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STABILITY ANALYSIS WITH APPLICATIONS OF A TWO-DIMENSIONAL DYNAMICAL SYSTEM ARISING FROM A STOCHASTIC MODEL FOR AN ASSET MARKET

by

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Stability analysis with applications of a two-dimensional dynamical system arising from a stochastic model for an asset market

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Abstract

We analyze the stability properties of equilibrium solutions and periodicity of orbits in a two-dimensional dynamical system whose orbits mimic the evolution of the price of an asset and the excess demand for that asset. The construction of the system is grounded upon a heterogeneous interacting agent model for a single risky asset market. An advantage of this construction procedure is that the resulting dynamical system becomes a macroscopic market model which mirrors the market quantities and qualities that would typically be taken into account solely at the microscopic level of modeling. The system's parameters correspond to: (a) the proportion of speculators in a market; (b) the traders' speculative trend; (c) the degree of heterogeneity of idiosyncratic evaluations of the market agents with respect to the asset's fundamental value; and (d) the strength of the feedback of the population excess demand on the asset price update increment. This correspondence allows us to employ our results in order to infer plausible causes for the emergence of price and demand fluctuations in a real asset market.

The employment of dynamical systems for studying evolution of stochastic models of socio-economic phenomena is quite usual in the area of heterogeneous interacting agent models. However, in the vast majority of the cases present in the literature, these dynamical systems are one-dimensional. Our work is among the few in the area that construct and study two-dimensional dynamical systems and apply them for explanation of socio-economic phenomena.

Key words and phrases: two-dimensional dynamical system, attractors, stability, omega-limit, periodic orbits, heterogeneous interacting agent model, a single risky asset market model, convergence and oscillation of market asset price and demand.

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1 Introduction

We investigate equilibria and stability properties of the discrete time dynamical system generated by planar map Ψ in the following manner

$$\begin{pmatrix} p_n \\ d_n \end{pmatrix} = \Psi \begin{pmatrix} p_{n-1} \\ d_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} + \lambda d_{n-1} \\ \alpha\big[1 - 2\Phi\big(p_{n-1} + (\lambda - J)d_{n-1}\big)\big] \\ \qquad + (1-\alpha)\big[1 - 2\Phi\big(p_{n-1} + \lambda d_{n-1}\big)\big] \end{pmatrix}. \tag{1}$$

In the dynamical system (1), $\alpha \in (0,1)$, λ and J are positive real numbers and Φ is a probability distribution function. Since we do not impose rigid constraints on this distribution function, the analysis of (1) becomes a nontrivial task. Nevertheless, our study of (1) was motivated by its potential applications rather than by intrinsic mathematical challenges yielded by the non-linearity. The applications stem from the link of the system to a stylized model of a single risky asset market represented by a stochastic process constructed by us.

The stochastic process just mentioned is a microscopic model of a market in the sense that it mirrors individual behavior of each market agent. The model will be explained in Section 3.1. We constructed it using ideas from the area of Heterogeneous Interacting Agent Models[1] (HIAMs). HIAMs have a long history, wide applications and employ methods from a variety of disciplines including Statistical Mechanics and Interacting Particle Systems (see [9]). The experience shows[2] that HIAMs are able to adequately describe macroscopic characteristics of socio-economic systems by means of mirroring system's microscopic components and their mutual interactions. Our HIAM was constructed with the aim to investigate evolution of two particular macroscopic characteristics: the asset price and the population excess demand for an asset in real world markets. In the constructed model, the evolutions of interest are represented by stochastic processes. These processes possess the following property: as the model's agent population increases, their trajectories converge to orbits of the dynamical system (1) (this convergence will be explained in Section 3.2). This convergence yields the following interpretation of the parameters and variables of (1) in terms of a single risky asset market:

- p_n corresponds to the price of the market asset at time n;
- d_n corresponds to the population excess demand for the asset at time n (in the sequel, it will be called excess demand for short);
- α corresponds to the proportion of speculators among the market traders;
- J corresponds to the traders' speculative trend;
- λ corresponds to the feedback of the excess demand on the increment of the update of the asset price;

[1]The works [3] and [6] provide an broad survey of the history of the HIAMs, as well as of the current state-of-the-art in the area. In particular, our HIAM shares many features in common with the HIAM from the seminal works of Kirman ([7]) and of Lux ([10]). Among the more recent studies that present HIAMs similar to ours, we wish to single out [4] and [11] because the problems addressed by them are related to the issues discussed by us here. However, we shall not pursue a detailed comparison of our model to the already existing ones since this is not essential for our presentation.

[2]See the review works [3] and [6], and citations therein.

Φ corresponds to the distribution of the deviations around a constant $\bar{v}$ of individual evaluations of the asset fundamental value, where $\bar{v}$ corresponds to the market fundamental value of the asset.

This interpretation will be explained in details and justified in Sections 3.1 and 3.2. It will allow us to re-phrase our results so that they explain whether and why an asset price and excess demand in an asset market would or would not oscillate as time goes on; this will be the contents of Section 3.3.

The above mentioned convergence of HIAM's trajectories to orbits of an appropriate dynamical system is a well known fact in the HIAM area. It is usually employed,[3] but almost always in such a way that the resulting dynamical system is of the dimension one. This is motivated by the wish to use the graphical analysis as a tool for study this system. Contrasting, in the present work, the limiting dynamical system is two-dimensional. The dimension increase happens to make a difference because it allows one to see socio-economic phenomena that are invisible through the lenses of one-dimensional dynamical systems. The price to pay for this is the growing complexity of the proofs.

The results of our study of (1) are formulated in Theorems 1, 3 and 4 of Section 2. Theorem 1 asserts that the point $(0,0)$ is the unique equilibrium state of the dynamical system (1) and determines when it is locally asymptotically stable and when it is unstable. An interesting feature of this theorem is that it characterizes the stability/instability via a relation between just two expressions involving the model's parameters (that are, we recall, three real numbers and a probability distribution function). This reduction of the parameters' space allows us to draw a phase diagram (Figure 1) representing the stability/instability of the unique equilibrium point. Our second result, Theorem 3, gives conditions on the parameter values for the appearance of a stable periodic orbit of the dynamical system. Generalizations of this theorem are indicated in Conjectures 1 and 2. Our third result, Theorem 4, identifies parameter values for which the point $(0,0)$ is a globally stable equilibrium.

2 Results

Here, we present results of our study of the dynamical system (1) in which

$$\alpha \in (0,1), \; J > 0, \; \lambda > 0, \tag{2}$$

and Φ is a probability distribution function that satisfies the following conditions:

$$\begin{array}{ll} (a) & \Phi \text{ is everywhere differentiable; we shall denote its derivative by } \Phi'; \\ (b) & \sup_x \{\Phi'(x)\} = \Phi'(0); \\ (c) & \Phi' \text{ is strictly increasing on } (-\infty,0)\text{, and strictly decreasing on } (0,\infty); \\ (d) & \Phi(0) = 1/2; \\ (e) & \text{a random variable with the distribution } \Phi \text{ has zero expectation;} \\ (f) & \text{a random variable with the distribution } \Phi \text{ has finite variance.} \end{array} \tag{3}$$

[3]See [8] for a general account on the employment of this convergence for revealing socio-economic phenomena.

We might have relaxed significantly the constraints (3) on Φ, but this would not have broadened the application of our results and would make their proofs more cumbersome without introducing essentially new ideas. We note also that for applications, it is reasonable to admit that Φ is a Gaussian zero mean distribution function. This function satisfies the above conditions.

2.1 The uniqueness of equilibrium and its basic stability properties

Theorem 1 (uniqueness of equilibrium and its stability).

(a) *The origin (i.e., the point* $(0,0)$*) is the unique equilibrium of the dynamical system (1).*

(b) *If* $\lambda\Phi'(0) > 1 + 2\alpha J\Phi'(0))$, *then the origin is an unstable equilibrium of (1), while if* $\lambda\Phi'(0) \leq 1 + 2\alpha J\Phi'(0)$, *then the stability of the origin depends on the value of* $2\alpha J\Phi'(0)$ *in the following manner:*

if $2\alpha J\Phi'(0) < 1$ *then the origin is locally asymptotically stable;*

if $2\alpha J\Phi'(0) > 1$ *then the origin is unstable.*

Proof of Thm 1. If $(p_n, d_n) = (p_0, d_0)$, for $n \in \mathbb{N}$, then in virtue of the first equation of (1), $\lambda d_0 = 0$. This implies that $d_0 = 0$ because $\lambda > 0$ by construction. On substituting $d_0 = 0$ in the second equation of (1), we get that $1 - 2\Phi(p_0) = 0$. This can hold only if $p_0 = 0$ because Φ is monotone and $\Phi(0) = 1/2$ (the properties ensured by the assumptions (3)). This completes the proof of item (a).

We proceed with the proof of (b) of the theorem. From (1) we easily get the Jacobian matrix of Ψ at the origin:

$$\begin{pmatrix} 1 & \lambda \\ -2\Phi'(0) & 2(\alpha J - \lambda)\Phi'(0) \end{pmatrix}. \tag{4}$$

Its characteristic polynomial is $\mu^2 - \left[1 + 2(\alpha J - \lambda)\Phi'(0)\right]\mu + 2\alpha J\Phi'(0)$ whose roots are

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}\left[1 + 2(\alpha J - \lambda)\Phi'(0) + \sqrt{\Delta}\right], \\ \mu_2 &= \tfrac{1}{2}\left[1 + 2(\alpha J - \lambda)\Phi'(0) - \sqrt{\Delta}\right], \end{aligned} \tag{5}$$

where

$$\Delta := \left[1 + 2(\alpha J - \lambda)\Phi'(0)\right]^2 - 8\alpha J\Phi'(0). \tag{6}$$

The assertions in (b) will all follow from the principle of linearized stability via analysis of the values of μ_1 and μ_2. The analysis are split into six cases. They may be seen from Fig. 1. This figure is a kind two dimensional phase diagram because – due to the form of the roots – the stability analysis relies on the relation between just two expressions: $2\alpha J\Phi'(0)$ and $2\lambda\Phi'(0)$.

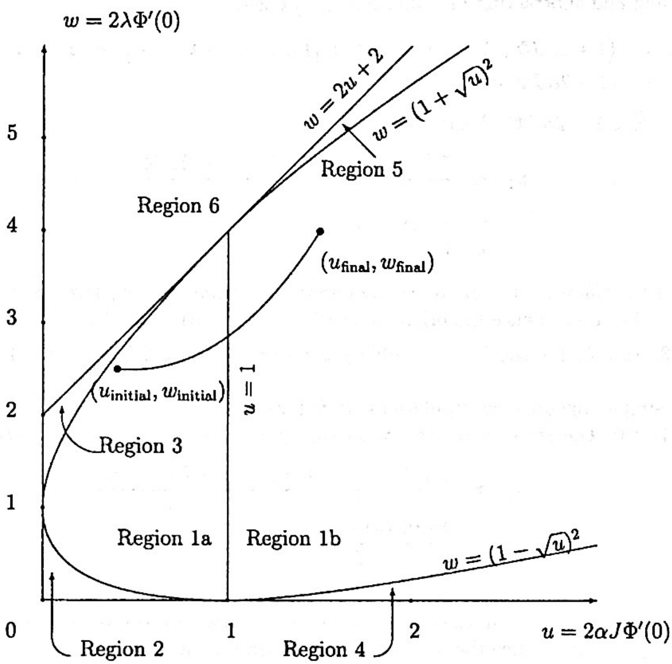

Figure 1: Phase diagram representing the stability of the unique equilibrium of the dynamical system (1).

All-throughout below, we shall usually use in our calculations that $\lambda > 0$ (the constraint on λ imposed by our construction) and that $\Phi'(0) > 0$ (the inequality that follows from (3-a, b, c)).

Case 1: $\Delta \leq 0$ (region 1 in Fig. 1).

We have:

$$|\mu_1|^2 = |\mu_2|^2 = \frac{1}{4}(1 + 2(\alpha J - \lambda)\Phi'(0))^2 + \left(\sqrt{-\Delta}\right)^2 = 2\alpha J\Phi'(0), \tag{7}$$

and therefore the origin is locally asymptotically stable, in case $2\alpha J\Phi'(0) < 1$ (region 1a), and is unstable, in case $2\alpha J\Phi'(0) > 1$ (region 1b).

Case 2: $\Delta > 0$, $2\lambda\Phi'(0) < 1 + 2\alpha J\Phi'(0)$ and $2\alpha J\Phi'(0) < 1$ (region 2 in Fig. 1).

From the hypotheses, we obtain immediately: $0 < 1+2(\alpha J-\lambda)\Phi'(0)) < 1+2\alpha J\Phi'(0)$. By adding and subtracting $(1+2\alpha J\Phi'(0))^2$, we get:

$$\begin{aligned}\Delta &= (1+2\alpha J\Phi'(0))^2 - 8\alpha J\Phi'(0) + [1+2(\alpha J-\lambda)\Phi'(0)]^2 - (1+2\alpha J\Phi'(0))^2 \\ &< (1-2\alpha J\Phi'(0))^2.\end{aligned}$$

Thus, $\sqrt{\Delta} < 1-2\alpha J\Phi'(0)$ and then:

$$\begin{aligned}\mu_1 &< \frac{1+2(\alpha J-\lambda)\Phi'(0)}{2} + \frac{(1-2\alpha J\Phi'(0))}{2} \\ &= 1-\lambda\Phi'(0) \\ &< 1.\end{aligned}$$

Since the coefficient of μ in the characteristic polynomial is negative we conclude that $|\mu_2| < |\mu_1| < 1$ and hence the origin is locally asymptotically stable.

Case 3: $\Delta > 0$, $1+2\alpha J\Phi'(0) < 2\lambda\Phi'(0) < 2+4\alpha J\Phi'(0)$ **and** $2\alpha J\Phi'(0) < 1$ (region 3 in Fig. 1).

By simple algebraic manipulations, it follows that $-(1+2\alpha J\Phi'(0)) < (1+2(\alpha J-\lambda)\Phi'(0)) < 0$. Therefore, as above, we obtain: $\sqrt{\Delta} < 1-2\alpha J\Phi'(0)$ and consequently

$$\begin{aligned}\mu_2 &> \frac{1+2(\alpha J-\lambda)\Phi'(0)}{2} - \frac{(1-2\alpha J\Phi'(0))}{2} \\ &= \frac{4\alpha J\Phi'(0) - 2\lambda\Phi'(0)}{2} \\ &> -1.\end{aligned}$$

Since the coefficient of μ in the characteristic polynomial now is positive we conclude that $|\mu_1| < |\mu_2| < 1$ and hence the origin is locally asymptotically stable.

Case 4: $\Delta > 0$, $2\lambda\Phi'(0) < 2\alpha J\Phi'(0) - 1$ **and** $2\alpha J\Phi'(0) > 1$ (region 4 in Fig. 1).

Now, we have $2(\alpha J-\lambda)\Phi'(0) > 1$ and, therefore:

$$\begin{aligned}\mu_1 &> \frac{1+2(\alpha J-\lambda)\Phi'(0)}{2} + \sqrt{\Delta} \\ &> \frac{1}{2} + \frac{2(\alpha J-\lambda)\Phi'(0)}{2} \\ &> 1,\end{aligned}$$

proving that the origin is unstable in this case.

Case 5: $\Delta > 0$, $1+2\alpha J\Phi'(0) < 2\lambda\Phi'(0) \leq 2+4\alpha J\Phi'(0)$ **and** $2\alpha J\Phi'(0) > 1$ (region 5 in Fig. 1).

Since $\Delta > 0$, we must have either $2\lambda\Phi'(0) > \left(1+\sqrt{2\alpha J\Phi'(0)}\right)^2$ or $2\lambda\Phi'(0) < \left(1-\sqrt{2\alpha J\Phi'(0)}\right)^2$. But, since $\left(1-\sqrt{2\alpha J\Phi'(0)}\right)^2 < 1+2\alpha J\Phi'(0) < 2\lambda\Phi'(0)$ then the second alternative is false. Thus:

$$\begin{aligned}2\lambda\Phi'(0) &> \left(1+\sqrt{2\alpha J\Phi'(0)}\right)^2 \\ &= 1+2\sqrt{2\alpha J\Phi'(0)} + 2\alpha J\Phi'(0) \\ &> 3+2\alpha J\Phi'(0).\end{aligned}$$

We then obtain $2(\alpha J - \lambda)\Phi'(0) < -3$, from which it follows that

$$\begin{aligned} \mu_2 &< \frac{1+2(\alpha J - \lambda)\Phi'(0)}{2} \\ &< -1, \end{aligned}$$

proving instability.

Case 6: $\Delta > 0$, $2\lambda\Phi'(0) > 2 + 4\alpha J\Phi'(0)$ (region 6 in Fig. 1).

In this case, since $1+2(\alpha J-\lambda)\Phi'(0) < 1+2\alpha J\Phi'(0)-2-4\alpha J\Phi'(0) = -1-2\alpha J\Phi'(0) < 0$, we have $\Delta = (1+2(\alpha J - \lambda)\Phi'(0))^2 - 8\alpha J > (1+2\alpha J\Phi'(0))^2 - 8\alpha J = (1-2\alpha J\Phi'(0))^2$.

If $2\alpha J\Phi'(0) < 1$, then $\sqrt{\Delta} > 1 - 2\alpha J\Phi'(0)$, and we obtain

$$\begin{aligned} \mu_2 &< \frac{1+2(\alpha J - \lambda)\Phi'(0)}{2} - \frac{1-2\alpha J\Phi'(0)}{2} \\ &= 2\alpha J\Phi'(0) - \lambda\Phi'(0) \\ &< 2\alpha J\Phi'(0) - 1 - 2\alpha J\Phi'(0) \\ &= -1. \end{aligned}$$

If $2\alpha J\Phi'(0) > 1$ then $\sqrt{\Delta} > 2\alpha J\Phi'(0) - 1$, and we obtain

$$\begin{aligned} \mu_2 &< \frac{1+2(\alpha J - \lambda)\Phi'(0)}{2} - \frac{-1+2\alpha J\Phi'(0)}{2} \\ &= 1 - \lambda\Phi'(0) \\ &< 1 - 1 - 2\alpha J\Phi'(0) \\ &< -1. \end{aligned}$$

Thus, $\mu_2 < 1$ in both case, and the instability follows. □

2.2 Hopf bifurcation and periodic orbits

The occurrence of Hopf bifurcation in the dynamical system (1) is the issue of Theorem 3 and Conjectures 1 and 2 of the present section. All they are based on Theorem 2 below. It contains a description of the Hopf bifurcation phenomenon and provides conditions that are sufficient for it to occur. This bifurcation implies the emergence of periodic orbits in the dynamical system (1) for particular set of its parameters' values.

Theorem 2 ([5], page 474, Poincaré-Hopf-Andronov theorem for maps).
Let

$$F : \mathbb{R} \times (\mathbb{R})^2 \to \mathbb{R}^2; \quad (\eta, \mathrm{x}) \to F(\eta, \mathrm{x})$$

be a C^4 map depending on a real parameter η satisfying the following conditions:

(i) $F(\eta, 0) = 0$ *for η near some fixed η_0;*

(ii) $DF(\eta, 0)$ *(that is, $D_{\mathrm{x}}F(\eta,\mathrm{x})\big|_{\mathrm{x}=0}$) has two non-real eigenvalues $\mu(\eta)$ and $\bar{\mu}(\eta)$ for η near η_0, with $|\mu(\eta_0)| = 1$;*

(iii) $\frac{d}{d\eta}|\mu(\eta)| > 0$ *at $\eta = \eta_0$;*

(iv) $\mu^k(\eta_0) \neq 1$ *for* $k = 1, 2, 3, 4$.

Then there is a smooth η*-dependent change of coordinates bringing* F *into the form*

$$F(\eta, \mathbf{x}) = \mathcal{F}(\eta, \mathbf{x}) + O(||\mathbf{x}||^5)$$

and there are smooth functions $A(\eta)$, $B(\eta)$ *and* $\Omega(\eta)$ *so that in polar coordinates the function* $\mathcal{F}(\eta, \mathbf{x})$ *is given by*

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} |\mu(\eta)|r - A(\eta)r^3 \\ \theta + \Omega(\eta) + B(\eta)r^2 \end{pmatrix}. \tag{8}$$

If $A(\eta_0) > 0$, *then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that, for* $|\eta - \eta_0| < \delta$ *and* $\mathbf{x} \in U$, *the* ω*−limit set of* $\mathbf{x}$ *is the origin if* $\eta < \eta_0$, *and belongs to a closed invariant* C^1 *curve* $\Gamma(\eta)$ *encircling the origin if* $\eta > \eta_0$. *Furthermore* $\Gamma(\eta_0) = 0$.[4]

If $A(\eta_0) < 0$, *then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that, for* $|\eta - \eta_0| < \delta$ *and* $\mathbf{x} \in U$, *the* ω*−limit set of* $\mathbf{x}$ *is the origin if* $\eta > \eta_0$, *and belongs to a closed invariant* C^1 *curve* $\Gamma(\eta)$ *encircling the origin if* $\eta < \eta_0$. *Furthermore* $\Gamma(\eta_0) = 0$.

Remark 1. We present here the method that we shall employ to calculate $A(\eta_0)$. The presentation follows [5]. If the linear part of map F at η_0 is written in the Jordan canonical form,

$$\mathbf{x} \to F(\eta_0, \mathbf{x}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} G_1(x_1, x_2) \\ G_2(x_1, x_2) \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2), \tag{9}$$

where a and b relate to the eigenvalue $\mu(\eta)$ via the equation $\mu(\eta_0) = a + ib$, then for $A(\eta)$ defined in (8), it holds that

$$A(\eta_0) = \mathrm{Re}\left[\frac{(1 - 2\mu(\eta_0))\bar{\mu}(\eta_0)^2}{1 - \mu(\eta_0)}\xi_{11}\xi_{20}\right] + \frac{1}{2}|\xi_{11}^2| + |\xi_{02}^2| - \mathrm{Re}\left[\bar{\mu}(\eta_0)\xi_{21}\right], \tag{10}$$

where (all the derivatives below are evaluated at $(x_1, x_2) = (0, 0)$)

$$\xi_{20} = \tfrac{1}{8}\left\{(G_1)_{x_1x_1} - (G_1)_{x_2x_2} + 2(G_2)_{x_1x_2} + i[(G_2)_{x_1x_1} - (G_2)_{x_2x_2} - 2(G_1)_{x_1x_2}]\right\},$$

$$\xi_{11} = \tfrac{1}{4}\left\{(G_1)_{x_1x_1} + (G_1)_{x_2x_2} + i[(G_2)_{x_1x_1} + (G_2)_{x_2x_2}]\right\},$$

$$\xi_{02} = \tfrac{1}{8}\left\{(G_1)_{x_1x_1} - (G_1)_{x_2x_2} - 2(G_2)_{x_1x_2} + i[(G_2)_{x_1x_1} - (G_2)_{x_2x_2} - 2(G_1)_{x_1x_2}]\right\},$$

and

$$\begin{aligned} \xi_{21} &= \frac{1}{16}\{(G_1)_{x_1x_1x_1} + (G_1)_{x_1x_2x_2} + (G_2)_{x_1x_1x_2} + (G_2)_{x_2x_2x_2} \\ &+ i[(G_2)_{x_1x_1x_1} + (G_2)_{x_1x_2x_2} - (G_1)_{x_1x_1x_2} - (G_1)_{x_2x_2x_2}]\}. \end{aligned} \tag{11}$$

[4]The phenomenon described in the last two sentences is called *Hopf bifurcation.* We add to this name the term *supercritical* in order to distinguish it from the case described in the next two sentences that will be called *the subcritical case.*

Theorem 3 below reveals the occurrence of Hopf bifurcation in the dynamical system (1) when it satisfies certain additional assumptions. How general these assumptions are will be discussed after the theorem's proof. The same discussion will expose at an intuitive lever the ideas behind the proof.

Theorem 3.
Suppose the dynamical system (1) satisfies the following assumptions:

Assumption 1. *The parameters λ, α, J and the distribution function $\Phi(\cdot)$ are all functions of a single real variable η that are defined on some (nonempty) interval $[\eta_{\text{initial}}, \eta_{\text{final}}]$; the functions $\lambda_\eta, \alpha_\eta, J_\eta$ and the distribution functions $\{\Phi_\eta(\cdot)\}$ are smooth enough to ensure that Ψ, the map that generates the system (1), is C^4 in (p_{n-1}, d_{n-1}) and η.*

Assumption 2. *The functions α_η, J_η and $\Phi'_\eta(0)$ (the latter means the value of $\partial\Phi_\eta(x)/\partial x$ at $x = 0$) are all increasing in η on the domain $[\eta_{\text{initial}}, \eta_{\text{final}}]$.*

Assumption 3. $(1+2(\alpha_\eta J_\eta - \lambda_\eta)\Phi'_\eta(0))^2 - 8\alpha_\eta J_\eta \Phi'_\eta(0)) < 0$, *for each* $\eta \in [\eta_{\text{initial}}, \eta_{\text{final}}]$.

Assumption 4. *There exists a unique $\eta_0 \in [\eta_{\text{initial}}, \eta_{\text{final}}]$ such that $2\alpha_{\eta_0} J_{\eta_0} \Phi'_{\eta_0}(0) = 1$, and $\eta_0 \neq \eta_{\text{initial}}$, $\eta_0 \neq \eta_{\text{final}}$.*

Suppose in addition that

$$(a)\ \lambda_{\eta_0} = \frac{1}{2}, \quad (b)\ \Phi'_{\eta_0}(0) = 1, \quad (c)\ \Phi''_{\eta_0}(0) = 0, \quad (d) \Phi'''_{\eta_0}(0) < 0. \tag{12}$$

Then the system undergoes the supercritical Hopf bifurcation when η passes through η_0.

Proof. In virtue of Thm. 2 and Remark 1, in order to prove the present theorem, it is sufficient to show that the mapping Ψ that generates the dynamical system (1) satisfies the conditions (i) – (iv) of Thm. 2 and that $A(\eta_0)$ is a positive number. We do so in Steps 1 – 5 below.

Step 1: Since Theorem 1 ensures that $(0,0)$ is an equilibrium point of (1) for any values of the parameters of this dynamical system, then Ψ satisfies (i) for each $\eta \in [\eta_{\text{initial}}, \eta_{\text{final}}]$.

Step 2: We start recalling facts and results from the proof of Thm. 1 that we shall need below: (a) the Jacobian matrix of the mapping Ψ at $(0,0)$ (i.e., $DF(\eta, 0)$, in the notations of Thm. 2) was calculated and the result is presented in (4); (b) the matrix' eigenvalues were calculated and their expressions are presented in (5); (c) it was proved that these eigenvalues are non-real numbers provided $\Delta < 0$, with Δ being defined by (6); (d) the modulus of each eigenvalue was found to be equal to $\sqrt{2\alpha J \Phi'(0)}$, provided $\Delta \leq 0$.

Now, since Ass. 3 ensures that $\Delta < 0$ for every η, then (c) above implies that Ψ satisfies the first part of the condition (ii). As for the second part of this condition, it is implied by Assumptions 3and 4 and the the fact (d).

Step 3: The fact (d) from the list of Step 2 and Assumptions 2 and 3 imply straightforwardly that the condition (iii) is satisfied by the eigenvalues of Ψ for every η, and in particular, for η_0.

Step 4: From the fact (b) of the list of Step 2 and from Ass. 3, it follows that

$$\mu(\eta_0) = \frac{1+2(\alpha_{\eta_0}J_{\eta_0}-\lambda_{\eta_0})\Phi'_{\eta_0}(0)}{2} + i\frac{\sqrt{8\alpha_{\eta_0}J_{\eta_0}\Phi'_{\eta_0}(0)) - (1+2(\alpha_{\eta_0}J_{\eta_0}-\lambda_{\eta_0})\Phi'_{\eta_0}(0))^2}}{2}. \tag{13}$$

Applying Ass. 4 and constraints (12) to (13), we get that $\mu(\eta_0) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. From this, $\mu^k(\eta_0) \neq 1, k = 1,2,3,4$, proving hence the validity of condition (iv).

Step 5: In this step, we shall prove that $A(\eta_0) > 0$. In the calculations that follow, all functions that depend on η will be evaluated at η_0. This allows us to omit η_0 in the notations throughout the proof, namely, we shall write A, λ, α, J and $\Phi'(0)$ for, respectively, $A(\eta_0)$, $\lambda_{\eta_0}, \alpha_{\eta_0}, J_{\eta_0}$ and $\Phi'_{\eta_0}(0)$.

Since we intend to use the expression (10) then we need to get a (9)-like form of the map Ψ. We shall get it from the Jacobian matrix at $(0,0)$ of the map Ψ that we've calculated in the proof of Thm. 1 (the matrix is presented in (4)). Since Ass. 4 imposes that $2\alpha J\Phi'(0) = 1$ then this matrix acquires (recall, at η_0) the following form:

$$M = \begin{pmatrix} 1 & \lambda \\ -2\Phi'(0) & 1-2\lambda\Phi'(0) \end{pmatrix}.$$

It is then easy to check that the matrix

$$P = \begin{pmatrix} 1 & 0 \\ -\Phi'(0) & \frac{\sqrt{1-\varepsilon^2}}{\lambda} \end{pmatrix}, \quad \text{with} \quad \varepsilon := 1-\lambda\Phi'(0),$$

puts M in the Jordan canonical form: $P^{-1}MP = \begin{pmatrix} \varepsilon & \sqrt{1-\varepsilon^2} \\ -\sqrt{1-\varepsilon^2} & \varepsilon \end{pmatrix}$. Therefore, in the new variables $\begin{pmatrix} \tilde{p}_n \\ \tilde{d}_n \end{pmatrix} := P^{-1}\begin{pmatrix} p_n \\ d_n \end{pmatrix}$, $n \in \mathbb{N}$, the map Ψ acquires a (9)-like form:

$$\begin{aligned} \begin{pmatrix} \tilde{p}_n \\ \tilde{d}_n \end{pmatrix} &= P^{-1}MP\begin{pmatrix} \tilde{p}_{n-1} \\ \tilde{d}_{n-1} \end{pmatrix} + P^{-1}(\Psi - M)P\begin{pmatrix} \tilde{p}_{n-1} \\ \tilde{d}_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon & \sqrt{1-\varepsilon^2} \\ -\sqrt{1-\varepsilon^2} & \varepsilon \end{pmatrix}\begin{pmatrix} \tilde{p}_{n-1} \\ \tilde{d}_{n-1} \end{pmatrix} + \begin{pmatrix} G_1(\tilde{p}_{n-1}, \tilde{d}_{n-1}) \\ G_2(\tilde{p}_{n-1}, \tilde{d}_{n-1}) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} G_1(x_1,x_2) &= 0, \\ G_2(x_1,x_2) &= \frac{\lambda}{\sqrt{1-\varepsilon^2}}\Big\{\alpha\left[1-2\Phi(K_1x_1+K_2x_2)\right] + (1-\alpha)\left[1-2\Phi(K_3x_1+K_4x_2)\right] \\ &\quad +\Phi'(0)\left(3-2\alpha\Phi'(0)\right)x_1 - \lambda^{-1}\left(1-2\alpha\Phi'(0)\right)\sqrt{1-\varepsilon^2}x_2\Big\}, \end{aligned}$$

and where

$$\begin{aligned} K_1 &= 1-(\lambda-J)\Phi'(0), & K_2 &= \lambda^{-1}\sqrt{1-\varepsilon^2}(\lambda-J), \\ K_3 &= 1-\lambda\Phi'(0), & K_4 &= \sqrt{1-\varepsilon^2}. \end{aligned}$$

From the expressions for G_1 and G_2, we shall now obtain $\xi_{\ell r}$'s following the formulas from Remark 1. Differentiating G_2 we get:

$$\frac{\partial^2}{\partial x_1^2}G_2(x_1,x_2) = -\frac{2\lambda}{\sqrt{1-\varepsilon^2}}\left(\alpha K_1^2\Phi''(K_1x_1+K_2x_2)+(1-\alpha)K_3^2\Phi''(K_3x_1+K_4x_2)\right).$$

From this result, in virtue of the assumption (12-c), we conclude that $\frac{\partial^2}{\partial x_1^2}G_2(0,0)=0$. By similar arguments, we conclude that all second order derivatives of G_2 vanish at $(x_1,x_2)=(0,0)$. These facts and the fact that $G_1\equiv 0$ imply that $\xi_{20}=\xi_{11}=\xi_{02}=0$. In order to find ξ_{21}, we calculate third order derivatives of G_2:

$$\begin{aligned}
\frac{\partial^3}{\partial x_1^3}G_2(0,0) &= -\frac{2\lambda\Phi'''(0)}{\sqrt{1-\varepsilon^2}}\left(\alpha K_1^3+(1-\alpha)K_3^3\right),\\
\frac{\partial^3}{\partial x_1^2\partial x_2}G_2(0,0) &= -\frac{2\lambda\Phi'''(0)}{\sqrt{1-\varepsilon^2}}\left(\alpha K_1^2K_2+(1-\alpha)K_3^2K_4\right),\\
\frac{\partial^3}{\partial x_1\partial x_2^2}G_2(0,0) &= -\frac{2\lambda\Phi'''(0)}{\sqrt{1-\varepsilon^2}}\left(\alpha K_1K_2^2+(1-\alpha)K_3K_4^2\right),\\
\frac{\partial^3}{\partial x_2^3}G_2(0,0) &= -\frac{2\lambda\Phi'''(0)}{\sqrt{1-\varepsilon^2}}\left(\alpha K_2^3+(1-\alpha)K_4^3\right).
\end{aligned}$$

Now, from the relations $2\alpha J\Phi'(0)=1$ (that holds because of Ass. 4) and $\Phi'(0)=1$ (ensured by (12-b)) we get that $\alpha=(2J)^{-1}$. This allows us to substitute α by $(2J)^{-1}$ in the expressions for the derivatives of G_2. We also substitute there λ and $\Phi'(0)$ by $\frac{1}{2}$ and 1, respectively (these substitutions are justified by (12-a, b)). The resulting simplified expressions for the derivatives lead, via the formula (11), to the following:

$$\begin{aligned}
\xi_{21} &= \tfrac{\Phi'''(0)}{64\sqrt{3}}[(-20J^2+(13\sqrt{3}-4)J+3\sqrt{3}-10)\\
&\quad +i(-10J^2+(4\sqrt{3}-7)J+2\sqrt{3}-5)].
\end{aligned}$$

Plugging in (10) the expressions for $\xi_{\ell r}$'s obtained above, we finally get that

$$A=-\mathrm{Re}(\bar{\mu}\xi_{21})=-\Phi'''(0)\frac{2+\sqrt{3}}{64\sqrt{3}}[5J^2+(1-2\sqrt{3})J+1].$$

This expression for A and the assumption (12-d) ensure that $A>0$ for every J. This completes Step 5 and the proof of Theorem 3. □

We proceed with the discussion of how general the Hopf bifurcation phenomenon is for the dynamical system (1).

In order to discuss Hopf bifurcation in a dynamical system with the help of Theorem 2, the minimal necessary condition is that all system's parameters be expressed as functions of a unique variable. In the framework of Theorem 3, this condition is ensured by Assumption 1. Let us accept it now and let us consider then two functions: $u_\eta := 2\alpha_\eta J_\eta\Phi'_\eta(0)$ and $w_\eta := 2\lambda_\eta\Phi'_\eta(0)$. We recall from the proof of Thm. 1 that (a) the quantity denoted there by Δ can be expresses as a function of solely u and w, (b) when $\Delta(u,w)<0$ then the eigenvalues of the linearization of Ψ at its fixed point $(0,0)$ are non-real numbers, and

(c) when $u = 1$ then the modulus of each eigenvalue is 1. The facts (a,b,c,) ensure that if the parameter η introduced in Assumption 1 is such that $\Delta(u_\eta, w_\eta) < 0$, for all η, and if $u_{\eta_0} = 1$, for some η_0, then the property (ii) of Thm. 2 is satisfied. These "if" conditions are provided by Assumptions 3 and 4. As for the Ass. 2 it ensures the validity of (iii) of Thm. 2. For this item to hold true, Assumption 2 may be not the minimal sufficient condition, but we did not search for such. Turning our attention to the condition (i) of Thm. 2, we easily see that it is valid for Ψ due to Thm.1(a) without the necessity for any additional assumptions.

We thus have showed that Assumptions 1– 4 ensure the validity of conditions (i) – (iii) of Thm. 2. It seems to us that these assumptions are also sufficient for the validity of the condition (iv) of Thm. 2 and the inequality $A(\eta_0) \neq 0$. Accordingly, we formulate the following:

Conjecture 1.
The dynamical system (1) exhibits periodic orbits, when the initial point is sufficiently close to $(0,0)$ *and when the parameters are such that the corresponding point* (u,w) *is sufficiently close to the right of the interval* $\{(u,w) : u = 1, 1 < w < 4\}$;[5] *precisely to state, there is a neighborhood* U *of* $(0,0)$ *such that the* ω*-limit of any orbit of (1) with initial condition in* U *belongs to a closed* C^1 *curve encircling* $(0,0)$, *provided* $2\alpha J\Phi'(0)$ *is slightly bigger than* 1 *and* $\lambda\Phi'(0)$ *is between* 0 *and* 2.

We did not pursue proofs in the degree of generality that would allow us to justify rigorously the generic property formulated in this conjecture. Rather, we considered two particular cases. The first case is presented in Thm. 3. There, we assumed an additional constraint (12) that helped a lot to simplify the calculations needed to establish the condition (iv) and the inequality $A(\eta_0) > 0$. The second case is presented in the statement below. This case attracted our attention because it arises in applications of our mathematical study of the dynamical system (1). The reason for this is explained in Section 3.3.

Conjecture 2.
If the parameters of the dynamical system (1) satisfy Assumptions 1 – 4 and Φ *is Normal Distribution with null mean then the system undergoes the supercritical Hopf bifurcation when* η *passes increasingly through* η_0.

In the case of Conjecture 2, the analytic verification of the condition (iv) and the inequality $A(\eta_0) \neq 0$ turned to be an extremely tedious task. To carry out this task, we resorted to numeric methods: we verified the condition and the inequality numerically for a grid of parameter values.

[5] If the point is close to this interval on its left, then $(0,0)$ is locally asymptotically stable equilibrium of the dynamical system (1) and hence the periodic orbits described in this statement cannot occur.

2.3 Global asymptotic stability for small values of the parameter λ

Theorem 4 (a sufficient condition for global asymptotic stability).
Suppose $2\alpha J\Phi'(0) < 1$. *Then, there exists a positive number* λ_c *such that if* $\lambda \in (0, \lambda_c]$ *then the equilibrium* $(0,0)$ *of the dynamical system (1) is globally asymptotically stable. (The numeric value of* λ_c *will be specified in the proof by (40).)*

Remark 2. On comparing the assertions of Thms. 1 and 4 one concludes that the assumptions of Thm. 4 must be a sub-case of the assumptions of Thm. 1. This inclusion can be established by simple calculations grounded on (40-(ii)), one of the conditions that will determine the value of λ_c. We shall omit these calculations.

We now proceed with the argument that will lead to the proof of Thm. 4. The arguments employ essentially the assumption (2) that ensures that $J > 0$ and $\alpha > 0$ and thus, allows us to divide by J and α. Actually, the theorem's assertion remains true even when either J and α is equal to 0, but this case requires a specific argument which will not be presented here.

Upon the substitution of $p_{n-1} + \lambda d_{n-1}$ by p_n in the second equation of the dynamical system (1), it acquires the form that makes it clear that the passage from (p_{n-1}, d_{n-1}) to (p_n, d_n) can be considered as a two-step procedure: first, p_n is obtained from p_{n-1} and d_{n-1} via the first equation of the system, and then, d_n is obtained from d_{n-1} and p_n via the second equation. The second equation, or better to say, its non-linearity is the principal cause for the difficulty of studying the evolution of the dynamical system (1). In the present proof, we overcome the difficulty by getting convenient estimates of two quantities (to be defined in (23) and (24) below) that relate to this equation and characterize its features that are important for our analysis. The presentation of these estimates becomes more transparent – in our opinion –, when we express the equation through an appropriate one-parameter $\mathbb{R} \to \mathbb{R}$ mapping in which p_n is the parameter and d_{n-1} is the argument. In order to do so, for the "parameter" $p \in \mathbb{R}$, we define $g_p(\cdot) : (-\infty, \infty) \to [-1, 1]$ as follows:[6]

$$g_p(d) = \alpha[1 - 2\Phi(p - Jd)] + (1 - \alpha)[1 - 2\Phi(p)], \quad d \in (-\infty, \infty). \tag{14}$$

Now, we can re-write (1) in the desired form:

$$\begin{aligned} p_n &= p_{n-1} + \lambda d_{n-1} && (15)\\ d_n &= g_{p_n}(d_{n-1}) && (16) \end{aligned}$$

From now on, we shall use the shorthand notation:

$$b := 2\alpha J\Phi'(0); \tag{17}$$

note that $b > 0$ (because of (3-(a), (c))) and $b < 1$ (because of the theorem's assumption).

[6] Although the argument of $g_p(\cdot)$ cannot exceed 1 in modulus – since it has been defined as the mathematical analogue of population excess demand –, it turns out to be convenient to extend the function's argument domain to the whole $\mathbb{R}$.

We start with a list of basic properties of $g_p(\cdot)$. For each p,

- $g_p(\cdot)$ is everywhere differentiable function and $g_p'(d) = 2\alpha J \Phi'(p - Jd)$; (18)
- $\sup_{d \in \mathbf{R}} \{g_p'(d)\} = g_p'\left(\frac{p}{J}\right) = 2\alpha J \Phi'(0) = b \in (0,1)$; (19)
- in particular, as a consequence of (18) and (19), $g_p(\cdot)$ is a contraction:
$$|g_p(x) - g_p(y)| \le b|x - y| < |x - y|; \quad (20)$$
- $g_p(\cdot)$ a monotone strictly increasing function; (21)
- the graph of $g_p(\cdot)$ passes through the point
$$(d^*, y^*) \text{ with } d^* = \frac{p}{J} \text{ and } y^* = (1-\alpha)[1 - 2\Phi(p)] \quad (22)$$
that satisfy: $\begin{cases} p \ge (>)\, 0 \\ p \le (<)\, 0 \end{cases} \Rightarrow \begin{cases} d^* \ge (>)\, 0,\ y^* \le (<)\, 0, \\ d^* \le (<)\, 0,\ y^* \ge (>)\, 0. \end{cases}$

All these properties stem from (14) in combination with the properties of Φ assumed in (3); exactly to state, we used (3-a) for (18), (3-b) for (19), (3-c) for (21), and (3-d) for (22).

We now define the quantities $\mathcal{D}(p)$ and $\mathcal{R}(p)$ that will play central role in our arguments:

$$\mathcal{D}(p) := \text{the solution of } g_p(d) = d, \text{ hence, } g_p(\mathcal{D}(p)) = \mathcal{D}(p), \quad (23)$$

$$\mathcal{R}(p) := \begin{cases} -\infty, \text{ when } g_p(d) > 0,\ \forall d, \\ +\infty, \text{ when } g_p(d) < 0,\ \forall d, \\ \text{the solution of } g_p(d) = 0, \text{ otherwise, and hence, } g_p(\mathcal{R}(p)) = 0. \end{cases} \quad (24)$$

Figure 2 helps the visualization of these definitions. We observe that the properties (19) and (21) ensure that $\mathcal{D}(p)$ and $\mathcal{R}(p)$ are well and uniquely defined.

We shall frequently use the following properties of $\mathcal{D}(p)$ and $\mathcal{R}(p)$:

- if $p \ge 0$ then $\mathcal{R}(p) \ge \frac{p}{J} \ge 0$, and if $p \le 0$ then $\mathcal{R}(p) \le \frac{p}{J} \le 0$; (25)
- if $p \le p'$ then $\mathcal{D}(p) \ge \mathcal{D}(p')$; (26)
- if $p \ge 0$ then $\mathcal{D}(p) \le 0$, and if $p \le 0$ then $\mathcal{D}(p) \ge 0$. (27)

(25) follows directly from (21) and (22). To prove (26), we argue as follows: First, from the monotonicity of Φ and the definition (14), we get:

- $g_p(\cdot)$ is decreasing in p, i.e., $g_p(d) > g_{p'}(d)\, \forall d$, when $p < p'$. (28)

Then, from this property and the fact that $|g_p'(d)| < 1$, we conclude that the first coordinate of the intersection point of the 45° line in the plane with $g_p(\cdot)$ decreases with the increase of p. This conclusion is exactly the property (26). As for (27), it follows from (26) because $\mathcal{D}(0) = 0$ (this equality holds because $g_0(\cdot)$ passes through $(0,0)$ as ensured by (3-d) and (14)).

There are two more properties of $\mathcal{D}(p)$ and $\mathcal{R}(p)$ that we shall need for the proof of Theorem 4. These are provided by Lemmas 1 and 2 below.

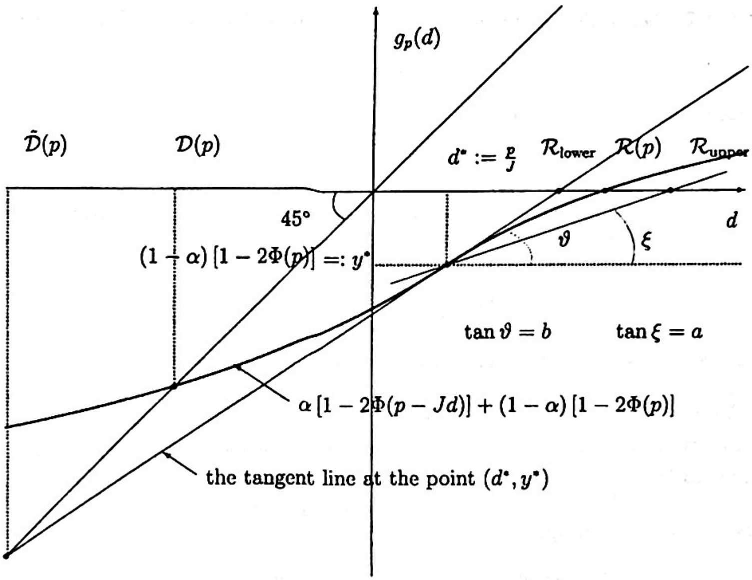

Figure 2: Illustration to the definition of the quantities $\mathcal{D}(p)$ and $\mathcal{R}(p)$ and the arguments that employ these quantities and their properties.

Lemma 1. *If $p \geq 0$ then $\mathcal{D}(p) \geq -\frac{b}{J\alpha(1-b)}p$, and if $p \leq 0$ then $\mathcal{D}(p) \leq -\frac{b}{J\alpha(1-b)}p$.*

Proof. We prove the case $p \geq 0$; in the case $p \leq 0$, the proof is similar.

We shall need the following estimate:

$$1 - 2\Phi(p) \overset{(3-(a),(d))}{=} -2\int_0^p \Phi'(x)dx \overset{(3-(b))}{\geq} -2\Phi'(0)p \overset{(17)}{=} -\frac{bp}{\alpha J}. \tag{29}$$

Now[7], we construct the tangent line to $g_p(\cdot)$ at the point (d^*, y^*). From (18), (19) and (22), we have that the slope of the line is b and the graph of $g_p(\cdot)$ to the left of (d^*, y^*) lies above this line. We also have that $d^* \geq 0$ and $y^* \leq 0$ (because of (22) and the assumption $p \geq 0$). Hence, $\mathcal{D}(p)$, the coordinate of the intersection of $g_p(\cdot)$ with the diagonal, is larger than $\tilde{\mathcal{D}}(p)$, the coordinate of the intersection of the tangent line with the diagonal. By simple calculations, we get that $\tilde{\mathcal{D}}(p) = (1-b)^{-1}\{(1-\alpha)[1-2\Phi(p)] - (bp)/J\}$. Combining this with the inequality $\mathcal{D}(p) \geq \tilde{\mathcal{D}}(p)$ just obtained and with the estimate (29), we complete the proof: $\mathcal{D}(p) \geq \tilde{\mathcal{D}}(p) \geq \frac{-\frac{1-\alpha}{\alpha}bp - bp}{J(1-b)} = -\frac{b}{J\alpha(1-b)}p$. □

[7]The following constructions are illustrated in Figure 2.

Lemma 2. *Let (a defined below might have been any other number from $(0, b)$):*

$$a := b/2. \tag{30}$$

Let next $x_\ell < 0$, $x_r > 0$ be such that:

$$2\alpha J\Phi'(x) \geq a, \ \ \forall x \in [x_\ell, x_r] \tag{31}$$

(the existence of x_ℓ and x_r follows from the properties (3-a, b, c) of Φ; in fact, x_ℓ and x_r are the negative and the positive solutions of the equation $2\alpha J\Phi'(x) = a$). Finally:

$$h := \min\left\{\frac{\alpha}{1-\alpha}\frac{a}{b},\ 1\right\} \times \min\{|x_\ell|, x_r\}. \tag{32}$$

Then, $\mathcal{R}(\cdot)$ satisfies the following properties:
(a) *If $p \geq 0$ then $\mathcal{R}(p) \geq \frac{1}{\alpha J}\frac{a}{b}p$, and if $p \leq 0$ then $\mathcal{R}(p) \leq -\frac{1}{\alpha J}\frac{a}{b}p$.*
(b) *If $0 \leq p \leq h$ then $\mathcal{R}(p) \leq \frac{1}{\alpha J}\frac{b}{a}p$, and if $-h \leq p \leq 0$ then $\mathcal{R}(p) \geq -\frac{1}{\alpha J}\frac{b}{a}p$.*

Proof. We shall prove both (a) and (b) for $p \geq 0$. For $p \leq 0$, the proofs are analogous.

Due to (32), if $0 \leq p \leq h$ then $x \in [-p, 0]$ implies $x \in [x_\ell, 0]$. This implication and (31) ensure that $\Phi'(x) \geq (2\alpha J)^{-1}a$, in case $x \in [-p, 0]$ and $p \in [0, h]$. We use this fact to derive the following inequality:

$$1 - 2\Phi(p) \overset{(3-(a),(d))}{=} -2\int_0^p \Phi'(x)dx \leq -\frac{a}{\alpha J}p, \qquad \text{for all } p \in [0, h]. \tag{33}$$

Recall now from (22) that the graph of g_p passes trough the point (d^*, y^*). We construct two straight lines passing through this point and having slopes b and a, and we denote by $\mathcal{R}_{\text{lower}}$ and $\mathcal{R}_{\text{upper}}$ the coordinates of their respective intersections with the d axis (see Figure 2). Directly from this construction, we have that $\mathcal{R}_{\text{lower}} = d^* - \frac{y^*}{b}$ and $\mathcal{R}_{\text{upper}} = d^* - \frac{y^*}{a}$. Since $y^* = (1-\alpha)[1 - 2\Phi(p)]$ then:

$$\mathcal{R}_{\text{lower}} \overset{(29)}{\geq} \frac{p}{J} + \frac{1-\alpha}{\alpha J}\frac{a}{b}p,\ \forall p \geq 0, \text{ and } \mathcal{R}_{\text{upper}} \overset{(33)}{\leq} \frac{p}{J} + \frac{1-\alpha}{\alpha J}\frac{b}{a}p,\ \forall p \in [0, h]. \tag{34}$$

Next, due to (32), if $0 \leq p \leq h$ then $x \in [0, \frac{1-\alpha}{\alpha}\frac{b}{a}p]$ implies that $x \in [0, x_r]$. This implication and (31) ensure that $2\alpha J\Phi'(x) \geq a$, in case $x \in [0, \frac{1-\alpha}{\alpha}\frac{b}{a}p]$ and $p \in [0, h]$. This conclusion and the relation (18) between g_p' and Φ' together ensure that if $p \in [0, h]$ then $g_p'(d) \geq a$ for each $d \in \left[\frac{p}{J}, \frac{p}{J} + \frac{1-\alpha}{\alpha}\frac{b}{a}p\right]$. From this inequality, it follows that the graph of $g_p(d)$, $d \in \left[\frac{p}{J}, \frac{p}{J} + \frac{1-\alpha}{\alpha}\frac{b}{a}p\right]$, lies above the line of slope a constructed by us, provided $p \in [0, h]$. This fact implies that $\mathcal{R}(p) \leq \mathcal{R}_{\text{upper}}$, when $p \in [0, h]$ (for this implication to be valid it is important that the interval on which $g_p(\cdot)$ lies above the line extends up to the upper bound of $\mathcal{R}_{\text{upper}}$ provided by (34)). Then, via simple algebraic manipulation with the r.h.s. of the estimate (34) for $\mathcal{R}_{\text{upper}}$, the assertion (b) of the lemma follows.

Finally, since $g_p'(\cdot) \leq b$ (as (19) ensures) then the graph of $g_p(d)$, $d \in \left[\frac{p}{J}, \infty\right)$, lies below the line with slope b constructed by us. Hence, $\mathcal{R}_{\text{lower}} \leq \mathcal{R}(p)$, and the assertion (a) of the lemma follows via simple algebraic manipulation with the r.h.s. of the lower bound for $\mathcal{R}_{\text{lower}}$ provided by (34). □

Lemma 3. *Let (p_n, d_n), $n \geq 0$, be an orbit of (1). Then, either $(p_n, d_n) \to (0,0)$ or there exists finite $r \geq 0$ such that*

$$\textit{either } 0 \leq p_{r+1} \leq \lambda d_r \textit{ or } \lambda d_r \leq p_{r+1} \leq 0. \tag{35}$$

Proof. *Case 1:* First, we consider the case when d_0 and p_0 have different signs. We shall conduct our arguments under the assumption that $p_0 \geq 0$ and $d_0 \leq 0$; in the opposite case, i.e., when $p_0 \leq 0$ and $d_0 \geq 0$, the proof follows by exactly the same argument with the obvious change of inequality directions.

The assumption $p_0 \geq 0$ and $d_0 \leq 0$ and the relation (15) imply that:

$$p_1 \leq p_0 \text{ and } p_1 \geq \lambda d_0. \tag{36}$$

For this reason, if $p_1 \leq 0$ then (35) holds with $r = 0$ and thus, the lemma is proved. Assume the contrary: $p_1 \geq 0$. Let us prove that $d_1 \leq 0$ under this assumption. Since

$$|d_1 - \mathcal{D}(p_1)| \overset{(16),(23)}{=} |g_{p_1}(d_0) - g_{p_1}(\mathcal{D}(p_1))| \overset{(20)}{\leq} |d_0 - \mathcal{D}(p_1)|, \tag{37}$$

then $|d_1 - \mathcal{D}(p_1)| \leq |d_0 - \mathcal{D}(p_1)|$. The inequality (37), the inequality $\mathcal{D}(p_1) \leq 0$ (that stems from (27) and the assumption $p_1 \geq 0$) and the inequality $d_0 \leq 0$ (the initial assumption) can hold simultaneously only if $d_1 \leq 0$. This is the conclusion we aimed for.

The argument of the above paragraph either finishes the proof of the lemma or implies that $p_1 \geq 0$ and $d_1 \leq 0$ hold. In the latter case, the argument can be repeated with these inequalities in the place of the argument's assumption "$p_0 \geq 0$ and $d_0 \leq 0$". Upon repeating this procedure, we either eventually find an r such that $p_{r+1} \leq 0$ or not. When the first alternative is the case, the last but one repetition of the argument ensured that $p_r \geq 0$. Combining this inequality with $p_{r+1} \leq 0$ and equation (15), we get that the second double inequality of (35) holds true for the found value of r, and thus the lemma's proof is finished. The second alternative is analyzed in the paragraph below.

Let us assume that the argument of the last but one paragraph can be repeated infinitely. Since, after n-th repetition, we have that $p_n \geq 0$ (otherwise, r would have been equal to $n-1$) and $d_n \leq 0$ (this inequality is the final conclusion in each repetition) then, in virtue of (15), $p_{n+1} \leq p_n$ and therefore, the sequence $\{p_n\}_{n=0}^{\infty}$ must have a non-negative limit; we denote it by $\bar{p}$. The convergence $p_n \to \bar{p}$ and the relation $d_{n-1} = \lambda^{-1}(p_n - p_{n-1})$ (that follows from (15)) ensure then that $d_n \to 0$. Therefore, $(\bar{p}, 0)$ is an equilibrium of (1). Due to Theorem 1, $\bar{p} = 0$. This proves the lemma's assertion.

Case 2: Now we consider the case where d_0 and p_0 have the same sign. We shall assume that $p_0 \geq 0$ and $d_0 \geq 0$; the opposite case can be treated by exactly the same argument.

We observe initially that

$$\text{if for some } n \geq 1,\ d_{n-1} \geq 0 \text{ and } p_{n-1} \geq 0 \text{ then } p_n \geq p_{n-1} \text{ and } d_n \leq d_{n-1}. \tag{38}$$

The first conclusion in the implication (38) stems directly from its assumption and from (15). For the second one, we argue as follows. If $\mathcal{R}(p_n) \neq \pm\infty$ then the following relations hold:

$$|d_n| \overset{(16),(24)}{=} |g_{p_n}(d_{n-1}) - g_{p_n}(\mathcal{R}(p_n))| \overset{(20)}{<} |d_{n-1} - \mathcal{R}(p_n)|. \tag{39}$$

From (39), we obtain that $|d_n| < |d_{n-1} - \mathcal{R}(p_n)|$. Next, from $p_n \geq 0$ and (25), we obtain that $\mathcal{R}(p_n) \geq 0$. The two inequalities just obtained and the assumption $d_{n-1} \geq 0$ of (38) can hold altogether only if $d_n \leq d_{n-1}$. This completes the proof of (38) in case $\mathcal{R}(p_n) \neq \pm\infty$. Let us analyse the opposite case. First, we note that the inequality $p_n \geq 0$ (which is the consequence of the assumption $p_{n-1} \geq 0$ and our conclusion $p_n \geq p_{n-1}$) and the property (25) altogether imply that $\mathcal{R}(p_n)$ can only be $+\infty$. This conclusion and property (24) ensure then that $g_{p_n}(d) < 0$ for all d, and consequently, that $d_n = g_{p_n}(d_{n-1}) < 0$. Since $d_{n-1} \geq 0$ has been assumed then $d_n \leq d_{n-1}$ follows and (38) is established.

Consider now an orbit $\{(p_n, d_n)\}_{n=0}^{\infty}$ satisfying our assumption $p_0 \geq 0$ and $d_0 \geq 0$. Obviously, either $d_n \geq 0$ for all n, or we shall find $m < \infty$ for which $d_m \leq 0$; there may be several such numbers, let m denote the smallest of them. For this m it holds that $p_m \geq 0$. This is ensured by recursive application of (38) for all $n \leq m$. Hence, the portion $\{(p_n, d_n)\}_{n=m}^{\infty}$ of the considered orbit can be treated by the argument of *Case 1*. By this argument, the lemma's assertion follows. It remains only to prove the lemma for the case when $d_n \geq 0$ for all n. We do this below.

Assume (in addition to the assumption $p_0 \geq 0$ and $d_0 \geq 0$ already made) that $d_n \geq 0$ for all n. The relation (38) implies that in this case, $\{p_n\}$ is a non-decreasing sequence of non-negative numbers. If it is unbounded from above, then so is the sequence $\{\mathcal{R}(p_n)\}$ in virtue of Lemma 2 (a). But since, by the very definition, $|d_n| \leq 1$ for all n, then there must exist k such that $d_k < \mathcal{R}(p_{k+1})$. For this k,

$$d_{k+1} \overset{(16)}{=} g_{p_{k+1}}(d_k) \begin{cases} \overset{(21),\, d_k<\mathcal{R}(p_{k+1})}{<} \quad g_{p_{k+1}}(\mathcal{R}(p_{k+1})) \overset{(24)}{=} 0, \text{ when } \mathcal{R}(p_{k+1}) \text{ is finite,} \\ < 0, \text{ since } g_{p_{k+1}}(\cdot) < 0 \text{ when } \mathcal{R}(p_{k+1}) = +\infty \end{cases}$$

(we used above that $\mathcal{R}(p_{k+1})$ cannot be $-\infty$ when $p_{k+1} \geq 0$) and therefore, $d_{k+1} < 0$. This contradicts the current assumption that $d_n \geq 0$ for all n. Hence, $\{p_n\}$ must be bounded and thus, converges to a finite limit $\bar{p}$. From this convergence we deduce that $(p_n, d_n) \to (0,0)$ in the same way as it has been done four paragraphs above. □

Lemma 4. *Let*[8] *k be the minimal positive integer for which $b^k \leq \frac{1}{2}\frac{J\alpha(1-b)}{b}$ (the condition $b \in (0,1)$ guarantees the existence of such k). Let λ_c be defined as the maximal positive real number satisfying the following inequalities:*

$$(i)\ \lambda_c k \leq \frac{1}{2}\frac{J\alpha(1-b)}{b}, \quad (ii)\ \lambda_c \leq \alpha J \frac{a}{b}, \quad (iii)\ \lambda_c \leq h. \tag{40}$$

Suppose $\lambda \in (0, \lambda_c]$. Let $(p_n, d_n), n \geq 0$, be an orbit of (1). Then,
if $0 \leq p_1 \leq \lambda d_0$, then either $(p_n, d_n) \to (0,0)$ or there exists a finite $\ell \geq 1$ such that

$$\begin{aligned} &(i)\ d_{j-1} \geq \mathcal{R}(p_j),\ (ii)\ 0 \leq d_j \leq b d_{j-1},\ (iii)\ p_{j+1} \geq p_j,\ \text{for all } j = 1, \cdots, \ell, \\ &\text{and } (iv)\ d_\ell \leq \mathcal{R}(p_{\ell+1}); \end{aligned} \tag{41}$$

if $\lambda d_0 \leq p_1 \leq 0$, then either $(p_n, d_n) \to (0,0)$ or there exists a finite $\ell \geq 1$ such that

$$\begin{aligned} &(i)\ d_{j-1} \leq \mathcal{R}(p_j),\ (ii)\ b d_{j-1} \leq d_j \leq 0,\ (iii)\ p_{j+1} \leq p_j,\ \text{for all } j = 1, \cdots, \ell, \\ &\text{and } (iv)\ d_\ell \geq \mathcal{R}(p_{\ell+1}). \end{aligned} \tag{42}$$

[8] b, a and h employed here for the definition of k and λ_c have been defined in (17), (30) and (32).

Proof. We shall conduct the proof under the assumption $0 \le p_1 \le \lambda d_0$; in the case $\lambda d_0 \le p_1 \le 0$ the argument is analogous and will not be presented.

Step 1: We claim that the following inequality stems from lemma's assumptions:

$$d_0 \ge \mathcal{R}(p_1). \tag{43}$$

Indeed, by the very definition (68), $|d_0| \le 1$ and hence the assumption $0 \le p_1 \le \lambda d_0$ implies that $0 \le p_1 \le \lambda$. This, together with the assumption $\lambda \le \lambda_c$ and the constraint (40-(iii)) yield that $0 \le p_1 \le h$ and thus we can apply Lemma 2(b). It gives:

$$\mathcal{R}(p_1) \le \frac{1}{\alpha J}\frac{b}{a}p_1. \tag{44}$$

On the other hand, the assumption $p_1 \le \lambda d_0$ together with the constraint (40-(ii)) imply that $d_0 \ge \frac{1}{\alpha J}\frac{b}{a}p_1$. This inequality and (44) yield (43).

Step 2: We get:

$$d_1 \overset{(16)}{=} g_{p_1}(d_0) \overset{(21),(43)}{\ge} g_{p_1}(\mathcal{R}(p_1)) \overset{(24)}{=} 0, \tag{45}$$

and

$$|d_1| \overset{(16),(24)}{=} |g_{p_1}(d_0) - g_{p_1}(\mathcal{R}(p_1))| \overset{(20)}{\le} b\,|d_0 - \mathcal{R}(p_1)| \le bd_0, \tag{46}$$

where the last passage in (46) is valid because $d_0 \ge 0$ (this is one of the inequalities assumed in the beginning of the proof) and $0 \le \mathcal{R}(p_1) \le d_0$ (here, the first inequality is provided by (25) and the assumption that $p_1 \ge 0$, while the second inequality is identical to (43)); note also that this double inequality ensures that $\mathcal{R}(p_1) \ne \pm\infty$ and hence, the last passage in both (45) and (46) is legitimate.

Step 3: Since $d_1 \ge 0$, as ensured by (45), then (15) yields that

$$p_2 \ge p_1. \tag{47}$$

This is the final conclusion of the third step.

The relations (45), (46) and (47) prove the validity of (i), (ii) and (iii) of (41) for $j = 1$.

Now, if $d_1 \le \mathcal{R}(p_2)$ then the lemma is proved with $\ell = 2$. Thus, we continue the proof assuming the contrary: $d_1 > \mathcal{R}(p_2)$. Obviously, this assumption implies that $d_1 \ge \mathcal{R}(p_2)$. Taking this relation in the place of (43) and repeating the second and the third steps of the argument presented above[9] we deduce the relations $0 \le d_2 \le bd_1$ and $p_3 \ge p_2$. These relations and the inequality $d_1 \ge \mathcal{R}(p_2)$ imply that (i), (ii) and (iii) of (41) hold for $j = 2$.

It is obvious that the argument of the above paragraph can be repeated for $j > 2$ provided $d_j > \mathcal{R}(p_{j+1}$. However, the repetition process cannot last forever, unless $p_1 = 0$. The reason for this is the following. After j consecutive repetitions, we would have that $d_j \le b^j d_0$ (in virtue of (41-ii)) and $p_{j+1} \ge p_1 \ge 0$ (in virtue of the assumption

[9]Note that we do not need to repeat the argument's first step since its conclusion (43) is now a direct consequence of our assumption. Hence, we will not employ Lemma 2(b) in the current and the consequent repetitions. Getting rid of the necessity for the use of this lemma is here, because we cannot guarantee that $|p_j| \le h$, for $j \ge 2$, and therefore, we cannot ensure the validity of lemma's assumptions at j-th repetition for $j \ge 2$.

$p_1 \geq 0$ and the property (41-iii)). Thus, the sequence $\{d_j\}$ decreases to zero, while the sequence $\{\mathcal{R}(p_j)\}$ possesses – in virtue of Lemma 2(a) – the following property: $\mathcal{R}(p_{j+1}) \geq \frac{1}{\alpha J}\frac{a}{b}p_{j+1} \geq \frac{1}{\alpha J}\frac{a}{b}p_1$. Consequently, if $p_1 \neq 0$ then there must be a finite ℓ for which $d_\ell \leq \mathcal{R}(p_{\ell+1})$, and the lemma is proved.

To finish the proof, we have to complete the argument of the above paragraph by analyzing the case $p_1 = 0$. Suppose in addition that $d_0 = 0$. It then follows directly from (1) that $(p_j, d_j) = (0,0)\,\forall j \geq 1$. This conclusion completes the proof since one of the alternatives in the lemma's assertion is that $(p_n, d_n) \to (0,0)$.

The last case to be considered is thus, $p_1 = 0$ and $d_0 > 0$. In this case, $g_{p_1}(\cdot) \equiv g_0(\cdot)$. We note that (17) and (3-d) ensure that $g_0(0){=}1 - 2\Phi(0){=}0$. Hence, we get:

$$d_1 \overset{(16)}{=} g_{p_1}(d_0) \overset{p_1=0}{=} g_0(d_0) \overset{d_0>0 \text{ and } (21)}{>} g_0(0) = 0. \tag{48}$$

From (15) and (48), we get that $p_2 = \lambda d_1 > 0$. This implies that $0 \leq p_2 \leq \lambda d_1$. With the latter double inequality in the place of $0 \leq p_1 \leq \lambda d_0$ we repeat the whole argument starting from the beginning of the proof and finishing at the end of the above paragraph. In the repetition, we shall not stumble upon the inconvenient possibility "$p_2 = 0$", since we have just shown that $p_2 > 0$. The result is the following conclusion: either $(p_n, d_n) \to (0,0)$ or (41) holds with $j = 2, \ldots, \ell$ for some finite $\ell \geq 2$. As for the validity of (41-i, ii, iii) for $j = 1$, it has been already established (read the sentence after (47)). This completes the proof of the lemma. □

Lemma 5. *Suppose $\lambda \in (0, \lambda_c]$ where λ_c satisfies (40). Let $(p_n, d_n), n \geq 0$, be an orbit of (1). Then,*
if $0 \leq d_0 \leq \mathcal{R}(p_1)$ then either $(p_n, d_n) \to (0,0)$ or there exists $m \geq 1$ such that

$$\text{(i) } \mathcal{D}(p_1) \leq d_j \leq 0, \text{ for } j = 1, \cdots, m, \text{ (ii) } 0 \leq p_j \leq p_{j-1}, \text{ for } j = 2, \cdots, m, \text{ and (iii) } \lambda d_m \leq p_{m+1} \leq 0; \tag{49}$$

and if $0 \geq d_0 \geq \mathcal{R}(p_1)$ then either $(p_n, d_n) \to (0,0)$ or there exists $m \geq 1$ such that

$$\text{(i) } \mathcal{D}(p_1) \geq d_j \geq 0, \text{ for } j = 1, \cdots, m, \text{ (ii) } 0 \geq p_j \geq p_{j-1}, \text{ for } j = 2, \cdots, m, \text{ and (iii) } \lambda d_m \geq p_{m+1} \geq 0. \tag{50}$$

Proof. We shall conduct the proof under the assumption $0 \leq d_0 \leq \mathcal{R}(p_1)$; in the case $0 \geq d_0 \geq \mathcal{R}(p_1)$ the argument is analogous and hence will not be presented.

First, we conclude that $d_1 \leq 0$ reasoning as follows. If $\mathcal{R}(p_1) \neq +\infty$, then

$$d_1 \overset{(16)}{=} g_{p_1}(d_0) \overset{(21), \text{ and the assumption } d_0 \leq \mathcal{R}(p_1)}{\leq} g_{p_1}(\mathcal{R}(p_1)) \overset{(24)}{=} 0. \tag{51}$$

If $\mathcal{R}(p_1) = +\infty$ then in virtue of (24), $g_{p_1}(d) < 0$ for all d, and hence $d_1 = g_{p_1}(d_0) < 0$.

Second, from (25) and the assumption $\mathcal{R}(p_1) \geq 0$ we get that $p_1 \geq 0$ and therefore, in virtue of Lemma 1, $\mathcal{D}(p_1) \leq 0$. The latter and the assumption $d_0 \geq 0$ imply that $d_0 \geq \mathcal{D}(p_1)$. We use this relation in the calculations below:

$$d_1 \overset{(16)}{=} g_{p_1}(d_0) \overset{(21), d_0 \geq \mathcal{D}(p_1)}{\geq} g_{p_1}(\mathcal{D}(p_1)) \overset{(23)}{=} \mathcal{D}(p_1), \tag{52}$$

and we conclude thus, that $d_1 \geq \mathcal{D}(p_1)$.

At the third step, we turn our attention to p_2, but consider separately the cases $p_2 \leq 0$ and $p_2 > 0$.

Let us assume that $p_2 \leq 0$. Our aim is to prove that in this case (49) is true for $m = 1$. Since it has been proved in the previous two steps that $\mathcal{D}(p_1) \leq d_1 \leq 0$ and since (49-(ii)) is void for $m = 1$, then our aim is achieved as soon as we prove that $\lambda d_1 \leq p_2 \leq 0$. In this relation, the second inequality is exactly our current assumption, and the first inequality is valid because $p_2 = p_1 + \lambda d_1$ (the relation (15)) and because $p_1 \geq 0$ (proved at the second step above). The lemma is therefore proved in case $p_2 \leq 0$.

Let us now assume that $p_2 > 0$. Our objective is to show that (49-(i)) and (49-(ii)) are satisfied with $j = 2$.

We start by establishing (49-(ii)) for $j = 2$. The first inequality is a direct consequence of our current assumption. The second inequality stems from the relation $p_2 = p_1 + \lambda d_1$ (the eq. (15)) and the inequality $d_1 \leq 0$ that has been proved in the first step.

We proceed by proving (49-(i)) for $j = 2$. Using $d_1 \leq 0$ (proved in the first step) and $p_2 > 0$ (the current assumption) we get:

$$d_2 \overset{(16)}{=} g_{p_2}(d_1) \overset{(21),\, d_1 \leq 0}{\leq} g_{p_2}(0) \overset{(21),(22),\, p_2>0}{\leq} g_{p_2}\left(\frac{p_2}{J}\right) \overset{(22)}{\leq} 0,$$

that proves the second inequality in (49-(i)) for $j = 2$. Thus, to achieve our objective, we have only to show that

$$d_2 \geq \mathcal{D}(p_1). \tag{53}$$

We split the argument in two parts. First we prove (53) under the assumption that $d_1 \geq \mathcal{D}(p_2)$, and then under the assumption that $d_1 \leq \mathcal{D}(p_2)$.

Assume that $d_1 \geq \mathcal{D}(p_2)$. This yields the desired relation (53) as follows:

$$d_2 \overset{(16)}{=} g_{p_2}(d_1) \overset{(21),\, d_1 \geq \mathcal{D}(p_2)}{\geq} g_{p_2}(\mathcal{D}(p_2)) \overset{(23)}{=} \mathcal{D}(p_2) \overset{(26) \text{ and } 0 \leq p_2 \leq p_1}{\geq} \mathcal{D}(p_1) \tag{54}$$

(note that the inequality $0 \leq p_2 \leq p_1$ used in the last passage is provided by (49-(ii)) for $j = 2$ that we have proved above).

Suppose now that $d_1 \leq \mathcal{D}(p_2)$. Then

$$d_2 \overset{(16)}{=} g_{p_2}(d_1) \overset{(21), \text{ and the assumption } d_1 \leq \mathcal{D}(p_2)}{\leq} g_{p_2}(\mathcal{D}(p_2)) \overset{(23)}{=} \mathcal{D}(p_2) \tag{55}$$

that ensures that $d_2 \leq \mathcal{D}(p_2)$. Next, we note that the assumption $p_2 > 0$ and the property (27) imply that $\mathcal{D}(p_2) \leq 0$. Because of this inequality, the following two relations are valid in the present case: $d_1 \leq \mathcal{D}(p_2) \leq 0$ and $d_2 \leq \mathcal{D}(p_2) \leq 0$. These relations and the inequality

$$|d_2 - \mathcal{D}(p_2)| \overset{(16)}{=} |g_{p_2}(d_1) - g_{p_2}(\mathcal{D}(p_2))| \overset{(20)}{<} |d_1 - \mathcal{D}(p_2)| \tag{56}$$

can hold altogether only if $d_1 \leq d_2$. The latter and the relation $\mathcal{D}(p_1) \leq d_1$ imply (53).

We can now conclude that if $p_2 > 0$ then (i) and (ii) of (49) hold for $j = 2$. Recall that in the first two steps of the proof we have already concluded that (49-(i)) is valid for $j = 1$. Combining these conclusions, we close our argument's third step: when $p_2 > 0$ then (i) and (ii) of (49) hold for $m = 2$.

The third step of our argument, i.e., the proof that started right after (52) and finished above, can be repeated for p_3, p_4, etc. until we find m such that $p_{m+1} \leq 0$. The argument will then ensure that (49-(iii)) holds for the value of m founded, and the previous recursion steps will ensure that (49-(i)) and (49-(ii)) hold for this value of m. This finishes the proof of the lemma in case m is founded. If, to the contrary, such an m does not exist then the recursion ensures – via (49-(ii)) – that $\{p_n\}_{n=1}^{\infty}$ is a monotone non-increasing sequence bounded below by 0. Let $\bar{p}$ denote its limit. Then, from the equation (15) we get that $d_n = (\lambda)^{-1}(p_n - p_{n-1})$, and therefore, $d_n \to 0$ as $n \to \infty$. But, in virtue of Theorem 1, the $n \to \infty$ limit of (p_n, d_n) can only be $(0,0)$. This completes the proof of the lemma. □

Proof of Thm. 4. In virtue of Lemma 3, in order to establish the theorem, it is sufficient to prove that $(p_n, d_n) \to (0,0)$, in case when either $\lambda d_0 \geq p_1 \geq 0$ or $\lambda d_0 \leq p_1 \leq 0$. We shall assume that

$$\lambda d_0 \geq p_1 \geq 0 \tag{57}$$

for the rest of the proof. In the case $\lambda d_0 \leq p_1 \leq 0$, the proof is the same up to obvious changes of the inequality directions.

The assumption (57) allows us to apply Lemma 4. Let ℓ be the integer provided by it (if ℓ does not exist then $(p_n, d_n) \to (0,0)$ in accordance to the lemma and thus, the theorem is proved).

First, let us prove that

$$p_\ell \leq \frac{1}{2}\frac{J\alpha(1-b)}{b}d_0. \tag{58}$$

For the proof, we shall need the fact that $p_\ell \geq 0$. It follows from the assumption $p_1 \geq 0$ via a recurrent application of the relation (41-iii) (that is valid due to Lemma 4). The continuation of our argument depends on whether $(\ell-1) \geq k$ or $(\ell-1) < k$, where, recall, k has been defined in Lemma 4 as the minimal positive integer for which $b^k \leq \frac{1}{2}\frac{J\alpha(1-b)}{b}$; note that k depends solely on the parameters of the studied dynamical system, and hence can be applied to any its orbit.

If $(\ell-1) \geq k$ we argue as follows. The inequality $p_\ell \geq 0$ just proved and the property (25) yield the inequality $p_\ell \leq J\mathcal{R}(p_\ell)$, from which we get (58) via the following chain of estimates:

$$\begin{aligned}
p_\ell \leq J\mathcal{R}(p_\ell) &\leq Jd_{\ell-1} && \text{(due to (41-i) of Lemma 4 for } j=\ell)\\
&\leq Jb^{\ell-1}d_0 && \text{(from (41-ii) of Lemma 4 applied for } j=1,\cdots,\ell-1)\\
&\leq Jb^k d_0 && \text{(because } (\ell-1)\geq k \text{ and } b<1)\\
&\leq \tfrac{1}{2}\tfrac{J\alpha(1-b)}{b}d_0 && \text{(by the definition of } k).
\end{aligned}$$

If $(\ell-1) < k$, then (58) follows from the following chain of inequalities:

$$\begin{aligned}
p_\ell &= p_1 + \lambda d_1 + \cdots + \lambda d_{\ell-1} && \text{(by recurrent application of (15))}\\
&\leq \lambda d_0 + \lambda d_1 + \cdots + \lambda d_{\ell-1} && \text{(using the first inequality of (57))}\\
&\leq \ell\lambda d_0 && \text{(using (41-ii) of Lemma 4 for } j=1,\cdots,\ell-1,\\
& && \quad\text{and taking into account that } b<1)\\
&\leq k\lambda d_0 && \text{(because } (\ell-1)<k \text{ in the considered case)}\\
&\leq \tfrac{1}{2}\tfrac{J\alpha(1-b)}{b}d_0 && \text{(by the condition (40-(i)) on } \lambda).
\end{aligned}$$

Second, from the inequality $p_\ell \geq 0$ just established and Lemma 1 we get that

$$\mathcal{D}(p_\ell) \geq -\frac{b}{\alpha(1-b)} p_\ell. \tag{59}$$

Third, we combine the conclusions (58) and (59) to deduce that

$$\mathcal{D}(p_\ell) \geq -\frac{1}{2} d_0. \tag{60}$$

The fourth step is based on the double inequality $0 \leq d_\ell \leq \mathcal{R}(p_{\ell+1})$; its first part is ensured by (41-ii) with $j = \ell$ and the second part by (41-iv). We take this inequality in the place of the assumption of Lemma 5 and derive then from the lemma that either $(p_n, d_n) \to (0, 0)$ and therefore the theorem is proved, or

$$\text{there exists } m \geq 1 \quad \text{such that} \quad \mathcal{D}(p_{\ell+1}) \leq d_j \leq 0, \text{ for } j = \ell+1, \cdots, \ell+m, \tag{61}$$

$$\text{and} \qquad \lambda d_{\ell+m} \leq p_{\ell+m+1} \leq 0. \tag{62}$$

Combining (60) and (61), we get that

$$-\frac{1}{2} d_0 \leq d_{\ell+j} \leq 0, \text{ for } j = 1, \ldots, m. \tag{63}$$

The inequalities (63) just derived, the inequalities (41-ii) for $j = 1, \ldots, \ell$, and the inequality $b < 1$ lead altogether to the following conclusion:

in the block $d_1, \ldots, d_\ell, d_{\ell+1}, \ldots, d_{\ell+m}$ of the sequence $\{d_n\}_{n=0}^\infty$,
the values of the first $\ell (\ell \geq 1)$ members are between 0 and d_0 (64)
while the values of the last $m (m \geq 1)$ members are between $-\frac{1}{2} d_0$ and 0.

The argument that started at (57) and finished at (64) can be repeated with the inequality (62) in the place of (57). The respective conclusion is that the block $d_1, \ldots, d_{\ell+m}$ is followed by another block – of the size 2 at least – that consists of two non-empty parts such that: the members of the first part are all between $d_{\ell+m}$ and 0, and the members of the second part are all between 0 and $-\frac{1}{2} d_{\ell+m}$. This implies in virtue of the inequality $|d_{\ell+m}| \leq \frac{1}{2} d_0$ (that stems from (64)), that the absolute value of each members of the block does not exceed $\frac{1}{2} d_0$.

It is obvious that the argument can be repeated yielding at the i-th step the conclusion that the members of the corresponding block of the sequence $d_1, d_2, \ldots$ do not exceed $\left(\frac{1}{2}\right)^{i-1} d_0$, in modulus. From this, $d_n \to 0$ follows.

To complete the proof, it is only left to show that $p_n \to 0$. Suppose that this is not the case. Then, there exists $\varepsilon > 0$ and a sequence of integers $\{n_j\}_{j \in \mathbb{N}}$ such that $n_j \uparrow \infty$ and $|p_{n_j}| \geq \varepsilon$ for each j. We suppose, without loss of generality that $p_{n_j} \geq \varepsilon$. Thus:

$$d_{n_j} \overset{(16)}{=} g_{p_{n_j}}(d_{n_j - 1}) \overset{(28)}{\leq} g_\varepsilon(d_{n_j - 1}). \tag{65}$$

The inequality (65), the continuity of $g_p(\cdot)$,[10] and our conclusion $d_n \to 0$ imply altogether that $0 \leq g_\varepsilon(0)$. But, on the other hand, (21) and (22) ensure that $0 > g_\varepsilon(0)$. Hence, by contradiction, we conclude that $p_n \to 0$.

We thus have proved that $d_n \to 0$ and $p_n \to 0$ for arbitrary p_0 and d_0. This is the theorem's assertion. □

[10] $g_p(\cdot)$ is continuous because it is differentiable – see (18).

3 Application

3.1 Our HIA model for a single risky asset market

The bridge that leads from our results to their applications is grounded on the Heterogeneous Interacting Agent Model to be presented now. We shall use the abbreviation "HIAM". When we need to distinguish it from other HIAMs, we shall refer to it as "our HIAM".

The model's *time set* is $\{0, 1, 2, \ldots\}$. The model's parameter denoted by K and called *the number of model's agents* is an arbitrary natural number. There are abstract *agents* in the model; they are labeled by the numbers $1, 2, \ldots, K$. The other model's parameter, denoted by α and called *the proportion of speculators*, is an arbitrary rational number in the interval $(0, 1)$. It plays the following role in the model: the agents numbered $1, 2, \ldots, \alpha K$ are called *speculators*, and the rest of the agent population, i.e., the the agents numbered $\alpha K + 1, \alpha K + 2, \ldots, K$, are called *fundamentalists.*[11] The choice of these names will be justified a few paragraphs below.

To each time $n \geq 1$ and to each agent labeled k, the model associates a random variable denoted by $d_n(k)$. Its definition is as follows (the quantities involved in (66) and (67) will be specified below; in particular, note that $v_n(k)$ will be a random variable – this is the source of the randomness of $d_n(k)$):

$$d_n(k) = \begin{cases} +1, \text{ if } & J\bar{d}_{n-1} - \{p_n - (\bar{v} + v_n(k))\} > 0 \\ -1, \text{ if } & J\bar{d}_{n-1} - \{p_n - (\bar{v} + v_n(k))\} \leq 0 \end{cases} \quad \begin{array}{r} \text{if } k = 1, 2, \ldots, \alpha K, \\ \text{i.e., if } k \text{ is a speculator,} \end{array} \tag{66}$$

$$d_n(k) = \begin{cases} +1, \text{ if } & -\{p_n - (\bar{v} + v_n(k))\} > 0 \\ -1, \text{ if } & -\{p_n - (\bar{v} + v_n(k))\} \leq 0 \end{cases} \quad \begin{array}{r} \text{if } k = \alpha K + 1, \alpha K + 2, \ldots, K, \\ \text{i.e., if } k \text{ is a fundamentalist.} \end{array} \tag{67}$$

$d_n(k)$ is called *the decision of agent k at time n*; when $d_n(k)$ is $+1$ we say that "the agent wishes to buy an asset share", and when it is -1 we say that "he wishes to sell an asset share".[12] This "buy/sell" interpretation helps to understand the link of the model with the asset market and the names that we shall attribute to the model's parameters and variables introduced below.

Let us specify the quantities involved in (66) and (67).

First,

$$\bar{d}_n := \frac{1}{K} \sum_{k=1}^{K} d_n(k), \quad n = 1, 2, \ldots, \tag{68}$$

and in accordance with this definition and with the interpretation of each $d_n(k)$, the real world analogue of $\bar{d}_n$ is the population relative excess demand for the asset at time n; the name for the analogue will be the name for $\bar{d}_n$ in the model. However, when possible, we shall shorten this name to just the *excess demand.* Observe that (68) defines $\bar{d}_n$ for $n \geq 1$.

[11] If α is settled before K – as it will happen in Thm. 5 and Corollary 1 below – then αK of the present definition should be substituted by $\lfloor \alpha K \rfloor$. When this modification is adapted, α may be any real from $(0, 1)$ and not neccessarely rational.

[12] In our HIAM, there is no an equivalent of the market clearing condition. Therefore, the one who wishes to buy/sell will not always accomplish this wish. By this reason, we say "wishes to buy/sell" and not simply "buys/sells".

As for $\bar{d}_0$, it is one of the model's parameters called *the initial excess demand*; its value range is $[-1, 1]$. The value of $\bar{d}_0$ as well as of p_0 to be defined below, must be specified in order to "switch on" the model's evolution.

Second, p_n is the model's quantity that corresponds to the price of one share of the asset in the market mimicked by the model.[13] We call p_n *the asset price at time* n, or simply *price*. The value range of p_n is $\mathbb{R}$ for each time $n = 0, 1, 2, \ldots$.

The model's price evolves in time. The evolution rule is as follows: p_0 is one of the model's parameters (it is called *the price initial value*), and all others p_n's are determined by the recurrent relation

$$p_{n+1} = p_n + \lambda \bar{d}_n, \quad n = 0, 1, 2, \ldots, \tag{69}$$

where λ is a positive real model's parameter called *the feedback of the excess demand on price increment*. The recurrent relation (69) called *price update rule* mirrors the demand-and-supply law of real markets. Indeed, when $\bar{d}_n$ is positive [resp., negative] – which means that there are more buyers than sellers [resp., sellers than buyers] – the rule rises [resp., lowers] the price.

Third, we specify and interpret $\bar{v}$ and $v_n(k)$'s. $\bar{v}$ is a real value which is one of the model's parameters, and $\{v_n(k),\ n = 1, 2, \ldots,\ k = 1, 2, \ldots, K\}$ are independent among themselves and of all other random variables present in the model, with a common distribution function that will be denoted by Φ; this function is another model's parameter. We assume that Φ satisfies the the conditions (3). The value $\bar{v} + v_n(k)$ is called *the evaluation of the asset's fundamental value by agent* k *at time* n, or, simply, *individual evaluation*. The name reflects the quantity that we want to mimic by $\bar{v} + v_n(k)$ in our model, namely: what the agent k thinks the asset worths at time n, or as in traditional economical terms, the asset's fundamental value evaluated by agent k at time n.

In the real world, asset's fundamental value is usually calculated from the official balance reports of the company that issued the asset. However, an individual evaluation of this value may not coincide with the officially calculated one because an individual may have privileged information (about the future company development, say), or may use his own way to deduce the fundamental value from the report data. This situation is mirrored in our model in the following way: $\bar{v}$ corresponds to the official asset's fundamental value that may be simply the average taken over all individual evaluations, and differences between this value and individual evaluations are modeled by the random variables v's with the distribution function Φ. Accordingly, the name for $\bar{v}$ in our model is *the market fundamental value of the asset*, and the name for Φ is *the distribution of deviation around* $\bar{v}$ *of individual evaluations of the fundamental value of the asset*.

Up to now, we have introduced and interpreted all the quantities involved in the decision rule (67). This allows us to make the following observation: an agent that "acts" due to this rule, will wish to buy one asset share if he thinks that the current price (i.e., p_n) is less than what the asset share worths in accordance to his current evaluation (i.e., $\bar{v} + v_n(k)$); otherwise, he will wish to sell one asset share. The one who behaves in this

[13] Actually, it corresponds to the to the logarithm of the price of one share of asset. By using the logarithm, we allow the model's counterpart of price be negative. This simplifies mathematical treatment of the model. Nevertheless, the name for p_n in the model will be "price" rather than "logarithm of price".

way in the real market, is called *fundamentalist*. This explains why we use this name for our model agents that obey the rule (67).

Finally, we explain the role of J. This model's parameter is a positive real number that mirrors *the social susceptibility*, or simply *susceptibility* of market agents. It models "susceptibility" because its use in the rule (66) makes it to correspond to the strength with which the population excess demand influences an individual to align his decision in accordance with this excess. Now observe that the population excess demand is the average of decisions taken over the whole population and hence the alignment means the coincidence of an individual decision with that of the majority of the population. Hence, the "susceptibility" mirrored by J has a "social" aspect of an individual behavior.

The parameter J is also interpreted as the *traders' speculative trend*. The justification for this is as follows. In our model, the population excess demand is proportional to the price trend (by the price trend we mean $p_{n+1} - p_n$, and the proportionality is yielded by (69)). Consequently, the agent who aligns his decision with the population excess demand acts as an asset market speculator (i.e., an individual that wishes to buy [resp., sell] when the trend indicates the price will rise [resp., fall]). This fact explains why the model's agents that "use" the decision rule (66) are called *speculators*. However, these agents are not pure speculators because the rule (66) mixes the speculative behavior and the fundamentalist behavior (explained two paragraphs above). In this mixture, the weight of the speculative component is given by J. This role of J suggests its interpretation as the market traders' speculative trend.

3.2 Why and how the studied dynamical system mimics evolution of asset price and excess demand in asset markets

Corollary 1 will answer the question posed in this section title. It grounds upon Theorem 5 and Lemma 6, both to be stated below.

Theorem 5 (point-wise in time almost sure convergence of the price and the excess demand of our HIAM.)
Choose a distribution function Φ satisfying the conditions (3) and real numbers $\alpha \in (0,1)$, $\lambda > 0$, $\bar{v} \in \mathbb{R}$, $J > 0$, $p_0 \in \mathbb{R}$ and $d_0 \in [-1,1]$.

For each $K = 1, 2, \ldots$, and the entities chosen above construct the HIA model as defined in Section 3.1 obeying $\bar{d}_0 = d_0$; denote the price and the excess demand at time n of the constructed HIAM by $p_n^{st,K}$ and $\bar{d}_n^{st,K}$, respectively.[14]

Let $\{(p_n\,,\ d_n)\,,\quad n = 0, 1, 2 \ldots\}$ denote the orbit of the dynamical system

$$\begin{cases} p_n = p_{n-1} + \lambda d_{n-1} \\ d_n = \alpha\big[1 - 2\Phi(p_n - \bar{v} - J d_{n-1})\big] + (1-\alpha)\big[1 - 2\Phi(p_n - \bar{v})\big] \end{cases} \tag{70}$$

where p_0 is the number chosen above and $d_0 = \bar{d}_0$.

Then, for any "time" $n \in \mathbb{N}$, it holds that

$$\bar{d}_n^{st,K} \to d_n \text{ and } p_n^{st,K} \to p_n \text{ almost surely, as } K \to \infty.$$

[14]The purpose of these modifications is to emphasize that the price and the demand in the HIA model constructed in Section 3.1 are <u>stochastic</u> quantities and that they depend on K, the model's population size.

Proof.[15] In order to simplify the notations, we shall accept that α is a rational number and that K tends to ∞ assuming the numbers such that αK is an integer (the footnote 11 helps to understant why this simplifies notations and how to adapt the proof to the generic situation).

We recall from the construction of our HIAM, that $v_n(k)$ denotes the random variable with the distribution function Φ and that it is used in our model to determine the decision of agent k at time n. All these random variables were postulated to be independent across the agents and time; this independence is essential for the derivation of (71) below. Let, as usual,

$$\mathrm{I}_{\{v_n(k)\le x\}} := \begin{cases} 1, & \text{if } v_n(k) \le x \\ 0, & \text{if } v_n(k) > x \end{cases}, \qquad x \in \mathbb{R},\ n = 1,2,\dots,\ k = 1,2,\dots,$$

and let next, for each time n and every population size K,

$$F_n^{st,K}(x) := \frac{1}{\alpha K}\sum_{k=1}^{\alpha K} \mathrm{I}_{\{v_n(k)\le x\}}, \qquad S_n^{st,K}(x) := \frac{1}{(1-\alpha)K}\sum_{k=\alpha K+1}^{K} \mathrm{I}_{\{v_n(k)\le x\}}, \qquad x \in \mathbb{R}.$$

Since $\mathbb{E}[\mathrm{I}_{\{v_n(k)\le x\}}] = \Phi(x)$, then $\forall x$ and $\forall n$, $F_n^{st,K}(x) \to \Phi(x)$ and $S_n^{st,K}(x) \to \Phi(x)$ almost surely as $K \to \infty$. Applying to these convergence results the argument that establishes the Glivenko-Cantelli Theorem (see [2], p. 59-60), we get the following almost sure uniform convergence result that will be used to close the proof:

$$\forall n, \quad \lim_{K\to\infty}\sup_{x\in\mathbb{R}}\left|F_n^{st,K}(x) - \Phi(x)\right| = 0, \quad \lim_{K\to\infty}\sup_{x\in\mathbb{R}}\left|S_n^{st,K}(x) - \Phi(x)\right| = 0 \text{ a.s.} \tag{71}$$

We shall now prove the theorem's statement by induction in time n.

The induction basis. By the assumptions of the theorem, the value of p_0 is attributed to $p_0^{st,K}$, for every K, and the value of $\bar{d}_0$ is attributed both to d_0 and to $\bar{d}_0^{st,K}$, for every K. Hence, the theorem's assertion is valid in the case $n = 0$.

The induction step. We assume that $p_{n-1}^{st,K} \to p_{n-1}$ and $\bar{d}_{n-1}^{st,K} \to d_{n-1}$ almost surely as $K \to \infty$.

Since $p_n^{st,K}$ is given by (69) and p_n is given by the first equation of the system (70) then the similarity of the equation and the induction assumption lead immediately to the conclusion that $p_n^{st,K} \to p_n$ almost surely as $K \to \infty$. To prove the convergence of $\bar{d}_n^{st,K}$, we shall need the following expansion (that stems straightforwardly from the above constructions and (66-68)):

$$\begin{aligned}
\bar{d}_n^{st,K} &= \frac{\alpha}{\alpha K}\sum_{k=1}^{\alpha K} d_n(k) + \frac{1-\alpha}{(1-\alpha)K}\sum_{k=\alpha K+1}^{K} d_n(k) \\
&= \frac{\alpha}{\alpha K}\sum_{k=1}^{\alpha K}\left(1 - 2\mathrm{I}_{\{v_n(k)\le p_n^{st,K} - \bar{v} - J\bar{d}_{n-1}^{st,K}\}}\right) + \frac{1-\alpha}{(1-\alpha)K}\sum_{k=\alpha K+1}^{K}\left(1 - 2\mathrm{I}_{\{v_n(k)\le p_n^{st,K} - \bar{v}\}}\right) \\
&= \alpha\left(1 - 2F_n^{st,K}\left(p_n^{st,K} - \bar{v} - J\bar{d}_{n-1}^{st,K}\right)\right) + (1-\alpha)\left(1 - 2S_n^{st,K}\left(p_n^{st,K} - \bar{v}\right)\right), \ \forall n, \forall K.
\end{aligned}$$

[15]The ideas of this proof have been employed already in [1], but there, the dynamical system is one-dimensional.

This expansion and the triangular inequality altogether yield that:

$$\begin{aligned}\left|\bar{d}_n^{st,K} - d_n\right| \leq\ & 2\alpha\left|F_n^{st,K}\left(p_n^{st,K} - \bar{v} - J\bar{d}_{n-1}^{st,K}\right) - \Phi\left(p_n^{st,K} - \bar{v} - J\bar{d}_{n-1}^{st,K}\right)\right| \\ & +2(1-\alpha)\left|S_n^{st,K}\left(p_n^{st,K} - \bar{v}\right) - \Phi\left(p_n^{st,K} - \bar{v}\right)\right| \\ & +2\alpha\left|\Phi\left(p_n^{st,K} - \bar{v} - J\bar{d}_{n-1}^{st,K}\right) - \Phi\left(p_n - \bar{v} - Jd_{n-1}\right)\right| \\ & +2(1-\alpha)\left|\Phi\left(p_n^{st,K} - \bar{v}\right) - \Phi\left(p_n - \bar{v}\right)\right|\end{aligned}$$

As $K \to \infty$, the first two absolute values of the r.h.s. of the above inequality converge to 0 almost surely due to the property (71), whereas the second two absolute values converge to 0 because Φ is a continuous function and because $\bar{d}_{n-1}^{st,K} \to d_{n-1}$ (the induction assumption) and $p_n^{st,K} \to p_n$ (the relation proved above). Hence, $\bar{d}_n^{st,K} \to d_n$ almost surely, as $K \to \infty$, and the proof of the induction step, and thus, of the theorem, is completed.□

It can be easily noted that (70), the dynamical system that figures in Thm. 5, is slightly different from (1), the dynamical system that we study in Section 2: the former has an "extra" parameter $\bar{v}$. However, $\bar{v}$ appears always in the expression $p_n - \bar{v}$ in the second equation of (70). Thus, if we re-write the first equation of (70) as $p_n - \bar{v} = p_{n-1} - \bar{v} + \lambda d_{n-1}$ then we get immediately the assertion of the lemma below. It allows us to "transport" any result about (1) into the framework of (70).

Lemma 6 (on the link between the dynamical systems (1) and (70)).
Suppose the dynamical systems (70) and (1) have the same values for the parameters α, λ, J, and the same distribution function Φ and let the value of the parameter $\bar{v}$ from (70) be arbitrary. Let us add the superscript $*$ to p_n and d_n in the system (70) so that they might be distinguished from p_n and d_n in the system (1). Suppose that p_0^* and d_0^*, the initial conditions for an orbit of (70), relate to p_0 and d_0, the initial conditions for an orbit of (1), as follows: $p_0 = p_0^* - \bar{v}$, and $d_0 = d_0^*$.

Then, for every $n \geq 1$, it holds that $p_n = p_n^* - \bar{v}$ and $d_n = d_n^*$.

Thm. 5 and Lemma 6 yield straightforwardly the following result.

Corollary 1 (why and how the dynamical system (1) mimics evolution of asset price and excess demand).
Let $p_0 \in \mathbb{R}$, $d_0 \in [-1, 1]$, $\alpha \in (0, 1)$, $J > 0$, $\lambda > 0$, $\bar{v} \in \mathbb{R}$, and a probability distribution function Φ satisfying the conditions (3) correspond to characteristics of a single risky asset market as specified in the Introduction.

Consider the dynamical system (1) determined by α, J, λ and Φ, and let (p_0, d_0) be its initial point; denote its orbit by $\{(p_n, d_n)\, n = 0, 1, \ldots,\}$.

Let $p_n^{st,K}$ and $\bar{d}_n^{st,K}$ be as defined in Theorem 5.

Let T be an arbitrarely fixed integer.

Then, as K, the population size from our HIAM, tends to ∞, the trajectory $\{(p_n^{st,K} - \bar{v}, \bar{d}_n^{st,K}),\ n = 0, 1, \ldots, T\}$ conveges almost surely to $\{(p_n, d_n)\, n = 0, 1, \ldots, T\}$, precisely to state:

$$\sup_{n\in[0,T]} \left|\left(p_n^{st,K} - \bar{v}\right) - p_n\right| \to 0 \text{ and } \sup_{n\in[0,T]} \left|\bar{d}_n^{st,K} - d_n\right| \to 0 \text{ almost surely, as } K \to \infty.$$

This convergence means that for any time horizon T, there exists K such that $p_0, \dots, p_T$ may be interpreted as the time evolution – up to time T – of the differences between the price for an asset and the mean evaluation of this asset's fundamental value in a single risky asset market model with the agent population size larger than K, while $d_0, \dots, d_T$ may be interpreted as the time evolution – up to T – of the excess demand for the asset in this model.

3.3 Revealing and explaining market price and population excess demand dynamics properties

In the present section, we employ Corollary 1 to interpret the results of Section 2 in terms of a real world asset market. For the interpretations to be valid, it is necessary that T and the market agent population size (both from Corollary 1) be large. We take it for granted that both are as large as necessary, when necessary.

Before we proceed, it must be noted that assumptions of the results of Section 2 do not concern the whole distribution function Φ but rather solely $\Phi'(0)$ (that is, the value of the derivative of Φ at 0). This is an interesting virtue of our results. However, when one wants to apply them to explain real world market properties this virtue becomes an obstacle since there is no a generic interpretation for $\Phi'(0)$ in terms of real world markets (that is, an interpretation that would suit any Φ). To overcome this obstacle, we shall accept the following assumption:

$$\Phi \text{ is a Normal distribution function with zero mean.} \tag{72}$$

The point here is that if σ^2 denotes the variance of Φ then this assumption implies that $\sigma^2 = \left(\sqrt{2\pi}\Phi'(0)\right)^{-1}$ which, in turn, allows us for the following interpretation:

$(\Phi'(0))^{-1}$ corresponds to the heterogeneity of the distribution of individual evaluations of the asset's fundamental value, in the sense that larger value of $(\Phi'(0))^{-1}$ corresponds to higher dispersion of evaluations over the population. (73)

The interpretation (73) is extremely convenient in re-phrasing the results of Section 2 in terms of real world asset markets. This is a strong motivation for accepting the assumption (72). There are other motivations. One of them stems from our belief that an individual is influenced by diverse factors and information streams, when he makes up his mind in respect to the fundamental value of an asset. Consequently, due to the Central Limit Theorem, individual evaluations of this value should be distributed over a population in accordance to a Normal Law. Another motivation comes from the fact that the Normal Law would be the most convenient and robust theoretical model for a populational distribution obtained by sampling from a real world population. In other words, the Normal Law would likely be chosen for Φ when fitting our HIAM to a real world socio-economic process.

Corollary 2 (of Thm. 1 obtained with the help of Cor. 1).
(a) *The market state in which*

the market asset price coincides with the asset's fundamental value and the excess demand for the asset is zero (74) *is the only possible equilibrium state for the asset price and the excess demand of an asset market.*

This means, in particular, that if the asset price and the excess demand converge to some value as time goes on, then their limit values are respectively, the asset's fundamental value and 0.

(b) (valid under additional assumption (72)) *Whether the state (74) is a locally stable or an unstable equilibrium is determined by a relation between two expressions that we present below and denote by u and w:*

$$
\begin{aligned}
u &:= 2 \times \textit{(proportion of speculators)} \times \textit{(traders' speculative trend)} \times \\
&\quad \times \textit{(the heterogeneity of the individual evaluations of the asset's fundamental value)}^{-1} \\
\textit{and} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (75) \\
w &:= 2 \times \textit{(the feedback of the excess demand on price increment)} \times \\
&\quad \times \textit{(the heterogeneity of the individual evaluations of the asset's fundamental value)}^{-1}.
\end{aligned}
$$

The determinant relation is as follows (it is illustrated in Figure 1): if

$w \leq 2u + 2$ *and* $u \leq 1$

(together with $u > 0, w > 0$ which are implicit from (75) and the natural constraints that impose that each entity in the definition (75) is positive)

then (74) is locally stable, otherwise, it is unstable.

Corollary 3 (of Thm. 4 obtained with the help of Cor. 1).
There exists a strictly positive threshold λ_0 such that when the feedback of the excess demand on the price increment is smaller than λ_0 then the local equilibrium described in Corollary 2 turns to be global, that means that the long time limit values for the asset price and the excess demand for this asset are, respectively, the asset's fundamental value and 0, whatever the initial values are.

Corollary 4 (of Thm. 3 and Conjecture 2 obtained with the help of Cor. 1 and valid under additional assumption (72)).
Suppose that, as a result of a smooth change of the parameters' values, an asset market is transferred from the parameter values region in which its equilibrium is stable to the region in which its equilibrium is unstable. Suppose that the change is such that

(i) *(a) the traders' speculative trend (i.e., the value of parameter J) increases, or*
(b) the proportion of speculators (i.e., the value of parameter α) increases, or
(c) the heterogeneity of the distribution of the individual evaluations of the asset's fundamental value diminishes (i.e., the value of $\Phi'(0)$ increases), or
(d) any combination of (a)–(c).

(ii) the feedback of the excess demand on the price increment (i.e., the parameter λ) is kept fixed, and its value is sufficiently small so that $\lambda/(\alpha J) < 2$ for all values of α and J during the process of change.

Then, in the transfer course, there will be a (maybe short) interval of time during which the market asset price and excess demand exhibit regular oscillation with non damped amplitude.

We would like to close the presentation with several comments:
(a) The last statement of Corollary 2 is important because a real world asset market always experiences endogenous and/or exogenous shocks, and therefore the stability properties of the equilibrium (74) become an important issue.
(b) In real world markets the price update increments are small, hence Corollary 3 suggests that, in the real world, when the state (74) is stable, it is actually globally stable.
(c) One can conclude from Theorem 3 and Conjecture 2 that for a specific combination of market parameters'. values, there will appear oscillations of asset price and excess demand for that asset. This conclusion has limited application since the precise estimation of market's parameters is a very difficult task. Contrasting, the conclusion presented in Corollary 4 has a practical value because it describes a qualitative aspect of market behavior. From this description, one indeed can derive useful results, like the following one: Let one know that the corollary's assumption (i), (ii) hold true and suppose one observes that a market exhibits oscillations of the asset price and the excess demand. Then, even though the oscillations might have disappeared, one can affirm that the market is in an unstable state.

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