag{4.44}$$

and by E^r an element of $U_q(\overline{\mathfrak{u}})$ defined by

$$E^r = E_{n+1,1}^{r_{1,1}} E_{n+1,2}^{r_{1,2}} \cdots E_{n+1,n}^{r_{1,n}} \cdots E_{n+m,1}^{r_{m,1}} \cdots E_{n+m,n}^{r_{m,n}}.$$

Since the $\mathbb{C}$-algebra $\mathbb{C}_q[\overline{\mathfrak{u}}^*]$ has a basis $\{x^r;\ r \in M_{m,n}(\mathbb{N}_0)\}$ we can find a family of isomorphisms $\varphi_q \colon \mathbb{C}[\overline{\mathfrak{u}}^*] \to \mathbb{C}_q[\overline{\mathfrak{u}}^*]$ of vector spaces such that $\varphi_q \to \mathrm{id}$ for $q \to 1$. Let us define $\varphi_q \colon \mathbb{C}[\overline{\mathfrak{u}}^*] \to \mathbb{C}_q[\overline{\mathfrak{u}}^*]$ by

$$\varphi_q(x^r) = x^r \tag{4.45}$$

for all $r \in M_{m,n}(\mathbb{N}_0)$. Furthermore, we denote by $1_{i,j} \in M_{m,n}(\mathbb{N}_0)$ the $(m \times n)$-matrix having 1 at the intersection of the i-th row and j-th column and 0 elsewhere. Then the corresponding $U_q(\mathfrak{l})$-module structure on $\mathbb{C}[\overline{\mathfrak{u}}^*]$ is given through the homomorphism

$$\rho_q \colon U_q(\mathfrak{l}) \to \mathcal{A}_{\overline{\mathfrak{u}}^*}^q \tag{4.46}$$

of associative $\mathbb{C}$-algebras, where $\mathcal{A}_{\overline{\mathfrak{u}}^*}^q$ is the quantum Weyl algebra of the vector space $\overline{\mathfrak{u}}^*$, defined by

$$\rho_q(a) = \varphi_q^{-1} \circ \tau_q(a) \circ \varphi_q \tag{4.47}$$

for all $a \in U_q(\mathfrak{l})$.

Let $\gamma_{i,j}$ be the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[\overline{\mathfrak{u}}^*]$ given by $\gamma_{i,j} \colon x_{k,\ell} \mapsto q^{\delta_{ik}\delta_{j\ell}} x_{k,\ell}$ and $\partial_{i,j}$ the corresponding twisted derivation of $\mathbb{C}[\overline{\mathfrak{u}}^*]$ relative to $\gamma_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Theorem 4.1 *We have*

$$\rho_q(e_i) = -\sum_{k=1}^{m} \prod_{t=k}^{m} \gamma_{t,i}\gamma_{t,i+1}^{-1} x_{k,i+1}\partial_{k,i}, \qquad \rho_q(f_i) = -\sum_{k=1}^{m} x_{k,i}\partial_{k,i+1} \prod_{t=1}^{k} \gamma_{t,i}^{-1}\gamma_{t,i+1},$$
$$\rho_q(k_i) = \prod_{t=1}^{m} \gamma_{t,i}^{-1}\gamma_{t,i+1} \tag{4.48}$$

for $i = 1, 2, \ldots, n-1$,

$$\rho_q(k_n) = \prod_{t=1}^{n} \gamma_{1,t}^{-1} \prod_{s=1}^{m} \gamma_{s,n}^{-1} \tag{4.49}$$

and

$$\rho_q(e_{n+i}) = \sum_{k=1}^{n} x_{i,k}\partial_{i+1,k} \prod_{t=k+1}^{n} \gamma_{i,t}\gamma_{i+1,t}^{-1}, \qquad \rho_q(f_{n+i}) = \sum_{k=1}^{n} \prod_{t=1}^{k-1} \gamma_{i,t}^{-1}\gamma_{i+1,t} x_{i+1,k}\partial_{i,k},$$
$$\rho_q(k_{n+i}) = \prod_{t=1}^{n} \gamma_{i,t}\gamma_{i+1,t}^{-1} \tag{4.50}$$

for $i = 1, 2, \ldots, m-1$.

Proof The proof is a straightforward computation. Using (4.26) and (4.20) we have

$$\begin{aligned}\tau_q(e_i)(x^r) &= -q^{-1}\sum_{k=1}^m\sum_{j=1}^{r_{k,i}} q^{\sum_{t=k}^m(r_{t,i}-r_{t,i+1})-j}\, x_{1,1}^{r_{1,1}}\cdots x_{k,i}^{j-1}x_{k,i+1}x_{k,i}^{r_{k,i}-j}\cdots x_{m,n}^{r_{m,n}}\\ &= -q^{-1}\sum_{k=1}^m\sum_{j=1}^{r_{k,i}} q^{\sum_{t=k}^m(r_{t,i}-r_{t,i+1})+r_{k,i}-2j}\, x_{1,1}^{r_{1,1}}\cdots x_{k,i}^{r_{k,i}-1}x_{k,i+1}^{r_{k,i+1}+1}\cdots x_{m,n}^{r_{m,n}}\\ &= -\sum_{k=1}^m q^{\sum_{t=k}^m(r_{t,i}-r_{t,i+1})-2}[r_{k,i}]_q\, x_{1,1}^{r_{1,1}}\cdots x_{k,i}^{r_{k,i}-1}x_{k,i+1}^{r_{k,i+1}+1}\cdots x_{m,n}^{r_{m,n}},\end{aligned}$$

$$\begin{aligned}\tau_q(f_i)(x^r) &= -q\sum_{k=1}^m\sum_{j=1}^{r_{k,i+1}} q^{\sum_{t=1}^k(r_{t,i+1}-r_{t,i})-j}\, x_{1,1}^{r_{1,1}}\cdots x_{k,i+1}^{r_{k,i+1}-j}x_{k,i}x_{k,i+1}^{j-1}\cdots x_{m,n}^{r_{m,n}}\\ &= -q\sum_{k=1}^m\sum_{j=1}^{r_{k,i+1}} q^{\sum_{t=1}^k(r_{t,i+1}-r_{t,i})+r_{k,i+1}-2j}\, x_{1,1}^{r_{1,1}}\cdots x_{k,i}^{r_{k,i}+1}x_{k,i+1}^{r_{k,i+1}-1}\cdots x_{m,n}^{r_{m,n}}\\ &= -\sum_{k=1}^m q^{\sum_{t=1}^k(r_{t,i+1}-r_{t,i})}[r_{k,i+1}]_q\, x_{1,1}^{r_{1,1}}\cdots x_{k,i}^{r_{k,i}+1}x_{k,i+1}^{r_{k,i+1}-1}\cdots x_{m,n}^{r_{m,n}}\end{aligned}$$

and

$$\tau_q(k_i)(x^r) = q^{\sum_{t=1}^m(r_{t,i+1}-r_{t,i})}x^r$$

for all $r\in M_{m,n}(\mathbb{N}_0)$ and $i=1,2,\ldots,n-1$, which gives us (4.48). Analogously, from (4.28) and (4.22) we obtain

$$\begin{aligned}\tau_q(e_{n+i})(x^r) &= \sum_{k=1}^n\sum_{j=1}^{r_{i+1,k}} q^{-\sum_{t=k+1}^n r_{i+1,k}-j+1}\, x_{1,1}^{r_{1,1}}\cdots x_{i+1,k}^{r_{i+1,k}-j}x_{i,k}x_{i+1,k}^{j-1}\cdots x_{m,n}^{r_{m,n}}\\ &= \sum_{k=1}^n\sum_{j=1}^{r_{i+1,k}} q^{\sum_{t=k+1}^n(r_{i,k}-r_{i+1,k})+r_{i+1,k}-2j+1}\, x_{1,1}^{r_{1,1}}\cdots x_{i,k}^{r_{i,k}+1}\cdots x_{i+1,k}^{r_{i+1,k}-1}\cdots x_{m,n}^{r_{m,n}}\\ &= \sum_{k=1}^n q^{\sum_{t=k+1}^n(r_{i,k}-r_{i+1,k})}[r_{i+1,k}]_q\, x_{1,1}^{r_{1,1}}\cdots x_{i,k}^{r_{i,k}+1}\cdots x_{i+1,k}^{r_{i+1,k}-1}\cdots x_{m,n}^{r_{m,n}},\end{aligned}$$

$$\begin{aligned}\tau_q(f_{n+i})(x^r) &= \sum_{k=1}^n\sum_{j=1}^{r_{i,k}} q^{-\sum_{t=1}^{k-1} r_{i,t}-j+1}\, x_{1,1}^{r_{1,1}}\cdots x_{i,k}^{j-1}x_{i+1,k}x_{i,k}^{r_{i,k}-j}\cdots x_{m,n}^{r_{m,n}}\\ &= \sum_{k=1}^n\sum_{j=1}^{r_{i,k}} q^{\sum_{t=1}^{k-1}(r_{i+1,t}-r_{i,t})+r_{i,k}-2j+1}\, x_{1,1}^{r_{1,1}}\cdots x_{i,k}^{r_{i,k}-1}\cdots x_{i+1,k}^{r_{i+1,k}+1}\cdots x_{m,n}^{r_{m,n}}\\ &= \sum_{k=1}^n q^{\sum_{t=1}^{k-1}(r_{i+1,t}-r_{i,t})}[r_{i,k}]_q\, x_{1,1}^{r_{1,1}}\cdots x_{i,k}^{r_{i,k}-1}\cdots x_{i+1,k}^{r_{i+1,k}+1}\cdots x_{m,n}^{r_{m,n}}\end{aligned}$$

and

$$\tau_q(k_{n+i})(x^r) = q^{\sum_{t=1}^n(r_{i,t}-r_{i+1,t})}x^r$$

for all $r\in M_{m,n}(\mathbb{N}_0)$ and $i=1,2,\ldots,m-1$, which implies (4.50). Finally, using (4.27) and (4.21) we get

$$\tau_q(k_n)(x^r) = q^{-\sum_{t=1}^n r_{1,t}-\sum_{s=1}^m r_{s,n}}x^r$$

for all $r\in M_{m,n}(\mathbb{N}_0)$, which finishes the proof.

Lemma 4.3 *We have*

$$\rho_q(E_{j,i}) = -\sum_{1\le k_1\le\cdots\le k_s\le m}(q^{-1}-q)^{\tau(k_1,\ldots,k_s)-1}x_{k_1,i}\theta_{k_1,\ldots,k_s}\partial_{k_s,j}\prod_{a=1}^s\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1} \tag{4.51}$$

for $1 \le i < j \le n$, *where* $s = j - i$, $\tau(k_1, \dots, k_s)$ *is the number of distinct integers in the* s*-tuple* $(k_1, k_2, \dots, k_s)$,

$$\theta_{k_1,\dots,k_s} = \beta_{k_1}\beta_{k_2}\cdots\beta_{k_{s-1}} \tag{4.52}$$

with $\beta_{k_t} = \partial_{k_t,i+t}x_{k_{t+1},i+t}$ *if* $k_t \neq k_{t+1}$ *and* $\beta_{k_t} = \gamma^{-1}_{k_t,i+t}$ *if* $k_t = k_{t+1}$.

Proof We prove the statement by induction on $j - i$. The case $j - i = 1$ follows from Theorem 4.1. Further, for $s = j - i > 1$ we have $\rho_q(E_{j,i}) = \rho_q(E_{j,j-1})\rho_q(E_{j-1,i}) - q^{-1}\rho_q(E_{j-1,i})\rho_q(E_{j,j-1})$. By induction assumption we have

$$\rho_q(E_{j,j-1}) = -\sum_{k=1}^{m} x_{k,j-1}\partial_{k,j}\prod_{t=1}^{k}\gamma_{t,j}\gamma^{-1}_{t,j-1}$$

and

$$\rho_q(E_{j-1,i}) = -\sum_{1\le k_1\le\cdots\le k_{s-1}\le m}(q^{-1}-q)^{\tau'-1}x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1},$$

where $\tau' = \tau(k_1, \dots, k_{s-1})$ for greater clarity, which gives us

$$\rho_q(E_{j,i}) = \sum_{k=1}^{m}\sum_{1\le k_1\le\cdots\le k_{s-1}\le m}(q^{-1}-q)^{\tau(k_1,\dots,k_{s-1})-1}\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}},$$

where $\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}}$ denotes the expression

$$x_{k,j-1}\partial_{k,j}\prod_{t=1}^{k}\gamma_{t,j}\gamma^{-1}_{t,j-1}x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}$$
$$-q^{-1}x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}x_{k,j-1}\partial_{k,j}\prod_{t=1}^{k}\gamma_{t,j}\gamma^{-1}_{t,j-1}$$

for $1 \le k_1 \le \cdots \le k_{s-1} \le m$ and $k = 1, 2, \dots, m$.

If $k < k_{s-1}$, we have $\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}} = 0$. For $k = k_{s-1}$ we may write

$$\begin{aligned}
\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}} &= q x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}x_{k_{s-1},j-1}\partial_{k_{s-1},j-1}\partial_{k_{s-1},j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}\prod_{t=1}^{k_{s-1}}\gamma_{t,j}\gamma^{-1}_{t,j-1}\\
&\quad -x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}x_{k_{s-1},j-1}\partial_{k_{s-1},j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}\prod_{t=1}^{k_{s-1}}\gamma_{t,j}\gamma^{-1}_{t,j-1}\\
&= -x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\gamma^{-1}_{k_{s-1},j-1}\partial_{k_{s-1},j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}\prod_{t=1}^{k_{s-1}}\gamma_{t,j}\gamma^{-1}_{t,j-1}\\
&= -x_{k_1,i}\theta_{k_1,\dots,k_{s-1},k_{s-1}}\partial_{k_{s-1},j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma^{-1}_{t,i+a-1}\prod_{t=1}^{k_{s-1}}\gamma_{t,j}\gamma^{-1}_{t,j-1},
\end{aligned}$$

where the second equality follows from $\partial_{i,j}x_{i,j} - qx_{i,j}\partial_{i,j} = \gamma_{i,j}^{-1}$. Finally, if $k > k_{s-1}$, we obtain

$$\begin{aligned}\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}} &= qx_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}x_{k,j-1}\partial_{k,j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1}\prod_{t=1}^{k}\gamma_{t,j}\gamma_{t,j-1}^{-1}\\ &\quad -q^{-1}x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}x_{k,j-1}\partial_{k,j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1}\prod_{t=1}^{k}\gamma_{t,j}\gamma_{t,j-1}^{-1}\\ &= (q-q^{-1})x_{k_1,i}\theta_{k_1,\dots,k_{s-1}}\partial_{k_{s-1},j-1}x_{k,j-1}\partial_{k,j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1}\prod_{t=1}^{k}\gamma_{t,j}\gamma_{t,j-1}^{-1}\\ &= (q-q^{-1})x_{k_1,i}\theta_{k_1,\dots,k_{s-1},k}\partial_{k,j}\prod_{a=1}^{s-1}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1}\prod_{t=1}^{k}\gamma_{t,j}\gamma_{t,j-1}^{-1}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\rho_q(E_{j,i}) &= \sum_{k=k_{s-1}}^{m}\sum_{1\le k_1\le\dots\le k_{s-1}\le m}(q^{-1}-q)^{\tau(k_1,\dots,k_{s-1})-1}\rho_q(E_{j,i})_{k,k_1,\dots,k_{s-1}}\\ &= -\sum_{1\le k_1\le\dots\le k_s\le m}(q^{-1}-q)^{\tau(k_1,\dots,k_s)-1}x_{k_1,i}\theta_{k_1,\dots,k_s}\partial_{k_s,j}\prod_{a=1}^{s}\prod_{t=1}^{k_a}\gamma_{t,i+a}\gamma_{t,i+a-1}^{-1},\end{aligned}$$

which gives the required statement. □

Now, let (σ_q, V) be a $U_q(\mathfrak{p})$-module. Then we can identify $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ with $\mathbb{C}_q[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ and obtain a $U_q(\mathfrak{g})$-module structure on $\mathbb{C}_q[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$. Further, using the isomorphism $\varphi_q \colon \mathbb{C}[\overline{\mathfrak{u}}^*] \to \mathbb{C}_q[\overline{\mathfrak{u}}^*]$ of vector spaces, we can transfer the $U_q(\mathfrak{g})$-module structure even on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$.

The main result of the present article is an explicit realization of the induced $U_q(\mathfrak{g})$-module structure on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ using quantum differential operators through the homomorphism

$$\pi_{q,V} \colon U_q(\mathfrak{g}) \to \mathcal{A}^q_{\overline{\mathfrak{u}}^*} \otimes_{\mathbb{C}} \operatorname{End} V \tag{4.53}$$

of $\mathbb{C}$-algebras defined by

$$((\psi_q \circ \varphi_q) \otimes \mathrm{id}_V)(\pi_{q,V}(a)(x^r \otimes v)) = a(E^r \otimes v) \tag{4.54}$$

for all $r \in M_{m,n}(\mathbb{N}_0)$ and $v \in V$. This is the content of the following theorem. Let us recall that $\mathbb{C}[\overline{\mathfrak{u}}^*]$ has a canonical structure of an $\mathcal{A}^q_{\overline{\mathfrak{u}}^*}$-module.

Since the $\mathbb{C}$-algebra isomorphism $\psi_q \colon \mathbb{C}_q[\overline{\mathfrak{u}}^*] \to U_q(\overline{\mathfrak{u}})$ is a homomorphism of $U_q(\mathfrak{l})$-modules by construction of $\mathbb{C}_q[\overline{\mathfrak{u}}^*]$ and the vector space isomorphism $\varphi_q \colon \mathbb{C}[\overline{\mathfrak{u}}^*] \to \mathbb{C}_q[\overline{\mathfrak{u}}^*]$ is a homomorphism of $U_q(\mathfrak{l})$-modules by definition of the $U_q(\mathfrak{l})$-module structure on $\mathbb{C}[\overline{\mathfrak{u}}^*]$, the identification of $U_q(\overline{\mathfrak{u}}) \otimes_{\mathbb{C}} V$ with $\mathbb{C}_q[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ is also an isomorphism of $U_q(\mathfrak{l})$-modules and we obtain

$$\pi_{q,V}(a) = \sum \rho_q(a_{(1)}) \otimes \sigma_q(a_{(2)}), \tag{4.55}$$

where

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}, \tag{4.56}$$

for all $a \in U_q(\mathfrak{l})$. If V is the trivial $U_q(\mathfrak{p})$-module, then we have $\pi_{q,V}(a) = \rho_q(a)$ for all $a \in U_q(\mathfrak{l})$. In that case, we shall denote $\pi_{q,V}$ by ρ_q.

Theorem 4.2 *Let (σ_q, V) be a $U_q(\mathfrak{p})$-module. Then the induced $U_q(\mathfrak{sl}_{n+m}(\mathbb{C}))$-module structure on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ is defined through the homomorphism*

$$\pi_{q,V} : U_q(\mathfrak{sl}_{n+m}(\mathbb{C})) \to \mathcal{A}^q_{\overline{\mathfrak{u}}^*} \otimes_{\mathbb{C}} End\, V \tag{4.57}$$

of $\mathbb{C}$-algebras by

$$\begin{aligned} \pi_{q,V}(f_i) &= \rho_q(f_i) \otimes id_V + \rho_q(k_i) \otimes \sigma_q(f_i), \\ \pi_{q,V}(e_i) &= \rho_q(e_i) \otimes \sigma_q(k_i^{-1}) + 1 \otimes \sigma_q(e_i), \\ \pi_{q,V}(k_i) &= \rho_q(k_i) \otimes \sigma_q(k_i) \end{aligned} \tag{4.58}$$

for $i = 1, 2, \dots, n-1$,

$$\begin{aligned} \pi_{q,V}(f_{n+i}) &= \rho_q(f_{n+i}) \otimes id_V + \rho_q(k_{n+i}^{-1}) \otimes \sigma_q(f_{n+i}), \\ \pi_{q,V}(e_{n+i}) &= \rho_q(e_{n+i}) \otimes \sigma_q(k_{n+i}) + 1 \otimes \sigma_q(e_{n+i}), \\ \pi_{q,V}(k_{n+i}) &= \rho_q(k_{n+i}) \otimes \sigma_q(k_{n+i}) \end{aligned} \tag{4.59}$$

for $i = 1, 2, \dots, m-1$, and

$$\begin{aligned} \pi_{q,V}(f_n) &= x_{1,n} \prod_{t=1}^{n-1} \gamma_{1,t} \otimes id_V, \\ \pi_{q,V}(e_n) &= \sum_{k=1}^{n-1} \pi_{q,V}(E_{n,k}k_n)\Bigg(\prod_{t=1}^{k} \gamma_{1,t}\partial_{1,k} \otimes id_V\Bigg) - \sum_{k=2}^{m}\prod_{t=k}^{m} \gamma_{t,n}\partial_{k,n} \otimes \sigma_q(k_n^{-1}E_{n+k,n+1}) \\ &\quad - \sum_{k=1}^{m}\prod_{t=1}^{k-1}\gamma_{t,n}\prod_{t=k+1}^{m}\gamma_{t,n}^{-1}\, x_{k,n}\partial_{k,n}\partial_{1,n} \otimes \sigma_q(k_n) + 1 \otimes \sigma_q(e_n) \\ &\quad + \prod_{t=1}^{m}\gamma_{t,n}\partial_{1,n} \otimes \frac{\sigma_q(k_n) - \sigma_q(k_n^{-1})}{q - q^{-1}} \\ \pi_{q,V}(k_n) &= \rho_q(k_n) \otimes \sigma_q(k_n), \end{aligned} \tag{4.60}$$

where

$$\begin{aligned} \pi_{q,V}(E_{n,k}) \;=\; &\rho_q(E_{n,k}) \otimes id_V + \rho_q(K_{k,n}) \otimes \sigma_q(E_{n,k}) \\ &+(q - q^{-1}) \sum_{k<\ell<n} \rho_q(E_{\ell,k}K_{\ell,n}) \otimes \sigma_q(E_{n,\ell}) \end{aligned}$$

for $k = 1, 2, \dots, n-1$.

Proof From the previous considerations we know that the action of the Levi quantum subgroup $U_q(\mathfrak{l})$ on $\mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ is given through the homomorphism

$$\pi_{q,V} : U_q(\mathfrak{l}) \to \mathcal{A}^q_{\overline{\mathfrak{u}}^*} \otimes \mathrm{End}\, V$$

of $\mathbb{C}$-algebras by the formula

$$\pi_{q,V}(a) = \sum \rho_q(a_{(1)}) \otimes \sigma_q(a_{(2)}),$$

where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, for all $a \in U_q(\mathfrak{l})$. Hence, by using Theorem 4.1 we get the corresponding expressions for all generators of $U_q(\mathfrak{g})$ except e_n and f_n.

By Lemma 4.1 we have

$$E_{n+1,n}E^r = q^{\sum_{t=1}^{n-1} r_{1,t}} E^{r+1_{1,n}},$$

which together with (4.54) gives us

$$\pi_{q,V}(E_{n+1,n})(x^r \otimes v) = q^{\sum_{t=1}^{n-1} r_{1,t}} x^{r+1_{1,n}} \otimes v = \left(x_{1,n} \prod_{t=1}^{n-1} \gamma_{1,t} \otimes \mathrm{id}_V\right)(x^r \otimes v)$$

for all $r \in M_{m,n}(\mathbb{N}_0)$ and $v \in V$. Further, using Lemma 4.1 we may write

$$\begin{aligned}
[E_{n,n+1}, E_{n+1,1}^{r_{1,1}} \cdots E_{n+m,n}^{r_{m,n}}] &= E_{n+1,1}^{r_{1,1}} \cdots [E_{n,n+1}, E_{n+1,n}^{r_{1,n}}] \cdots E_{n+m,n}^{r_{m,n}} \\
&\quad + \sum_{k=1}^{n-1} E_{n+1,1}^{r_{1,1}} \cdots [E_{n,n+1}, E_{n+1,k}^{r_{1,k}}] \cdots E_{n+m,n}^{r_{m,n}} \\
&\quad + \sum_{k=2}^{m} E_{n+1,1}^{r_{1,1}} \cdots [E_{n,n+1}, E_{n+k,n}^{r_{k,n}}] \cdots E_{n+m,n}^{r_{m,n}} \\
&= [r_{1,n}]_q E^{r-1_{1,n}} \frac{q^{-\sum_{t=1}^{m} r_{t,n}+1} K_{n,n+1} - q^{\sum_{t=1}^{m} r_{t,n}-1} K_{n,n+1}^{-1}}{q-q^{-1}} \\
&\quad + \sum_{k=1}^{n-1} [r_{1,k}]_q q^{-\sum_{t=k+1}^{n} r_{1,t} - \sum_{t=1}^{m} r_{t,n}} E_{n,k} E^{r-1_{1,k}} K_{n,n+1} \\
&\quad - \sum_{k=2}^{m} q^{-2} [r_{k,n}]_q q^{\sum_{t=k}^{m} r_{t,n}} E^{r-1_{k,n}} E_{n+k,n+1} K_{n,n+1}^{-1},
\end{aligned}$$

which together with (4.54) implies

$$\begin{aligned}
\pi_{q,V}(E_{n,n+1})(x^r \otimes v) &= x^r \otimes \sigma_q(E_{n,n+1}) v \\
&\quad + [r_{1,n}]_q x^{r-1_{1,n}} \otimes \frac{q^{-\sum_{t=1}^{m} r_{t,n}+1} \sigma_q(K_{n,n+1}) - q^{\sum_{t=1}^{m} r_{t,n}-1} \sigma_q(K_{n,n+1}^{-1})}{q-q^{-1}} v \\
&\quad + \sum_{k=1}^{n-1} [r_{1,k}]_q q^{-\sum_{t=k+1}^{n} r_{1,t} - \sum_{t=1}^{m} r_{t,n}} \pi_{q,V}(E_{n,k})(x^{r-1_{1,k}} \otimes \sigma_q(K_{n,n+1}) v) \\
&\quad - \sum_{k=2}^{m} q^{-2} [r_{k,n}]_q q^{\sum_{t=k}^{m} r_{t,n}} x^{r-1_{k,n}} \otimes \sigma_q(E_{n+k,n+1} K_{n,n+1}^{-1}) v
\end{aligned}$$

for all $r \in M_{m,n}(\mathbb{N}_0)$ and $v \in V$. Furthermore, using

$$\begin{aligned}
\frac{q^{-\sum_{t=1}^{m} r_{t,n}+1} K_{n,n+1} - q^{\sum_{t=1}^{m} r_{t,n}-1} K_{n,n+1}^{-1}}{q-q^{-1}} &= q^{\sum_{t=1}^{m} r_{t,n}-1} \frac{K_{n,n+1} - K_{n,n+1}^{-1}}{q-q^{-1}} \\
&\quad - \frac{q^{\sum_{t=1}^{m} r_{t,n}-1} - q^{-\sum_{t=1}^{m} r_{t,n}+1}}{q-q^{-1}} K_{n,n+1},
\end{aligned}$$

we get

$$\pi_{q,V}(E_{n,n+1})(x^r \otimes v) = x^r \otimes \sigma_q(E_{n,n+1})v + \prod_{t=1}^{m} \gamma_{t,n}\partial_{1,n}x^r \otimes \frac{\sigma_q(K_{n,n+1}) - \sigma_q(K_{n,n+1}^{-1})}{q - q^{-1}} v$$
$$- \frac{\prod_{t=1}^m \gamma_{t,n} - \prod_{t=1}^m \gamma_{t,n}^{-1}}{q - q^{-1}} \partial_{1,n}x^r \otimes \sigma_q(K_{n,n+1})v$$
$$+ \sum_{k=1}^{n-1} \pi_{q,V}(E_{n,k}) \Big(\rho_q(K_{n,n+1}) \prod_{t=1}^{k} \gamma_{1,t}\partial_{1,k}x^r \otimes \sigma_q(K_{n,n+1})v \Big)$$
$$- \sum_{k=2}^{m} \prod_{t=k}^{m} \gamma_{t,n}\partial_{k,n}x^r \otimes \sigma_q(K_{n,n+1}^{-1}E_{n+k,n+1})v$$

for all $r \in M_{m,n}(\mathbb{N}_0)$ and $v \in V$. Finally, using the formula

$$\frac{\prod_{t=1}^m \gamma_{t,n} - \prod_{t=1}^m \gamma_{t,n}^{-1}}{q - q^{-1}} = \sum_{k=1}^{m} \prod_{t=1}^{k-1} \gamma_{t,n} \prod_{t=k+1}^{m} \gamma_{t,n}^{-1} x_{k,n}\partial_{k,n},$$

we obtain the required statement. Moreover, from Lemma 4.2 we have

$$\pi_{q,V}(E_{n,k}) = \rho_q(E_{n,k}) \otimes \mathrm{id}_V + \rho_q(K_{k,n}) \otimes \sigma_q(E_{n,k}) + (q - q^{-1}) \sum_{k<\ell<n} \rho_q(E_{\ell,k}K_{\ell,n}) \otimes \sigma_q(E_{n,\ell})$$

for $k = 1, 2, \dots, n-1$. This finishes the proof. □

Theorem 4.2 together with Theorem 4.1 and Lemma 4.3 give us an explicit realization of induced modules $M^{\mathfrak{g}}_{q,\mathfrak{p}}(V) \simeq \mathbb{C}[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ by quantum differential operators for any $U_q(\mathfrak{p})$-module V. In fact, we have a stronger result. As $\pi_{q,V} : U_q(\mathfrak{g}) \to \mathcal{A}^q_{\overline{\mathfrak{u}}^*} \otimes_{\mathbb{C}} \operatorname{End} V$ is a homomorphism of associative $\mathbb{C}$-algebras, we may take another $\mathcal{A}^q_{\overline{\mathfrak{u}}^*}$-module $\mathcal{M}$ instead of $\mathbb{C}[\overline{\mathfrak{u}}^*]$ and we obtain a $U_q(\mathfrak{g})$-module structure on $\mathcal{M} \otimes_{\mathbb{C}} V$ through the homomorphism $\pi_{q,V}$. Moreover, since the classical limit of $\mathcal{A}^q_{\overline{\mathfrak{u}}^*}$ via the specialization $q \to 1$ is the Weyl algebra $\mathcal{A}_{\overline{\mathfrak{u}}^*}$, it would be interesting to consider such $\mathcal{A}^q_{\overline{\mathfrak{u}}^*}$-modules for $\mathcal{M}$ that the corresponding $U_q(\mathfrak{g})$-module $\mathcal{M} \otimes_{\mathbb{C}} V$ is a flat deformation of a twisted induced module.

A finite-dimensional module V over $U_q(\mathfrak{l})$ has the classical limit $\widetilde{V}$ over $\mathfrak{l}$. As it was mentioned at the end of the previous section, extending properly V to a module over $U_q(\mathfrak{p})$ we can guarantee that it still admits the classical limit and that $\mathbb{C}_q[\overline{\mathfrak{u}}^*] \otimes_{\mathbb{C}} V$ is a flat deformation of a generalized Verma module for $\mathfrak{g}$. Moreover, from Theorem 4.2 we easily get the classical limit by the specialization $q \to 1$.

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