RT-MAT 2001-09

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Setembro 2001

GLOBAL EXISTENCE OF PERIODIC TRAVELLING WAVES OF AN INFINITE NON-LINEARLY SUPPORTED BEAM I. CONTINUOUS MODEL

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ABSTRACT. The paper deals with problems of existence of periodic travelling wave solutions with non-small amplitudes of a PDE describing oscillations of an infinite beam, which lies on a non-linearly elastic support. Such solutions are in fact critical points of a functional on a suitable functional space. By means of a minimax variational technique, the authors found a domain in the parameter space for which there exist periodic travelling waves of a certain fixed period Σ.

1. INTRODUCTION

Motions of an infinite beam lying on a dense support obeying a non-linear deformation law can be described by means of the following PDE

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \lambda \frac{\partial^2 y}{\partial x^2} + k^2 y - \beta y^3 = 0. \tag{1.1}$$

Here parameter λ is responsible for pre-stress effects and k^2, β describe properties of the support. If β is positive than the support is called softening and it is called hardening otherwise. The problem of existence of travelling waves of different shapes for equation (1.1) was posed by a well-known specialist in non-linear mechanics, Professor of the University Colleage of London J.M.T.Thompson and up to now has been considered only from the local point of view in papers [1, 2]. In Part II of this paper we will show how one can obtain the above equation (1.1) as a limit of a chain of ODEs describing oscillations of material particles connected by means of elastic springs. Here we will not pay special attention to a physical model directly resulting in (1.1). The usual substitution $y(x,t) = u(x - vt)$ gives us travelling wave solutions

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Date: September 17th, 2001.
Key words and phrases. Elastic beam, periodic travelling wave, minimax approach.
The research was supported by the FAPESP Foundation (Brazil) and partially supported by the Alexander von Humboldt Foundation (Federal Republic of Germany).

of equation (1.1), where v is a prescribed speed of the wave. The function u obtained obeys a simple ODE of 4th order

$$\frac{d^4u}{ds^4} + \sigma\frac{d^2u}{ds^2} + k^2u - \beta u^3 = 0, \tag{1.2}$$

where a new parameter $\sigma = \lambda + v^2$, not necessarily positive, is introduced.

We will search for periodic travelling waves, i.a. for smooth Σ- periodic solutions $u(s)$, $u(s+\Sigma) = u(s)$ of equation (1.2) with a certain prescribed period Σ. After scaling the independent variable and parameters of the problem, we can look for only 2π-periodic solutions of (1.2).

Any 2π-periodic function can be developed into Fourier series

$$u(s) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos ns + b_n \sin ns) .$$

It is obvious that if $\beta > 0$, (1.2) has evident constant solutions

$$u_\pm = \pm\frac{k}{\sqrt{\beta}} .$$

Further, (1.2) is invariant under translations $s \mapsto s+S$ and if $u(s)$ is a desired solution, then $u(s+S)$ is also a desired solution. So, it would be reasonable to fix the nodes of the wave in the s-space. That equation is also invariant under mirror-symmetry reflections $u \mapsto -u$ and $s \mapsto -s$ which makes it possible to consider either even or odd solutions. That is why to avoid constant solutions and those obtained by means of all the above transforms, we will deal only with solutions which can be developed into a Fourier series with respect to sinusoidal harmonics

$$u(s) = \sum_{n=1}^{\infty} u_n \sin ns \tag{1.3}$$

In [2] the authors investigated local behaviour of periodic travelling waves and showed that all the periodic waves bifurcating from the trivial solution $u(s) = 0$ can be obtained by means of a spatial shift from those which can be developed into the Fourier series (1.3).

It is easy to show that equation (1.2) admits a variational formulation. Indeed, since $u(0) = u(\pi) = u''(0) = u''(\pi) = 0$, classical solutions of (1.2) are critical points of the functional

$$I[u] = \frac{1}{2}\int_0^\pi \left(u''^2 - \sigma u'^2 + k^2u^2 - \frac{\beta}{2}u^4\right) ds \to \text{extr.} \tag{1.4}$$

It is worth noting that critical points of $I[u]$ appear symmetrically: if $\hat{u}$ is a critical point, then $-\hat{u}$ is also a critical point.

The main result can be formulated as follows.

Theorem 1. *For a softening support ($\beta > 0$) the functional $I[u]$ has an infinite number of critical points which are classical solutions of (1.2) and can be developed into the Fourier series (1.3) for any values of parameters σ, k, β. If the support is hardening ($\beta < 0$) then the functional $I[u]$ has at least N 'symmetric' critical points for any fixed k, β if $\sigma > n^2 + k^2/n^2$, $n = 1, 2, ..., N$.*

2. Functional Setting

Since we are interested only in solutions of the variational problem (1.4) and correspondingly of equation (1.2) which can be expanded into the Fourier series (1.3), we have to construct a suitable functional space which a desired solution belongs to. Let us consider a scale of Hilbert spaces $\mathbf{H}^0_\theta[S^1]$ with $\theta \in \mathbf{R}$ obtained by closing the space of functions from $\mathbf{C}^\infty[S^1]$ which can be developed into the series (1.3) with respect to the norms associated with the following scalar products

$$\langle u, v\rangle_\theta = \sum_{n=1}^{\infty} n^{2\theta} u_n v_n, \tag{2.1}$$

where u_n, v_n are the corresponding Fourier coefficients for a pair of functions $u(s)$, $v(s)$.

It is not difficult to note that for positive integers $\theta = m$ the norms generated by scalar products (2.1) can be rewritten as follows

$$\|u\|_\theta = \left(\sum_{n=1}^{\infty} n^{2\theta} u_n^2\right)^{1/2} = \left(\frac{2}{\pi}\int_0^\pi \left(u^{(m)}(s)\right)^2 ds\right)^{1/2}. \tag{2.2}$$

It is easy to see that for any positive integer even m and $u \in \mathbf{H}^0_\theta[S^1]$ the m-th derivative $u^{(m)}$ belongs to $\mathbf{H}^0_{\theta-m}[S^1]$ and

$$\|u^{(m)}\|_{\theta-m} = \|u\|_\theta. \tag{2.3}$$

In fact, we identify our spaces of functions with the spaces of sequences of their Fourier coefficients. We will use also spaces of odd functions on S^1 $\mathbf{L}^{0,q}[S^1] = \mathbf{H}^0_0[S^1] \cap \mathbf{L}^q[S^1]$, $q \geq 2$ with standard norms

$$|||u|||_q = \left(\int_0^\pi |u(s)|^q ds\right)^{1/q} \tag{2.4}$$

and $\mathbf{C}_0[S^1] = \mathbf{H}^0_0[S^1] \cap \mathbf{C}[S^1]$ with the norm

$$|||u|||_\infty = \sup_{[0,\pi]} |u(s)|. \tag{2.5}$$

To proceed further, we need to use some properties of the spaces introduced. They are formulated in the following lemmas.

Lemma 1. *For any $\theta_1, \theta_2 \in \mathbf{R}$, $\theta_1 < \theta_2$, $\mathbf{H}^0_{\theta_2}[S^1] \hookrightarrow \mathbf{H}^0_{\theta_1}[S^1]$, where the sign $\hookrightarrow$ means compact embedding, and for any $u \in \mathbf{H}^0_{\theta_2}[S^1]$ the following inequality holds*

$$\|u\|_{\theta_2} \le \|u\|_{\theta_1}.$$

Lemma 2. *For any $\theta \in \mathbf{R}$ spaces $\mathbf{H}^0_\theta[S^1]$ and $\mathbf{H}^0_{-\theta}[S^1]$ are reciprocally conjugate $\left((\mathbf{H}^0_\theta[S^1])^* = \mathbf{H}^0_{-\theta}[S^1]\right)$.*

Lemma 3. *For any $\theta \ge 1$ and $q \ge 2$ the following embeddings take place $\mathbf{L}^{0,q}[S^1] \hookrightarrow \mathbf{H}^0_\theta[S^1]$ and $\mathbf{C}_0[S^1] \hookrightarrow \mathbf{H}^0_\theta[S^1]$. There exist also positive constants $C_{\theta,q}$ and $C_{\theta,\infty}$ such that for any $u \in \mathbf{H}^0_\theta[S^1]$ the following inequalities hold*

$$|||u|||_q \le C_{\theta,q}\|u\|_\theta,$$

$$|||u|||_\infty \le C_{\theta,\infty}\|u\|_\theta.$$

Lemma 4. *For any $\theta \ge 0$ and $q \ge 2$ the following embeddings take place $\mathbf{H}^0_{-\theta}[S^1] \hookrightarrow \mathbf{L}^{0,q}[S^1]$ and $\mathbf{H}^0_{-\theta}[S^1] \hookrightarrow \mathbf{C}_0[S^1]$. There exist also positive constants $B_{-\theta,q}$ and $B_{-\theta,\infty}$ such that for any $u \in \mathbf{L}^{0,q}[S^1]$ or $u \in \mathbf{C}_0[S^1]$ the following inequalities hold*

$$\|u\|_{-\theta} \le B_{-\theta,q}|||u|||_q,$$

$$\|u\|_{-\theta} \le B_{-\theta,\infty}|||u|||_\infty.$$

The statements of Lemmas 1 - 4 easily result from the Sobolev embedding theorems on a segment [3] (see also [4]).

We will look for critical points of the functional $I[u]$ belonging to the space $\mathbf{H}^0_2[S^1]$.

3. Analysis of Critical Points

To prove that the variational problem (1.4) possesses critical points different from the trivial one $u(s) \equiv 0$, we need to use some results related to minimax methods in critical point theory [5].

Theorem 2 (Generalized Mountain Pass Theorem). *Let $\mathbf{H}$ be a real Banach space and $I[u]$ be a real functional from $\mathbf{C}^1[\mathbf{H}, \mathbf{R}]$. Let $I[0] = 0$, $u = 0$ be a critical point, I be even ($I[-u] = I[u]$) satisfy the Palais-Smale condition. If additionally the following two conditions hold:*

(i) *the space $\mathbf{H}$ can be decomposed into a sum $\mathbf{H} = \mathbf{V} \oplus \mathbf{X}$ so that $\dim \mathbf{V} <$ ∞ and there are two positive constants ρ and α such that for any $u \in \partial\mathbf{B}_\rho \cap \mathbf{X}$*

$$I[u] > \alpha,$$

where $\mathbf{B}_\rho = \{u \in \mathbf{H} : \|u\| \le \rho\}$ *is a centered closed ball in* $\mathbf{H}$ *of a radius* ρ*;*

(ii) *for any finite-dimensional subspace* $\widetilde{\mathbf{H}} \subset \mathbf{H}$ *there exists a positive constant* $R(\widetilde{\mathbf{H}})$ *such that for any* $u \in \widetilde{\mathbf{H}} \setminus \mathbf{B}_{R(\widetilde{\mathbf{H}})}$

$$I[u] \le 0,$$

then the functional $I[u]$ *possesses an infinite number of distinct pairs of nontrivial critical points.*

Minimax methods in the variational calculus is a powerful tool in situations when functionals are not bounded from below. For bounded functionals we can use another result which states the existence of nontrivial critical points and their multiplicity [5].

Theorem 3 (Generalized Clark's Theorem). *Let* $I[u]$ *be as above and additionally bounded from below . Suppose that*

(iii) *there is a set* $\mathbf{K} \subset \mathbf{H}$ *homeomorphic to a finite-dimensional sphere* S^{N-1} *by an odd map, and*

$$\sup_{u \in \mathbf{K}} I[u] < 0,$$

then the functional $I[u]$ *possesses at least* N *distinct pairs of nontrivial critical points.*

We will apply those general results to the variational problem (1.4). Let us consider some important properties of the functional $I[u]$ required in the above theorems. Obviously, $I[0] = 0$ and the functional I is even.

Lemma 5. $I \in \mathbf{C}^1[\mathbf{H}_2^0[S^1], \mathbf{R}]$

Proof. The functional I can be rewritten in the following simple form $I = I^{(1)} + I^{(2)}$, where

$$I^{(1)}[u] = \frac{1}{\pi}\left(\|u\|_2^2 - \sigma\|u\|_1^2 + k^2\|u\|_0^2\right), \; I^{(2)}[u] = -\frac{\beta}{4}|||u|||_4^4.$$

Due to Lemmas 1 and 3 we have two chains of compact inclusions:

$$\mathbf{H}_2^0[S^1] \hookrightarrow \mathbf{H}_1^0[S^1] \hookrightarrow \mathbf{H}_0^0[S^1] = \mathbf{L}^{0,2}[S^1]$$

and

$$\mathbf{H}_2^0[S^1] \hookrightarrow \mathbf{H}_1^0[S^1] \hookrightarrow \mathbf{L}^{0,4}[S^1].$$

That means that the functional I is well defined on $\mathbf{H}_2^0[S^1]$. Let us now calculate the Gateaux derivative of the functional $I^{(1)}$, $\nabla_u^{weak} I^{(1)}[u] = u^{IV} + \sigma u'' + k^2 u$. Since for any $h \in \mathbf{H}_2^0[S^1]$

$$\left|I^{(1)}[u+h] - I^{(1)}[u] - \nabla_u^{weak} I^{(1)}[u]h\right| \equiv 0,$$

the quadratic functional $I^{(1)}$ is Frechet differentiable and its strong derivative $\nabla_u I^{(1)}$ coincides with the Gateaux derivative. Moreover, using properties of spaces $\mathbf{H}^0_\theta[S^1]$, it is easy to prove that for any fixed $u \in \mathbf{H}^0_2[S^1]$, the element $\nabla_u I^{(1)}[u] \in \mathbf{H}^0_{-2}[S^1]$. Below we show that the strong derivative is bounded for any fixed u in the sense of $\mathbf{H}^0_{-2}[S^1]$. Thus, as a linear operator $\nabla_u I^{(1)}[u] \in \mathbf{L}\left[\mathbf{H}^0_2[S^1], \mathbf{R}\right]$. Here it is worth reminding that $\mathbf{H}^0_{-2}[S^1] = \left(\mathbf{H}^0_2[S^1]\right)^*$ in accordance with Lemma 2. Further, let us prove the continuity of $\nabla_u I^{(1)}[u]$ with respect to u as an operator function from $\mathbf{H}^0_2[S^1]$ to $\mathbf{H}^0_{-2}[S^1]$. Indeed, let $h \in \mathbf{H}^0_2[S^1]$ and let

$$h(s) = \sum_{n=1}^{\infty} h_n \sin ns$$

be its Fourier expansion. We need to consider the difference between two operators

$$\nabla_u I^{(1)}[u+h] - \nabla_u I^{(1)}[u] = h^{\mathrm{IV}} + \sigma h'' + k^2 h\,.$$

Then

$$\|\nabla_u I^{(1)}[u+h] - \nabla_u I^{(1)}[u]\|_{-2} = \left(\sum_{n=1}^{\infty} n^{-4}(n^4 - \sigma n^2 + k^2)^2 h_n^2\right)^{1/2} =$$

$$\left(\sum_{n=1}^{\infty} n^4 \left(1 - \frac{\sigma}{n^2} + \frac{k^2}{n^4}\right)^2 h_n^2\right)^{1/2} \le (1 + |\sigma| + k^2)\|h\|_2.$$

Hence, $\nabla_u I^{(1)}[u]$ is continuous.

The Gateaux derivative of $I^{(2)}$ is obviously equal to $-\beta u^3$. Then for any $h \in \mathbf{H}^0_2[S^1]$

$$\left|I^{(2)}[u+h] - I^{(2)}[u] - \nabla_u^{weak} I^{(2)}[u]h\right| =$$

$$\frac{|\beta|}{4}\left|\int_0^\pi (6u^2h^2 + 4uh^3 + h^4)\,ds\right| \le$$

$$\frac{|\beta|}{4}\left(6\int_0^\pi u^2h^2 ds + 4\int_0^\pi |u||h|^3 ds + \int_0^\pi h^4 ds\right) \le$$

$$\frac{|\beta|}{4}\left(6\left(\int_0^\pi u^4 ds\right)^{1/2}\left(\int_0^\pi h^4 ds\right)^{1/2} + 4\left(\int_0^\pi u^4 ds\right)^{1/4}\left(\int_0^\pi h^4 ds\right)^{3/4} +\right.$$

$$\left.\int_0^\pi h^4 ds\right) = \frac{|\beta|}{4}\left(6|||u|||_4^2\,|||h|||_4^2 + 4|||u|||_4\,|||h|||_4^3 + |||h|||_4^4\right) \le$$

$$\frac{|\beta|}{4} C_{2,4}^4 \left(6\|u\|_2^2 + 4\|u\|_2\,\|h\|_2 + \|h\|_2^2\right)\|h\|_2^2 = O\left(\|h\|_2^2\right)$$

for any fixed $u \in \mathbf{H}_2^0[S^1]$ as $||h||_2 \to 0$. Here we used the Hölder inequality and the embedding properties from Lemma 3.

Thus, the functional $I^{(2)}$ is Frechet differentiable and its strong derivative coincides with the weak one. Obviously, the element $\nabla_u I^{(2)}[u]$ belongs to the conjugate space $\mathbf{H}^0_{-2}[S^1]$. Further we need to estimate the remainder

$$\nabla_u I^{(2)}[u+h] - \nabla_u I^{(2)}[u] = -\beta(3u^2 + 3uh + h^2)h.$$

Using Lemmas 3 and 4, we arrive at the following chain of inequalities

$$\begin{aligned}
&||\nabla_u I^{(2)}[u+h] - \nabla_u I^{(2)}[u]||_{-2} \leq B_{-2,\infty}|||\nabla_u I^{(2)}[u+h] - \nabla_u I^{(2)}[u]|||_\infty \leq \\
&B_{-2,\infty}|\beta| \left(3|||u^2h|||_\infty + 3|||uh^2|||_\infty + |||h^3|||_\infty\right) \leq \\
&B_{-2,\infty}|\beta| \left(3|||u|||_\infty^2 \, |||h|||_\infty + 3|||u|||_\infty \, |||h|||_\infty^2 + |||h|||_\infty^3\right) \leq \\
&B_{-2,\infty}|\beta| C_{2,\infty} \left(3||u||_2^2 + 3||u||_2||h||_2 + ||h||_2^2\right) ||h||_2 = O\left(||h||_2\right)
\end{aligned}$$

for any fixed $u \in \mathbf{H}_2^0[S^1]$ as $||h||_2 \to 0$.

The last thing we need to finish the proof is to check whether the operator

$$\nabla_u I[u] = u^{\mathrm{IV}} + \sigma u'' + k^2 u - \beta u^3$$

is continuous. But it is easy to see that the operator under consideration is bounded and consequently, it is continuous. Indeed, using the identity (2.3) and the embedding properties from Lemmas 3, 4, we obtain

$$\begin{aligned}
||\nabla_u I[u]||_{-2} \leq\; & ||u^{\mathrm{IV}}||_{-2} + |\sigma| \, ||u''||_{-2} + k^2||u||_{-2} + |\beta| \, ||u^3||_{-2} \leq \\
& ||u||_2 + |\sigma| \, ||u||_0 + k^2||u||_{-2} + |\beta| \, ||u^3||_{-2} \leq \\
& ||u||_2 + |\sigma| \, ||u||_2 + k^2||u||_2 + B_{-2,2}|\beta| \, |||u^3|||_2 = \\
& ||u||_2 + |\sigma| \, ||u||_2 + k^2||u||_2 + B_{-2,2}|\beta| \, |||u|||_6^3 \leq \\
& ||u||_2 + |\sigma| \, ||u||_2 + k^2||u||_2 + B_{-2,2}|\beta| C_{2,6}^3||u||_2^3.
\end{aligned}$$

Thus, the operator $\nabla_u I[u]$ is bounded. □

Let us pass to the most difficult stage of the proof of Theorem 1, i.e. we need to prove that the functional I satisfies the Palais-Smale condition.

Lemma 6. *Any sequence* $\{u_{(p)}\}_{p=1}^\infty \in \mathbf{H}_2^0[S^1]$ *such that the sequence* $|I[u_{(p)}]|$ *is bounded while the sequence* $\nabla_u I[u_{(p)}] \to 0$ *as* $p \to \infty$ *contains a convergent subsequence* $\{u_{(p_j)}\}_{j=1}^\infty$.

Proof. We can split the process of checking the Palais-Smale condition into two steps.

Step 1.

Lemma 7. *Any sequence with the above properties is bounded in* $\mathbf{H}_2^0[S^1]$.

Proof. Let $\{u_{(p)}\}_{p=1}^{\infty}$ be such that $|I[u_{(p)}]| < M$ for a certain $M > 0$. Obviously, $I[u_{(p)}] \le M$.

We first concentrate on a more complicated case $\beta > 0$.

Since $\nabla_u I[u_{(p)}] \to 0$ as $p \to \infty$ there is a number p_0 such that for any $p > p_0$ the norm $\|\nabla_u I[u_{(p)}]\|_{-2} < 1$ and

$$\left|\nabla_u I[u_{(p)}]u_{(p)}\right| \le \|u_{(p)}\|_2.$$

Consequently,

$$\nabla_u I[u_{(p)}]u_{(p)} \ge -\|u_{(p)}\|_2.$$

For a fixed $\mu \in (\frac{1}{4}, \frac{1}{2})$ we have the following inequality

$$\begin{gathered} M + \mu\|u_{(p)}\|_2 \ge I[u_{(p)}] - \mu\nabla_u I[u_{(p)}]u_{(p)} = \\ \left(\frac{1}{2} - \mu\right)\int_0^{\pi}\left((u''_{(p)})^2 - \sigma(u'_{(p)})^2 + (u_{(p)})^2\right)ds + \\ \beta\left(\mu - \frac{1}{4}\right)\int_0^{\pi}(u_{(p)})^4 ds = 2\left(\frac{1}{2} - \mu\right) I^{(1)}[u_{(p)}] - 4\left(\mu - \frac{1}{4}\right) I^{(2)}[u_{(p)}]. \end{gathered} \tag{3.1}$$

Lemma 8. *There exist two constants $\delta > 0$, $\Delta \ge 0$ such that for any $u \in \mathbf{H}_2^0[S^1]$ the following estimate holds*

$$I^{(1)}[u] \ge \frac{1}{2}\left(\delta\|u\|_2^2 - \Delta\|u\|_0^2\right). \tag{3.2}$$

Proof. Let us consider an element $u \in \mathbf{H}_2^0[S^1]$ which can be expanded into the Fourier series (1.3). Then

$$I^{(1)}[u] = \frac{\pi}{2}\sum_{n=1}^{\infty}(n^4 - \sigma n^2 + k^2)\, u_n^2.$$

To have the desired estimate (3.2), we need to choose δ and Δ so that for any n the following inequality holds

$$\pi(n^4 - \sigma n^2 + k^2) \ge \delta n^4 - \Delta.$$

That means that the discriminant of the following quadratic trinomial

$$(\pi - \delta)x^2 - \pi\sigma x + (k^2 + \pi\Delta)$$

should be positive as $\delta < \pi$ which immediately results in the inequality

$$\delta < \pi - \frac{\pi^2\sigma^2}{4(\pi k^2 + \Delta)}.$$

By choosing Δ large enough, we can satisfy the above inequality. It is worth noticing that if $|\sigma| < 2k$, i.e. the speed of the wave does not exceed its critical value (see [1, 2]), we can put $\Delta = 0$. □

Hence, using inequality (3.2), we can rewrite (3.1) as follows

$$
\begin{aligned}
M + \mu\|u_{(p)}\|_2 &\geq \left(\frac{1}{2} - \mu\right)\left(\delta\|u_{(p)}\|_2^2 - \Delta\|u_{(p)}\|_0^2\right) \\
&+ \beta\left(\mu - \frac{1}{4}\right)|||u_{(p)}|||_4^4 \geq \\
&\left(\frac{1}{2} - \mu\right)\delta\|u_{(p)}\|_2^2 + ay^2 - 2by\,,
\end{aligned} \tag{3.3}
$$

where the following notations are introduced

$$y = |||u_{(p)}|||_4^2, \quad a = \beta\left(\mu - \frac{1}{4}\right) > 0, \quad b = \frac{1}{2}\left(\frac{1}{2} - \mu\right)\Delta B_{0,4}^2 > 0.$$

Here we also used the embedding $\mathbf{H}_0^0[S^1] \hookrightarrow \mathbf{L}^{0,4}[S^1]$ (Lemma 4). There is a certain positive constant c such that for any real y

$$ay^2 - 2by \geq -c.$$

For instance, we can choose $c \geq b^2/a$. Then (3.3) gives

$$dz^2 - \mu z - f \leq 0,$$

where

$$z = \|u_{(p)}\|_2, \quad d = \delta\left(\frac{1}{2} - \mu\right) > 0, \quad f = M + c > 0.$$

Since z is non-negative, $0 \leq z \leq z_+$, where z_+ is the positive root of the equation $dz^2 - \mu z - f = 0$.

Thus, we have proved that the sequence $\{u_{(p)}\}_{p=1}^{\infty}$ is bounded.

The case $\beta < 0$ is much easier. Indeed, inequality $I[u_{(p)}] \leq M$ and Lemma 8 gives us

$$M \geq \frac{1}{2}\left(\delta\|u\|_2^2 - \Delta\|u\|_0^2\right) - \frac{\beta}{4}|||u|||_4^4 \geq \frac{\delta}{2}\|u\|_2^2 + a_1y^2 - 2b_1y, \tag{3.4}$$

where y is as above and

$$a_1 = -\frac{\beta}{4} > 0, \quad b_1 = \frac{\Delta}{4}B_{0,4}^2 > 0.$$

By choosing $c_1 \geq b_1^2/a_1$, from (3.4) we obtain the final estimate

$$\|u_{(p)}\|_2^2 \leq \frac{2}{\delta}(M + c_1).$$

Thus, the sequence $\{u_{(p)}\}_{p=1}^{\infty}$ is bounded. □

Step 2.

Lemma 9. *The sequence $\{u_{(p)}\}_{p=1}^{\infty}$ has a convergent subsequence $\{u_{(p_j)}\}_{j=1}^{\infty}$.*

Proof. We have to start with some technicalities. Let us consider the differential operator $D = \frac{d}{ds}$. Obviously, operator D^2 maps spaces $\mathbf{H}^0_\theta[S^1]$ in $\mathbf{H}^0_{\theta-2}[S^1]$.

Lemma 10. *Operator D^2 is invertible.*

Proof. Via the relation (2.3) the operator D^2 is an isometric one. Hence, it is invertible.

□

Of course, any iterations of D^2 are invertible in corresponding spaces. Below we will use operators D^{-2} and D^{-4} for which we can even give explicit expressions:

$$\left(D^{-2}u\right)(s) = \frac{1}{\pi}\left((s-\pi)\int_0^s \sigma u(\sigma)d\sigma + s\int_s^\pi (\sigma-\pi)u(\sigma)d\sigma,\right)$$

$$\left(D^{-4}u\right)(s) = \frac{1}{6\pi}\left(\int_0^\pi s\sigma(s^2+\sigma^2+2\pi^2)u(\sigma)d\sigma\right.$$

$$\left.-\pi\left(\int_0^s \sigma(\sigma^2+3s^2)u(\sigma)d\sigma + \int_s^\pi s(s^2+3\sigma^2)u(\sigma)d\sigma\right)\right).$$

It is worth noticing that the operators D^{-2} and D^{-4} treated as maps from $\mathbf{H}^0_\theta[S^1]$ into itself are compact because the embedding $\mathbf{H}^0_{\theta+m}[S^1] \hookrightarrow \mathbf{H}^0_\theta[S^1]$ for positive m is compact. Since

$$\nabla_u I[u_{(p)}] = \left(u_{(p)}\right)^{\mathrm{IV}} + \sigma\left(u_{(p)}\right)'' + k^2 u_{(p)} - \beta\left(u_{(p)}\right)^3 \to 0,$$

as $p\to\infty$ in the sense of $\mathbf{H}^0_{-2}[S^1]$,

$$D^{-4}\nabla_u I[u_{(p)}] = u_{(p)} - v_{(p)} \to 0$$

in the sense of $\mathbf{H}^0_{-2}[S^1]$, where

$$v_{(p)} = -\sigma D^{-2}u_{(p)} - k^2 D^{-4}u_{(p)} + \beta D^{-4}\left(u_{(p)}\right)^3.$$

The sequences $D^{-2}u_{(p)}$ and $D^{-4}u_{(p)}$ are bounded in spaces $\mathbf{H}^0_4[S^1]$ and $\mathbf{H}^0_6[S^1]$ respectively and compact in $\mathbf{H}^0_2[S^1]$. Due to Lemma 4, $\left(u_{(p)}\right)^3$ belongs at least to $\mathbf{H}^0_0[S^1]$ and bounded in that space. Consequently, $D^{-4}\left(u_{(p)}\right)^3$ is bounded in $\mathbf{H}^0_4[S^1]$ and compact in $\mathbf{H}^0_2[S^1]$. Hence, the whole sequence $\{v_{(p)}\}_{p=1}^{\infty}$ is compact and contains a subsequence $\{v_{(p_j)}\}_{j=1}^{\infty}$ converging to an element $v_* \in \mathbf{H}^0_2[S^1]$ as $j\to\infty$. Therefore, $u_{(p_j)} \to v_*$ as $j\to\infty$. □

Thus, the functional $I[u]$ satisfies the Palais-Smale condition. □

Let us start now checking conditions of Theorems 2 and 3. First, we will prove that the condition (i) holds. Indeed, let N be a non-negative integer number such that

$$n^4 - \sigma n^2 + k^2 \leq 0,\ n = 1, 2, ..., N, \quad n^4 - \sigma n^2 + k^2 > 0,\ n = N+1, N+1, ...$$

Let us decompose $\mathbf{H}_2^0[S^1]$ as a direct sum $\mathbf{H}_2^0[S^1] = \mathbf{V} \oplus \mathbf{X}$, where

$$\mathbf{V} = \text{span}\{\sin s, \sin 2s, ..., \sin Ns\},$$

which can be trivial, of course.

Lemma 11. *There exists a positive constant δ such that for any $u \in \mathbf{X}$ the following estimate holds*

$$I^{(1)}[u] \geq \frac{\delta}{2} ||u||_2^2. \tag{3.5}$$

Proof. Let

$$u(s) = \sum_{n=N+1}^{\infty} u_n \sin ns$$

be the Fourier expansion of a function $u(s)$ from $\mathbf{X}$.

$$I^{(1)}[u] = \frac{\pi}{2} \sum_{n=N+1}^{\infty} \left(n^4 - \sigma n^2 + k^2\right) u_n^2.$$

We can take

$$\delta < \frac{\pi}{2} \inf_{n \geq N+1} \left\{1 - \frac{\sigma}{n^2} + \frac{k^2}{n^4}\right\}$$

and obtain the desired estimate. □

In accordance with Lemma 3

$$I^{(2)}[u] = -\frac{\beta}{4} |||u|||_4^4 \geq -\frac{|\beta|}{4} C_{2,4}^4 ||u||_2^4. \tag{3.6}$$

Let $||u||_2 = \rho$ for a certain small ρ. Then using the inequalities (3.5) and (3.6), we obtain

$$I[u] \geq \alpha = \rho^2 \left(\frac{\delta}{2} - \frac{|\beta|}{4} C_{2,4}^4 \rho^2\right),$$

which is, of course, positive for any ρ small enough.

Let us now prove that for any fixed k, σ and $\beta > 0$ the condition (*ii*) of Theorem 2 holds. Indeed, let $\widetilde{\mathbf{H}}$ be a finite-dimensional subspace of $\mathbf{H}_2^0[S^1]$. Then it can be represented as follows

$$\widetilde{\mathbf{H}} = \left\{ u \,:\, u(s) = \sum_{q=1}^{Q} u_{n_q} \sin n_q s, \quad 1 \le n_1 \le \ldots \le n_Q \right\}.$$

Let us consider the following set

$$\partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}} = \left\{ u \,:\, \sum_{q=1}^{Q} n_q^4 u_{n_q}^2 = 1 \right\}.$$

Evidently,

$$\widetilde{\mathbf{H}} \setminus \mathbf{B}_R = \left\{ u = rv,\ R \le r < \infty,\ v \in \partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}} \right\},$$

and

$$I[u] = r^2 I^{(1)}[v] + r^4 I^{(2)}[v].$$

In fact, $I^{(1)}[v]$, $I^{(2)}[v]$ are continuous functions of coefficients $u_{n_1}, \ldots, u_{n_Q}$ and $\partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}}$ is compact as a closed and bounded set in a finite-dimensional space. Let us introduce the following notations

$$I_*^{(1)} = \min_{\partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}}} I^{(1)}[v], \quad I_*^{(2)} = \min_{\partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}}} I^{(2)}[v].$$

Thus,

$$I[u] \le r^2 \left(I_*^{(1)} + r^2 I_*^{(2)} \right). \tag{3.7}$$

Since $I^{(2)}[v] < 0$ on $\partial \mathbf{B}_1 \cap \widetilde{\mathbf{H}}$ if $\beta > 0$, then $I_*^{(2)} < 0$. In principle, $I_*^{(1)}$ may have both signs. If $I_*^{(1)}$ is non-positive then we can put $R(\widetilde{\mathbf{H}}) = 0$. Otherwise, we can take

$$R(\widetilde{\mathbf{H}}) = \sqrt{-\frac{I_*^{(1)}}{I_*^{(2)}}}.$$

In this case, it immediately follows from (3.7) that

$$I[u] \le 0,$$

which means that the condition (*ii*) of Theorem 2 holds.

Thus, all the conditions of Theorem 2 hold and the functional $I[u]$ possesses an infinite number of critical points for any values of parameters σ, k, β if $\beta > 0$.

It is easy to prove that for $\beta < 0$ the functional $I[u]$ is bounded from below. Indeed, by arguing as in Lemma 7, we can obtain inequality like (3.4):

$$I[u] \geq \frac{\delta}{2}\|u\|_2^2 + a_1|||u|||_4^4 - 2b_1|||u|||_4^4, \tag{3.8}$$

the estimate (3.8) obtained results in the following inequality

$$I[u] \geq -c_1,$$

where $c_1 \geq b_1^2/a_1$, which means that $I[u]$ is bounded from below.

Now we need to check the condition (*iii*) of Theorem 3. If $\sigma > n^2 + k^2/n^2$, $n = 1, 2, ..., N$, then

$$n^4 - \sigma n^2 + k^2 < 0, \; n = 1, 2, ..., N.$$

We will denote an N-dimensional subspace of $\mathbf{H}_2^0[S^1]$ spanned on functions $\{\sin s, \sin 2s, ..., \sin Ns\}$ as $\mathbf{W}$. Let $\mathbf{K} = \partial \mathbf{B}_r \cap \mathbf{W}$ for a certain small $r > 0$. It is easy to see that $\mathbf{K}$ can be obtained from the unit sphere S^{N-1} by contractions in basis directions. Then we can mirror-symmetrically map the sphere into itself. The composition of those two maps will be odd. Let us put

$$\delta_- = -\frac{\pi}{2} \sup_{n \leq N} \left\{ 1 - \frac{\sigma}{n^2} + \frac{k^2}{n^4} \right\} > 0.$$

Then for any $u \in \mathbf{K}$

$$I^{(1)}[u] \leq -\delta_- r^2.$$

By using the embedding properties from Lemma 3, we obtain the following estimate

$$I^{(2)}[u] \leq \frac{|\beta|}{4} C_{2,4}^4 r^4.$$

Hence, for a small but fixed $r > 0$ we have $I[u] < 0$ and

$$\sup_{u \in \mathbf{K}} I[u] < 0,$$

since $\mathbf{K}$ is a compact set.

Thus, the condition (*iii*) of Theorem 3 is satisfied and for a hardening support ($\beta < 0$) the functional $I[u]$ has at least N distinct pairs of critical points in $\mathbf{H}_2^0[S^1]$ for any values of parameters k, β and $\sigma > n^2 + k^2/n^2$, $n = 1, 2, ..., N$.

To finish up the proof of Theorem 1, we need to show that the obtained critical points correspond to classical solutions of (1.2). Let $u \in \mathbf{H}_2^0[S^1]$ be a critical point of the functional $I[u]$, then (1.2) holds in a weak sense, i.a. in the sense of the space $\mathbf{H}_{-2}^0[S^1]$. By applying the operator D^{-4}, we obtain that u must satisfy the following integral equation in the space $\mathbf{H}_2^0[S^1]$:

$$u = -\sigma D^{-2}u - k^2 D^{-4}u + \beta D^{-4}u^3. \tag{3.9}$$

Obviously, if $u \in \mathbf{H}_2^0[S^1]$, then $D^{-2}u \in \mathbf{H}_4^0[S^1]$, $D^{-4}u \in \mathbf{H}_6^0[S^1]$ and at least $D^{-4}u^3 \in \mathbf{H}_4^0[S^1]$, which means that the above solution is classical. Evidently, we can show that u even belongs to $\mathbf{C}^\infty[S^1]$.

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