RT-MAT 90-02

# Generation of Quantum-fields by non-deterministic fields.

R.G. LINTZ

junho 1990

# GENERATION OF QUANTUM-FIELDS BY NON-DETERMINISTIC FIELDS

*R.G. Lintz*

*University of São Paulo, Brazil*

**Summary**

**A method for construction of quantum-fields is presented by using concepts of non-deterministic mathematics. In particular it is shown how non-deterministic fields generate quantum-fields.**

## §I. Generalities

1. In this paragraph we introduce the basic mathematical tools to be used in the presente work. These ideas lying under the roof of the so-called non-deterministic mathematics have been developed by us together with students and collaborators since 1966 and their applications to physics are collected in a recent unpublished work, available in typewritten form, entitled "Non-deterministic foundation of mechanics" [1]. In the present paper we develop some ideas sketched therein connected with quantum-field theory. Our aim here is to illustrate the methods of non-deterministic mathematics in an important area of present day physics.

2. The basic philosophy of non-deterministic mathematics consists in looking to mathematics as a collection of sharply defined concepts and afterward trying to substitute them by non-sharply defined ones.

For example, take the concept of function: we usually say that a function $f : X \to Y$ is defined in a point $x \in X$ and its value is a well defined point $y \in Y$. The non-deterministic approach would say that $f$ is defined "around $x$" and its value is "around $y$", namely we look to non sharply defined quantities. The natural formalization of this idea is to start from functions taking open sets of $X$ into open sets of $Y$, instead of points into points. Since 1966 this idea has been developed by us and today we can say that we have a rather complete "calculus" on topological spaces with derivatives, integrals, differential equations, etc. without any compromise with metric or linear structures, namely, we only need topological structures. For more precise information on part of this material one can look at [2].

All this fits very well with our feelings about the world. Indeed, we have never a "point" and so a particle always occupies a certain volume and consequently it is more appropriate to represent it by a single open set. In the same way, instead of "instant of time" we consider "interval of time". Consequently, the motion of a particle can be represented by the association of open sets in a space $X$ to open intervals in the real line, what is formalized in a precise way by the concept of non-deterministic function. Afterwards by the introduction of Gauss structures in $X$ we can define velocity, acceleration, etc., in general topological spaces and this culminates in the construction of a mechanics in topological spaces without any appeal to metric or linear structures. All this is discussed in details in [1]. Some information can also be obtained from [2].

In this paper we only need the concept of non-deterministic fields, abbreviate n-fields, which constitute a particular case of non-deterministic functions.

Let $X$ be a topological space and $V$ a family of open coverings of $X$, i.e., an element $\sigma \in V$ is a collection of subsets of $X$ made up of open sets $A \in \sigma$. For the particular case when $X$ is the set of real numbers $R$ we allow some sets in $\sigma \in V$ to be reduced to single points or singletons as called by topologists. We use the notation $(X, V)$ to indicate a pair formed by a topological space $X$ and a family $V$ of collection of subsets of $X$. When $X$ is the set of real numbers with the usual topology we use the notation $[R, V_R]$ to recall that some elements of $\sigma_R \in V_R$ might be singletons instead of open intervals, that is, any element $A_R \in \sigma_R$ is either an open interval or a singleton.

**Definition I:** A $n$–field from the pair $(X, V)$ into the pair $[R, V_R]$

$$\Phi : (X, V) \to [R, V_R]$$

is:

a) a function

$$\Phi_V : V \to V_R$$

which associates to each collection $\sigma \in V$ a collection $\sigma_R \in V_R$;

b) for each $\sigma \in V$ a function

$$\Phi_\sigma : \sigma \to \sigma_R = \Phi_V(\sigma)$$

which associates to each set $A \in \sigma$ a set $A_R \in \sigma_R$. We assume also that if $A = \emptyset$ then $A_R = \emptyset$, where $\emptyset$ denotes the empty set.

**Definition II:** A $n$–field

$$\Phi : (X, V) \to [R, V_R]$$

is continuous if:

1) for any $\sigma, \tau \in V$, with $\tau > \sigma$, (where this notation means "$\tau$ refines $\sigma$", namely any $B \in \tau$ is contained in some $A \in \sigma$), we have also

$$\Phi_V(\tau) > \Phi_V(\sigma);$$

2) if $\tau > \sigma$ and $B \in \tau$, $A \in \sigma$ we have, if $B \subset A$,

a) $f_\tau(B) \subset f_\sigma(A)$, if both are either open sets or singletons.

b) $f_\tau(B) \subset \overline{f_\sigma(A)}$ if $f_\sigma(B)$ is a singleton and $f_\sigma(A)$ is an open set.

**Definition III:** A $n$–field

$$\Phi : (X, V) \to [R, V_R]$$

generates a usual function

$$\phi : X \to R$$

if for any $x \in X$ and any neighborhood $V[\phi(x)]$ in $R$ there is $\sigma \in V$ and $A \in \sigma$ such that

$$\forall\, x \in A \Rightarrow \phi(x) \in \overline{\Phi_\sigma(A)} \quad \text{and} \quad \phi_\sigma(A) \subset V[\phi(x)].$$

Intuitively this definition means that both "values" of $\phi_\sigma(A)$ and $\phi(x)$ are close to each other, when $x$ runs in $X$.

It can be shown that if $\Phi$ is continuous then $\phi$ is continuous as well and also that $\phi$ is uniquely defined by $\Phi$. Several theorems have been proved by A. Jensen [3] showing when a given non-deterministic function generates a usual function and many other important results in the area.

For our needs in this paper we have to introduce the idea of *complex n–field.* There are two ways of doing that; the first one, which will be adopted in this work is defined as follows: let $[C, V_C]$ be a pair where $C$ is the complex plane and $V_C$ is defined by

$$\forall\, \sigma_C \in V_C \Rightarrow \sigma_C = \sigma_R \otimes \sigma_R$$

where $\sigma_R \in V_R$ and

$$\sigma_C = \{A_R \times B_R : A_R, B_R \in \sigma_R\}.$$

Let $\Phi_1$ and $\Phi_2$ be $n$–fields as considered before such that

$$\forall\, \sigma \in V, \quad \Phi_{1V}(\sigma) = \Phi_{2V}(\sigma) = \sigma_R.$$

Now define

$$\Phi : (X, V) \to [C, V_C]$$

as follows:

$$\forall \sigma \in V, \quad \Phi_V(\sigma) = \Phi_{1V}(\sigma) \times \Phi_{2V}(\sigma) = \sigma_R \otimes \sigma_R \in V_C$$

and for a given $\sigma \in V$,

$$\forall A \in \sigma, \quad \Phi_\sigma(A) = \Phi_{1\sigma}(A) \times \Phi_{2\sigma}(A) \in \sigma_C = \sigma_R \otimes \sigma_R \in V_C.$$

We denote $\Phi$ by $\Phi = (\Phi_1, \Phi_2)$ calling $\Phi_1$ its real part and $\Phi_2$ its imaginary part.

The second point of view defines $\Phi$ as a $n$–function

$$\Phi : (X, V) \to [C, V_C]$$

where $V_C$ is now an arbitrary family of collections $\sigma_C$, with $\sigma_C$ made up of open sets or singletons in $C$. However, this point of view will be not adopted here.

For both ways of introducing a complex $n$–field we can define some operations with them, namely

a) addition of $n$–fields, where given $\Phi_1$ and $\Phi_2$ complex $n$–fields we define

$$\Phi = \Phi_1 + \Phi_2$$

by considering for each $\sigma \in V$ and $A \in \sigma$

$$\Phi_\sigma(A) = \{z \in C : z = z_1 + z_2, \ z_1 \in \Phi_{1\sigma}(A), \ z_2 \in \Phi_{2\sigma}(A)\}$$

b) multiplication by a complex number $\alpha$, in a similar way as for the addition.

With the operations a) and b) the notation

$$\Phi = \Phi_1 + i\Phi_2$$

has the usual meaning as a sum, namely $\Phi_1$ is identified with the complex $n$–field $(\Phi_1, 0)$ and $\Phi_2$ with $(0, \Phi_2)$ and

$$\Phi = (\Phi_1, 0) + (0, \Phi_2) = (\Phi_1, \Phi_2) = \Phi_1 + i\Phi_2.$$

Besides the concepts defined above we shall use concepts of topology and functional analysis without necessarily defining them and we refer the reader to standard books in those areas.

## §II. Construction of quantum fields

1. Among the several approaches to quantum field theory proposed by different authors we choose the traditional one as considered by A.S. Wightman in [4]. In forthcoming papers we shall deal with alternative ways of looking to the same question by using integration of $n$–functions which corresponds to the use of Feynman integrals as done, for instance in [5].

We start by recalling the axions of quantum field theory used by us in this paper. Let $\Gamma$ be a represention of dimension $r$ of $SU(2)$, the group of unitary transformations of the complex plane of dimension 2, namely if $A \in SU(2)$ then $A$ is a linear transformation of det 1 taking a pair of complex numbers $(z_1, z_2)$ into another pair of complex numbers $(w_1, w_2)$. Now, $\Gamma(A)$ will be a linear transformation of $R^r$, namely, associating to a point $x \in R^r$ another point $y \in R^r$. It is given by a matrix $\{\Gamma_{ij}(A)\}$, $i, j = 1, \ldots, r$. Let $\tilde{P}_{\bullet}$ be the group $SU(2)$ plus a translation, namely if $\tilde{A} \in \tilde{P}_{\bullet}$ then $\tilde{A}$ takes the point $(z_1, z_2)$ into $(a_1, a_2) + (w_1, w_2)$ where $(a_1, a_2)$ is a pair of complex numbers defined by $\tilde{A}$. Usually we indicate $\tilde{A}$ by $(a, A)$, where $A \in SU(2)$ and $a = (a_1, a_2)$. To complete our group theoretical requirements we assume an unitary representation of $\tilde{P}_{\bullet}$ in the state vector space $H$ of wave functions defining the states of the physical system under consideration, where we intend to introduce a relativistic quantum field. To clarify this last statement let us recall that $H$ could be for instance defined as the set of square summable complex valued functions $\Psi$ defined in some Euclidian space $R^m$ where in

$H$ the topology is given by the usual metric

$$\text{dist}(\Psi_1, \Psi_2) = \left(\int_{R^m} \Psi_1 \Psi_2^* dx\right)^{\frac{1}{2}}, \qquad x \in R^m,$$

of a Hilbert space where $\Psi^*$ is the complex conjugate of $\Psi$. Now to each $\tilde{A} \in \tilde{P}_o$ it is associated a transformation $U(a, A)$ of $H$ and this association is a homomorphism.

Consider now the set $S(R^4)$ of functions defined as follows: each $f \in S(R^4)$ is a complex function of the spacetime $R^4$ with derivatives of all orders, such that for all integer $N \geq 0$ and $x = (x_1, x_2, x_3, x_4) \in R^4$, we have

$$\lim_{x \to \infty} \frac{|f(x)|}{[x_1^2 + x_2^2 + x_3^2 + x_4^2]^{-\frac{N}{2}}} = 0.$$

The topology of $S(R^4)$ is given by the following sequence of norms, for integers $r, s > 0$:

$$\|f\|_{r,s} = \sum_{\substack{k \\ |k| \leq r}} \sum_{\substack{\ell \\ |\ell| \leq s}} \sup_x |x^k D^\ell f(x)|$$

where

$$x^k = x_1^{k_1} . x_2^{k_2} . x_3^{k_3} . x_4^{k_4}$$

$$|k| = k_1 + k_2 + k_3 + k_4$$

$$|\ell| = \ell_1 + \ell_2 + \ell_3 + \ell_4$$

$$D^\ell f(x) = \frac{\partial^{|\ell|} f(x)}{(\partial x_2)^{\ell_1} (\partial x_2)^{\ell_2} (\partial x_3)^{\ell_3} (\partial x_4)^{\ell_4}}$$

If instead of $R^4$ we want to consider cartesian products of $R^4$ the same expressions above are also good with the pertinent adjustments. However in this work we use only $R^4$.

The set $S(R^4)$ with the topology generated by this family of norms is called the *space of test functions.*

2. A quantum field theory is given by a Hilbert space $H$ of state function $\psi$, an unitary representation of $\tilde{P}_o$ in $H$ given by unitary linear operators $U(a, A)$ acting

on $H$ defined by elements $(a, A) \in \tilde{P}_o$ and linear vector-operators $T(f)$ acting on $H$ defined by each element $f \in S(R^4)$, satisfying the axioms:

I) The operators $T(f) = (T_1(f), \ldots, T_r(f))$ with $r$ components and their adjoints $T^*(f)$ act on $\psi \in H$ as

$$T(f)\psi = \sum_{i=1}^{r} T_i(f)\psi$$

and they have a common invariant domain $\Omega \subset H$, independent of $f$, dense in $H$ containing a vacuum vector $\psi_o$ such that

$$U(a, A)\psi_o = \psi_o$$

for all $(a, A) \in \tilde{P}_o$.

II) For any $f, g \in S(R^4)$ we have

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$$

with $\alpha, \beta$ complex numbers.

III) For any $\psi_1, \psi_2 \in H$ and for a given $f \in S(R^4)$, $(\psi_1, T(f)\psi_2)$ is continuous as a function of $f$ in the topology of $S(R^4)$.

IV) For a representation $U(a, A)$ of $\tilde{P}_o$, as indicated above, we have invariance of $\Omega$, i.e., $U(a, A)\Omega \subset \Omega$ and

$$\Gamma T(L(f)) = U(a, A)T(f)U^{-1}(a, A)$$

where $\Gamma T$ is the transform of the operator $T(f)$ to be explained below and $L(f)$ is defined by the action of the Lorentz transformation $L$ on $f$, namely

$$L(f)(x) = f(L^{-1}(x - a))$$

and $\Gamma$ and $L$ are related to each other as explained below.

Let us clarify in details the rather condensed form of this axiom, called the *axiom of relativistic covariance of the fields.*

Call $S$ the space $S(R^4)$ and $O(S)$ the set of all linear operators defined by associating to each $f \in S$ a vector operator $T(f)$ acting in the Hilbert space $H$. This association must satisfy the conditions to be discussed now.

Call $\alpha$ a function associating to each $f \in S$ an operator $T(f)$ such that:

1) If $(L,a)$ is a Lorentz transformation plus a translation of $R^4$ we define for each $f \in S$

$$Lf(x) = f(L^{-1}(x-a))$$

and we call $LS$ the space of all functions $Lf$ on $R^4$ for all $f \in S$. Then we assume that to the function

$$\alpha : S \to O(S)$$

it is associated another function

$$L\alpha : LS \to O(LS)$$

and a function

$$L^* : O(S) \to O(LS)$$

such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & O(S) \\ \downarrow L & & \downarrow L^* \\ LS & \xrightarrow{L}\alpha & O(LS) \end{array}$$

commutes.

2) If $L'$ is another Lorentz transformation and $(L',a')$ is given with the translation

$\alpha'$, then:

$$
\begin{array}{ccc}
S & \xrightarrow{\alpha} & O(S) \\
\downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L^*} \\
LS & \xrightarrow{L\alpha} & O(LS) \\
\downarrow{\scriptstyle L'} & & \downarrow{\scriptstyle L'^*} \\
L'LS & \xrightarrow{L'L\alpha} & O(L'LS)
\end{array}
$$

is commutative.

3) If $U$ is an unitary representation of $\tilde{P}_e$ and $UH$ is the set of all states $U\psi$ for $\psi \in H$, the diagram

$$
\begin{array}{ccccc}
S \times H & \xrightarrow{\alpha \times 1} & O(S) \times H & \xrightarrow{\beta} & H \\
\downarrow{\scriptstyle L \times U} & & & & \downarrow{\scriptstyle U} \\
LS \times UH & \xrightarrow{L\alpha \times 1} & O(LS) \times UH & \xrightarrow{L\beta} & UH
\end{array}
\qquad (1)
$$

is commutative, where $\beta$ takes the pair $(T(f), \psi)$ into $T(f)\psi \in H$ whenever $T(f)\psi$ is defined. By denoting $L^*T$ by $\Gamma T$ the commutativity of (1) means precisely that

$$UT(f)\psi = (\Gamma T)(Lf)U\psi \qquad (1')$$

which is the expression of the relativistic covariance of fields. In many circumstances the use of the diagram (1) is much better than the expression (1'). Of course, as $T(f)$ is a vector operator this expression represents a system of equations depending on the $r$-dimensional representation of $\tilde{P}_e$ if $T(f)$ has $r$ components.

V) Let the supports of $f, g \in S(R^4)$ be separated by a spacelike interval in $R^4$, namely

$$f(x).g(x) = 0 \quad \text{if} \quad (x-y)^2 \geq 0.$$

In this case we assume that $T(f)$ and $T(g)$ either commute or anticommute, i.e.,

$$[T(f), T(g)]_{\mp} = [T(f), T^*(g)]_{\mp} = 0$$

where [,] stands for the usual Poission bracket. This axiom is called *axiom of locality.*

Besides these axioms we usually consider another one called *completeness axiom*, stating that a certain algebra of operators of the type $T(f)$ applied to the vacuum state $\psi_o$ gives a dense subset of $H$. However we do not state this axiom here because a much stronger form of it will be discussed later.

Finally the so called *collision states* will be studied in a forthcoming paper and will not be considered here.

3. To study now the main question of this paper, i.e., the construction of a quantum field theory satisfying the axioms above by using $n$–fields we need several hypothesis and consequently we assume the following:

I) In $R^4$ we consider a family $V$ of open coverings with countably many coverings and each covering $\sigma \in V$ has also countably many bounded open sets. In this way we denote these coverings by $\sigma_1, \sigma_2, \ldots$ and for each $\sigma_i$ we ennumerate its open sets as $A_{ij}$ with the assumption that $j \geq i$, for reasons to be clarified later. We also choose the indexing of $\sigma_i$ in such a manner that $\sigma_j > \sigma_i$ if $j \geq i$. In few words, $V$ is a totally ordered set of open coverings of $R^4$.

II) Let $r$ be a positive integer, $r \geq 1$ and consider $r$ $n$–fields, which will be always assumed to be complex unless stated otherwise,

$$\Phi^i : (R^4, V) \rightarrow [C, V_C], \quad 1 \leq i \leq r$$

defined over the same pair $(R^4, V)$ with $V$ as in I).

For each $\Phi^i$ we introduce a countable set of complex functions defined for each $A_{k\ell} \in \sigma_k$, $k \geq 1$ and $\ell \geq 1$, $\ell \geq k$:

$$d^i_{k\ell} : C \rightarrow \Phi^i_{\sigma_k}(A_{k\ell})$$

with the only condition of being measurable relatively to the usual Lebesgue measure on $R^4$. For all the axioms, excepted the locality axiom, we do not need any additional condition on $d^i_{k\ell}$, however for this axiom we have to put strong restriction on the functions $d^i_{k\ell}$. We postpone the introduction of these conditions until we come to the discussion of the locality axiom.

We denote

$$d^i = \{d^i_{k\ell}\}, \quad \begin{matrix} \ell \geq 1 \\ k \geq 1 \end{matrix}, \quad \ell \geq k$$

and call the pair $(\Phi^i, d^i)$ a *weighted n-field* calling $d^i$ the *weight of* $\Phi^i$.

Let $\lambda_1 \ldots \lambda_r$ be complex numbers. We define

$$(\Phi, d) = \sum_{i=1}^{r} \lambda_i (\Phi^i, d^i)$$

as follows:

$$\Phi : \sum_{i=1}^{r} \lambda_i \Phi^i$$

in the sense considered before and

$$d = \{d_{k\ell}\} \quad \begin{matrix} \ell \geq 1 \\ k \geq 1 \end{matrix}, \quad \ell \geq k$$

where

$$d_{k\ell}(z) = \sum_{i=1}^{r} \lambda_i d^i_{k\ell}(z), \quad z \in C.$$

Therefore we have defined linear combination of weighted fields.

For all $n$-fields involved in this paper unless otherwise stated we shall assume the following boundness condition:

(B): for any $A_{k\ell} \in \sigma_k$ call $M_{k\ell}$ the least upper bound of the set $|\Phi_{\sigma_{k\ell}}(A_{k\ell})|$ in the real line, then assume that the matrix

$$\{M_{k\ell}\}_{k,\ell}$$

with $M_{\ell k} = M_{k\ell}$ is square summable, i.e., all rows and columns are square summable sequences.

Now let us show how we associate to each $f \in S(R^4)$ a linear operator in $H$, the Hilbert space of states of our system.

As easily seen each $f \in S(R^4)$ is square summable and hence we can assume that it belongs to the space of square summable complex valued functions on $R^4$, where we have an orthonormal base $\{\varphi_u\}$, $u \geq 1$, so that each $f \in S(R^4)$ can be written as

$$f = \sum_{u \geq 1} a_u \varphi_u$$

in the topology of that space.

Suppose now that we give $r \geq 1$ complex weighted $n$-fields $(\Phi^i, d^i)$ on $(R^4, V)$ and if supp$f$ is the support of $f$ in $R^4$, define for each $A_{k\ell} \in \sigma_k$, $\ell \geq k$, and each $(\Phi^i, d^i)$ the function

$$f^i_{k\ell}(x) \begin{cases} = \sum a_u (d^i_{k\ell} \circ \varphi_u(x)) 2^{-u}, & x \in A_{k\ell} \\ = 0 & \text{otherwise,} \end{cases}$$

if $A_{k\ell} \cap \text{supp} f \neq \emptyset$. But if $A_{k\ell} \cap \text{supp} f = \emptyset$ then $f^i_{k\ell}(x) = 0$ for all $x \in R^4$.

Now define for the give $f$

$$T^i_{k\ell}(f) = \frac{1}{m(A_{k\ell})} \int_{A_{k\ell}} f^i_{k\ell}(x) dx \qquad (\ell \geq k)$$

the integral understood in the sense of Lebesgue.

From the intuitive point of view each $d^i_{k\ell}$ brings the value of $\varphi_u$ at $x$ into the range of the $n$-field $\Phi^i$ in $A_{k\ell}$. When $\Phi^i$ generates an usual function $\lambda : R^4 \to C$, all $\varphi_u(x)$ will be taken to coincide with $\lambda(x)$ for each $x \in R^4$. In few words for the case of a classical field everything is sharply defined at $x$ as it will be discussed in more details in the end of this paper.

Let us now drop the restriction $\ell \geq k$ and define

$$A_{\ell k} = A_{k\ell}$$

$$d^i_{\ell k} = (d^i_{k\ell})^*, \quad 1 \leq i \leq r,$$

where the astherisk ($*$) denotes the conjugate of a complex number. Then

$$T^i_{\ell k}(f) = (T^i_{k\ell})^*$$

and

$$T^i(f) = \{T^i_{k\ell}(f)\}_{k,\ell}, \qquad k,\ell \geq 1$$

is then a hermitian matrix where rows and columns are square summable. Indeed, for each $k$,

$$\sum_{\ell\geq 1} |T^i_{k\ell}(f)|^2 \leq \sum_{\ell \geq 1} \frac{1}{m^2(A_{k\ell})} \left( \int_{A_{k\ell}} |f^i_{k\ell}(x) \mid dx \right)^2 \leq$$
$$\leq \sum_{\ell \geq 1} |\sup_s a_s|^2 |M_{k\ell}|^2 \leq \sup^2_s |a_s| \sum_{\ell\geq 1} |M_{k\ell}|^2 < +\infty,$$

due to the boundness condition (B).

We have called $H$ the Hilbert space of the states of the system but really these states are given by rays in $H$ which is also separable, i.e., has a countable base $\{e_k\}$. By a classical procedure in the theory of Hilbert spaces we can assume that $\{e_k\}$ is an orthonormal base and represent each state of the system as

$$\psi = \sum_{k\geq 1} b_k e_k$$

in the topology of $H$, with $\{b_k\}$ a square summable sequence according to the theorem of Fisher-Riesz.

Then we define the action of

$$T^i(f) = \{T^i_{k\ell}(f)\}_{k\ell}, \quad k,\ell \geq 1$$

in $H$ by looking to $\{b_k\}_{k\geq 1}$ for a given $\psi \in H$ as a column vector an applying $T^i(f)$ by matrix multiplication, row $\times$ column, what gives a new vector $\{b'_k\}_{k\geq 1}$ whose components are

$$b'_k = \sum_{s\geq 1} T^i_{ks}(f) b_s$$

and consequently we can define

$$T^i(f)\psi = \psi' = \sum_{k\geq 1} b'_k e_k.$$

Observe that $\{b'_k\}$ might not be square summable and in this case we say that $T^i(f)$ is not defined at $\psi$. This is the case of non-bounded operators.

Finally, we define the vector linear operator

$$T(f) = (T^1(f), \ldots, T^r(f))$$

by

$$T(f)\psi = \sum_{i=1}^{r} T^i(f)\psi$$

which is in general an unbounded operator in $H$.

Our task now for the rest of this paper is to show that the operators $T(f)$ satisfy the axioms of a quantum field theory as indicated before.

## §III - The axioms of quantum-field theory

1. For most of the axioms we do not need any special restrictions on the weights $d^i$ of the $n$-fields involved; however, for the axiom of locality we need to impose restrictions on the weights $d^i$ related with the physical interpretation of this axiom leading in many cases to philosophical discussions about our possibilities of knowledge of reality in points moving from each other with speed perhaps greater than light.

Another remark is the following: for certain axioms the proof for a general number $r \geq 1$ of $n$-fields is a straightforward extension of the proof for $r = 1$ and in this case, for simplicity, we shall consider only this case of $r = 1$. Again, for the axiom of locality the interpretation of the case $r = 1$ is essentially different from that of the general case and then both cases will be discussed separately.

2. **Axiom I.** Considering $r = 1$ we see that $T^*(f)$ is defined whenever $T(f)$ is and consequently it is enough to show that the domain of $T(f)$ is dense in $H$. For that, consider all $\psi \in H$ having only finitely many coefficients different from zero in their representations relatively to a base $\{e_k\}$ of $H$. As well known from the theory of Hilbert spaces this set $\Omega$ is dense in $H$ and clearly $T(f)$ is defined for $\psi \in \Omega$ with value $T(f)\psi$ also in $\Omega$.

For what concerns the unitary representation $U$ of $\tilde{P}_e$ and the vacuum vector $\psi_o$, this is independent of the definition of $T(f)$ and consequently we have to assume as given from the beginning the vector $\psi_o$, which depend on the nature of physical system under consideration, and $U$ satisfying the conditions:

$$U\,\Omega \subset \Omega \quad \text{and} \quad U\psi_o = \psi_o.$$

Later when we discuss the completness axiom we shall give some informations about the vacuum vector $\psi_o$.

**Axiom II.** It is immediate from the definition of $T(f)$.

**Axiom III.** It is enough to study the case $r = 1$.

Let $\psi_1, \psi_2 \in H$ be arbitrarily selected and consider the functional

$$F(f) = (\psi_1, T(f)\psi_2)$$

defined on $S(R^4)$. To say that $F(f)$ is continuous as a distribution on $S(R^4)$ it means that for any $\varepsilon > 0$ there is a neighborhood of the origin in $S(R^4)$,

$$U_{\sigma\delta}(0) = \{f : \|f\|_{r,s} < \delta\}$$

with $\sigma = \max(r,s)$, such that

$$\forall\, f \in U_{\sigma\delta}(0) \Rightarrow |F(f)| < \varepsilon.$$

Suppose that

$$\psi_1 = \sum_{k\geq 1} c_k \varphi_k \quad \text{and} \quad \psi_2 = \sum_{k\geq 1} b_k \varphi_k$$

are selected as before with $T(f)\psi_2$ defined, i.e., $\|T(f)\psi_2\|$ is finite. From this we have

$$\|T(f)\psi_2\|^2 = \sum_k (\sum_j T_{kj}(f) b_j)^2 < +\infty,$$

what implies

$$\sum_j |T_{kj}(f)|\ |b_j| < +\infty, \quad k = 1, 2, \ldots$$

Recalling the expression of $f_{kj}$ we have

$$|T_{kj}(f)| = \frac{1}{m(A_{kj})} \left| \int_{A_{kj}} \sum_{m\geq 1} a_m (d_{kj} \circ \varphi_m) 2^{-m} dx \right| < +\infty,$$

what gives

$$|(\psi_1, T(f)\psi_2)| = \left| \sum_k c_k (\sum_j T_{kj}(f) b_j) \right| < +\infty.$$

Now when $f \to 0$ in the topology of $S(R^4)$ considered before its Fourier coefficients $a_m$ also tend to zero and consequently as $(\psi_1, T(f)\psi_2)$ remains bounded we conclude that

$$T_{kj}(f) \to 0, \qquad k, j \geq 1.$$

Therefore

$$(\psi_1, T(f)\psi_2) \to 0$$

what proves axiom III.

**Axiom IV.** We shall use the approach of this axiom discussed in §II,2 with the corresponding notations introduced therein.

We start by discussing the case $r = 1$. Let $\psi \in H$ be arbitrarily selected in the domain of definition of $T(f)$, with

$$\psi = \sum_k b_k e_k.$$

We get, for any $(a, A) \in \tilde{P}_{*}$,

$$U(a, A)\psi(x) = \sum_{k} b_k U(a, A) e_k$$

because $U(a, A)$ is a continuous operator and also because it is unitary $U(a, A)e_k$, $k \geq 1$, is also an orthonormal base of $H$, or more properly of $UH$.

Now observe that because the determinant of the Lorentz transformation $L$ is equal to $1$ and consequently the transformation $(L, a)$ preserves the Lebesgue's measure, we have, for each $k$, $i \geq 1$,

$$T_{ki}(f) = T_{ki}(Lf) = (\Gamma T)_{ki}(Lf).$$

Furthermore if $A_{ij} \in \sigma_i$ by the action of $(L, a)$ this set is transformed in another one denoted by $LA_{ij}$ and for a $n$-field $\Phi$ we define the transformed $n$-field $L\Phi$ by looking to the covering $L\sigma_i = \{LA_{ij}\}$ and putting:

$$\begin{cases} L\Phi_{LV} = \Phi_V, \qquad LV = \{L\sigma_i\} \\ L\Phi_{L\sigma_i}(LA_{ij}) = \Phi_{\sigma_i}(A_{ij}) \end{cases}$$

Hence

$$\Gamma T(Lf)[U(a, A)\psi] = \sum_{k} (\sum_{i} T_{ki} b_i) U(a, A) e_k.$$

By another side we have

$$U(a, A)[T(f)\psi] = U(a, A) \sum_{k} (\sum_{i} T_{ki} b_i) e_k =$$

$$= \sum_{k} (\sum_{i} T_{ki} b_i) U(a, A) e_k,$$

what proves axiom IV for $r = 1$.

For the case $r > 1$ we start using $r$, weighted $n$-fields $(\Phi^1, d^1), (\Phi^2, d^2), \ldots, (\Phi^r, d^r)$ which we write as the $r$-components of a $r$-dimensional $n$-field

$$(\Phi, d) = \{(\Phi, d)^1, \ldots, (\Phi, d)^r\}$$

where $(\Phi, d)^i = (\Phi^i, d^i)$, $1 \le i \le r$ is called the $i$-component of $(\Phi, d)$. Given an element $(A, a) \in \tilde{P}_*$ and a $r$-dimensional representation $\Gamma$ of $\tilde{P}_*$ we denote by $\Gamma(A, a)$ its $r \times r$ matrix

$$\Gamma(A,a) = \{\Gamma_{ij}(A,a)\}_{i,j}, \qquad \begin{matrix} 1 \le i \le r \\ 1 \le j \le r \end{matrix}$$

and assume that the components of $(\Phi, d)$ transform accordingly to the formulas

$$\Gamma(\Phi,d)^j = \sum_{i=1}^{r} \Gamma_{ji}^{-1}(A,a)(\Phi^i,d^i), \qquad 1 \le j \le r,$$

where $\Gamma_{ji}^{-1}(A,a)$ is the adjoint of the $(i,j)$-element of the matrix representing $\Gamma^{-1}$. The reason for this sort of contravariant approach in the definition of $\Gamma(\Phi, d)$ will be clarified as we proceed.

To show the validity of axiom IV we suppose as before that

$$f = \sum_{k \ge 1} a_k \varphi_k$$

and define for each $i = 1, \dots, r$ and $A_{m,n} \in \sigma_m$

$$f_{m,n}^i(x) \begin{cases} = \sum 2^{-k} a_k (d_{m,n}^i \circ \varphi_k)(x), x \in A_{m,n} & \text{and } A_{m,n} \cap \operatorname{supp} f \neq \emptyset. \\ = 0, & \text{otherwise.} \end{cases}$$

where $d_{m,n}^i$ is given by the weight $d^i = \{d_{m,n}^i\}$ of $(\Phi^i, d^i)$. Again as before we put, in matrix form

$$T_i(f) = \left\{ \frac{1}{m(A_{m,n})} \int_{A_{m,n}} f_{m,n}^i dx \right\}_{m,n}, \qquad i = 1, \dots, r$$

and

$$T(f) = \sum_{i=1}^{r} T_i(f).$$

Let us define $\Gamma T$ given by the representation $\Gamma$ of $\tilde{P}_*$. Call $L$ the Lorentz transformation given by the element $(A, a) \in \tilde{P}_*$. The base $\{\varphi_k\}$ changes into a new base $\{L\varphi_k\}$ in $LR^4$, with

$$L\varphi_k(x) = \varphi_k(L^{-1}(x - a))$$

and we define

$$\Gamma f^j_{m,n}(x)\begin{cases} = \sum 2^{-k} a_k (\Gamma d^j_{m,n} \circ L\varphi_k)(x), x \in A^L_{m,n} & \text{and } A_{m,n} \cap \operatorname{supp} f \neq \emptyset \\ = 0, & \text{otherwise.}\end{cases}$$

where $A^L_{m,n}$ is the transform of $A_{m,n} \in \sigma_m$ by $L$, namely a point $x \in A^L_{m,n}$ is given by

$$y = L^{-1}(x-a) \qquad y \in A_{m,n}.$$

By the definition of $\Gamma d^j_{m,n}$ we have for $x \in A^L_{m,n}$

$$\sum_k a_k 2^{-k}(\Gamma d^j_{m,n} \circ L\varphi_k) = \sum_k 2^{-k} a_k (\sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) d^i_{m,n} \circ L\varphi_k =$$

$$= \sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) \sum_k 2^{-k} a_k d^i_{m,n} \circ L\varphi_k = \sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) Lf^i_{m,n}.$$

Now we put

$$\Gamma T_j(Lf) = \left\{ \frac{1}{m(A^L_{m,n})} \int_{A^L_{m,n}} \Gamma f^j_{m,n} dx \right\}_{m,n}$$

which by the invariance of the integrals by the transformation $x \to y = L^{-1}(x-a)$ gives

$$\Gamma T_j(Lf) = \left\{ \frac{1}{m(A^L_{m,n}j)} \int_{A^L_{m,n}} (\sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) Lf^i_{m,n} dx \right\}_{m,n} =$$

$$= \left\{ \sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) \frac{1}{m(A_{m,n})} \int_{A_{m,n}} f^i_{m,n} dy \right\}_{m,n}$$

or

$$\Gamma T_j(Lf) = \sum_{i=1}^r \Gamma^{-1}_{ji}(A,a) T_i(f)$$

and solving for $T_i(f)$,

$$T_i(f) = \sum_{j=1}^r \Gamma_{ij}(A,a) \Gamma T_j(Lf).$$

Finally if $U$ is the unitary representation of $\tilde{P}_e$ in $H$ we have for each $\psi \in H$, taking in consideration the case $r = 1$,

$$U(T_i(f)\psi) = \sum_{j=1}^r \Gamma_{ij}(A,a) \Gamma T_j(Lf) U\psi$$

which proves the axiom IV in the sense of the diagram (1) of §II,2 in the discussion of this axiom. Observe that the maps $\alpha$ and $Lf$ of that diagram are given by

$$\alpha(f) = T(f)$$

$$L\alpha(Lf) = \Gamma T(Lf).$$

**Axiom V.** This axiom needs special considerations and has to be adjusted to the spirit of the non-deterministic analysis whose basic elements are open sets instead of points.

First of all there is a philosophical question involved here related to observation and measurements of physical events. Indeed, suppose that a observation is made of a certain event at a point $P$ of coordinates $(x, y, z)$ at the instant $t$ and another observation of another event is made at $P'$ with coordinates $(x', y', z')$ at an instant $t'$. How those observers at $P$ and $P'$ could tell each other about their measurements? According to the theory of relativity this is possible through a signal travelling at a speed not greater than the speed of light, what requires for a large distance between $P$ and $P'$ that the difference $(t - t')$ should not be too small. Consequently to say that a $n$–field $\Phi$ has values $\Phi_\sigma(A)$ and $\Phi_\sigma(B)$ if $A$ and $B$, as subsets of $R^4$, are separated by a space-like interval can not have any physical meaning from the point of view of observation and measurement when we assume the possibility of communication between two observers at $P$ and $P'$ as above through signals travelling at speed higher than the speed of light.

Secondly, due to those requirements from the theory of relativity, all we could do is to assume to an observed value of $\Phi_\sigma(A)$ a non-observed value arbitrarily choosen for $\Phi_\sigma(B)$, namely, we assume that a measurement of $\Phi_\sigma(A)$ *instantly* defines a value for $\Phi_\sigma(B)$ and this clearly throws us right into the middle of the controversy related to the celebrated Einstein - Rosen - Podolsky paradox.

The only alternative out of this battleground is the philosophical attitude of assuming the existence of a physical event without necessarily compromising ourselves to effective measurements of those events, if at least there are some ways of verification a posteriori of what we assumed before. For instance, suppose we leave a gas in a container located at a point P at a certain temperature T which we know will be kept constant. Then we move to a far distant point P' from P where we have another gas in a container whose temperature we measure and is equal T' at a certain instant t. Then we make the statement: "the temperature of the gas at P is T and that at P' is T' at the *same* instant t". This is certainly a true statement eventhough we cannot measure the temperature at P. In this case the value of T at P was known a *priori* and we can also modify this example to have a verification a *posteriori*. For instance, if the temperature at P is not kept constant but all the time registered in a certain apparatus together with the time we could state, being at P': "right now at time t measured in my clock at P the temperature of the gas at P is T". Of course, later we should move to P and verify if at the time t the registered temperature was T or not. In any case we could decide if our original statement was true or false. If in all cases it comes always true we would say that we discovered a law connecting t with T at P by observations made at P' far from P.

Now at this point the axiom of locality can be regarded as a criterion to decide how to connect knowledge of an event at P with the knowledge of another event at P' such that the distance between P and P' is a space-like distance. Indeed, we can use that axiom to determine the value at P' of a measurement made at P by selecting that value in such a way that the axiom in question is verified. Of course, this attitude might clash with the point of view of many physicists and philosophers of science who do not accept the possibility of communication between two points in space through a signal travelling at speed higher than light. However, in this paper we shall adopt that attitude because in many cases it can actually be checked "a posteriori". More precisely we shall adopt the following philosophy: given a $n$–field

$$\Phi : (X, V) \to [C, V_C],$$

with $X = R^4$, if we know the value $\Phi_\sigma(A)$ for a certain $\sigma \in V$ and $A \in \sigma$ and if

$B \in \sigma$ is at a space-distance from $A$ then we shall assume a value for $\Phi_\sigma(B)$ such that the operators defined before satisfy the axiom of locality.

With all those facts discussed above in mind let us now introduce the notions needed for our study of the axiom of locality under the philosophy of non-deterministic analysis.

**Definition IV.** Let $V$ be an arbitrary family of open coverings of $R^4$. We say that two subsets $A, B$ of $R^4$ are $V$–space–like separated if:

a) $\forall\, x \in A,\ y \in B$ we have $(x-y)^2 < 0$.

b) $\forall\, \sigma \in V$ and $\forall\, E \in \sigma$ we have that if $E \cap A \neq \emptyset$ then $E \cap B = \emptyset$ and vice-versa.

Similar definition for $V$–time–like separated.

As it is well known the locality axiom is tied up with the statistics of the particles involved in the fields under consideration, in the sense that we have commutativity or anti-commutativity depending on the fact that we are dealing with fermions or bosons respectively. More precisely we can write

$$[T(f), T(g)]_{\pm} = T(f)T(g) + (-1)^{\xi} T(g)T(f) = 0$$

where $\xi = 0$ for fermions and $\xi = 1$ for bosons.

The definition IV above is consistent with the spirit of non-deterministic analysis in the sense that if the space-like distance between $A$ and $B$ is very small in absolute value our field might not be able to detect that separation.

Now taking in consideration our earlier discussion about the values of $n$–fields far apart and measured simultaneously we introduce some hypothesis about the $n$–fields to be used:

$H_I$) All $n$–fields involved have constant weights, namely, for $(\Phi, d)$, with

$$\Phi : (R^4, V) \to [C, V_C]$$

$$d = \{d_{ij}\}$$

we have, for each $A_{ij} \in \sigma_i \in V$ and $x \in C$

$$d_{ij}(x) = \text{barycenter of } \Phi_{\sigma_i}(A_{ij}),$$

where by barycenter of $\Phi_{\sigma_i}(A_{ij})$ we understand the barycenter of the rectangle in $C$ which represents the value of $\Phi_{\sigma_i}$ in the open set $A_{ij}$. We denote this barycenter by

$$\widehat{\Phi}_{\sigma_i}(A_{ij}).$$

$H_{II}$) Given any $n$–field

$$\Phi : (R^4, V) \to [C, V_C]$$

and a function

$$f : R^4 \to C$$

we call $\Phi(f)$ the $n$–field such that:

$$\Phi_V(f) = \Phi_V$$

and $\forall\, \sigma \in V$ and $A \in \sigma$:

$$\begin{cases} \Phi_\sigma(f)(A) = \Phi_\sigma(A) & \text{if } A \cap \text{supp} f \neq \emptyset \\ \Phi_\sigma(f)(A) = \{0\} & \text{otherwise} \end{cases}$$

Call $\widehat{\Phi}(f)$ the matrix whose elements are defined, under the hypothesis that $(R^4, V)$ is considered in §II,3,I), as follow

$$\widehat{\Phi}_{ij}(f) = \widehat{\Phi}_{\sigma_i}(f)(A_{ij}) \qquad \begin{matrix} i,j \geq 1 \\ i \leq j \end{matrix}$$

and

$$\widehat{\Phi}_{ji}(f) = \widehat{\Phi}_{ij}(f)^*$$

Then we assume that if $f, g \in S(R^4)$ and their supports are $V$–space–like separated we have:

$$\widehat{\Phi}(f)\widehat{\Phi}(g) \pm \widehat{\Phi}(g)\widehat{\Phi}(f) = 0$$

where + is used for fermions and - for bosons, where the $n$–field $\Phi$ is considered as before, namely

$$(\Phi, d) = \{(\Phi^1, d^1), \ldots, (\Phi^r, d^r)\}.$$

Let us show that the locality axiom is satisfied with the $n$–field $\Phi$ under the hypothesis $H_I$, $H_{II}$.

If $T(f)$ and $T(g)$ are the vector operators defined as before for $f, g \in S(R^4)$ with supports $V$–space–like separated we have:

$$T(f)T(g) \pm T(g)T(f) =$$
$$\sum_{i,j} T^i(f)T^j(g) \pm T^i(g)T^j(f).$$

In matrix notation we have:

$$T^i(f)T^j(g) \pm T^i(g)T^j(f) = \{\sum_s T^i_{ks}(f)T^j_{s\ell}(g) \pm T^i_{ks}(g)T^j_{s\ell}(f)\}_{k,\ell} \qquad (1)$$
$$(k, \ell \geq 1)$$

As introduced before we have

$$f = \sum_n a_n \varphi_n \qquad \text{and} \qquad g = \sum_m b_m \varphi_m$$

and

$$T^i_{ks}(f) = \frac{1}{m(A_{ks})} \int_{A_{ks}} f^i_{ks}(x)dx, \qquad x \in A_{ks}$$

or

$$T^i_{ks}(f) = \frac{1}{m(A_{ks})} \int_{A_{ks}} 2^{-n} a_n (d^i_{ks} \circ \varphi_n)(x)dx = 2^{-n} a_n \widehat{\Phi}_{sk}(f)(A_{ks}).$$

Therefore, expression (1) above becomes

$$T^i(f)T^j(g) \pm T^i(g)T^j(f) =$$
$$= \{\sum_{n,m} 2^{-(m-n)} a_n b_m \sum_s \widehat{\Phi}_{sk}(f)(A_{ks})\widehat{\Phi}_{sr}(g)(A_{s\ell}) \pm \widehat{\Phi}_{sk}(g)(A_{ks})\widehat{\Phi}_{sr}(f)(A_{s\ell})\}_{k,\ell}$$

By $H_{II}$ we get that all terms in the "s" summation are zero and this clearly proves that the locality axiom is satisfied.

**Remarks.**

1) Essentially what we have proved is that if the locality axiom is true for the $n$–field involved it is also true for the operators derived from it, under hypothesis $H_I$ of constant weights. The practical question is, of course, that of deciding if a given $n$–field satisfies or not the locality axiom. This will depend on what conventions we make for measurements in different points of $R^4$, $V$–space–like–separated. One trivial case is the following: fix a light cone with vertex in the initial location of the observer and assume that for any $A_{ks} \in \sigma_k$ outside this cone we have:

$$\Phi^{i}_{\sigma_k}(A_{ks}) = \{0\} \qquad i = 1, \ldots, r.$$

As we can check without difficulty in this case the Poisson bracket for the $n$–fields $\widehat{\Phi}(f)$ and $\widehat{\Phi}(g)$ is identically zero and we cannot distinguish between fermions and bosons. However, it is interesting to observe that contrary to the classical results stated in [4] we do not have the operators identically zero. That is because we assume $V$–space–like–separation instead of the usual space–like–separation.

2) There is a possibility that we can extend the validity of the locality axiom to the general case where the weights of the $n$–fields involved are arbitrarily measurable functions. Indeed, we could go from the case of constant weights to that when the weights are step-functions and from that by limiting process get the case of measurable functions. However, this project might depend on the theory of limiting $n$–fields which is still in a very sketchy and primitive stage and consequently we believe that there is still a long way ahead to obtain the desired result.

Finally for the *completness axiom* we prove the much stronger result below:

**Theorem 1.** There are vectors $\psi_k \in H$ such that for any given $\varepsilon > 0$ and any

$\theta \in H$ there is $f \in S(R^4)$ such that

$$\|T(f)\psi_b - \theta\| < \varepsilon$$

**Proof.** Let

$$\theta = \sum_k c_k e_k$$

and recall that for any $\psi \in H$ and $f \in S(R^4)$

$$\psi = \sum_k b_k e_k \quad \text{and} \quad f = \sum_k a_k \varphi_k$$

we have

$$T(f)\psi = \sum_k \Big(\sum_j T_{kj}(f) b_j\Big) e_k.$$

We have

$$T_{kj}(f) = \frac{1}{m(A_{kj})} \int_{A_{kj}} f_{kj} d_m$$

with

$$f_{kj} \begin{cases} \sum_m a_m (d_{kj} \circ \varphi_m)(x) 2^{-m}; & \text{for } x \in A_{kj} \text{ and } A_{kj} \cap \operatorname{supp} f \neq \emptyset \\ = 0 & \text{otherwise} \end{cases}$$

Then

$$T_{kj}(f) = \sum_m a_m \frac{2^{-m}}{m(A_{kj})} \int_{A_{kj}} (d_{kj} \circ \varphi_m) d_m = \sum_m a_m \Delta_{kj}^m$$

with

$$\Delta_{kj}^m = \frac{2^{-m}}{m(A_{kj})} \int_{A_{kj}} (d_{kj} \circ \varphi_m) d_m$$

which depend only on the $n$-field $\Phi$ and the selected base for $S(R^4)$.

This gives

$$\sum_j T_{kj}(f) b_j = \sum_m a_m \sum_j \Delta_{kj}^m b_j = \sum_m a_m \beta_{km}$$

where

$$\beta_{km} = \sum_j \Delta_{kj}^m b_j \tag{1}$$

which is a convergent series by using Schwartz inequality and recalling property 2)a) in §2,2 of the $n$–field $\Phi$.

Now select the $b_j$, $j = 1, 2, \ldots$ such that all determinants

$$|\beta_{km}| \quad \begin{matrix} 1 \leq k \leq n \\ 1 \leq m \leq n \end{matrix}$$

for $n = 1, 2, \ldots$ are different from zero. That is easy to do by induction. Then call

$$\psi_b = \sum_k b_k e_k$$

some particular state with the $b_k$, $k = 1, 2, \ldots$ selected as above.

Take now an arbitrary number $\eta > 0$ and select $p$ so that

$$\| \sum_{i=1}^{p} c_i e_i - \theta \| < \eta$$

and

$$\| \sum_{k>p} ( \sum_{m=1} a_m \beta_{km} ) e_k \| < \eta.$$

Now look to the linear system in $a_m$,

(2) $$\sum_{m=1}^{p} a_m \beta_{km} = c_k \qquad k = 1, \ldots, p$$

whose determinant is different from zero, as discussed above, which allows us to solve it for $a_1, \ldots, a_p$. Considering $a_m = 0$ for $m > p$ we can write

$$f = \sum_{k=1}^{p} a_k \varphi_k$$

Finally with a proper choice for $\eta$ we get

$$\| T(f) \psi_k - \theta \| < \varepsilon.$$

To end the discussion about the axioms we want to clarify the question of the vaccum state $\psi_o$.

As we observe so far the unitary representation $U(a, A)$ of the Poincaré group has been left completely unrestricted. Now we have to put some restriction on it in the following way: as seen above the vectors $\psi_b$ are selected only under the condition that the system (2) is solvable for all $p$ and this gives in general a rather large set of vectors $\psi_b$. Then we restrict $U(a, A)$ in such a way that at least one of these vectors is invariant, namely,

$$U(a, A)\psi_b = \psi_b$$

for all $(a, A)$ in the Poincaré group. As the vectors $\psi_b$ depend only on the $n$–field $\Phi$ and the base selected for $S(R^4)$ we see that the unitary representation depends only on these elements and, of course, on the base $\{e_k\}$ selected for the Hilbert space of the states of the theory.

To conclude, there is enough room left in the theory for a proper selection of the representation of the Poincaré groups and for the vacuum state. On top of that we can also satisfy in most cases the spectral condition for $U(a, A)$.

## §IV.

In this last paragraph of our paper we make some general comments and remarks to indicate the direction of possible developments of the ideas discussed in this work.

1. We start by looking into the question of $n$–fields which generate usual fields. For simplicity we restrict ourselves to the case of $r = 1$.

Supose that the $n$–field

$$\Phi : (R^4, V) \to [C, V_C]$$

generates, in the sense defined in §I, 2, Def. III, an usual continuous functions

$$\varphi : R^4 \to C.$$

For each $i \geq 1$ let us define $\|\Phi_V(\sigma_i)\|$ as the supremum of the diameters of the sets $\Phi_{\sigma_i}(A_{ij})$ for $j \geq 1$, which is finite due to the boundness condition (B) of §II, 3, II). As $\Phi$ generates an usual function we have

$$\lim_{i\to\infty} \|\Phi_V(\sigma_i)\| = 0.$$

Now let $x \in R^4$ be given and for each $i \geq 1$ call $N(i,x)$ the set of integers $j \geq 1$ such that $x \in A_{ij}$ with $A_{ij} \in \sigma_i$.

From the definition of the functions $d_{ij}$ we see that for each $y \in C$ we have, by associating to each $i$ a $j = \min N(i,x)$,

$$\lim_{i\to\infty} d_{ij}(y) = \varphi(x).$$

Consider any test function $f \in S(R^4)$

$$f = \sum_{k\geq 1} a_k \varphi_k$$

and

$$f_{ij}(x) = \sum_{k\geq 1} 2^{-k} a_k (d_{ij} \circ \varphi_k)(x) \qquad x \in A_{ij} \qquad \text{and} \quad A_{ij} \cap \operatorname{supp} f \neq \emptyset$$

and equal zero otherwise as seen before. Then we have

$$\lim_{i\to\infty} f_{ij}(x) = \sum_{k\geq 1} 2^{-k} a_k \varphi(x) = K(f)\varphi(x)$$

where

$$K(f) = \sum_{k\geq 1} 2^{-k} a_k$$

is a constant depending only on $f$. Therefore to each $f \in S(R^4)$ we can associate the function $K(f)\varphi(x)$ which is equal to $\varphi(x)$ up to a constant depending on $f$. Therefore, we can regard $\varphi(x)$ as a classical field and all test functions are just equal to $\varphi(x)$ times a constant. Of course the matrices $T(f)$ will have all terms $T_{ij}(f)$ changed accordingly and they provide a quantization of the classical field $\varphi(x)$ by operators acting on the states $\psi \in H$. More precisely:

$$T_{ij}^{f}(\varphi) = \frac{1}{m(A_{ij})} \int_{A_{ij}} K(f)\varphi(x)dx$$

namely the classical field $K(f)\varphi(x)$ is quantized by the operator

$$T^{f}(\varphi) = \{T_{ij}^{f}(\varphi)\}_{ij}$$

acting on $\psi \in H$.

2. As can be seen at [2] we can have the motion of particle defined by a non-deterministic function and also equation of motion, etc. Then suppose that we give a $n$-field

$$\Phi : (R^4, V) \to [C, V_C]$$

with weight $d$ defining a quantum field theory as considered before and assume we introduce particles in $R^4$ which will interact with $\Phi$ accordingly to the equations of motion introduced in [2]. Again to such particles we can associate a wave function $\psi$, as discussed in details in [2] and we can define for each $f \in S(R^4)$ the new state $T(f)\psi$ which in turn will provide a $n$-function representing a particle in $R^4$ in the non-deterministic sense. Then it might be interesting to study the connection of motions of non-deterministic particles in $R^4$ and actions of the operators $T(f)$ on $H$.

3. Finally we could also look to the project of extending the present theory developed in this paper to the case of $n$-fields defined in topological spaces in general. Here instead of Lorentz invariance we have to use the idea of Gauss isomorphism as introduced in [1]. Unfortunately all those comments are rather vague without the knowledge of [1] and so we find advisable to come back to them after the eventual publication of [1] in a regular journal.

## References

[1] Lintz, R.G. *Non-deterministic foundation of mechanics*, McMaster University, typewritten notes, 227 pages, 1985.

[2] Lintz, R.G. with Buonomano, V. *Differential equations in topological spaces and generalized mechanics.* Journal für die reine und angewandte Mathematik, Bd 265 (1974), pp. 31-60.

[3] Jensen, A. *Some Mapping and Homological Properties of g–functions.* Ph.D. Thesis, McMaster University (1970).

[4] A.S. Wightman, R.F. Streater. *PCT, Spin and Statistics, and all that.* Mathematics Physics Monograph Series, Benjamin Publ. (1964).

## TRABALHOS DO DEPARTAMENTO DE MATEMATICA

## TITULOS PUBLICADOS

80-01 PLETCH, A. Local freeness of profinite groups. 10 p.

80-02 PLETCH, A. Strong completeness in profinite groups. 8 p.

80-03 CARNIELLI, W.A. & ALCANTARA, L.P. de Transfinite induction on ordinal configurations. 22 p.

80-04 JONES RODRIGUES, A.R. Integral representations of cyclic p-groups. 13 p.

80-05 CORRADA, M. & ALCANTARA, L.P. de Notes on many-sorted systems. 25 p.

80-06 POLCINO MILIES, F.C. & SEHGAL, S.K. FC-elements in a group ring. 10 p.

80-07 CHEN, C.C. On the Ricci condition and minimal surfaces with constantly curved Gauss map. 10 p.

80-08 CHEN, C.C. Total curvature and topological structure of complete minimal surfaces. 21 p.

80-09 CHEN, C.C. On the image of the generalized Gauss map of a complete minimal surface in R4. 8 p.

81-10 JONES RODRIGUEZ, A.R. Units of ZCpn. 7 p.

81-11 KOTAS, J. & COSTA, N.C.A. da Problems of model and discussive logics. 35 p.

81-12 BRITO, F.B. & GONCALVES, D.L. Algebras não associativas, sistemas diferenciais polinomiais homogêneos e classes características. 7 p.

81-13 POLCINO MILIES, F.C. Group rings whose torsion units from a subgroup II. Iv. (não paginado).

81-14 CHEN, C.C. An elementary proof of Calabi's theorems on holomorphic curves. 5 p.

81-15 COSTA, N.C.A. da & ALVES, E.H. Relations between paraconsistent logic and many-valued logic. 8 p.

81-16 CASTILLA, M.S.A.C. On Przymusinski's theorem. 6 p.

81-17 CHEN, C.C. & GOES, C.C. Degenerate minimal surfaces in R4. 21 p.

81-18 CASTILLA, M.S.A.C. Imagens inversas de algumas aplicações fechadas. 11 p.

81-19 ARAGONA VALLEJO, A.J. & EXEL FILHO, R. An infinite dimensional version of Hartogs' extension theorem. 9 p.

81-20 GONÇALVES, J.Z. Groups rings with solvable unit groups. 15p.

81-21 CARNIELLI, W.A. & ALCANTARA, L.P. de Paraconsistent algebras. 16 p.

81-22 GONÇALVES, D.L. Nilpotent actions. 10 p.

81-23 COELHO, S.P. . Group rings with units of bounded exponent over the center. 25 p.

81-24 PARMENTER, M.M. & POLCINO MILIES, F.C. A note on isomorphic group rings. 4 p.

81-25 MERKLEN GOLDSCHMIDT, H.A. Hereditary algebras with maximum spectra are of finite type. 10 p.

81-26 POLCINO MILIES, F.C. Units of group rings: a short survey. 32 p.

81-27 CHEN, C.C. & GACKSTATTER, F. Elliptic and hyper-elliptic functions and complete minimal surfaces with handles. 14 p.

81-28 POLCINO MILIES, F.C. A glance at the early history of group rings. 22 p.

81-29 FERRER SANTOS, W.R. Reductive actions of algebraic groups on affine varieties. 52 p.

81-30 COSTA, N.C.A. da The philosophical import of paraconsistent logic. 26 p.

81-31 GONÇALVES, D.L. Generalized classes of groups, spaces c-nilpotent and "the Hurewicz theorem". 30 p.

81-32 COSTA, N.C.A. da & MORTENSEN, Chris. Notes on the theory of variable binding term operators. 18 p.

81-33 MERKLEN GOLDSCHMIDT, H.A. Homogenes l-hereditary algebras with maximum spectra. 32 p.

81-34 PERESI, L.A. A note on semiprime generalized alternative algebras. 10 p.

81-35 MIRAGLIA NETO, F. On the preservation of elementary equivalence and embedding by filtred powers and structures of stable continuous functions. 9 p.

81-36 FIGUEIREDO, G.V.R. Catastrophe theory: some global theory a full proof. 91 p.

82-37 COSTA, R.C.F. On the derivations of gametic algebras. 17 p.

82-38 FIGUEIREDO, G.V.R. A shorter proof of the Thom-Zeeman global theorem for catastrophes of cod <= 5. 7 p.

82-39 VELOSO, J.M.M. Lie equations and Lie algebras: the intrasitive case. 97 p.

82-40 GOES, C.C. Some results about minimal immersions having flat normal bundle. 37 p.

82-41 FERRER SANTOS, W.R. Cohomology of comodules II. 15 p.

82-42 SOUZA, V.H.G. Classification of closed sets and diffeos of one-dimensional manifolds. 15 p.

82-43 GOES, C.C. The stability of minimal cones of codimension greater than one in Rn. 27 p.

82-44 PERESI, L.A. On automorphisms of gametic algebras. 27 p.

82-45 POLCINO MILIES, F.C. & SEHGAL, S.K. Torsion units in integral group rings of metacyclic groups. 18 p.

82-46 GONÇALVES, J.Z. Free subgroups of units in group rings. 8 p.

82-47 VELOSO, J.M.M. New classes of intransitive simple Lie pseudogroups. 8 p.

82-48 CHEN, C.C. The generalized curvature ellipses and minimal surfaces. 10 p.

82-49 COSTA, R.C.F. On the derivation algebra of zygotic algebras for polyploidy with multiple alleles. 24 p.

83-50 GONÇALVES, J.Z. Free subgroups in the group of units of group rings over algebraic integers. 3 p.

83-51 MANDEL, A. & GONÇALVES, J.Z. Free k-triples in linear groups. 7 p.
83-52 BRITO, F.G.B. A remark on closed minimal hypersurfaces of S4 with second fundamental form of constant length. 12 p.
83-53 KIIHL, J.C.S. U-structures and sphere bundles. 8 p.
83-54 COSTA, R.C.F. On genetic algebras with prescribed derivations. 23 p.
83-55 SALVITTI, R. Integrabilidade das distribuições dadas por subálgebras de Lie de codimensão finita no gh(n,C). 4 p.
83-56 MANDEL, A. & GONÇALVES, J.Z. Construction of open sets of free k-Tuples of matrices. 18 p.
83-57 BRITO, F.G.B. A remark on minimal foliations of codimension two. 24 p.
83-58 GONÇALVES, J.Z. Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings. 9 p.
83-59 BELOQUI, J.A. Modulus of stability for vector fields on 3-manifolds. 40 p.
83-60 GONÇALVES, J.Z. Some groups not subnormal in the group of units of its integral group ring. 8 p.
84-61 GOES, C.C. & SIMOES, P.A.Q. Imersões mínimas nos espaços hiperbólicos. 15 p.
84-62 GIANBRUNO, A.; MISSO, P. & POLCINO MILIES, F.C. Derivations with invertible values in rings with involution. 12 p.
84-63 FERRER SANTOS, W.R. A note on affine quotients. 6 p.
84-64 GONÇALVES, J.Z. Free-subgroups and the residual nilpotence of the group of units of modular and p-adic group rings. 12 p.
84-65 GONÇALVES, D.L. Fixed points of S1-fibrations. 18 p.
84-66 RODRIGUES, A.A.M. Contact and equivalence of submanifolds of homogenous spaces. 15 p.
84-67 LOURENÇO, M.L. A projective limit representation of (DFC)-spaces with the approximation property. 20 p.
84-68 FORNARI, S. Total absolute curvature of surfaces with boundary. 25 p.
84-69 BRITO, F.G.B. & WALCZAK, P.G. Totally geodesic foliations with integral normal bundles. 6 p.
84-70 LANGEVIN, R. & POSSANI, C. Quase-folheações e integrais de curvatura no plano. 26 p.
84-71 OLIVEIRA, M.E.G.G. de Non-orientable minimal surfaces in RN. 41 p.
84-72 PERESI, L.A. On baric algebras with prescribed automorphisms. 42 p.
84-73 MIRAGLIA NETO, F. & ROCHA FILHO, G.C. The measurability of Riemann integrable-function with values in Banach spaces and applications. 27 p.
84-74 MERKLEN GOLDSCHMIDT, H.A. Artin algebras wich are equivalent to a hereditary algebra modulo preprojectives. 38 p.
84-75 GOES, C.C. & SIMOES, P.A.Q. The generalized Gauss map of minimal surfaces in H3 and H4. 16 p.

84-76 GONÇALVES, J.Z. Normal and subnormal subgroups in the group of units of a group rings. 13 p.

85-77 ARAGONA_VALLEJO, A.J. On existence theorems for the $\partial$-operator on generalized differential forms. 13 p.

85-78 POLCINO MILIES, C.; RITTER, J. & SEHGAL, S.K. On a conjecture of Zassenhaus on torsion units in integral group rings II. 14 p.

85-79 JONES RODRIGUEZ, A.R. & MICHLER, G.O. On the structure of the integral Green ring of a cyclic group of order p2. The Jacobson radical of the integral Green ring of a cyclic group of order p2. 26 p.

85-80 VELOSO, J.M.M. & VERDERESI, J.A. Three dimensional Cauchy-Riemann manifolds. 19 p.

85-81 PERESI, L.A. On baric algebras with prescribed automorphisms II. 18 p.

85-82 KNUDSEN, C.A. O impasse aritmo-geométrico e a evolução do conceito de número na Grécia antiga. 43p.

85-83 VELOSO, J.M.M. & VERDERESI, J.A. La géometrie, le probleme d'équivalence et le classification des CR-varietés homogenes en dimension 3. 30 p.

85-84 GONÇALVES, J.Z. Integral group rings whose group is solvable, an elementary proof. 11 p.

85-85 LUCIANO, O.O. Nebuleuses infinitesimelement fibrées. 5 p.

85-86 ASPERTI, A.C. & DAJCZER, M. Conformally flat Riemannian manifolds as hypersurfaces of the ligth cone. 8 p.

85-87 BELOQUI, J.A. A quasi-transversal Hopf bifurcations. 11 p.

85-88 POLCINO MILIES, F.C. & RAPHAEL, D.M. A note on derivations with power central values in prime rings. 7 p.

85-89 POLCINO MILIES, F.C. Torsion units in group rings and a conjecture of H.J.Zassenhaus. 14 p.

86-90 LOURENÇO, M.L. Riemann domains over (DFC) spaces. 32 p.

86-91 ARAGONA VALLEJO, A.J. & FERNANDES,J.C.D. The Hartogs extension theorem for holomorphic generalized functions. 9 p.

86-92 CARRARA ZANETIC, V.L. Extensions of immersions in dimension two. 27 p.

86-93 PERESI, L.A. The derivation algebra of gametic and zygotic algebras for linked loci. 25 p.

86-94 COELHO, S.P. A note on central idempotents in group ring. 5 p.

86-95 PERESI, L.A. On derivations of baric algebras with prescribed automorphisms. 21 p.

86-96 COELHO, F.U. A generalization of a theorem of Todorov on preprojectives partitions. 18 p.

86-97 ASPERTI, A.C. A note on the minimal immersions of the two-sphere. 11 p.

86-98 COELHO, S.P. & POLCINO MILIES, F.C. A note on central idempotents in group rings II. 8 p.

86-99 EXEL FILHO, R. Hankel matrices over right ordered amenable groups. 18 p.

## NOVA SERIE

86-01 GOODAIRE, E.G. & POLCINO MILIES, F.C. Isomorphisms of integral alternative loop rings. 11 p.

86-02 WALCZAK, P.G. Foliations which admit the most mean curvature functions. 11 p.

86-03 OLIVEIRA, M.E.G.G. Minimal Klein bottles with one end in R3 and R4. 12 p.

86-04 MICALI, A. & VILLAMAYOR, O.E. Homologie de Hochschild de certaines algebres de groupes. 11 p.

86-05 OLIVEIRA, M.E.G.G. Minimal Klein bottles in R3 with finite total curvature. 9 p.

86-06 CARRARA ZANETIC, V.L. Classification of stable maps between 2-manifolds with given singular set image. 22 p.

87-01 BRITO, F.G.B. & WALCZAK, P.G. Total curvature of orthogonal vector fields on three-manifolds. 4 p.

87-02 BRITO, F.G.B. & LEITE, M.L. A remark on rotational hypersurfaces of Sn. 13 p.

87-03 GONÇALVES, J., RITTER, J. & SEHGAL, S. Subnormal sub groups in U(ZG). 13 p.

87-04 ARAGONA VALLEJO, A.J. & COLOMBEAU, J.F. The interpolation theorem for holomorphic generalized functions. 12 p.

87-05 ALMEIDA, S.C. de & BRITO, F.G.B. Immersed hypersurfaces of a space form with distinct principal curvatures. 9 p.

87-06 ASPERTI, A.C. Generic minimal surfaces. 21 p.

87-07 GOODAIRE, E.G. & POLCINO MILIES, F.C. Torsion units in alternative group rings. 17 p.

87-08 MERKLEN GOLDSCHMIDT, H.A. Algebras which are equivalent to a hereditary algebra modulo preprojectives II. 27 p.

87-09 REYNOL FILHO, A. P-localization of some classes of groups. 30 p.

88-01 CARLSON, J.F. & JONES, A. An exponential property of lattices over group rings. 22 p.

88-02 CARLSON, J.F. & JONES, A. Wild categories of periodic modules. 6 p.

88-03 SALLUM, E.M. The nonwandering set of flows on a Reeb foliation. 13 p.

88-04 ALMEIDA, R. Cohomologie des suites d'Atiyah. 14 p.

88-05 GUZZO JR., H. Alguns teoremas de caracterização para álgebras alternativas à direita. 16 p.

88-06 HARLE, C.E. Subvariedades isoparamétricas homogêneas. 5 p.

88-07 LANGEVIN, R. Vers une classification des diffeomorphismes Morse-Smale d'une surface. 18 p.

88-08 ARAGONA VALLEJO, A. J. & BIAGIONI, H. A. An intrinsic definition of the Colombeau algebra of generalized functions. 48 p.

88-09 BORSARI, H.D. A Cohomological characterization of redutive algebraic groups. 20 p.

88-10 COELHO, S.P. & POLCINO MILIES, F.C. Finite conjugacy in group rings. 20 p.

88-11 FERNANDEZ, R. Characterization of the dual of an Orlicz Space. 19 p.

89-01 GONÇALVES, J.Z. & MANDEL, A. Embedding the Free Group Ring into Formal Series. 4 p.

89-02 REYNOL Fº, A. Nilpotent spaces: some inequalities on nilpotency degrees. 17 p.

89-03 GOUVÊA, F.Q. Deforming Galois Representations: Controlling the Conductor. 22 p.

89-04 CORDARO, P. & TREVES, F. Homology & Cohomology in Hypo-Analytic Structures of the Hypersurfaces Type. 108p.

89-05 ARAGONA VALLEJO, A. J. & VILLARREAL ALVARADO, F. Colombeau's theory and shock Waves in a problem of Hydrodynamics. 48 p.

90-01 MANDEL, A. & GONÇALVES, J.Z. Semigroup laws and free semigroups in the group of units of group rings. 11 p.

90-02 LINTZ, R.G. Generation of Quantum-fields by non-deterministic fields. 32p.