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CHARACTERIZATION OF THE INTENSITY FOR A CLASS OF POINT PROCESSES

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ABSTRACT. For point processes that satisfy $P\{N(\Delta) = n\} \leq K_\delta|\Delta|^n$, for $n \geq 2$, or more generally that satisfy $EN(\Delta) = P\{N(\Delta) = 1\} + o(\Delta)$, we have established the relationship between the existence of the Radon-Nikodym derivative of the expectation measure and that of the single occurrence intensity. For such processes the intensity and the single occurrence intensity are equal a.e.[ℓ]. For point processes that satisfy $E(\prod_{i=1}^m N)(\prod_{i=1}^m \Delta_i) = P\{N(\Delta_i) = 1,\ 1 \leq i \leq m\} + o_{t,\prod_{i=1}^m \Delta_i}(\Delta)$ we have shown that the existence of the joint single occurrence intensity implies that of the Radon-Nikodym derivative of product measures, i.e., of product densities. Under some asumptions on the form of the product densities we have established the reciprocal implication, namely the existence of the joint single occurrence intensity, and, in both cases we have the equality of these quantities. The uniformity of infinitesimals o is shown to be related not only to the Riemann integrability of the joint single occurrence intensity and to the essential Riemann integrability of the product densities but also to the continuity of the single occurrence intensity and to the essential continuity of the intensity.

1. INTRODUCTION

In this article we study the single occurrence intensity (from now on referred to as s.o. intensity) of point processes on the real line. By s.o. intensity we mean the limit of the ratio of the probability of occurrence of exactly one event in an interval of the real line by the length of this interval as this length goes to zero. It is usual, see Brillinger (1977), to assume that point processes under study satisfy $P\{N(\Delta) = n\} \leq K_\delta|\Delta|^n$ for $n \geq 1$. Here we assume less than this. As a matter of fact we will only assume that $EN(\Delta) = P\{N(\Delta) = 1\} + o_{t,\Delta}(|\Delta|)$ and, as a particular case, the former power relation for $n > 1$ which contrary to the case $n \geq 1$ allows unbounded s.o. intensity functions to occur. Similar structures for joint probabilities are assumed for the study of joint single occurrence intensities (briefly joint s.o. intensities). We have established the relationship between the existence of the s.o. intensity and that of the Radon-Nikodym derivative of the expectation measure in Theorem 3.1 and that among uniform boundedness of infinitesimals and the Riemann integrability of the joint s.o. intensities and continuity of the s.o. intensity. Conditions on the Radon-Nikodym derivative of moment measures, i.e., product densities, that ensure the existence of joint s.o. intensities are given. For point processes that satisfy the former probability assumptions we

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have characterized a.e.[ℓ], their intesnity and drawn conclusions about their product densities.

In this section we will present some definitions, assumptions and notation. In section 2 we show that the infinitesimals associated to the power law assumption on the probability of event occurrence are uniformly bounded. In section 3 we prove the main results and in section 4 we present a conclusion.

We will work with Lebesgue measurable functions, $h : \mathbb{R}^m \to \mathbb{R}$ that are bounded over bounded $\mathbb{R}^m$-intervals or, equivalently, that are Lebesgue integrable and bounded over bounded $\mathbb{R}^m$- intervals. We will call this class of functions $\mathcal{L}^m$. We will denote by $\overline{\mathcal{L}}^m$ the class of Lebesgue integrable functions over bounded $\mathbb{R}^m$-intervals. The class of Riemann integrable functions $h : \mathbb{R}^m \to \mathbb{R}$ over bounded $\mathbb{R}^m$-intervals will be denoted by $\mathcal{R}^m$. Since all Riemann integrable functions over bounded intervals are bounded over these intervals, we have $\mathcal{R}^m \subset \mathcal{L}^m$.

Lebesgue measure on $\mathbb{R}^m$ will be simply denoted by ℓ independently of the dimension m. When it is necessary or to emphasize dimension, we will write ℓ_m. Functions that only differ over zero measure subsets of their common domain or of a common extension of their domains are naturally considered identical. All functions that we consider are assumed to be measurable.

We denote by $N(A)$ the number of events of a certain sort that occur in a Borel set $A \subset \mathbb{R}$. If $A = (\alpha, \beta]$, we write $N(\alpha, \beta]$ instead of $N((\alpha, \beta])$. We also denote by N the integer valued function defined by the equalities $N(t) = N(0, t]$, if $t > 0, N(0) = 0$ and $N(t) = -N(t, 0]$ if $t < 0$. Clearly $N(\alpha, \beta] = N(\beta) - N(\alpha)$. Let $\{\cdots, \tau_{-2} \leq \tau_{-1} \leq \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots\}$ denote the times at which the events occur. Then $N(t) = n$, if and only if $\tau_{n-1} \leq t < \tau_n$.

Provided probabilities of the form

$$P(N(\alpha_1, \beta_1] = n_1, \ldots, N(\alpha_k, \beta_k] = n_k)$$

are defined and consistent, for all $k \in \mathbb{N}^* = \{1, 2, \ldots\}$, and all $n_1, \ldots, n_k$ non-negative integers, we can define an appropriate probability space $(\Omega, \mathcal{A}, P)$, such that there exists a measurable mapping from this space into $(\mathbb{R}^Z, \mathcal{B}_{\mathbb{R}^Z})$, the set of sequences of real numbers, that is, of events in time, defining then a stochastic point process that will also be called N.

Define $dN(t) = N(t + dt) - N(t)$. A basic assumption is that there exist boundedly finite measures M_k such that

$$E\{dN(t_1) \cdots dN(t_k)\} = M_k(dt_1, \ldots, dt_k),$$

that is, $E(\prod_{i=1}^k N) = M_k$. We emphasize that from now on all the point processes that we consider are assumed to have boundedly finite expectation measures.

It is usual to deal with integrals of the form

$$\int \varphi(t) dN(t) = \sum_j \varphi(\tau_j).$$

Suppose that $\varphi_i, 1 \leq i \leq k$, are (essentially) bounded measurable functions with compact support. Then

$$E\left\{\int \varphi_1(t_1) dN(t_1) \cdots \int \varphi_k(t_k) dN(t_k)\right\} = \int \varphi_1(t_1) \cdots \varphi_k(t_k) dM_k(t_1, \ldots, t_k).$$

We will assume that there exists a positive real number δ and a constant $K_\delta > 0$ such that for all intervals $\Delta \subset I\!R$ with length $|\Delta| < \delta$, all integers $n > 1$ and all $t \in I\!R$, not only the relation

$$P\{N(\Delta) = n\} \leq K_\delta |\Delta|^n \tag{1}$$

holds, but also the limit

$$\lim_{|\Delta| \to 0, t \in \Delta} \frac{1}{|\Delta|} P\{N(\Delta) = 1\} = p_N(t) \tag{2}$$

exists uniformly in t. Inequality (1) implies that

$$P\{N(\Delta) > 1\} \leq K_\delta (\sum_{j \geq 2} |\Delta|^j) = O(|\Delta|^2).$$

Notice that if inequality (1) were valid for $n = 1$ then we would have $P\{N(\Delta) = 1\}/|\Delta| \leq K_\delta$ and hence, if it would exist, $p_N(t)$ would be a bounded function on $\mathbb{R}$. Notice also that (2) implies that $\forall x \in \mathbb{R}$, $P\{N(\{x\}) = 1\} = 0$, otherwise there would exist $t \in \mathbb{R}$ for which the limit $p_N(t)$ would be infinite.

Due to uniformity, relation (2) is equivalent to

$$P\{N(\Delta) = 1\} = p_N(t)|\Delta| + o_{t,\Delta}(|\Delta|),$$

for an infinitesimal $o_{t,\Delta}(z)$ with the following properties:
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall t \in \mathbb{R}\ \forall \Delta \subset \mathbb{R}, t \in \Delta,\ (0 < |\Delta| < \delta) \to |o_{t,\Delta}(|\Delta|)| \leq \frac{\varepsilon}{2}|\Delta|$ and $o_{t,\Delta}(0) = 0$,
that is,
$\forall \varepsilon > 0\ \exists \delta > 0\ \ (0 < z < \delta) \to \sup_{\substack{t \in \mathbb{R}, \Delta \subset \mathbb{R} \\ t \in \Delta, |\Delta| = z}} |o_{t,\Delta}(z)| \leq \frac{\varepsilon}{2} z < \varepsilon z$ and $o_{t,\Delta}(0) = 0$.

The quantity $\sup_{\substack{t \in \mathbb{R}, \Delta \subset \mathbb{R} \\ t \in \Delta, |\Delta| = z}} |o_{t,\Delta}(z)| = o(z)$ is a non-negative infinitesimal independent of t and Δ. In this sense, we also write $|o_{t,\Delta}(|\Delta|)| \leq o(|\Delta|)$ and say that $o_{t,\Delta}$ is a uniformly bounded infinitesimal.

More generally we may substitute

$$EN(\Delta) = P\{N(\Delta) = 1\} + o_{t,\Delta}(|\Delta|) \tag{3}$$

by (1).

For the easy of notation, we will write o_t instead of $o_{t,\Delta}$.

We say that $p_N(t)$ is the **s.o. intensity** of events at time t.

We remark that, in general, s.o. intensity is not Khinchin-Leadbetter intensity, which is given by

$$i_{N+}(t) = \lim_{h \downarrow 0} \frac{P\{N((t, t+h]) \geq 1\}}{h}.$$

The next proposition shows that, for point processes under (3), they are essentially the same object.

Proposition 1.1. *For point process that satisfy relation (3), we have : (i) there exists p_N a.e.$[\ell]$; (ii) there exists i_{N+} a.e.$[\ell]$ and (iii) $p_N = i_{N+}$ a.e. $[\ell]$.*

Proof Since EN is boundedly finite, M defined by $M(t) = EN([0, t])$ for $t \geq 0$ and $M(t) = -EN([t, 0])$ for $t < 0$, is a *real-valued monotone non-decreasing* function defined on the whole line. Thus, the limit $M'(t) = \lim_{h\to 0} \frac{M(t+h)-M(t)}{h}$ exists a.e.[ℓ]. Now, whenever $M'(t)$ exists, the limit $\lim_{|\Delta|\to 0, t\in\Delta} \frac{EN(\Delta)}{|\Delta|}$ exists and it is equal to $M'(t)$. Thus the latter limit exists a.e.[ℓ]. Observe that

$$p_N(t) = \lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{N(\Delta)=1\}}{|\Delta|} \leq \lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{N(\Delta)\geq 1\}}{|\Delta|} \equiv i_N(t)$$

$$\leq \lim_{|\Delta|\to 0, t\in\Delta} \frac{EN(\Delta)}{|\Delta|} = \lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{N(\Delta)=1\}}{|\Delta|} + \lim_{|\Delta|\to 0, t\in\Delta} \frac{o_{t,\Delta}(\Delta)}{|\Delta|} = p_N(t)$$

implies that the three limits exist whenever one of them exists and, in this situation, they are equal.

Now, since $\lim_{|\Delta|\to 0, t\in\Delta} \frac{EN(\Delta)}{|\Delta|}$ exists a.e.[ℓ] we have that p_N exists a.e.[ℓ], i_N exists a.e.[ℓ] and $p_N(t) = i_N(t)$ a.e.[ℓ]. Also notice that the existence of $i_N(t)$ implies that there exists $i_{N+}(t)$ and $i_{N+}(t) = i_N(t)$. This completes the proof. ∎

In particular, we see that $\{t \mid \exists i_{N+}(t) \wedge i_{N+}(t) \neq p_N(t)\} \subset \{t \mid \exists i_{N+}(t) \wedge \nexists i_N(t)\} \subset \{t \mid \nexists i_N(t)\} = \{t \mid \nexists M'(t)\}$, which is a set of zero Lebesgue measure.

Suppose now that there exists a positive real number δ and a constant $k_{\delta,m}$ such that for all intervals $\Delta_1, \ldots, \Delta_m$ of the real line with lengths $0 < |\Delta_i| < \delta$, $1 \leq i \leq m$, all integers $n_i \geq 1$ and all vectors $(t_1, \ldots, t_n) \in \mathbb{R}^m$ with $t_i \neq t_j$ for $i \neq j$, $1 \leq i \leq m$, $1 \leq j \leq m$, both properties below are valid:

(4) if $(n_1, \ldots, n_m) \neq (1, \ldots, 1)$ then $P\{N(\Delta_i) = n_i, 1 \leq i \leq m\} \leq k_{\delta,m} \prod_{i=1}^{m} |\Delta_i|^{n_i}$

and for$\Delta = (|\Delta_1|, \ldots, |\Delta_m|) \in (\mathbb{R}_+^*)^m$, $t_i \in \Delta_i$, $1 \leq i \leq m$, there exists the limit

(5) $$\lim_{\Delta\to 0} \frac{1}{\prod_{i=1}^{m} |\Delta_i|} P\{N(\Delta_i) = 1, 1 \leq i \leq m\} = p_m(t_1, \ldots, t_m),$$

uniformly in $t = (t_1, \ldots, t_m)$.

Observe that for $m = 1$ the symbol Δ has two different meanings, the interval and the length, but this will be of no harm.

The limit above measures the intensity of the joint occurrence of events in the distinct instants $t_1, \ldots, t_m$. We will call it the joint single occurrence intensity of order m (for short, joint intensity). Relation (5) implies that

(6) $$P\{N(\Delta_i) = 1, 1 \leq i \leq m\} = p_m(t_1, \ldots, t_m) \prod_{i=1}^{m} |\Delta_i| + o_{t,\prod_{i=1}^{m} \Delta_i}(\Delta)$$

for $o_{t,\prod_{i=1}^{m} \Delta_i}(\Delta)$ an infinitesimal such that

$$\sup_{\substack{t\in\mathbb{R}^m - \mathcal{E}^m, \prod_{i=1}^{m} \Delta_i \subset \mathbb{R}^m \\ t\in\prod_{i=1}^{m} \Delta_i, |\Delta_i| = z_i, 1\leq i\leq m}} |o_{t,\prod_{i=1}^{m} \Delta_i}(z)| = o(z),$$

where $z = (z_1, \ldots, z_m) \in (\mathbb{R}_+^*)^m$, is another infinitesimal which is independent of $t \in \mathbb{R}^m - \mathcal{E}^m$ and $\prod_{i=1}^{m} \Delta_i \subset \mathbb{R}^m$ which satisfies $\frac{o(\Delta)}{\prod_{i=1}^{m} |\Delta_i|} \to 0$ when $\Delta \to 0$.

We denote by $\mathcal{E}^m$ the set

$$\{(t_1,\ldots,t_m)\in \mathbb{R}^m | t_i = t_j \text{ for some } i \neq j\}.$$

Similarly to what is done for $m = 1$ we may assume that

$$E(\prod_{i=1}^{m} N)(\prod_{i=1}^{m} \Delta_i) = P\{N(\Delta_i) = 1, 1 \le i \le m\} + o_{t,\prod_{i=1}^{m}\Delta_i}(\Delta) \tag{7}$$

with $|o_{t,\prod_{i=1}^{m}\Delta_i}(\Delta)| \le o(\Delta)$ where $o(\Delta)/\prod_{i=1}^{m}|\Delta_i| \to 0$ as $\Delta \to 0$.

Again, for the easy of notation, we write o_t instead of $o_{t,\prod_{i=1}^{m}\Delta_i}$.

We can also define cumulants for $N(t)$; and in particular, we define the limit covariance, for $u \neq v$, by

$$q_2(u,v) = \lim_{\Delta\to 0}\frac{\mathrm{Cov}\,(N,N)(\Delta_1\times\Delta_2)}{|\Delta_1||\Delta_2|}.$$

For finite point processes, Janossy densities and product moment densities are related (see Daley and Vere-Jones, 1988). The latter will be shown to coincide a.e. $[\ell]$ with joint s.o. intensities for some classes of processes.

We refer to Daley and Vere-Jones (1988) for further information on point processes.

2. Point Processes and Infinitesimals

In this section we prove some preliminary results, some of which will be used in section 3.

Proposition 2.1. *Under conditions (1) and (2), we have*

$$P\{N(\Delta)=1\} \le E\{N(\Delta)\} \le P\{N(\Delta)=1\} + O(|\Delta|^2),$$
$$P\{N(\Delta)=1\} - A \le Var\{N(\Delta)\} \le P\{N(\Delta)=1\} + B,$$

where A and B are $O(|\Delta|^2)$ whenever $\sup_{t\in\Delta} p_N(t)$ is finite.

Therefore we can write

$$E\{N(\Delta)\} = p_N(t)|\Delta| + o_t(|\Delta|)$$

and

$$Var\{N(\Delta)\} = p_N(t)|\Delta| + o_t(|\Delta|).$$

Proof For all Δ such that $|\Delta| < \min\{\delta, 1\}$ we have

$$E\{N(\Delta)\} = P\{N(\Delta)=1\} + \sum_{j\ge 2} jP\{N(\Delta)=j\} \le$$
$$\le P\{N(\Delta)=1\} + K_\delta \sum_{j\ge 2} j|\Delta|^j.$$

Since

$$\sum_{j\geq 2} j|\Delta|^j = \sum_{k\geq 2}\sum_{j\geq k} |\Delta|^j = \sum_{k\geq 2}(|\Delta|^k \sum_{j\geq 0} |\Delta|^j) =$$

$$\sum_{k\geq 2} |\Delta|^k(1/(1-|\Delta|)) = \frac{|\Delta|^2}{1-|\Delta|}\sum_{k\geq 0}|\Delta|^k = |\Delta|^2\left(\frac{1}{1-|\Delta|}\right)^2,$$

and $\lim_{|\Delta|\to 0}(1/(1-|\Delta|)^2) = 1$, it follows that

$$P\{N(\Delta)=1\} \leq E\{N(\Delta)\} \leq P\{N(\Delta)=1\} + O(|\Delta|^2).$$

Then,

$$E\{N(\Delta)\} = P\{N(\Delta)=1\} + o_t(|\Delta|).$$

From $P\{N(\Delta)=1\} = p_N(t)|\Delta| + o_t(|\Delta|)$, it follows that $E\{N(\Delta)\} = p_N(t)|\Delta| + o_t(|\Delta|)$ and the first part of the proof is complete. We notice that the latter $o_t(|\Delta|)$ which appears in the expression for the expectation of $N(\Delta)$ is bounded in modulus by an infinitesimal independent of t and Δ, since it is a sum of that related to the probability $P\{N(\Delta)=1\}$, which bears this property, and an infinitesimal independent of t and Δ, namely $O(|\Delta|^2)$.

Now, to estimate the variance, observe that

$$E\{N(\Delta)^2\} = P\{N(\Delta)=1\} + \sum_{j\geq 2} j^2 P\{N(\Delta)=j\}$$

$$\leq P\{N(\Delta)=1\} + K_\delta \sum_{j\geq 2} j^2|\Delta|^j.$$

Let $f(z)$ be the analytic function over the disc $|z|<1$, defined by $f(z) = \sum_{n\geq 2} z^{n+2}$. Thus, $f(z) = z^4/(1-z)$ and

$$\frac{d^2}{dz^2}f(z) = \frac{12z^2}{(1-z)} + \frac{2(4-3z)z^3}{(1-z)^3}.$$

Moreover,

$$\frac{d^2}{dz^2}f(z) = \frac{d^2}{dz^2}\sum_{n\geq 2} z^{n+2} = \sum_{n\geq 2}\frac{d^2}{dz^2}z^{n+2} = \sum_{n\geq 2}(n+2)(n+1)z^n.$$

Therefore, for $0\leq z<1$ get

$$\frac{d^2}{dz^2}f(z) = \sum_{n\geq 2}(n+2)(n+1)z^n \geq \sum_{n\geq 2} n^2 z^n,$$

and write

$$E\{N(\Delta)^2\} \leq P\{N(\Delta)=1\} + K_\delta \frac{d^2}{dz^2}f(z)|_{z=|\Delta|} =$$

$$= P\{N(\Delta)=1\} + \frac{12|\Delta|^2}{1-|\Delta|} + \frac{2(4-3|\Delta|)|\Delta|^3}{(1-|\Delta|)^3}.$$

Since, when $|\Delta|\to 0$, $12/(1-|\Delta|)\to 12$ and $2(4-3|\Delta|)/(1-|\Delta|)^3 \to 8$, we obtain

$$E\{N(\Delta)^2\} \leq P\{N(\Delta)=1\} + O(|\Delta|^2) + O(|\Delta|^3).$$

Then

$$\begin{aligned}\mathrm{Var}\{N(\Delta)\} &\leq P\{N(\Delta)=1\} + O(|\Delta|^2) + O(|\Delta|^3) - (p_N(t)|\Delta| + o_t(|\Delta|))^2 \\ &\leq P\{N(\Delta)=1\} + O(|\Delta|^2).\end{aligned}$$

It is also true that

$$\begin{aligned}\mathrm{Var}\{N(\Delta)\} &= E\{N(\Delta)^2\} - (E\{N(\Delta)\})^2 \geq P\{N(\Delta)=1\} - (p_N(t)|\Delta| + o_t(|\Delta|))^2 \\ &\geq P\{N(\Delta)=1\} - (\sup_{t\in\Delta} p_N(t)|\Delta| + o(|\Delta|))^2 = P\{N(\Delta)=1\} - O(|\Delta|^2).\end{aligned}$$

Combining the two inequalities we have proved the second part of the Proposition, and immediately,

$$\mathrm{Var}\{N(\Delta)\} = p_N(t)|\Delta| + o_t(|\Delta|).$$

■

These $o_t = o_{t,\Delta}$ may depend on t and Δ but their absolute values are bounded by other o's which are independent of t.

Proposition 2.2. *Under the hypotheses (4) and (5) we have, for $m \geq 1$,*

$$\begin{aligned}&P\{N(\Delta_i)=1,\ 1\leq i\leq m\} \leq E\left\{\prod_{i=1}^m N(\Delta_i)\right\} \\ &\leq P\{N(\Delta_i)=1,\ 1\leq i\leq m\} + k_{\delta,m}\left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\}\prod_{i=1}^m |\Delta_i|.\end{aligned}$$

Proof Let $n=(n_1,\ldots,n_m)\in \mathbb{N}^m$, $\mathbb{N}^* = \mathbb{N} - \{0\}$. Then

$$\begin{aligned}E\left\{\prod_{i=1}^m N(\Delta_i)\right\} &= \sum_{n\in(\mathbb{N}^*)^m}\left((\prod_{i=1}^m n_i)P\{N(\Delta_i)=n_i, 1\leq i\leq m\}\right) \\ &\leq P\{N(\Delta_i)=1, 1\leq i\leq m\} + \sum_{n\in(\mathbb{N}^*)^m-\{(1,\ldots,1)\}}\left(\prod_{i=1}^m n_i\right)k_{\delta,m}\prod_{i=1}^m |\Delta_i|^{n_i} \\ &= P\{N(\Delta_i)=1, 1\leq i\leq m\} + \sum_{n\in(\mathbb{N}^*)^m} k_{\delta,m}\prod_{i=1}^m n_i|\Delta_i|^{n_i} - k_{\delta,m}\prod_{i=1}^m |\Delta_i|.\end{aligned}$$

Now,

$$\sum_{n\in(\mathbb{N}^*)^m}\prod_{i=1}^m n_i|\Delta_i|^{n_i} = \prod_{i=1}^m\left(\sum_{n_i\in\mathbb{N}^*} n_i|\Delta_i|^{n_i}\right) = \prod_{i=1}^m\left(|\Delta_i|\left(\frac{1}{1-|\Delta_i|}\right)^2\right).$$

Therefore,

$$E\left\{\prod_{i=1}^m N(\Delta_i)\right\} \leq P\{N(\Delta_i)=1, 1\leq i\leq m\} + k_{\delta,m}\prod_{i=1}^m |\Delta_i|\left(\prod_{i=1}^m\left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right)$$

■

Proposition 2.3. *If N satisfies (4) and (5) then there is an infinitesimal $o_t^*(\Delta) = o_{t,\prod_{i=1}^m \Delta_i}^*(\Delta)$ such that*

$$E\left(\prod_{i=1}^m N(\Delta_i)\right) = p_m(t)\prod_{i=1}^m |\Delta_i| + o_t^*(\Delta),$$

and for which $o^* = \sup_{t,\prod \Delta_i} o_{t,\prod \Delta_i}^*$ satisfies $\dfrac{o^*(\Delta)}{\prod_{i=1}^m |\Delta_i|} \to 0$ when $\Delta \to 0$.

Proof We will use the notation $p_N = p_1$, $k_\delta = k_{\delta,1}$, $t = (t_1, \dots, t_m)$, $\Delta = (|\Delta_1|, \dots, |\Delta_m|)$.

For all $m \geq 1$, by Proposition (2.2)

$$P\{N(\Delta_i) = 1,\ 1 \leq i \leq m\} \leq E\left(\prod_{i=1}^m N(\Delta_i)\right)$$

$$\leq P\{N(\Delta_i) = 1,\ 1 \leq i \leq m\} + K_{\delta,m}\prod_{i=1}^m |\Delta_i| \left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\}.$$

From (6),

$$E\left(\prod_{i=1}^m N(\Delta_i)\right) \leq p_m(t)\prod_{i=1}^m |\Delta_i| + o_t(\Delta) + K_{\delta,m}\prod_{i=1}^m |\Delta_i| \left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\}.$$

Now,

$$\sup_{t\in\mathbb{R}^m - \mathcal{E}^m} \left| o_t(\Delta) + K_{\delta,m}\prod_{i=1}^m |\Delta_i| \left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\}\right|$$

$$\leq o(\Delta) + K_{\delta,m}\prod_{i=1}^m |\Delta_i| \left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\} = o^1(\Delta).$$

$$\lim_{\Delta\to 0}\frac{o^1(\Delta)}{\prod_{i=1}^m |\Delta_i|} = \lim_{\Delta\to 0}\frac{o(\Delta)}{\prod_{i=1}^m |\Delta_i|} + K_{\delta,m}\lim_{\Delta\to 0}\left\{\prod_{i=1}^m \left(\frac{1}{1-|\Delta_i|}\right)^2 - 1\right\} = 0.$$

Thus, there is an infinitesimal $o_t^*(\Delta) = o_{t,\prod_{i=1}^m \Delta_i}^*$ such that

$$E\left(\prod_{i=1}^m N(\Delta_i)\right) = p_m(t)\prod_{i=1}^m |\Delta_i| + o_t^*(\Delta),$$

with $o^* = \sup_{t,\prod \Delta_i} o_{t,\prod \Delta_i}$ that satisfies $\dfrac{o^*(\Delta)}{\prod_{i=1}^m |\Delta_i|} \to 0$ when $\Delta \to 0$. ∎

From now on we understand the joint s.o. intensity p_m, $m \geq 1$, as being the function that associates to each point $t \in \mathbb{R}^m - \mathcal{E}^m$, for which the joint intensity defining limit exists, the value of this limit, that is, $p_m(t)$ as given by equation (5) without necessity of this limit being uniform.

Theorem 2.1. *Let* $\mathcal{E}^m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m | x_i = x_j$ *for some pair* $i, j, i \neq j\}$, φ *an* $E\left(\prod_{i=1}^{m} \mathrm{d}N(t_i)\right)$*-integrable function over* $\mathbb{R}^m - \mathcal{E}^m$, p_m *the* m*-th order joint s.o. intensity and* $p_1 = p_N$ *the s.o. intensity function of point process* N *that satisfies (7). Then, if* $p_m \in \overline{\mathcal{L}}^m$, $m \geq 1$, *we have*

$$\int_{\mathbb{R}^m - \mathcal{E}^m} \varphi E\left(\prod_{i=1}^{m} \mathrm{d}N(t_i)\right) = \int_{\mathbb{R}^m - \mathcal{E}^m} \varphi p_m \prod_{i=1}^{m} \mathrm{d}t_i \ .$$

We observe that this theorem shows that if the s.o. intensity function or the joint intensity p_m is a.e.[ℓ] defined and it is Lebesgue-integrable over bounded $\mathbb{R}^m$-intervals, then it is also the Radon-Nikodym derivative of $E\left(\prod_{i=1}^{m} N\right)$ with respect to ℓ.

Proof We remind the following results. (See Rudin, 1981). Let μ be a real Borel measure on $\mathbb{R}^m$, $D\mu(x) = \lim_{n\to\infty} \mu(E_n)/\ell(E_n)$ where $\{E_n\}_{n\in\mathbb{N}}$ is any sequence of Borel sets that shrink nicely to x, $C_{n,x}$ the set of all open cubes $C \in \mathbb{R}^m$, $x \in C$, whose edges are parallel to the coordinate axes and have length less than $1/n$. We have:

(i) If $\mu \geq 0$ and for all $x \in \mathbb{R}^m$ $\lim_{n\to\infty} \sup_{C\in C_{n,x}} (\mu(C)/\ell(C)) < \infty$ then $\mu \ll \ell$

(ii) $\mu \ll \ell$ if and only if $D\mu(x)$ exists a.e.[ℓ] and $\forall E \in \mathcal{B}_{\mathbb{R}^m}$ $\mu(E) = \int_E (D\mu)\mathrm{d}\ell$. In this case, $D\mu = \mathrm{d}\mu/\mathrm{d}\ell$ a.e.[ℓ].

Let us apply these results for $\mu = E\prod_{i=1}^{m} N$. The existence of $p_m(x)$, $p_m(x) \in \mathbb{R}$, for all $x \in \mathbb{R}$, implies that $\lim_{n\to\infty} \sup_{C\in C_{n,x}} (\mu(C)/\ell(C)) < \infty$. Since $E\prod_{i=1}^{m} N \geq 0$, from (i) we have $E\prod_{i=1}^{m} N \ll \ell$. On the other hand, $E\prod_{i=1}^{m} N \ll \ell$ implies the existence of $\mathrm{d}E\prod_{i=1}^{m} N/\mathrm{d}\ell$ and (ii) guarantees not only the existence of $D\mu$ but also $D\mu = \mathrm{d}E\prod_{i=1}^{m} N/\mathrm{d}\ell$ a.e.[ℓ].

Now, a.e.[ℓ], if we choose a sequence of $\mathbb{R}^m$ intervals $\{(\prod_{i=1}^{m} \Delta_i)_j\}_{j\in\mathbb{N}}$ that shrinks nicely to x, we have $D\mu(x) = \lim_{j\to\infty} \frac{E\prod_{i=1}^{m} N((\prod_{i=1}^{m}\Delta_i)_j)}{\ell((\prod_{i=1}^{m}\Delta_i)_j)} = p_m(x)$. In this way we finally obtain $p_m = \mathrm{d}E\prod_{i=1}^{m} N/\mathrm{d}\ell$ a.e.[ℓ]. ∎

3. Characterization of the intensity

Initially we observe that, as a consequence of Theorem 2.1, if the limit that defines the joint intensity p_m ($m \geq 1$) exists and if the joint intesity function (or s.o. intensity) belongs to $\overline{\mathcal{L}}^m$, then it is the Radon-Nikokym derivative of $E\prod_{i=1}^{m} N$, that is, $p_m = \mathrm{d}E\prod_{i=1}^{m} N/\mathrm{d}\ell$ a.e.[ℓ]. An important question is if there exists some kind of reciprocal statement, that is, under what conditions could we say that if the Radon-Nikodym derivative exists, then it will be the joint intensity p_m. This question leads us to propose Theorems 3.1, 3.2, 3.3 and 3.4 as well as Corollaries 3.1, 3.2 and 3.3. We recall Lebesgue's differentiation theorems for $\mathbb{R}$ and $\mathbb{R}^m$.

Proposition 3.1 (Lebesgue's differentiation theorem for $\mathbb{R}$). *Let f be Lebesgue integrable on $\mathbb{R}$ and let $\varphi(x) = \int_{-\infty}^{x} f(y)\mathrm{d}y$. Then we have $\frac{\mathrm{d}\varphi}{\mathrm{d}x} = f$ a.e.[ℓ].*

Proposition 3.2 (Lebesgue's differentiation theorem for $\mathbb{R}^m$). *Let f be Lebesgue integrable on $\mathbb{R}^m$. Let $Q(x,r)$ be a hypercube with center x and edge $2r > 0$. Then*

$$\lim_{r\to 0} \frac{1}{\ell(Q(x,r))} \int_{Q(x,r)} f(y)\mathrm{d}y = f(x) \quad a.e.[\ell].$$

If we substitute $B(x,r)$ for $Q(x,r)$ we still obtain a theorem but, if we substitute an arbitrary $\mathbb{R}^m$-interval that contains x, $R(x)$ for $Q(x,r)$ and take the limit as the diameters tend to zero we will not obtain a true statement (See Fernandez, 1976).

Definition 1. *We say that a measurable function $f : A \subset \mathbb{R}^m \to \mathbb{R}$, $\ell(\mathbb{R}^m - A) = 0$ essentially belongs to $\mathcal{L}^m$, equivalently, f is essentially bounded over bounded intervals, when there is a set D, $\ell(D) = 0$ such that the function $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$. defined by $\tilde{f}|_{\mathbb{R}^m-D} = f|_{\mathbb{R}^m-D}$ and $\tilde{f}|_D = 0$. belongs to $\mathcal{L}^m$. We will denote such fact by $f \in \operatorname{ess}\mathcal{L}^m$ and we will call $\operatorname{ess}\mathcal{L}^m$ the set of essentially bounded functions over bounded intervals.*

Definition 2. *We will say that the function $f : A \subset \mathbb{R}^m \to \mathbb{R}$ is essentially continuous on A when there is a set D, $\ell(D) = 0$ such that $f|_{A-D}$ is a continuous function. The set of essentially continuous functions will be called $\operatorname{ess}\mathcal{C}^m$.*

For example, the function $f : \mathbb{R}^m \to \mathbb{R}$, $f(\mathbb{Q}^m) = \{1\}$, $f(\mathbb{R}^m - \mathbb{Q}^m) = \{0\}$, is a function that is not continuous at any point; for all A, set with non-void interior, it is not Riemann-integrable over this set since its discontinuity points contains the interior of A, which being non-void and open, has measure greater than zero. Meanwhile this function is essentially continuous since $\ell(\mathbb{Q}^m) = 0$ and over $\mathbb{R}^m - \mathbb{Q}^m$ it is identically 0. It is also Lebesgue-integrable over all measurable A.

Theorem 3.1 below gives us a "characterization" of the s.o. intensity $p_N = p_1$ understood as its defining limit, without the necessity of uniformity, for point processes that satisfy (1) or more generally (3). We have written the quotation marks because the characterization is still dependent on some hypothesis that N must fulfill.

Theorem 3.1. *(***Characterization of the s.o. intensity***). Let N be a point processes that satisfy (3) and EN be its expectation measure. Then we have the equivalences:*

(i) *The s.o. intensity function p_N exists and is a function in $\overline{\mathcal{L}}^1$ if and only if the Radon-Nikodym derivative $\mathrm{d}EN/\mathrm{d}\ell$ exists. In this situation we have $p_N = \mathrm{d}EN/\mathrm{d}\ell$ a.e. [ℓ].*
(ii) *$p_N \in \operatorname{ess}\mathcal{L}^1$ if and only if $\mathrm{d}EN/\mathrm{d}\ell \in \operatorname{ess}\mathcal{L}^1$.*

Proof (i) By Theorem 2.1, $\exists p_N \wedge p_N \in \overline{\mathcal{L}}^1 \to \exists \frac{\mathrm{d}EN}{\mathrm{d}\ell} = p_N$ a.e.[ℓ].

For all $t \in \mathbb{R}$, using (3), we compute the defining limit $p_N(t)$:

$$p_N(t) = \lim_{\substack{t\in\Delta\\|\Delta|\to 0}} \frac{P\{N(\Delta)=1\}}{|\Delta|} = \lim_{\substack{t\in\Delta\\|\Delta|\to 0}} \frac{EN(\Delta) - o_t(|\Delta|)}{|\Delta|} = \lim_{\substack{t\in\Delta\\|\Delta|\to 0}} \frac{EN(\Delta)}{|\Delta|},$$

observe that, for all N satisfying (1), we have

$$P\{N(\Delta)=1\} \le EN(\Delta) \le P\{N(\Delta)=1\} + \sum_{j\ge 2} j|\Delta|^j = P\{N(\Delta)=1\} + O(|\Delta|^2)$$

and so $EN(\Delta) = P\{N(\Delta)=1\} + o_t(|\Delta|)$.

Let $f = \frac{\mathrm{d}EN}{\mathrm{d}\ell}$, $\varphi(x) = \int_c^x f(y)\mathrm{d}y$, $\Delta = |a,b|$, $a < b$, $h_1 = b - t$ and $h_2 = t - a$. Thus,

$$p_N(t) = \lim_{\substack{h_1\to 0\\h_2\to 0}} \frac{\varphi(t+h_1) - \varphi(t-h_2)}{h_1+h_2}.$$

Now,

$$\begin{aligned}\frac{\varphi(t+h_1)-\varphi(t-h_2)}{h_1+h_2} &= \frac{\varphi(t+h_1)-\varphi(t)}{h_1}\frac{h_1}{h_1+h_2} + \frac{\varphi(t-h_2)-\varphi(t)}{-h_2}\frac{h_2}{h_1+h_2}\\ &= (f(t)+o_t(h_1))\frac{h_1}{h_1+h_2} + (f(t)+o_t(-h_2))\frac{h_2}{h_1+h_2}\\ &= f(t) + \left(o_t(h_1)\frac{h_1}{h_1+h_2} + o_t(-h_2)\frac{h_2}{h_1+h_2}\right),\end{aligned}$$

where, by Lebesgue differentiation theorem, o_t is an infinitesimal a.e.[ℓ] (this means that the set of t's such that o_t is not an infinitesimal has zero Lebesgue measure).

Since $0 \le \frac{h_1}{h_1+h_2} \le 1$ and $0 \le \frac{h_2}{h_1+h_2} \le 1$, we have

$$\lim_{\substack{h_1\to 0\\h_2\to 0}} \frac{\varphi(t+h_1)-\varphi(t-h_2)}{h_1+h_2} = f(t) + 0 \;\text{ a.e.}[\ell].$$

Thus, $p_N(t) = \frac{\mathrm{d}EN}{\mathrm{d}\ell}$ a.e.[ℓ].

(ii) If $\frac{\mathrm{d}EN}{\mathrm{d}\ell} \in \operatorname{ess}\mathcal{L}^1$, then there is D_1, $\ell(D_1) = 0$ such that for the set $\mathbb{R} - D_1$ we have that $\frac{\mathrm{d}EN}{\mathrm{d}\ell}$ is bounded over bounded sets in $\mathbb{R} - D_1$. Since $\frac{\mathrm{d}EN}{\mathrm{d}\ell} = p_N$ a.e.[ℓ], $\exists D_2$, $\ell(D_2) = 0$ such that $\frac{\mathrm{d}EN}{\mathrm{d}\ell} = p_N$ over $(\mathbb{R} - D_1) - D_2$.

Let $D = D_1 \cup D_2$, $\ell(D) = 0$.

Thus, $\tilde{p}_N$ defined by $\tilde{p}_N|_{\mathbb{R}-D} = \frac{\mathrm{d}EN}{\mathrm{d}\ell}|_{\mathbb{R}-D}$ and $\tilde{p}_N|_D = 0$ is such that $\tilde{p}_N \in \mathcal{L}^1$, since $\frac{\mathrm{d}EN}{\mathrm{d}\ell}|_{\mathbb{R}-D}$ is integrable and bounded over bounded intervals. Thus, $p_N \in \operatorname{ess}\mathcal{L}^1$.

Analogously $(\frac{\mathrm{d}EN}{\mathrm{d}\ell} = p_N \text{ a.e.}[\ell] \wedge p_N \in \operatorname{ess}\mathcal{L}^1) \to \frac{\mathrm{d}EN}{\mathrm{d}\ell} \in \operatorname{ess}\mathcal{L}^1$. ∎

Note that ess$\mathcal{C}^1$ is not contained in ess$\mathcal{L}^1$ since $f(x) = 1/x$ for $x \neq 0$, $f(0) = 0$ is such that $f \in$ ess$\mathcal{C}^1$ and $f \notin$ ess$\mathcal{L}^1$.

Since all point processes that we consider have boundedly finite expectation measure, we have the following

Corollary 3.1. (Characterization of the intensity). *Let N be a point processes that satisfy (3) and EN be its expectation measure. Then we have the equivalences:*

(i) *The intensity function i_{N+} exists and is a function in $\overline{\mathcal{L}}^1$ if and only if the Radon-Nikodym derivative $\mathrm{d}EN/\mathrm{d}\ell$ exists. In this situation we have $i_{N+} = \mathrm{d}EN/\mathrm{d}\ell$ a.e. [ℓ].*

(ii) *$i_{N+} \in \operatorname{ess}\mathcal{L}^1$ if and only if $\mathrm{d}EN/\mathrm{d}\ell \in \operatorname{ess}\mathcal{L}^1$.*

Proof Direct consequence of Proposition 1.1 and Theorem 3.1. ∎

For point processes that do not satisfy (3) it may occur that $p_N(t)$ and $i_{N+}(t)$ are different and both differ from the Radon-Nikodym derivative of the expectation measure. As an example, let N be a homogeneous Poisson process with intensity $\lambda > 0$ and then form the new process $*N$ defined by, $\forall \omega \in \Omega, \forall t \in \mathbb{R}$, $*N_\omega(\{t\}) = 2N_\omega(\{t\})$. For this process we have, for all $t \in \mathbb{R}$,

$$\lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{*N(\Delta) = 1\}}{|\Delta|} = 0 < \lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{*N(\Delta) \geq 1\}}{|\Delta|} =$$

$$= \lim_{|\Delta|\to 0, t\in\Delta} \frac{P\{N(\Delta) \geq 1\}}{|\Delta|} = \lambda < \lim_{|\Delta|\to 0, t\in\Delta} \frac{E(*N(\Delta))}{|\Delta|}$$

$$= \lim_{|\Delta|\to 0, t\in\Delta} \frac{2E(N(\Delta))}{|\Delta|} = 2\lambda,$$

that is, for all $t \in \mathbb{R}$,

$$0 = p_{*N}(t) < i_{*N+}(t) = \lambda < \frac{dE(*N)}{d\ell} = 2\lambda.$$

This example shows the opposite situation to that of point processes under (3), namely, the single occurrence intensity, the intensity and the Radon-Nikodym derivative of the expectation measure are *nowhere* equal.

Theorem 3.2. *Let N be a point process that satisfy*

$$E\left(\prod_{i=1}^{m} N(\Delta_i)\right) = p_m(t) \prod_{i=1}^{m} |\Delta_i| + o_{t,\prod \Delta_i}(\Delta),$$

with $o^* = \sup_{t,\prod \Delta_i} o_{t,\prod \Delta_i}$ that satisfies $\dfrac{o^*(\Delta)}{\prod_{i=1}^{m} |\Delta_i|} \to 0$ when $\Delta \to 0$. Then we have

(i) If $p_m \in \mathcal{L}^m$ then $p_m \in \mathcal{R}^m$, for all $m \geq 1$.

(ii) If $p_1 \in \overline{\mathcal{L}^1}$ then $p_1 \in C(\mathbb{R}, \mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Proof See de Miranda (2003), Theorems 3.1 and 3.2.

(i) The measure $\mu = E \prod_{i=1}^{m} N$ is under the hypothesis of Theorem 3.1 where $h = p_m$ and $o_{a,b} = o_{t, \prod_{i=1}^{m} |\Delta_i|}$, so that we conclude $p_m \in \mathcal{R}^m$.

(ii) If $m = 1$ we can also apply Theorem 3.2 and conclude $p_1 \in C(\mathbb{R}, \mathbb{R})$. ■

Definition 3. *We will say that the function $f : A \subset \mathbb{R}^m \to \mathbb{R}$ is essentially Riemann integrable on A when there is a set D, $\ell(D) = 0$ and a function $c : D \to \mathbb{R}$ such that $\tilde{f} : A \to \mathbb{R}$, defined by $\tilde{f}(x) = f(x)$, for all $x \in A - D$ and $\tilde{f}(x) = c(x)$ for $x \in D$, is Riemann integrable on A. The set of essentially Riemann integrable functions will be called* ess$\mathcal{R}^m$.

Corollary 3.2. *Under the hypothesis of Theorem 3.2 we have:*
(i)*If $p_m \in \mathcal{L}^m$ then $(dE\prod_{i=1}^m N)/d\ell \in$ ess$\mathcal{R}^m$.*
(ii) *If $p_1 \in \overline{\mathcal{L}^1}$ (equivalently if $\frac{dEN}{d\ell} \in \overline{\mathcal{L}^1}$) then $\frac{dEN}{d\ell} \in$ ess$C(\mathbb{R}, \mathbb{R})$.*

Proof Direct consequence of Theorems 2.1 and 3.2. ■

Theorem 3.3. *Let N be a point process that satisfies (4) or, in a more general way, that satisfies*

$$P\{N(\Delta_i) = 1,\ 1 \leq i \leq m\} = E\prod_{i=1}^m N\left(\prod_{i=1}^m \Delta_i\right) - o_t\left(\prod_{i=1}^m |\Delta_i|\right),\ \text{i.e. (7)}.$$

Let $f = \mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell$ be the Radon-Nikodym derivative of $E\prod_{i=1}^m N$. If f is essentially continuous on $\mathbb{R}^m$, that is, $\exists D \subset \mathbb{R}^m$, $\ell(D) = 0$, such that $f|_{\mathbb{R}^m - D}$ is continuous, then there exists $p_m : \mathbb{R}^m - D \to \mathbb{R}$, m-th order joint s.o. intensity of N, and $p_m = f$ over $\mathbb{R}^m - D$. So, $p_m \in$ ess$\mathcal{C}^m$.

Proof Let $t \in (\mathbb{R}^m - \mathcal{E}^m) - D$.

$$\begin{aligned} p_m(t) &= \lim_{\substack{t\in\prod_{i=1}^m \Delta_i \\ |\Delta_i|\to 0 \\ 1\leq i\leq m}} \frac{P\{N(\Delta_i) = 1, 1 \leq i \leq m\}}{\prod_{i=1}^m |\Delta_i|} \\ &= \lim_{\substack{t\in\prod_{i=1}^m \Delta_i \\ |\Delta_i|\to 0 \\ 1\leq i\leq m}} \frac{E\prod_{i=1}^m N\left(\prod_{i=1}^m \Delta_i\right)}{\prod_{i=1}^m |\Delta_i|} = \lim_{\substack{t\in\prod_{i=1}^m \Delta_i \\ |\Delta_i|\to 0 \\ 1\leq i\leq m}} \frac{\int_{\prod_{i=1}^m \Delta_i - D} f\mathrm{d}\ell}{\prod_{i=1}^m |\Delta_i|}. \end{aligned}$$

Since f is essentially continuous,

$$\forall t \in \mathbb{R}^m - D\ \ \forall \varepsilon > 0\ \ \exists \delta > 0\ \ \forall x \in \mathbb{R}^m - D\ \ \|x - t\| < \delta \to \|f(x) - f(t)\| < \varepsilon.$$

Thus, when diam $(\prod_{i=1}^m \Delta_i) < \delta$,

$$(f(t) - \varepsilon)\ell\left(\prod_{i=1}^m \Delta_i - D\right) \leq \int_{\prod_{i=1}^m \Delta_i - D} f\mathrm{d}\ell \leq (f(t) + \varepsilon)\ell\left(\prod_{i=1}^m \Delta_i - D\right).$$

Then,

$$\lim_{\substack{t\in\prod_{i=1}^m \Delta_i \\ |\Delta_i|\to 0 \\ 1\leq i\leq m}} \frac{\int_{\prod_{i=1}^m \Delta_i - D} f\mathrm{d}\ell}{\ell\left(\prod_{i=1}^m \Delta_i - D\right)} = f(t)$$

and so, $p_N(t) = f(t)$ ■

Lemma 3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be essentially Riemann- integrable on bounded intervals of $\mathbb{R}^m$. Then f is essentially continuous on $\mathbb{R}^m$.*

Proof Let $z = (z_1, \dots, z_m) \in \mathbb{R}^m$, $1 = (1, \dots, 1) \in \mathbb{Z}^m$, $\prod_{i=1}^m (z_i, z_{i+1}) = (z, z+1)$ and write $\mathbb{R}^m = (\cup_{z\in\mathbb{Z}^m}(z, z+1)) \cup (\cup_{z\in\mathbb{Z}^m}\partial(z, z+1))$. Clearly, $\ell(\cup_{z\in\mathbb{Z}^m}\partial(z, z+1)) = 0$.

Since f is essentially Riemann-integrable on each $(z, z+1)$ we have that $\forall z \in \mathbb{Z}^m \exists \tilde{f}_z : (z, z+1) \to \mathbb{R}$, $\tilde{f}_z$ a Riemann-integrable function, $\exists D_z$, $\ell(D_z) = 0$, $f|_{(z,z+1)-D_z} = \tilde{f}_z|_{(z,z+1)-D_z}$.

Now, for all $z \in \mathbb{Z}^m$, since $\tilde{f}_z$ is Riemann-integrable, the set formed by its descontinuity points, $\tilde{D}_z$, is such that $\ell(\tilde{D}_z) = 0$. So

$$f|_{(z,z+1)-(D_z\cup\tilde{D}_z)} = \tilde{f}_z|_{(z,z+1)-(D_z\cup\tilde{D}_z)}$$

is continuous and, consequently, the function

$$f|_{(\mathbb{R}^m-\cup_{z\in\mathbb{Z}^m}(D_z\cup\tilde{D}_z))-\cup_{z\in\mathbb{Z}^m}\partial(z,z+1)}$$

is continuous, i.e., f is essentially continuous.

■

Corollary 3.3. *Let N be a point process that satisfy (7). If $f = d(E\prod_{i=1}^m N)/d\ell$ is essentially Riemann-integrable on bounded intervals of $\mathbb{R}^m$ then there exists $D \subset \mathbb{R}^m$, $\ell(D) = 0$, $p_m = f$ on $\mathbb{R}^m - D$.*

Proof Direct consequence of Lemma 3.1 and Theorem 3.3. ■

Once again, we observe that for a point process that satisfies (3) or (7), for $m \geq 1$, if the limit for p_m exists and $p_m \in \overline{\mathcal{L}}^m$, (if this limit is uniform and $p_m \in \mathcal{L}^m$ we directly have $p_m \in \mathcal{R}^m$) then p_m is the Radon-Nikodym derivative of $E\prod_{i=1}^m N$, $p_m = \mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell$ a.e.$[\ell]$. On the other hand, for $m > 1$, for a point process that satisfies (7), if the derivative $\mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell$ exists, it is not clear that p_m will also exist. We can not use Lebesgue differentiation theorem for $m \geq 2$ as we have done for $m = 1$ on Theorem 3.1 because the p_m limit to be calculated is a limit that uses $\mathbb{R}^m$-intervals that contain x. Thus, in case that $m \geq 2$, the veracity of the searched reciprocal proposition will depend on the form of the Radon-Nikodym derivative. Theorem 3.3 shows that for the essentially continuous functions class the referred reciprocal holds for any point process that satisfy (7). We observe that it is crucial, in case $m \geq 2$, to have the existence of p_m assured. Thus, we propose the following theorem:

Theorem 3.4. *If a point process satisfies to condition (7) and p_m, the joint s.o. intensity function* **exists**, *then $\mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell$ exists and $p_m = \mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell$ a.e.$[\ell]$. In particular, $\mathrm{d}E\prod_{i=1}^m N/\mathrm{d}\ell \in \mathrm{ess}\,\mathcal{L}^m$ if and only if $p_m \in \mathrm{ess}\,\mathcal{L}^m$.*

Observe that we can have $dE\prod_{i=1}^{n} N/d\ell \in (\overline{\mathcal{L}}^m - \text{ess}\,\mathcal{L}^m)$, situation under which the same occurs to p_m and in if $p_m \in \mathcal{L}^m$, under uniformity condition, we directly have $p_m \in \mathcal{R}^m$.

Proof By hypothesis, the existence of $p_m : \mathbb{R}^m - \mathcal{E}^m \to \mathbb{R}$ is already assured, that is, the defining limit of p_m exists for all points t of $\mathbb{R}^m - \mathcal{E}^m$. The existence of $dE\prod_{i=1}^{n} N/d\ell$ is guaranteed by Theorem 2.1. Since the p_m limit exists, we can calculate it through any path. We will calculate it using hypercubes with center t, since for such path we can use the Lebesgue's differentiation theorem for $\mathbb{R}^m$.

Then,

$$\begin{aligned}
p_m(t) &= \lim_{\substack{t\in\prod_{i=1}^{n}\Delta_i \\ |\Delta_i|\to 0 \\ 1\le i\le m}} \frac{P\{N(\Delta_i)=1,\ 1\le i\le m\}}{\prod_{i=1}^{m}|\Delta_i|} = \lim_{\substack{t\in\prod_{i=1}^{n}\Delta_i \\ |\Delta_i|\to 0 \\ 1\le i\le m}} \frac{E\prod_{i=1}^{m} N\left(\prod_{i=1}^{m}\Delta_i\right)}{\prod_{i=1}^{m}|\Delta_i|} \\
&= \lim_{\substack{t\in Q(t,r) \\ r\to 0}} \frac{E\prod_{i=1}^{m} N(Q(t,r))}{\ell(Q(t,r))} = \lim_{\substack{t\in Q(t,r) \\ r\to 0}} \frac{\int_{Q(t,r)}(dE\prod_{i=1}^{n} N/d\ell)d\ell}{\ell(Q(t,r))} \\
&= \frac{dE\prod_{i=1}^{m} N}{d\ell}(t) \ \text{ a.e.}[\ell].
\end{aligned}$$

■

4. Conclusion

In this work we obtained the following results. For a point process that satisfies (3) or (7), for $m \geq 1$, if the limit for p_m exists, $p_m \in \overline{\mathcal{L}}^m$, then p_m is the Radon-Nikodym derivative of $E\prod_{i=1}^{m} N$ i.e., $p_m = dE\prod_{i=1}^{m} N/d\ell$ a.e.$[\ell]$. Moreover if the uniform limit for p_m exists and $p_m \in \mathcal{L}^m$ we directly have $p_m \in \mathcal{R}^m$ and for $m = 1$ in addition we have that $p_N = p_1$ is continuous.

Also, for point process that satisfy (7), if the Radon-Nikodym derivative of $E\prod_{i=1}^{m} N$ exists and it is essentially continuous, then p_m also exists and both are equal a.e.$[\ell]$.

If the point process satisfies (7) and the **defining limit of p_m exists**, then the Radon-Nikodym derivative of $E\prod_{i=1}^{m} N$, $dE\prod_{i=1}^{m} N/d\ell$ exists and we have $p_m = dE\prod_{i=1}^{m} N/d\ell$ a.e.$[\ell]$. Note that we don't have necessarily $p_m \in \mathcal{L}^m$, or better, $p_m \in \mathcal{R}^m$ since we may not have the uniformity of the limit p_m.

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