ON THE REGULARITY OF THE SOLUTIONS OF LINEAR STIELTJES-INTEGRAL EQUATIONS

by

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ABSTRACT - We prove that if $A:[a,b] \longrightarrow L(X)$ is such that for every $g \in C([a,b],X)$ the function

$$t \in [a,b] \longmapsto \int_a^t dA(s)\cdot g(s) \in X$$

is continuous (respectively, regulated or of bounded variation [i.e, if $A \in SVC^{\sigma}([a,b],L(X))$ (respectively, $A \in SVG^{\sigma}([a,b],L(X))$ or $A \in BV([a,b],L(X)))$ - see §1 for the notations]) then for any $f: [a,b] \longrightarrow X$ that is continuous (respectively, regulated or of bounded variation) every solution $y: [a,b] \longrightarrow X$ of the Stieltjes-integral equation

$$y(t) - x + \int_a^t \cdot dA(s)\cdot y(s) = f(t) - f(a), \qquad a \le t \le b$$

is continuous too (respectively, regulated or of bounded variation).

§1 - INTRODUCTION

We follow the notations of [1] to [6] but for some small changes.

X and Y always denote Banach spaces. $L(X,Y)$ denotes the space of all linear continuous mappings $u: X \longrightarrow Y$. By I_X we denote the identical automorphism of X.

We say that a function $f:[a,b] \rightarrow X$ is *regulated,* we write $f \in G([a,b],X)$, if for every $t \in [a,b[$ $(t \in]a,b])$ there exists $f(t+)$ $(f(t-))$.

We say that a function $A: [a,b] \longrightarrow L(X,Y)$ is *simply regulated* (*simply continuous*), we write $A \in G^{\sigma}([a,b],L(X,Y))$ $(A \in C^{\sigma}([a,b],L(X,Y)))$, if for every $x \in X$ the function

$$Ax: t \in [a,b] \longmapsto (Ax)(t) = A(t)x \in Y$$

is regulated (continuous).

I - If $A \in G^{\sigma}([a,b],L(X,Y))$, for every $t \in [a,b[$ $(t \in]a,b])$ there exists $A(t+) \in L(X,Y)$ $(A(t-) \in L(X,Y))$ such that for every $x \in X$ we have $(Ax)(t+) = A(t+)x$ $((Ax)(t-) = A(t-)x)$. See [3; p. 176].

A *division* of an interval $[a,b]$ is a finite sequence

$$d: t_0 = a < t_1 < t_2 < \ldots < t_n = b.$$

We write $|d| = n$ and $\Delta d = \sup_{1 \le i \le n} |t_i - t_{i-1}|$; we denote by $D_{[a,b]}$ or simply by D the set of all divisions of $[a,b]$. We say that a division $\bar{d}$ is *finer* than a division d if $\bar{d} \supset d$. Given points $x, (x_d)_{d \in D}$ of a topological space E, we write $x = \lim_{d \in D} x_d$ if for every neighborhood V of x there exists a $d_V \in D$ such that for $d \supset d_V$ we have $x_d \in V$. The meaning of $x = \lim_{\Delta d \to 0} x_d$ is obvious.

We say that $f\colon [a,b] \longrightarrow X$ is a *function of bounded variation*, we write $f \in BV([a,b],X)$, if we have $V[f] < \infty$ where

$$V[f] = V_{[a,b]}[f] = \sup_{d\in D} V_d[f].$$

with

$$V_d[f] = \sum_{i=1}^{|d|} \| f(t_i) - f(t_{i-1}) \| .$$

For $a \le r < t \le b$ we define $V_{]r,t]}[f] = \lim_{s \downarrow r} V_{[s,t]}[f]$.

II - Every function of bounded variation f is regulated and we have $\lim_{t\downarrow r} V_{]r,t]}[f] = 0$. See [1; p. 28 and 26].

We say that $A\colon [a,b] \longrightarrow L(X,Y)$ is a *function of bounded semivariation*, we write $A \in SV([a,b],L(X,Y))$, if we have $SV[A] < \infty$ where

$$SV[A] = SV_{[a,b]}[A] = \sup_{d\in D} SV_d[A]$$

with

$$SV_d[A] = \sup\left\{ \left\| \sum_{i=1}^{|d|} [A(t_i) - A(t_{i-1})] x_i \right\| \,\middle|\, x_i \in X_i, \|x_i\| \le 1 \right\}.$$

For $A\colon [a,b] \longrightarrow L(X,Y)$ and $f\colon [a,b] \longrightarrow X$ we define the *interior integral*

$$\int_a^b \cdot\, dA(t) \cdot f(t)$$

and the *Riemann-Stieltjes integral*

$$\int_a^b dA(t) \cdot f(t),$$

respectively, by

$$\int_a^b \cdot\, dA(t) \cdot f(t) = \lim_{d\in D} \sum_{i=1}^{|d|} [A(t_i) - A(t_{i-1})] \cdot f(\xi_i^{\bullet}), \text{ where } \xi_i^{\bullet} \in \,]t_{i-1}, t_i[,$$

and

$$\int_a^b dA(t)\cdot f(t) = \lim_{\Delta d\to 0} \sum_{i=1}^{|d|} [A(t_i)-A(t_{i-1})]\cdot f(\xi_i), \text{ where } \xi_i\in[t_{i-1},t_i],$$

when these limits exist.

III - If there exists

$$\int_a^b \cdot dA(t)\cdot f(t)$$

and if A and f have no common points of discontinuity (for instance if A or f is continuous) then there exists

$$\int_a^b dA(t)\cdot f(t) = \int_a^b \cdot dA(t)\cdot f(t).$$

See [2,p. 9].

IV - For $A\in SV([a,b],L(X,Y))$ and $f\in G([a,b],X)$ $(f\in C([a,b],X))$ there exists

$$\int_a^b \cdot dA(t)\cdot f(t) \quad \left(\int_a^b dA(t)\cdot f(t)\right)$$

and we have

$$\left\|\int_a^b \cdot dA(t)\cdot f(t)\right\| \leq SV[A]\|f\|.$$

See [2; p. 26 and 24].

We define

$$SVG^\sigma([a,b],L(X,Y)) = SV([a,b],L(X,Y))\cap G^\sigma([a,b],L(X,Y))$$

$$SVC^\sigma([a,b],L(X,Y)) = SV([a,b],L(X,Y))\cap C^\sigma([a,b],L(X,Y))$$

V - Take $A:[a,b] \longrightarrow L(X,Y)$. For every $f\in C([a,b],X)$, the function

$$t\in[a,b] \longmapsto \int_a^t dA(s)\cdot f(s)\in Y$$

is regulated (respectively, continuous or of bounded variation) iff we have $A\in SVG^{\sigma}([a,b], L(X,Y))$ (respectively, $A\in SVC^{\sigma}([a,b],L(X,Y))$ or $A\in BV([a,b],L(X,Y))$).

PROOF: See [5; Lemma 3.6] for the first two cases; the third one is easy to prove (by contradiction).

VI - Let $A\in SVG^{\sigma}([a,b],L(X))$ be such that there exists $d\in D_{[a,b]}$ with $SV_{]t_{i-1},t_i[}[A] < 1$ for $i = 1,2,\ldots,|d|$. Then the following properties are equivalent:

i) For every $x\in X$ and $f\in G([a,b],X)$ there exists one and only one $y\in G([a,b],X)$ that satisfies the linear Stieltjes-integral equation

$$y(t) - x + \int_a^t \cdot dA(s)\cdot y(s) = f(t) - f(a), \qquad a \le t \le b$$

ii) For every $t\in[a,b[$ the operator $I_X+A(\dot{+})-A(t)$ is invertible (in $L(X)$).

PROOF: see [4; Theorem 4.7] or [5; Theorem 3.8].

REMARK 1 - The hipothesis of VI is satisfied for *every* $A\in SV([a,b],L(X))$ iff $X \not\supset c_0(N)$ (i.e., X contains no subspace isomorphic to $c_0(N)$); see [7; p. 73].

From VI follow easily VII and VIII.

VII - Let $A\in SVC^{\sigma}([a,b],L(X))$ be such that there exists

$d \in D_{[a,b]}$ with $SV_{]t_{i-1},t_i[}[A] < 1$ for $i = 1,2,\ldots,|d|$. Then for $x \in X$ and $f \in C([a,b],X)$ there exists one and only one $y \in C([a,b],X)$ that satisfies

$$y(t) - x + \int_a^t dA(s)\cdot y(s) = f(t) - f(a), \qquad a \le t \le b.$$

VIII - For $A \in BV([a,b],L(X))$ the following properties are equivalent:

i) For every $x \in X$ and $f \in BV([a,b],X)$ there exists one and only one $y \in BV([a,b],X)$ that satisfies

$$y(t) - x + \int_a^t \cdot dA(s)\cdot y(s) = f(t) - f(a), \qquad a \le t \le b$$

ii) For every $t \in [a,b[$ the operator $I_X + A(t+) - A(t)$ is invertible (in $L(X)$).

A priori however it could happen that the equation in VI (respectively, VII or VIII) admits also (non-bounded) solutions that are non-regulated (respectively, non-continuous or of non bounded variation). The main result of this paper asserts that no such other solutions exist.

THEOREM - If $A \in SVG^\sigma([a,b],L(X))$ (respectively, $A \in SVC^\sigma([a,b],L(X))$ or $A \in BV([a,b],L(X))$) and $f \in G([a,b],X)$ (respectively, $f \in C([a,b],X)$ or $f \in BV([a,b],X)$) then for every solution $y:[a,b] \longrightarrow X$ of the Stieltjes-integral equation

$$(1) \qquad y(t) - x + \int_a^t \cdot dA(s)\cdot y(s) = f(t) - f(a), \qquad a \le t \le b$$

we have $y \in G([a,b],X)$ (respectively, $y \in C([a,b],X)$ or $y \in BV([a,b],X)$)

REMARK 2 - An analogous result is not true for linear Volterra Stieltjes-integral equations

$$(2) \qquad y(t) + \int_a^t \cdot d_s K(t,s)\cdot y(s) = f(t) - f(a), \qquad a \leq t \leq b$$

even in the numerical case (i.e., $X = \mathbb{R}$).

EXAMPLE: We take $K(t,s) = \chi_{[a,t[}(s)$; the integral in (2) exists and is a regulated function of t for every regulated function y. However for any function $g\colon [a,b] \longrightarrow \mathbb{R}$ that has a left limit $g(t-)$ at every point $t \in]a,b]$ we have

$$\int_a^t \cdot d_s K(t,s)\cdot g(s) = -g(t-).$$

Hence for $x = 0$ and $f = 0$ any function $y\colon [a,b] \longrightarrow \mathbb{R}$ such that $y(a) = 0$ and $y(t) = y(t-)$ for $a < t \leq b$, is a solution of (2).

REMARK 3 - If (2) has only one *regulated* solution, we ignore if it can have further non-regulated solutions.

§2 - PROOF OF THE THEOREM

In order to prove the THEOREM we need the following

LEMMA - Let $A\colon [a,b] \longrightarrow L(X,Y)$ and $f\colon [a,b] \longrightarrow X$ be such that there exists

$$\int_a^b \cdot dA(s)\cdot f(s) \in Y.$$

Then for every $t \in [a,b]$ there exists

$$\int_a^t \cdot dA(s)\cdot f(s)$$

and given $\varepsilon>0$, there exists a step-function $f_\varepsilon:[a,b] \longrightarrow X$ such that for every $t\in[a,b]$, but for a finite number of points, we have

$$\text{(3)} \qquad \left\| \int_a^t \cdot dA(s)\cdot f(s) - \int_a^t \cdot dA(s)\cdot f_\varepsilon(s) \right\| \leq \varepsilon .$$

PROOF: By the Cauchy criterion the existence of

$$\int_a^b \cdot dA(s)\cdot f(s)$$

implies that given $\varepsilon>0$ there exist $d_\varepsilon \in D_{[a,b]}$ such that for every $d\supset d_\varepsilon$ we have

$$\left\| \sum_{j=1}^{|d|} [A(t_j) - A(t_{j-1})]\cdot f(\eta_j^{\bullet}) - \sum_{i=1}^{|d_\varepsilon|} [A(t_i^\varepsilon) - A(t_{i-1}^\varepsilon)]\cdot f(\xi_i^{\bullet}) \right\| \leq \frac{\varepsilon}{2}.$$

We define

$$f_\varepsilon = \sum_{i=1}^{|d_\varepsilon|} \chi_{]t_{i-1}^\varepsilon, t_i^\varepsilon]} f(\xi_i^{\bullet}).$$

For $t\in[a,b]$, $t\neq a$, there exists $k\in\{1,2,\ldots,|d_\varepsilon|\}$ such that $t\in]t_{k-1}^\varepsilon, t_k^\varepsilon]$; we consider two cases:

a) $t_{k-1}^\varepsilon < \xi_k^{\bullet} < t \leq t_k^\varepsilon$ and b) $t_{k-1}^\varepsilon < t < \xi_k^{\bullet}$.

a) We take the division $d_\varepsilon^t = d_\varepsilon \cup \{t\}_{|[a,t]}$ of $[a,t]$ and take any $d^t \in D_{[a,t]}$ with $d^t \supset d_\varepsilon^t$. We complete the divisions d_ε^t and d^t to divisions $\bar{d}_\varepsilon^t$ and $\bar{d}^t$ of $[a,b]$ in such a way that $\bar{d}_\varepsilon^t$ and $\bar{d}^t$ are identical in $[t,b]$ and $\bar{d}_\varepsilon^t \supset d_\varepsilon$, $\bar{d}^t \supset d_\varepsilon$.

Hence we have

$$\|\sum_{j=1}^{|d^t|}[A(t_j)-A(t_{j-1})]\cdot f(\dot{\eta}_j) - \sum_{i=1}^{|d_\varepsilon^t|}[A(t_i^\varepsilon)-A(t_{i-1}^\varepsilon)]\cdot f(\dot{\xi}_i)\| =$$

$$= \sum_{i=1}^{|\bar{d}^t|}[A(\bar{t}_j)-A(\bar{t}_{j-1})]\cdot f(\dot{\bar{\eta}}_j) - \sum_{i=1}^{|\bar{d}_\varepsilon^t|}[A(\bar{t}_i^\varepsilon)-A(\bar{t}_{i-1}^\varepsilon)]\cdot f(\dot{\bar{\xi}}_i)\| \leq \frac{\varepsilon}{2}$$

and since we have

$$\sum_{i=1}^{|d_\varepsilon^t|}[A(t_i^\varepsilon)-A(t_{i-1}^\varepsilon)]\cdot f(\dot{\xi}_i) = \int_a^t \cdot dA(s)\cdot f_\varepsilon(s)$$

it follows (3) in the case a). In the case b) we have

$$\|\int_a^t \cdot dA(s)\cdot[f(s)-f_\varepsilon(s)]\| =$$

$$= \|\int_a^b \cdot dA(s)\cdot[f(s)-f_\varepsilon(x)] - \int_t^b \cdot dA(s)\cdot[f(s)-f_\varepsilon(s)]\|$$

$$\leq \|\int_a^b \cdot dA(s)\cdot[f(s)-f_\varepsilon(s)]\| + \|\int_t^b \cdot dA(s)\cdot[f(s)-f_\varepsilon(s)]\| ;$$

by a) the first summand is $\leq \frac{\varepsilon}{2}$ and the proof that the second summand is $\leq \frac{\varepsilon}{2}$ is analogous to the proof given in a). Since (3) is obvious for $t = a$, we proved (3) for all $t \in [a,b]$ but for $t = \dot{\xi}_1, \dots, \dot{\xi}_{|d_\varepsilon|}$.

REMARK 4 - The following example *sugests* that under the hypothesis of the Lemma we cannot assure (3) for all points $t \in [a,b]$, even if A is bounded: we take $X = Y = H$ a Hilbert space and $\alpha : [a,b] \longrightarrow H$ such that for $a \leq s < t \leq b$ we have $(\alpha(s)|\alpha(t)) = 0$ (for instance, take $H = \ell_2([a,b])$ and $\alpha(x) = \chi_{\{s\}}$); we take $f = \alpha$, fix an element $e \in H$ and define $A : [a,b] \longrightarrow L(H)$ by $A(t)x = (\alpha(t)|x)e$; it is immediate that for every $t \in [a,b]$ we have

$$\int_a^t \cdot dA(s)\cdot f(s) = 0.$$

However if we give a division d of $[a,b]$ and take the step-function

$$f_\varepsilon = \sum_{i=1}^{|d|} \chi_{]t_{i-1},t_i]} f(\dot{\xi}_i),$$

where $t_{i-1} < \dot{\xi}_i < t_i$, then we have

$$\int_a^{\dot{\xi}_i} \cdot\, dA(s)\cdot f_\varepsilon(s) = \|\alpha(\dot{\xi}_i)\|^2 e, \qquad i = 1,2,\dots,|d|.$$

REMARK 5 - Under the hypothesis of the Lemma the function

$$t\in[a,b] \longmapsto I(t) = \int_a^t \cdot dA(s)\cdot f(s)$$

may not be bounded even in the numerical case $(X = Y = \mathbb{R})$. Exemple: we take $[a,b] = [0,1]$ and $\alpha(0) = \alpha(1) = 0$, $\alpha(t) = n$ if $1 - \frac{1}{n} < t \leq$ $\leq 1 - \frac{1}{n+1}$; for $f \equiv 1$ there exists

$$\int_0^1 \cdot d\alpha(s)\cdot f(s) = 0$$

but for $1 - \frac{1}{n} < t \leq 1 - \frac{1}{n+1}$ we have

$$\int_0^t \cdot d\alpha(s)\cdot f(s) = n.$$

COROLLARY 1 - Under the hypothesis of the Lemma, if A is bounded so is the function

$$t\in[a,b] \longmapsto I(t) = \int_a^t \cdot dA(s)\cdot f(s).$$

Indeed, since f_ε is bounded it follows immediatelly that the function

$$t\in[a,b] \longmapsto I_\varepsilon(t) = \int_a^t \cdot dA(s)\cdot f_\varepsilon(s)$$

is bounded if A is bounded. The result follows from (3).

COROLLARY 2 - Under the hypothesis of the Lemma, if $A\in G^\sigma([a,b],L(X,Y))$ then I is regulated.

Indeed, it is immediate that under the hypothesis the function I_ε is regulated; hence by (3) I is regulated too as a uniform limit of regulated functions (at the finite number of points $s\in[a,b]$ where we do not have (3) we define $I_\varepsilon(s) = I(s)$).

COROLLARY 3 - Under the hypothesis of the Lemma, if $A\in C^\sigma([a,b],L(X,Y))$ then I is continuous.

Indeed, in this case I_ε is continuous and we have (3) at all point $t\in[a,b]$ (since in the corresponding proof we replace

$$\int_a^t \cdot \text{ by } \int_a^t ,$$

hence we may take $t_{k-1}^\varepsilon \leq \xi_k \leq t_k^\varepsilon$ and consider the two cases

a) $\xi_k \leq t \leq t_k^\varepsilon$ and b) $t_{k-1}^\varepsilon \leq t \leq \xi_k$).

Hence I is continuous too as the uniform limit of the continuous functions I_ε, $\varepsilon>0$.

PROPOSITION 4 - Under the hypothesis of the Lemma, if $A\in SV([a,b],L(X,Y))$ and f is bounded we have $I\in BW([a,b],Y) = SV([a,b],L(\mathbb{R},Y))$.

Indeed, given a division d of $[a,b]$ and $\lambda_i\in\mathbb{R}$ with $|\lambda_i|\leq 1$

we have (Cf. IV)

$$\Big\| \sum_{i=1}^{|d|} \lambda_i [I(t_i)-I(t_{i-1})] \Big\| = \Big\| \sum_{i=1}^{|d|} \int_{t_{i-1}}^{t_i} \cdot\, dA(s)\cdot \lambda_i f(s) \Big\| \le SV[A] \| f \| .$$

In an analogous way we prove the

PROPOSITION 5 - Under the hypothesis of the Lemma, if $A \in BV([a,b],L(X,Y))$ and f is bounded then we have $I \in BV([a,b],Y)$.

PROOF OF THE THEOREM: By Corollary 1 it follows that the integral in (1) is bounded. If $A \in SVG^{\sigma}([a,b],L(X))$ $(A \in SVC^{\sigma}([a,b],L(X)))$ it follows by Corollary 2 (Corollary 3) that the integral in (1) is a regulated (continuous) function of t, hence the result if $f \in G([a,b],X)$ $(f \in C([a,b],X))$. If $A \in BV([a,b],L(X))$ and $y: [a,b] \longrightarrow X$ satisfies (1), by Corollary 1 the integral in (1) is a bounded function of t, hence since $f \in BV([a,b],X)$ the solution y of (1) is bounded too. Hence by Proposition 5 the function

$$t \in [a,b] \longmapsto \int_a^t \cdot\, dA(s)\cdot y(s) \in X$$

is a function of bounded variation and so is then y by (1).

REMARK 6 - More generally let $y: [a,b] \longrightarrow X$ be a solution of (1):

a) If $A \in G^{\sigma}([a,b],L(X))$ and $f \in G([a,b],X)$ then $y \in G([a,b],X)$

b) If $A \in C^{\sigma}([a,b],L(X))$ and $f \in C([a,b],X)$ then $y \in C([a,b],X)$

c) If $A \in SV([a,b],L(X))$ and $f \in BW([a,b],X)$ then $y \in BW([a,b],X)$

In these cases however the integral in (1) may not be meaningfull for every y of the corresponding class (Cf.V of

§1) nor do we have results corresponding to VI, VII or VIII of §1.

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