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**Almost Automorphic Solutions to Some Partial
Neutral Functional Differential Equations**

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Soluções Quase Automorfas para Equações Diferenciais Funcionais do Tipo Neutro.

Eduardo Hernández & Toka Diagana

Resumo

Neste trabalho estudamos a existência e unicidade de soluções quase automorfas para o problema de Cauchy abstrato

$$\begin{aligned}\frac{d}{dt} [u(t) + f(t, u_t)] &= Au(t) + g(t, u_t), & t \in [\sigma, \sigma + a], \\ u_\sigma &= \phi \in \mathcal{B}.\end{aligned}$$

No sistema anterior, A é gerador infinitesimal de um semigrupo uniformemente estável de operadores lineares limitados definido sobre um espaço de Banach X ; $\mathcal{B}$ é um espaço de fase definido de maneira axiomática e f, g são funções contínuas.

Almost Automorphic Solutions to Some Partial Neutral Functional Differential Equations

Eduardo Hernández and Toka Diagana

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Abstract

This paper is concerned with the existence and uniqueness of almost automorphic mild solutions to some classes of first-order partial neutral functional-differential equations. Sufficient conditions for the existence and uniqueness of almost automorphic mild solutions to the above-mentioned equations are obtained. As an application, a first-order boundary value problem arising in control of systems is considered.

1 Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. This paper deals with the existence and uniqueness of *almost automorphic* mild solutions to the first-order abstract partial neutral functional-differential equations of the form

$$\frac{d}{dt} [u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \quad t \in [\sigma, \sigma + a), \quad (1)$$

$$u_\sigma = \phi \in \mathcal{B}, \quad (2)$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is the infinitesimal generator of a uniformly exponentially stable semigroup of linear operators on $\mathbb{X}$, the history u_t defined by $u_t(\theta) := u(t + \theta)$ for each $\theta \in (-\infty, 0]$, belongs to an abstract phase space $\mathcal{B}$ that is defined axiomatically, and f, g are some continuous functions.

The existence of almost automorphic, asymptotically almost automorphic, almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is among the most attractive topics in the qualitative theory of differential equations due to their significance and applications in physical sciences.

The concept of almost automorphy which is the central issue in this paper was first initiated by Bochner in his landmark paper [2]. Since then the theory of almost automorphic functions has found several developments and applications in the theory of abstract differential equations, partial differential equations, functional-differential equations, integro-differential equations, and others. For more on these and related issues,

see, e.g., [3], [17], [18], [24], [25], and [26] and the references therein. To deal with delay and related issues, it is more convenient to consider the so-called compact almost automorphy [3, 13] rather than the classical almost automorphy [17].

Existence results related to almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations have recently been established in [16, 19], and [11], respectively. A few investigations on the existence of almost automorphic solutions to functional-differential equations with delay exist in the literature, see for instance [13] for abstract partial differential equations, and [4] for ordinary neutral differential equations. The existence of this type of solutions for abstract partial differential equations remains an untreated topic in the literature and this is the main motivation of this paper. Note that neutral differential equations arise in many areas and sub-areas of applied mathematics. For these reasons, they have largely been studied during the last few decades. The literature related to ordinary neutral differential equations is very extensive, thus, we refer to [6] only, which contains a comprehensive description of such equations. Similarly, for more on partial neutral functional differential equations and related issues we refer to Hale [7], Wu [21, 22, 23], and Adimy[1] for finite delay equations, and Hernández and Henríquez [8, 9] and Hernández [10] for unbounded delays.

In what follows we recall some definitions and notations that we need in the sequel. From now on, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ denotes the infinitesimal generator of an uniformly asymptotically stable semigroup of linear operators $(T(t))_{t \geq 0}$ on $\mathbb{X}$ and M, w are positive constants such that $\|T(t)\| \leq Me^{-wt}$ for each $t \geq 0$.

To deal with almost automorphic solutions we will need to introduce some classical notions. Throughout the rest of the paper, $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$, $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ stand for abstract Banach spaces. In addition to that $C(\mathbb{R}, \mathbb{Z})$ and $BC(\mathbb{R}, \mathbb{Z})$ denote respectively the collection of continuous functions and the collection of bounded continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ equipped with the sup norm defined by $\|u\|_{\infty} := \sup_{t \in \mathbb{R}} \|u(t)\|_{\mathbb{Z}}$. Similar definitions apply for both $C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$.

Definition 1 A (strongly) continuous function F defined from $\mathbb{R}$ into $\mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ of $(s'_n)_{n \in \mathbb{N}}$ such that $G(t) := \lim_{n \rightarrow \infty} F(t + s_n)$ is well defined for each $t \in \mathbb{R}$, and $F(t) = \lim_{n \rightarrow \infty} G(t - s_n)$ for each $t \in \mathbb{R}$.

The range of an almost automorphic function is relatively compact on $\mathbb{X}$, therefore it is bounded. We denote the space of almost automorphic functions $F : \mathbb{R} \mapsto \mathbb{X}$ by $AA(\mathbb{X})$. It is well-known that $(AA(\mathbb{X}), \|\cdot\|_{\infty})$ is a Banach space, see, e.g., [17].

Definition 2 A continuous function $F : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $z \in \mathbb{Z}$ if for every sequence of real numbers $(\sigma_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ of $(\sigma_n)_{n \in \mathbb{N}}$ such that

$$G(t, z) := \lim_{n \rightarrow \infty} F(t + s_n, z) \quad \text{in } \mathbb{W}$$

is well defined for each $t \in \mathbb{R}$ and each $z \in \mathbb{Z}$ and

$$F(t, u) = \lim_{n \rightarrow \infty} G(t - s_n, z) \quad \text{in } \mathbb{W}$$

for each $t \in \mathbb{R}$ and for every $z \in \mathbb{Z}$. The collection of such functions will be denoted by $AA(\mathbb{Z}, \mathbb{W})$.

Using [17, Theorem 2.2.6, p. 22] one easily obtains the following.

Theorem 1 *Let $F : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$ be an almost automorphic function in $t \in \mathbb{R}$ for each $z \in \mathbb{Z}$ and assume that F satisfies a Lipschitz condition in z uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \mapsto \mathbb{Z}$ be almost automorphic. Then the function $\Phi : \mathbb{R} \mapsto \mathbb{W}$ defined by $\Phi(t) = F(t, \phi(t))$ is almost automorphic.*

In addition to the above-mentioned notions, the present setting requires the introduction of the concept of compact almost automorphy, see, e.g., [13].

Definition 3 *A continuous function $F : \mathbb{R} \mapsto \mathbb{Z}$ is said to be compact almost automorphic if for every sequence of real numbers $(\sigma_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}} \subset (\sigma_n)_{n \in \mathbb{N}}$ such that $G(t) := \lim_{n \rightarrow \infty} F(t + s_n)$, and $F(t) = \lim_{n \rightarrow \infty} G(t - s_n)$ uniformly on compact subsets of $\mathbb{R}$. The collection of those functions will be denoted by $AA_c(\mathbb{Z})$.*

Let $AP(\mathbb{Z})$ denote the space of almost periodic functions $f : \mathbb{R} \mapsto \mathbb{Z}$. It is well-known that $AP(\mathbb{Z})$, $AA_c(\mathbb{Z})$, and $AA(\mathbb{Z})$ are closed subsets of $(BC(\mathbb{R}, \mathbb{Z}), \|\cdot\|_\infty)$ with

$$AP(\mathbb{Z}) \subset AA_c(\mathbb{Z}) \subset AA(\mathbb{Z}) \subset BC(\mathbb{R}, \mathbb{Z}).$$

In view of the above, the proof of the next lemma is straightforward.

Lemma 1 *The space $AA_c(\mathbb{Z})$ endowed with the sup norm is a Banach space.*

Definition 4 *A continuous function $F : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$, $(t, u) \mapsto F(t, u)$ is said to be compact almost automorphic in $t \in \mathbb{R}$, if for every sequence of real numbers $(\sigma_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}} \subset (\sigma_n)_{n \in \mathbb{N}}$ such that*

$$G(t, z) := \lim_{n \rightarrow \infty} F(t + s_n, z), \quad \text{and} \quad F(t, z) = \lim_{n \rightarrow \infty} G(t - s_n, z) \quad \text{in } \mathbb{W},$$

where the limits are uniform on compact subset of $\mathbb{R}$, for each $z \in \mathbb{Z}$. The space of such functions will be denoted by $AA_c(\mathbb{Z}, \mathbb{W})$.

In this work we will define the phase space $\mathcal{B}$ axiomatically, using ideas and notations developed in [15]. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty, 0]$ into $\mathbb{X}$ endowed with a seminorm denoted $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold:

- A. If $x : (-\infty, \sigma + b) \rightarrow \mathbb{X}$ with $b > 0$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for each $t \in [\sigma, \sigma + b)$ the following conditions hold:
 - (i) x_t is in $\mathcal{B}$,
 - (ii) $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}}$,
 - (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant, and $K, M : [0, \infty) \mapsto [1, \infty)$ are functions such that $K(\cdot)$ and $M(\cdot)$ are respectively continuous and locally bounded, and H, K, M are independent of $x(\cdot)$.

A1. If $x(\cdot)$ is a function as in **(A)**, then x_t is a $\mathcal{B}$ -valued continuous function on $[\sigma, \sigma + b)$.

B. The space $\mathcal{B}$ is complete.

C2. If $(\varphi^n)_{n \in \mathbb{N}}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into $\mathbb{X}$, which converges to φ uniformly on compact subsets of $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$.

Remark 1 Throughout the rest of the paper, $\mathfrak{L} > 0$ denotes a constant such that $\|\varphi\|_{\mathcal{B}} \leq \mathfrak{L} \cdot \sup_{\theta \leq 0} \|\varphi(\theta)\|$ for every $\varphi \in BC((-\infty, 0]; \mathbb{X})$, see [15, Proposition 7.1.1].

Definition 5 Let $S(t) : \mathcal{B} \mapsto \mathcal{B}$ be the C_0 -semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\|S(t)\varphi\|_{\mathcal{B}} \rightarrow 0$ as $t \rightarrow \infty$ for each $\varphi \in \mathcal{B}$ with $\varphi(0) = 0$.

Remark 2 In this work we assume the existence of $\mathcal{K} > 0$ such that $\max\{K(t), M(t)\} \leq \mathcal{K}$ for each $t \geq 0$. Observe that this condition is verified, for example, if $\mathcal{B}$ is a fading memory, see, e.g., [15, Proposition 7.1.5] for details.

This paper is organized as follows: In Section 2 we study the existence of almost automorphic solutions to the neutral system (1), and Section 3 is devoted to applications.

2 Existence Results

In this section we establish the existence of almost automorphic solutions to the neutral system (1). For that, we first prove some preliminary technical Lemmas, which are needed for the proof of our main result. From now on, $\mathbb{Y}$ denotes an arbitrary Banach space continuously embedded into $\mathbb{X}$. In this event, $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ and $\mathcal{L}(\mathbb{Y})$ stand respectively for the class of bounded linear operators which go from $\mathbb{Y}$ into $\mathbb{X}$ and the class of bounded linear operators from $\mathbb{Y}$ into itself.

Our setting requires the following assumption:

H₁ The function $s \mapsto T(s)y$ belongs to $C([0, \infty), \mathbb{Y})$ for each $y \in \mathbb{Y}$. Moreover, the function $s \mapsto AT(s)$ defined from $(0, \infty)$ into $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ is strongly measurable, and there exist a non-decreasing function $H : [0, \infty) \mapsto [0, \infty)$ and $\delta > 0$ such that $e^{-\delta s}H(s) \in L^1([0, \infty))$ with $\|AT(s)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \leq e^{-\delta s}H(s)$ for each $s > 0$.

Lemma 2 Let $F \in AA_c(\mathbb{Z}, \mathbb{W})$ and let $u \in AA_c(\mathbb{Z})$. Assume that there exists a bounded function $L_F : \mathbb{R} \mapsto [0, \infty)$ such that

$$\|F(t, x) - F(t, y)\|_{\mathbb{W}} \leq L_F(t) \|x - y\|_{\mathbb{Z}}, \quad \forall t \in \mathbb{R}, \forall x, y \in \mathbb{Z}. \quad (3)$$

Then the $\mathbb{W}$ -valued function G defined by $G(t) := F(t, u(t))$ is in $AA_c(\mathbb{W})$.

Proof: Let $(s'_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. Now, fix a subsequence $(s_n)_{n \in \mathbb{N}}$ of $(s'_n)_{n \in \mathbb{N}}$, $G \in BC(\mathbb{R} \times \mathbb{Z}; \mathbb{W})$ and $v \in BC(\mathbb{R}, \mathbb{Z})$ so that the pair $G, (s_n)_{n \in \mathbb{N}}$ is associated with F as in Definition 4 and the pair $v, (s_n)_{n \in \mathbb{N}}$ is associated with u as in Definition 3. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Since $\mathcal{R}(v) = \{v(t) : t \in \mathbb{R}\}$ is relatively compact, there exist points $x_i \in \mathbb{Z}, i = 1, \dots, n$, such that for each $t \in \mathbb{R}$ one can find $i(t) \in \{1, \dots, n\}$ with

$$\|v(t) - x_{i(t)}\|_{\mathbb{Z}} \leq \varepsilon.$$

Let N_ε be a natural number such that $\|F(s + s_n, x_i) - G(s, x_i)\|_{\mathbb{W}} \leq \varepsilon, \forall s \in K$ and for all $i = 1, \dots, n$, whenever $n \geq N_\varepsilon$. In view of the above, for each $s \in K$, and $n \geq N_\varepsilon$,

$$\begin{aligned} & \|F(t + s_n, u(t + s_n)) - G(t, v(t))\|_{\mathbb{W}} \\ & \leq \|F(t + s_n, u(t + s_n)) - F(t + s_n, v(t))\|_{\mathbb{W}} + \|F(t + s_n, v(t)) - F(t + s_n, x_{i(t)})\|_{\mathbb{W}} \\ & \quad + \|F(t + s_n, x_{i(t)}) - G(t, x_{i(t)})\|_{\mathbb{W}} + \|G(t, x_{i(t)}) - G(t, v(t))\|_{\mathbb{W}} \\ & \leq L_F(t + s_n)\|u(t + s_n) - v(t)\|_{\mathbb{Z}} + L_F(t + s_n)\|v(t) - x_{i(t)}\|_{\mathbb{Z}} \\ & \quad + \varepsilon + L_F(t)\|x_{i(t)} - v(t)\|_{\mathbb{Z}} \end{aligned}$$

which proves that the convergence is uniform on K .

Arguing as previously it follows that $G(t - s_n, u(t - s_n)) - F(t, u(t))$ converges uniformly to 0 on compact sets of $\mathbb{R}$. This completes the proof. ■

Lemma 3 Let $u \in AA_c(\mathbb{Y})$. Under assumption $\mathbf{H}_1$, if w is the function defined by

$$w(t) := \int_{-\infty}^t AT(t - s)u(s)ds, \quad \forall t \in \mathbb{R},$$

then $w \in AA_c(\mathbb{X})$.

Proof: For a given sequence $(\sigma_n)_{n \in \mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n \in \mathbb{N}}$, and a continuous functions $v \in BC(\mathbb{R}; \mathbb{Y})$ such that $u(t + s_n)$ converges to $v(t)$ in $\mathbb{Y}$, and $v(t - s_n)$ converges to $u(t)$ in $\mathbb{Y}$, uniformly on compact sets of $\mathbb{R}$. From the Bochner's criterion related to integrable functions and the estimate

$$\begin{aligned} \|AT(t - s)u(s)\|_{\mathcal{L}(\mathbb{Y}; \mathbb{X})} &= \|AT(t - s)\|_{\mathcal{L}(\mathbb{Y}; \mathbb{X})}\|u(s)\|_{\mathbb{Y}} \\ &\leq e^{-\delta(t-s)}H(t - s)\|u(s)\|_{\mathbb{Y}} \end{aligned} \quad (4)$$

it follows that the function $s \mapsto AT(t - s)u(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Furthermore, since

$$w(t + s_n) = \int_{-\infty}^t AT(t - s)u(s + s_n)ds, \quad t \in \mathbb{R}, \quad n \in \mathbb{N},$$

using the estimate (4) and the Lebesgue Dominated Convergence Theorem, it follows that $w(t + s_n)$ converges to $z(t) = \int_{-\infty}^t AT(t - s)v(s)ds$ for each $t \in \mathbb{R}$.

The remaining task is to prove that the convergence is uniform on all compact sets in $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Fix $L > 0$ and $N_\varepsilon \in \mathbb{N}$ such that $K \subset [-\frac{L}{2}, \frac{L}{2}]$ with

$$\begin{aligned} \int_{\frac{L}{2}}^{\infty} e^{-\delta s}H(s)ds &< \varepsilon, \\ \|u(s + s_n) - v(s)\|_{\mathbb{Y}} &\leq \varepsilon, \quad n \geq N_\varepsilon, \quad s \in [-L, L]. \end{aligned}$$

Using the notation $\|u\|_{\mathbb{Y},\infty} = \sup_{s \in \mathbb{R}} \|u(s)\|_{\mathbb{Y}}$, for each $t \in K$, one has:

$$\begin{aligned}
\|w(t + s_n) - z(t)\| &\leq \int_{-\infty}^t \|AT(t-s)\|_{\mathcal{L}(\mathbb{Y};\mathbb{X})} \|u(s + s_n) - v(s)\|_{\mathbb{Y}} ds \\
&\leq \int_{-\infty}^{-L} e^{-\delta(t-s)} H(t-s) \|u(s + s_n) - v(s)\|_{\mathbb{Y}} ds \\
&\quad + \int_{-L}^t e^{-\delta(t-s)} H(t-s) \|u(s + s_n) - v(s)\|_{\mathbb{Y}} ds \\
&\leq 2\|u\|_{\mathbb{Y},\infty} \int_{t+L}^{\infty} e^{-\delta s} H(s) ds + \varepsilon \int_0^{\infty} e^{-\delta s} H(s) ds \\
&\leq 2\|u\|_{\mathbb{Y},\infty} \int_{\frac{L}{2}}^{\infty} e^{-\delta s} H(s) ds + \varepsilon \int_0^{\infty} e^{-\delta s} H(s) ds \\
&\leq \varepsilon \left(2\|u\|_{\mathbb{Y},\infty} + \int_0^{\infty} e^{-\delta s} H(s) ds \right),
\end{aligned}$$

which proves that the convergence is uniform on K , by the fact that the last estimate is independent of $t \in K$. Proceeding as previously, one can similarly prove that $z(t - s_n)$ converges to w uniformly on compact sets in $\mathbb{R}$. This completes the proof. ■

Using the similar steps as in the proof of Lemma 3, one obtains the following:

Lemma 4 *Let $u \in AA_c(\mathbb{X})$. Under assumption $\mathbf{H}_1$, if w is the function defined by*

$$w(t) := \int_{-\infty}^t T(t-s)u(s)ds, \quad \forall t \in \mathbb{R},$$

then $w \in AA_c(\mathbb{X})$.

To study the system (1) we need the following result:

Lemma 5 *If $u \in AA_c(\mathbb{X})$, then the function $s \rightarrow u_s$ belongs to $AA_c(\mathcal{B})$.*

Proof: For a given sequence $(s'_n)_{n \in \mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n \in \mathbb{N}}$ of $(s'_n)_{n \in \mathbb{N}}$ and a function $v \in BC(\mathbb{R}, \mathbb{X})$ such that $u(s + s_n) \rightarrow v(s)$ uniformly on compact subsets of $\mathbb{R}$. Since $\mathcal{B}$ satisfies axiom $\mathbf{C}_2$, from [15, Proposition 7.1.1], we infer that $u_{s+s_n} \rightarrow v_s$ in $\mathcal{B}$ for each $s \in \mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $L > 0$ such that $K \subset [-L, L]$. For $\varepsilon > 0$, fix $N_{\varepsilon,L} \in \mathbb{N}$ such that

$$\begin{aligned}
\|u(s + s_n) - v(s)\| &\leq \varepsilon, & s \in [-L, L], \\
\|u_{-L+s_n} - v_{-L}\| &\leq \varepsilon,
\end{aligned}$$

whenever $n \geq N_{\varepsilon,L}$.

In view of the above, for $t \in K$ and $n \geq N_{\varepsilon,L}$ we get

$$\begin{aligned}
\|u_{t+s_n} - v_t\|_{\mathcal{B}} &\leq M(L+t)\|u_{-L+s_n} - v_{-L}\|_{\mathcal{B}} + K(L+t) \sup_{\theta \in [-L,L]} \|u(\theta + s_n) - v(\theta)\| \\
&\leq 2K\varepsilon,
\end{aligned}$$

where $\mathcal{K}$ is the constant appearing in Remark 2.

In view of the above, u_{t+s_n} converges to v_t uniformly on K . Similarly, one can prove that v_{t-s_n} converges to u_t uniformly on K . Thus, the function $s \mapsto u_s$ belongs to $AA_c(\mathcal{B})$. ■

The rest of this section is devoted to the existence of almost automorphic solutions to the neutral system (1). For that, we require the following assumption:

H₂ The functions $f : \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{Y}, g : \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{X}$ are continuous, and there are continuous and bounded functions $L_f, L_g : \mathbb{R} \rightarrow (0, \infty)$ such that

$$\begin{aligned} \|f(t, x) - f(t, y)\|_{\mathbb{Y}} &\leq L_f(t) \cdot \|x - y\|, \\ \|g(t, x) - g(t, y)\| &\leq L_g(t) \cdot \|x - y\|, \end{aligned}$$

for all $t \in \mathbb{R}, x, y \in \mathbb{X}$.

We adopt the notion of mild solutions for (1)-(2) from the one given in [8].

Definition 6 A continuous function $u : [\sigma, \sigma + a) \rightarrow \mathbb{X}$, $a > 0$, is called a mild solution for the neutral system (1)-(2) on $[\sigma, \sigma + a)$, if $u_s \in \mathcal{B}$ for every $s \in \mathbb{R}$, the function $s \rightarrow AT(t-s)f(s, u_s)$ is integrable on $[0, t]$ for every $\sigma < t < \sigma + a$, and

$$\begin{aligned} u(t) &= T(t-\sigma)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) - \int_{\sigma}^t AT(t-s)f(s, u_s)ds \\ &\quad + \int_{\sigma}^t T(t-s)g(s, u_s)ds, \quad t \in [\sigma, \sigma + a). \end{aligned}$$

Remark 3 Let $x \in BC(\mathbb{R}, \mathbb{X})$. Let us mention that the assumption on f is heavily linked to the integrability of the function $s \rightarrow AT(t-s)f(s, x_s)$. In general, except trivial cases, the operator function $t \rightarrow AT(t)$ is not integrable over $[0, a]$. If $\mathcal{L}, \mathcal{K}$ are the constants appearing in Remarks 1, 2, and f satisfies assumption **H₂**, then from the Bochner's criterion for integrable functions and the estimate

$$\begin{aligned} \|AT(t-s)f(s, x_s)\| &= \|AT(t-s)\|_{\mathcal{L}(\mathbb{Y}; \mathbb{X})} \cdot \|f(s, x_s)\|_{\mathbb{Y}} \\ &\leq e^{-\delta(t-s)} H(t-s) L_f(s) (M(s) \|x_s\|_{\mathcal{B}} + K(s) \|x\|_{\infty}) \\ &\quad + e^{-\delta(t-s)} H(t-s) \|f(s, 0)\| \\ &\leq e^{-\delta(t-s)} H(t-s) [L_f(s)(\mathcal{L} + 1) \mathcal{K} \|x\|_{\infty} + \|f(s, 0)\|], \end{aligned}$$

it follows that $s \mapsto AT(t-s)f(s, x_s)$ is integrable over $(-\infty, t)$ for each $t > 0$. For additional remarks related to this types of conditions in connection with partial neutral differential equations, see, e.g., [1, 8, 9], in particular, [10].

Definition 7 A function $u \in AA_c(\mathbb{X})$ is an almost automorphic mild solution to the neutral system (1)-(2) provided that the function $s \rightarrow AT(t-s)f(s, u_s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = -f(t, u_t) - \int_{-\infty}^t AT(t-s)f(s, u_s)ds + \int_{-\infty}^t T(t-s)g(s, u_s)ds, \quad t \in \mathbb{R}.$$

Theorem 2 Under assumptions $\mathbf{H}_1$ - $\mathbf{H}_2$, there exist a unique almost automorphic solution to (1)-(2) whenever

$$\Theta = \left(L_f + \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\delta(t-s)} H(t-s) L_f(s) ds + M \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-w(t-s)} L_g(s) ds \right) \mathcal{L} < 1,$$

where $L_f = \sup_{t \in \mathbb{R}} L_f(t)$ and $\mathcal{L}$ is the constant appearing in Remark 1.

Proof: In $AA_c(\mathbb{X})$, define the operator $\Gamma : AA_c(\mathbb{X}) \rightarrow C(\mathbb{R}, \mathbb{X})$ by setting

$$\Gamma u(t) := -f(t, u_t) - \int_{-\infty}^t AT(t-s)f(s, u_s)ds + \int_{-\infty}^t T(t-s)g(s, u_s)ds, \quad t \in \mathbb{R}.$$

From previous assumptions one can easily see that Γu is well-defined and continuous. Moreover, from Lemmas 2, 3, 4, and 5 we infer that Γ maps $AA_c(\mathbb{X})$ into itself. Next, we prove that Γ is a strict contraction on $AA_c(\mathbb{X})$. Indeed, if $\mathcal{L}$ is the constant appearing in Remark 1, for $u, v \in AA_c(\mathbb{X})$, we get

$$\begin{aligned} \|\Gamma u(t) - \Gamma v(t)\| &\leq L_f(t) \|u_t - v_t\|_{\mathcal{B}} + \int_{-\infty}^t L_f(s) e^{-\delta(t-s)} H(t-s) \|u_s - v_s\|_{\mathcal{B}} ds \\ &\quad + M \int_{-\infty}^t e^{-w(t-s)} L_g(t) \|u_s - v_s\|_{\mathcal{B}} ds \\ &\leq \left(L_f + \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-\delta(t-s)} H(t-s) L_f(s) ds \right) \cdot \mathcal{L} \cdot \|u - v\|_{\infty} \\ &\quad + \left(M \sup_{t \in \mathbb{R}} \int_{-\infty}^t e^{-w(t-s)} L_g(s) ds \right) \cdot \mathcal{L} \cdot \|u - v\|_{\infty} \\ &\leq \Theta \cdot \|u - v\|_{\infty}. \end{aligned}$$

The assertion is now a consequence of the classical Banach fixed-point principle.

3 Examples

In this section we provide with an example to illustrate our main result. For that, we first introduce the related background.

Let $\mathbb{X} = (L^2[0, \pi]; \|\cdot\|_2)$. Define the linear operator A by:

$$D(A) := \{u \in \mathbb{X} : u'' \in \mathbb{X}, u(0) = u(\pi) = 0\}, \quad \text{and,} \quad Au := u'', \quad \forall u \in D(A).$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $\mathbb{X}$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, whose corresponding (normalized) eigenfunctions are given by: $z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi)$. In addition, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{X}$;

(b) For $u \in \mathbb{X}$, $T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n$ and $Au = -\sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n$, $\forall u \in D(A)$;

(c) It is possible to define the fractional power $(-A)^\alpha$, $0 < \alpha \leq 1$ of A , as a closed linear operator over its domain $D((-A)^\alpha)$. More precisely, the operator $(-A)^\alpha : D((-A)^\alpha) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is explicitly given by $(-A)^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n$, for all $u \in D((-A)^\alpha)$, where $D((-A)^\alpha) = \{u(\cdot) \in \mathbb{X} : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in \mathbb{X}\}$;

(e) If $\mathbb{X}_\alpha$ denotes the space $D((-A)^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$, then $\mathbb{X}_\alpha$ is a Banach space. Moreover, $\mathbb{X}_\alpha \rightarrow \mathbb{X}_\beta$ is continuous for $0 < \beta \leq \alpha \leq 1$ and there exist some constants $C_\alpha, \delta_\alpha > 0$ such that $\|T(t)\|_{\mathcal{L}(\mathbb{X}_\alpha, \mathbb{X})} \leq \frac{C_\alpha e^{-\delta_\alpha t}}{t^\alpha}$ for $t > 0$.

For the phase space, we choose the space $\mathcal{B} = C_r \times L^p(\rho, \mathbb{X})$, $r \geq 0$, $1 \leq p < \infty$. We next follow the terminology of [15], where g is replaced by ρ to avoid confusions with the function g that we previously considered. Briefly, this means that $\rho : (-\infty, -r) \mapsto \mathbb{R}$ is a positive (Lebesgue) integrable function and that there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure equal to zero. The space $C_r \times L^p(\rho, \mathbb{X})$ denotes the collection of functions $\varphi : (-\infty, 0] \rightarrow \mathbb{X}$ such that $\varphi(\cdot)$ is continuous on $[-r, 0]$, (Lebesgue) measurable and $\rho\|\varphi\|_p^p$ is (Lebesgue) integrable on $(-\infty, -r)$.

The seminorm on $\mathcal{B}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\|_2 + \left(\int_{-\infty}^{-r} \rho(\theta) \|\varphi(\theta)\|_2^p d\theta \right)^{1/p}.$$

In what follows we assume that $\rho(\cdot)$ is a continuous function satisfying assumptions (g-5)-(g-7) of [15, Theorem 1.3.8]. Under the previous assumptions, $\mathcal{B}$ is a fading memory space, which satisfies the axioms **A**, **A1**, **B** and **C2**, see [15, Theorem 1.3.8] and [15, Example 7.1.8] for details.

Throughout the rest of this section, we suppose that $r = 0$, $p = 2$, and use the notations of Remarks 1 and 2.

Consider the first-order boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} \left[u(t, \xi) + \int_{-\infty}^0 \int_0^\pi b(s, \eta, \xi) u(t+s, \eta) d\eta ds \right] &= \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(t) u(t, \xi) \\ &+ \int_{-\infty}^0 a(s) u(t+s, \xi) ds, \end{aligned} \quad (5)$$

$$u(t, 0) = u(t, \pi) = 0, \quad (6)$$

for $(t, \xi) \in \mathbb{R} \times [0, \pi]$.

Let us mention that equations of type (5)-(6) arise for instance in control of systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-differential law, see [8, Examples 4.2] for details.

In addition to the above-mentioned assumptions, we suppose that a, a_0, a_1 are continuous functions and that the following conditions hold:

- (i) The functions $b(\cdot), \frac{\partial^i}{\partial \zeta^i} b(\tau, \eta, \zeta), i = 1, 2$ are (Lebesgue) measurable, $b(\tau, \eta, \pi) = 0$, $b(\tau, \eta, 0) = 0$ for every (τ, η) , and

$$N_1 := \max\left\{\int_0^\pi \int_{-\infty}^0 \int_0^\pi \rho^{-1}(\tau) \left(\frac{\partial^i}{\partial \zeta^i} b(\tau, \eta, \zeta)\right)^2 d\eta d\tau d\zeta : i = 0, 1\right\} < \infty;$$

- (ii) The function $a(\cdot)$ is continuous and $L = \left(\int_{-\infty}^0 \frac{a^2(s)}{\rho(s)} ds\right)^{\frac{1}{2}} < \infty$.

Define $f, g : \mathbb{R} \times \mathcal{B} \rightarrow \mathbb{X}$ by setting

$$\begin{aligned} f(t, \psi)(\xi) &:= \int_{-\infty}^0 \int_0^\pi b(s, \eta, \xi) \psi(s, \eta) d\eta ds \\ g(t, \psi)(\xi) &:= a_0(t) \psi(\xi) + \int_{-\infty}^0 a(s) \psi(s, \xi) ds. \end{aligned}$$

In view of the above, it is clear that the system (5)-(6) can be rewritten as an abstract system of the form (1). By a straightforward estimation that uses (i) one can show that f has values in $\mathbb{Y} = [D(-A)^{\frac{1}{2}}]$ and that $f(t, \cdot) : \mathcal{B} \mapsto \mathbb{X}_{\frac{1}{2}}$ is a bounded linear operator with $\|f(t, \cdot)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \leq \sqrt{N_1}$ for each $t \in \mathbb{R}$. Furthermore, for all $t \in \mathbb{R}$, $g(t, \cdot)$ is a bounded linear operator on $\mathbb{X}$ with $\|g(t, \cdot)\| \leq L$. The next result is a straightforward consequence of Theorem 2.

Proposition 1 *Under previous assumptions, the system (5)-(6) has a unique almost automorphic solution whenever $\Theta = \sqrt{N_1} \left(1 + 2C_{\frac{1}{2}} + \delta_{\frac{1}{2}}^{-1}\right) + L < 1$.*

References

- [1] M. ADIMY AND K. EZZINBI, Strict solutions of nonlinear hyperbolic neutral differential equations. *Appl. Math. Lett.* **12** (1999), no. 1, pp. 107–112.
- [2] S. BOCHNER, A New Approach to Almost Periodicity. *Proc. Nat. Acad. Sci. U.S.A.* Vol. **48**(1962), pp. 2039–2043.
- [3] T. DIAGANA, G. M. N'GUÉRÉKATA, AND N. V. MINH, Almost Automorphic Solutions of Evolution Equations, *Proc. Amer. Math. Soc.*, **132** (2004), no. 11, pp. 3289–3298.
- [4] K. EZZINBI AND G. M. N'GUÉRÉKATA, A Massera Type Theorem for Almost Automorphic Solutions of Functional Differential Equations of Neutral Type. *Journal of Mathematical Analysis and Applications*, 2005 (in press).
- [5] A. M. FINK, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics, Vol. **377**, Springer-Verlag, New York-Berlin, 1974.

- [6] J. K. HALE AND S. M. VERDUYN LUNEL, *Introduction to functional-differential equations*. Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.
- [7] J. K. HALE, Partial neutral functional-differential equations. *Rev. Roumaine Math. Pures Appl.* **39** (1994), no. 4, pp. 339–344.
- [8] E. HERNÁNDEZ AND H. R. HENRÍQUEZ, Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* **221** (1998), no. 2, pp. 452–475.
- [9] E. HERNÁNDEZ AND H. R. HENRÍQUEZ, Existence of periodic solutions of partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* **221** (1998), no. 2, pp. 499–522.
- [10] E. HERNÁNDEZ, Existence Results for Partial Neutral Integrodifferential Equations with Unbounded Delay. *J. Math. Anal. Appl.* **292** (2004), no. 1, pp. 194–210.
- [11] E. M. HERNÁNDEZ AND M. L. PELICER, Asymptotically Almost Periodic and Almost Periodic Solutions for Partial Neutral Differential Equations, *Applied Mathematics Letters*, 2005 (in press).
- [12] H. R. HENRÍQUEZ AND C. H. VÁSQUEZ, Almost periodic solutions of abstract retarded functional-differential equations with unbounded delay. *Acta Appl. Math.* **57** (1999), no. 2, pp. 105–132.
- [13] Y. HINO AND S. MURAKAMI, Almost automorphic solutions for abstract functional differential equations. *J. Math. Anal. Appl.* **286** (2003), pp. 741–752.
- [14] Y. HINO AND S. MURAKAMI, Limiting equations and some stability properties for asymptotically almost periodic functional differential equations with infinite delay. *Tôhoku Math. J. (2)* **54** (2002), no. 2, pp. 239–257.
- [15] Y. HINO, S. MURAKAMI, AND T. NAITO, *Functional-differential equations with infinite delay*. Lecture Notes in Mathematics. 1473. Springer-Verlag, Berlin, 1991.
- [16] N. MINH MAN AND N. VAN MINH, On the existence of quasi periodic and almost periodic solutions of neutral functional differential equations. *Commun. Pure Appl. Anal.* **3** (2004), no. 2, pp. 291–300.
- [17] G. M. N'GUÉRÉKATA, *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic / Plenum Publishers, New York-London-Moscow, 2001.
- [18] G. M. N'GUÉRÉKATA, *Topics in Almost Automorphy*, Springer, New York, Boston, Dordrecht, London, Moscow 2005.

- [19] R. YUAN, Existence of almost periodic solutions of neutral functional differential equations via Liapunov-Razumikhin function. *Z. Angew. Math. Phys.* **49** (1998), no. 1, pp. 113–136.
- [20] T. YOSHIZAWA, *Stability theory and the existence of periodic solutions and almost periodic solutions*. Applied Mathematical Sciences, Vol. **14**. Springer-Verlag, New York-Heidelberg, 1975.
- [21] J. WU AND H. XIA, Rotating waves in neutral partial functional-differential equations. *J. Dynam. Differential Equations* **11** (1999), no. 2, pp. 209–238.
- [22] J. WU AND H. XIA, Self-sustained oscillations in a ring array of coupled lossless transmission lines. *J. Differential Equations* **124** (1996), no. 1, pp. 247–278.
- [23] J. WU, *Theory and applications of partial functional-differential equations*. Applied Mathematical Sciences **119** Springer-Verlag, New York, 1996.
- [24] W. A. VEECH, Almost Automorphic Functions. *Proc. Nat. Acad. Sci. U.S.A.* Vol. **49**(1963), pp. 462–464.
- [25] S. ZAIDMAN, Almost Automorphic Solutions of Some Abstract Evolution Equations. II. *Istit. Lombardo. Accad. Sci. Lett. Rend. A* **111**(1977), no. 2, pp. 260–272.
- [26] M. ZAKI, Almost Automorphic Solutions of Certain Abstract Differential Equations. *Ann. Mat. Pura Appl.* Vol. **101** (1974), no. 4, pp. 91–114.

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