DEPARTAMENTO DE MATEMÁTICA APLICADA

Relatório Técnico

RT-MAP-8710

TOPICS ON HAMILTONIAN SYSTEMS

W.M. Oliva, J.C.F. de Oliveira and M.S.A. Castilla

Dedicated to Jack K. Hale on the occasion of his 60^{th} birthday

DEZEMBRO 1987

TOPICS ON HAMILTONIAN SYSTEMS

by

W.M. Oliva, J.C.F. de Oliveira and M.S.A.C.Castilla

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20.570 (Agência Iguatemi)
São Paulo (SP) - Brasil

Dedicated to Jack K. Hale on the occasion of his 60th birthday.

São Paulo

Dezembro / 1987

CONTENTS

§ 1 - INTRODUCTION TO HAMILTONIAN SYSTEMS. THE THEOREM OF ARNOLD AND LIOUVILLE

Let $M = M^{2n}$ be an even dimensional differentiable manifold. A symplectic manifold is a pair $(M;\omega)$, where ω is an alternate nondegenerate and closed 2-form on M. (We will assume enough differentiability).

If (M,ω) and (N,ν) are symplectic manifolds and $f : M \to N$ is a diffeomorphism such that $f^*\nu = \omega$, that is, f is a symplectic preserving diffeomorphism, f is said to be a canonical transformation.

Example 1.1: $M = \mathbb{R}^{2n} = \{(q,p)\}$ with the natural 2-form

$$\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$$

Example 1.2: $M = T^*Q$, the cotangent bundle of any differentiable manifold Q, is a symplectic manifold. The 2-form ω will be, in this case, the derivative $d\theta$ of a 1-form θ described below. Let $\tau : T^*Q \to Q$ be the canonical projection and, for all $p_x \in T^*Q$, $\sigma_{p_x} \in T_{p_x}(T^*Q)$ one defines

$$\theta(p_x)(\sigma_{p_x}) = p_x(d\tau(\sigma_{p_x})).$$

Any local coordinate system $U(q_1,\ldots,q_n)$ on Q induces naturally a system of coordinates $(\tau)^{-1}(U)(\tilde{q},p)$, $\tilde{q}_i = q_i \circ \tau$, $i=1,\ldots,n$. For these coordinates, $\theta = \sum_{i=1}^{n} p_i d\tilde{q}_i$ so that $\omega = d\theta = \sum_{i=1}^{n} dp_i \wedge d\tilde{q}_i$.

A smooth function $H : M \to \mathbb{R}$ defines a <u>Hamiltonian vector field</u> X_H on M by the formula

$$\omega(\eta, X_H) = dH(\eta)$$

for all vector fields η on M. X_H is well defined since ω is non-degenerate.

An important result of Darboux gives us local canonical coordinates for which ω has a useful expression:

Theorem 1.1: <u>Let</u> (M,ω) <u>be a symplectic manifold</u>. <u>Every point</u> $x \in M$ <u>has a coordinate neighbourhood</u> $U = U(q_1,\ldots,q_n,p_1,\ldots,p_n)$ <u>such that</u>

$$\omega\Big|_U = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$$

(For a proof see [A],[A-M]).

Using these coordinates, called <u>canonical coordinates</u>, the local expression of the vector field X_H assumes the classical form:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} \ , \quad i=1,\ldots,n.$$

Remarks: The following facts will be mentioned without proofs . For details, see [A],[A-M].

1 - Every symplectic manifold (M,ω) is orientable since it admits the following volume form

$$\Omega = \omega \wedge \ldots \wedge \omega$$

2 - If ϕ_H^t denotes the one-parameter pseudogroup of local diffeomorphisms generated by X_H, then $(\phi_H^t)^*\omega = \omega$, i.e., the local flow ϕ_H^t preserves the symplectic structure. In particular, ϕ_H^t preserves the volume form Ω.

3 - The Hamiltonian function H is constant along the trajectories of X_H, that is,

$$dH(X_H) = \omega(X_H,X_H) = 0.$$

This is the so called <u>conservation of energy</u> law.

4 - The <u>Poisson bracket</u> (H,G) of two C^∞-functions H and G on (M,ω) is the C^∞-function defined by:

$$(H,G) = \omega(X_G,X_H).$$

This operation turns $C^\infty(M)$ into a Lie algebra since the Jacobi identity

$$((F,G),H) + ((G,H),F) + ((H,F),G) = 0$$

holds true. Moreover, the map $H \to X_H$ is a homomorphism of Lie algebras since $(H,G) \mapsto [X_H,X_G]$, where $[\cdot,\cdot]$ is the Lie-bracket for two vector fields on M. When $(H,G) = 0$ the functions H and G are said to be <u>in involution</u> and, since $[X_H,X_G]=0$, X_H and X_G are commuting vector fields. This also means that the local flows ϕ_H^t and ϕ_G^s satisfies

$$\phi_H^t \circ \phi_G^s = \phi_G^s \circ \phi_H^t .$$

When ϕ_H^t is defined for all $t \in \mathbb{R}$, X_H is said to be <u>complete</u>.

Example 1.3: According to Newton's law, the motion of a particle under a time independent potential $V = V(x)$ is given by the second order equation $\ddot{x} = \frac{-\partial V}{\partial x}$, $x \in \mathbb{R}^n$, equivalent to $\dot{x} = y$, $\dot{y} = \frac{-\partial V}{\partial x}$, $(x,y) \in \mathbb{R}^{2n}$. This system is associated to the Hamiltonian function

$$H(x,y) = \frac{1}{2}|y|^2 + V(x).$$

The theorem of Arnold and Liouville

Let (M,ω) be a symplectic manifold and X_H be the Hamiltonian vector field corresponding to H. A smooth function $F:M \to \mathbb{R}$ is a <u>first integral</u> of X_H if F is constant along the trajectories of X_H, that is, $dF(X_H) = 0$. Since $\{F,H\} = \omega(X_H,X_F) = dF(X_H)$, one sees that F is a first integral of X_H if and only if F and H are in involution. The Jacobi identity shows that the set of all first integrals is a Lie subalgebra of $C^\infty(M)$.

Theorem 1.2: <u>Let</u> (M^{2n},ω) <u>be a symplectic manifold and</u> $F_1,\dots,F_n$ <u>be functions in involution</u>. <u>Consider a connected component</u> M_λ <u>of the level set</u> $\{x \in M^{2n} | F_i(x) = \lambda_i,\ i=1,\dots,n\}$ for $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$. <u>Assume</u> $dF_1(x),\dots,dF_n(x)$ <u>are linearly independent for any</u> $x \in M_\lambda$. <u>Then</u>:

1 - M_λ <u>is a n-dimensional submanifold of</u> M^{2n} invariant under the flows of the <u>commuting vector fields</u> $X_{F_1},\dots,X_{F_n}$ <u>and</u> $\omega(X_{F_i},X_{F_j})=0$.

2 - <u>If</u> $X_{F_1},\dots,X_{F_n}$ <u>are complete on</u> M_λ <u>this manifold is diffeomorphic to a product of</u> $\mathbb{R}^{n-k}$ <u>by a torus</u> T^k, <u>for some integer</u> k, $0\le k\le n$. <u>Furthermore</u>, <u>if</u> M_λ <u>is compact then</u> M_λ <u>is diffeomorphic to a torus</u> $T^n = \{(\varphi_1,\dots,\varphi_n) \bmod 2\pi\}$.

3 - <u>If</u> M_λ <u>is compact and under the same hypothesis in</u> 2), <u>the flow of the vector field</u> X_H, <u>for</u> $H = F_1$, <u>defines on</u> M_λ <u>quasiperiodic</u>

- motions $\varphi(t)$ given, in the angular coordinates $\varphi = (\varphi_1,\dots,\varphi_n)$ by

$$\frac{d\varphi}{dt} = \nu \ , \quad \nu = \nu(\lambda).$$

4 - Under the same hypothesis on M_λ in 3), it is possible to find functions $I_1,\dots,I_n$ depending only on $F_1,\dots,F_n$, called action coordinates such that $(I_1,\dots,I_n,\varphi_1,\dots,\varphi_n)$ are canonical coordinates in a neighbourhood of M_λ and then X_H is given by:

$$\frac{dI}{dt} = 0 \quad \frac{d\varphi}{dt} = \nu(I) \ ,$$

so that $H = H(I)$ and $\nu(I) = \frac{\partial H}{\partial I}(I)$.

Proof. The statement 1) is an immediate consequence of the fact that λ is a regular value of the function

$$(F_1,F_2,\dots,F_n) : M^{2n} \to \mathbb{R}^n$$, and that $F_1,\dots,F_n$ are in involution. To prove 2), note that the condition $(F_i,F_j) = 0$ implies $[X_{F_i},X_{F_j}] = 0$, so, by completeness of X_{F_i}, their flows satisfy $\phi_i^t \circ \phi_j^s = \phi_j^s \circ \phi_i^t$, $t,s \in \mathbb{R}$, and one can define an action A of the Abelian additive group $\mathbb{R}^n$ on M_λ in the following way

$$A((t_1,\dots,t_n),x) = \phi_1^{t_1} \circ \dots \circ \phi_n^{t_n}(x) \ ,$$

for all $x \in M_\lambda$ and all $(t_1,\dots,t_n) \in \mathbb{R}^n$. This action is transitive on M_λ, that is M_λ is an orbit of $\mathbb{R}^n$ under the action A. In fact, for a fixed $x \in M_\lambda$, the map

$$A_x : (t_1,\dots,t_n) \in \mathbb{R}^n \mapsto A((t_1,\dots,t_n),x) \in M_\lambda$$

is a local diffeomorphism since $dA_x\left(\frac{\partial}{\partial t_i}\bigg|_{t=0}\right) = X_{F_i}(x)$ and the vectors $X_{F_i}(x)$, $i=1,\dots,n$, are linearly independent. The inverse function theorem implies that each orbit of $\mathbb{R}^n$ in M_λ is open, and since M_λ is connected, the full orbit is M_λ. Let $x \in M_\lambda$ and $G = G_x = \{t=(t_1,\dots,t_n) \in \mathbb{R}^n | A(t,x) = x\}$ be the isotropy group at the point x. It can be easily shown that G does not depend on the choice of x and that G is a discrete subgroup of $\mathbb{R}^n$. Therefore, there exist k vectors $e_1,\dots,e_k \in G$ such that

$$G = \left\{\sum_{i=1}^{k} m_i e_i \,\middle|\, m_i \in \mathbb{Z}\ ,\ i=1,\dots,k\right\} .$$

Since M_λ is diffeomorphic to the quotient $\mathbb{R}^n/G$, it follows that M_λ is diffeomorphic to $\mathbb{R}^{n-k} \times T^k$. The statement 2) is thus proved. Now, if M_λ is compact, it is clear that M_λ is diffeomorphic to a torus $T^n = \{(\varphi_1,\dots,\varphi_n) \bmod 2\pi\}$. The map $(\varphi_1,\dots,\varphi_n) \mapsto A\left(\sum_{i=1}^{n}\varphi_i e_i, x\right)$ is a diffeomorphism of the unit rectangle $0 < \varphi_i < 1$, $i=1,\dots,n$, into the torus M_λ; these are the angular coordinates. In coordinates $(t_1,\dots,t_n)$ the integral curve $\phi_1^t(x)$ of X_H through x is the line $t_2 = t_3 = \dots = t_n = 0$. Since the change of coordinates $t \mapsto \varphi$ is linear, then the integral curves of X_H, in angular coordinates, are given by $\varphi(t) = \nu t + \varphi(0)$. To prove 4) one sees that the compactness of M_λ and the implicit function theorem imply that there exist a ball B in R^n and a neighborhood W of M_λ, diffeomorphic to $T^n \times B$, such that $T^n \times \{0\}$ is the image of M_λ under this diffeomorphism. One sees that $(\varphi_1,\ldots,\varphi_n,F_1,\ldots,F_n)$ is a global system of coordinates for W under which X_H is given by

$$\frac{dF}{dt} = 0 , \frac{d\varphi}{dt} = \nu(F).$$

These coordinates may not be canonical; but, as can be seen in [A, pag 276], it is possible to find n other functions $I_1,\ldots,I_n$, depending only on $F_1,\ldots F_n$, called the action coordinates such that $(I_1,\ldots,I_n,\varphi_1,\ldots,\varphi_n)$ is a global system of canonical coordinates for a neighborhood $\tilde{W}$ of M_λ, diffeomorphic to $T^n \times \tilde{B}$, $\tilde{B}$ a ball in R^n, and M_λ is the image of $T^n \times \{0\}$ under this diffeomorphism. The system X_H in these coordinates (I,φ) is given by $\frac{dI}{dt} = 0, \frac{d\varphi}{dt} = \nu(I)$, so that, $H = H(I)$ and $\nu(I) = \frac{\partial H}{\partial I}(I)$.

§ 2 - INTEGRABLE SYSTEMS

In the present notes an <u>integrable system</u> will mean a Hamiltonian system defined on a symplectic manifold (M^{2n},ω) which admits n first integrals $F_1,\dots,F_n$ in involution and <u>independent</u> that is $dF_1(x),\dots,dF_n(x)$ are linearly independent at all $x \in M$.

It will be presented now some examples of integrable systems. The symplectic manifold will be $\mathbb{R}^{2n}$, or a proper open set of it, whith the canonical 2-form, unless mention in contrary.

Example 2.1: <u>Harmonic oscillators</u>

The Hamiltonian function $H = \frac{1}{2}\sum_{i=1}^{n} \alpha_i\left(p_i^2 + q_i^2\right)$, $\alpha_i \in \mathbb{R}$, $i=1,\dots,n$, defines the differential equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \alpha_i p_i \ , \ \dot{p}_i = -\frac{\partial H}{\partial q_i} = -\alpha_i q_i$$

which are equivalent to $\ddot{q}_i = -\alpha_i^2 q_i$, $i=1,\dots,n$. The functions $F_i = q_i^2 + p_i^2$, $i=1,\dots,n$, are first integrals independent and in involution.

Example 2.2:

Consider a Hamiltonian function H depending only on the variables $p_1,\ldots,p_n$, so that $\frac{\partial H}{\partial q_i} = 0$, $i=1,\ldots,n$. The functions $F_i = p_i$, $i=1,\ldots,n$, prove the integrability of the system.

Example 2.3: Toda lattice for finitely many points

Consider n points on the line with coordinates $q_1,\ldots,q_n$, satisfying the differential equations

$$\ddot{q}_i = -\frac{\partial U}{\partial q_i}, \quad i=1,\ldots,n,$$

where the potential is given by $U = \sum_{k=1}^{n-1} \exp\left(q_k - q_{k+1}\right)$. The corresponding Hamiltonian system has Hamiltonian $H = \frac{1}{2}|p|^2 + U(q_1,\ldots,q_n)$. The integrability was discovered by Henon and Flaschka ([H],[F]) using different methods; it will be presented here the approachs of Flaschka and Moser (see also [M-1],[M-2],[M-3]).

Flaschka constructed the tridiagonal matrices

$$L = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a_1 & & \\ -a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & -a_{n-1} & 0 \end{bmatrix}$$

and showed that the Hamiltonian system

$$\dot{q}_i = p_i \ , \ \dot{p}_i = -\frac{\partial U}{\partial q_i} = \exp(q_{i-1} - q_i) - \exp(q_i - q_{i+1}),$$

$i=1,\ldots,n$ (here $q_0 = -\infty$ and $q_{n+1} = +\infty$), is equivalent to the system $\dot{L} = \frac{d}{dt} L = [B,L] = BL - LB$, if one defines

$$2b_i = -p_i \ , \ i=1,\ldots,n, \text{ and}$$

$$2a_k = \exp \frac{1}{2}(q_k - q_{k+1}) \ , \ k=1,\ldots,n-1.$$

The completeness of the Toda lattice system follows from the next lemma.

Lemma 2.1: <u>If</u> $U = U(q_1,\ldots,q_n)$ <u>is bounded below</u>, <u>the system</u>

$$\dot{q}_i = p_i \ , \ \dot{p}_i = -\frac{\partial U}{\partial q_i} \ , \ i=1,\ldots,n \ ,$$ <u>has all solutions defined on</u> $-\infty < t < +\infty$.

Proof. By hypothesis, there exists a real constant b such that $U(q) \geq b$. Therefore

$$\frac{1}{2}|p|^2 \leq H(p(t),q(t)) - b \ .$$

Then, $|p(t)| \leq K = (H(p(0),q(0)) - b)^{1/2}$. If (m,M) is the maximal interval of definition of the solution $(p(t),q(t))$ and t_0 and t are in (m,M), one has

$$|q(t)| \le |q(t_0)| + \left|\int_{t_0}^{t} |p(\tau)|d\tau\right| \le |q(t_0)| + K|t-t_0|.$$

If $M < +\infty$, one gets $|q(t)| \le |q(t_0)| + K(M - t_0)$ so that $(p(t),q(t))$ remains in a compact subset of $\mathbb{R}^{2n}$ for all $t \in [t_0,M)$, which is a contradiction. Therefore, $M = +\infty$. Analogously, one proves that $m = -\infty$. □

It is well known that $L(t)$ has real simple eigenvalues since $L(t)$ is a Jacobi matrix. The next lemma proves that these eigenvalues do not depend on t.

Lemma 2.2: <u>For any solution</u> $L(t)$ <u>of</u> $\dot{L} = [B,L]$, <u>there exists an orthogonal matrix</u> $Q(t)$ <u>such that</u> $L(t) = {}^TQ(t)\, L(0)\, Q(t)$ <u>with</u> $Q(t)$ <u>given by</u> $\dot{Q} = -QB$, $Q(0) = I$. <u>So, the eigenvalues of</u> $L(t)$ <u>do not depend on t</u>.

Proof. We have

$$\frac{d}{dt}(Q\,{}^TQ) = \dot{Q}\,{}^TQ + Q\,{}^T\dot{Q} = -QB\,{}^TQ - Q\,{}^TB\,{}^TQ$$

$$= -QB\,{}^TQ + QB\,{}^TQ = 0\ , \text{ since } {}^TB = -B.$$

Therefore, $Q(t)\,{}^TQ(t) = Q(0)\,{}^TQ(0) = I$.

Define $M(t) = {}^TQ(t)\, L(0)\, Q(t)$. It is easy to see that $M(t)$ satisfies $\dot{M} = [B,M]$ and $M(0) = L(0)$. By uniqueness of solution, it results $M(t) = L(t)$. Since $L(t)$ is similar to $L(0)$, it follows that the eigenvalues of $L(t)$ do not depend on time t. □

The next lemma proves that, as t goes to $+\infty$ or $-\infty$, the matrix $L(t)$ tends to a diagonal matrix with distinct eigenvalues.

Lemma 2.3: <u>The elements</u> $a_1(t),\dots,a_{n-1}(t)$, $b_1(t),\dots,b_n(t)$ <u>are bounded functions</u>, $\lim_{t\to\pm\infty} a_k(t) = 0$ <u>and</u> $\lim_{t\to\pm\infty} b_i(t) = b_i(\pm\infty)$.

<u>Moreover</u> $\quad -\infty < b_n(+\infty) < \dots < b_1(+\infty) < +\infty$

<u>and</u> $\quad -\infty < b_1(-\infty) < \dots < b_n(-\infty) < +\infty.$

Proof. By Lemma 2.2, $L(t) = {}^TQ(t)\, L(0)\, Q(t)$; them $|L(t)| \leq |L(0)|$ which implies that $a_k(t)$ and $b_i(t)$ are bounded functions. The differential equation $\dot{L} = [B,L]$ can be written as the system

$$\dot{a}_k = a_k(\dot{b}_{k+1} - \dot{b}_k)\ ,\ \dot{b}_j = 2(a_j^2 - a_{j-1}^2)\ ,$$

$k=1,\dots,(n-1)$, $j=1,\dots,n$, (here $a_0 = a_n = 0$), which admits the first integral $H = 4\sum_{k=1}^{n-1} a_k^2 + 2\sum_{j=1}^{n} b_j^2$. The integrals $\int_{-\infty}^{+\infty} a_k^2(t)dt$, $k=1,\dots,n-1$, converge; this is true for $k=1$ and $k=n-1$, because

$$\int_{-T}^{T} (2a_1^2(t) + 2a_{n-1}^2(t))dt = \int_{-T}^{T} (\dot{b}_1(t)-\dot{b}_n(t))dt = (b_1(t)-b_n(t))\Big|_{-T}^{T}$$

and the functions $b_j(t)$ are bounded. Analogously, using the equality

$$\int_{-T}^{T} (2a_1^2(t) - 2a_j^2(t) + 2a_{j-1}^2(t))dt = (b_1(t)-b_j(t))\Big|_{-T}^{T},$$

the argument follows by induction. The convergence of $\int_{-\infty}^{+\infty} a_k^2(t)dt$ assures that $\lim_{t\to\pm\infty} a_k(t) = 0$; also, $\dot{b}_j(t) = 2(a_j^2(t)-a_{j-1}^2(t))$ implies $\lim_{t\to\pm\infty} \dot{b}_j(t) = 0$.

Moreover,

$$b_j(T) - b_j(0) = \int_0^T 2(a_j^2(t) - a_{j-1}^2(t))dt$$

so that

$$\lim_{t\to+\infty} b_j(t) = 2\int_0^{+\infty} (a_j^2(t) - a_{j-1}^2(t))dt + b_j(0)$$

and

$$\lim_{t\to-\infty} b_j(t) = -2\int_{-\infty}^{0} (a_j^2(t) - a_{j-1}^2(t))dt + b_j(0).$$

Next, to prove that the diagonal elements of $L(\infty)$ are distinct, one uses the fact that the spectrum of $L(t)$ tends to the spectrum of $L(+\infty)$ as $t\to+\infty$ and the eigenvalues of $L(t)$ are distinct and do not depend on t. Finally, if $b_{k+1}(+\infty)-b_k(+\infty) > 0$ then $\dot{a}_k(t) > 0$ for all t big enough which contradicts the fact that $\lim_{t\to+\infty} a_k(t) = 0$. Therefore one obtains

$$-\infty < b_n(+\infty) < \dots < b_1(+\infty) < +\infty$$

and analogously $-\infty < b_1(-\infty) < \dots < b_n(-\infty) < +\infty$. □

The next theorem proves the integrability of the Toda lattice system. In order to do that, one considers the characteristic polynomial

$$\Delta_n(\lambda) = \lambda^n + I_1\lambda^{n-1} + \dots + I_n$$

of L(t), which does not depend on t; hence $I_1,\dots,I_n$ are first integrals.

Theorem 2.1: <u>The integrals</u> $I_1,\dots,I_n$ <u>are independent and in involution</u>, <u>so that the Toda lattice system is integrable</u>.

Proof. Consider the matrices L and $M = \mathrm{diag}(b_1,\dots,b_n)$. The first integrals $I_j = I_j(a_1,\dots,a_{n-1},b_1,\dots,b_n)$, $j=1,\dots,n$, tend, in the C^1-sense, to the symmetric functions $\sigma_1,\dots,\sigma_n$, $\sigma_j = \sigma_j(b_1,\dots,b_n)$, $j=1,\dots,n$, of the matrix M. In fact,

$$\frac{\partial I_j}{\partial a_i} \to 0 \text{ as } (a_1,\dots,a_{n-1}) \to 0.$$

The functions $b_1, \ldots, b_n$ are independent, i.e., $db_1, \ldots, db_n$ are linearly independent everywhere in the phase space. Also, the determinant of the coefficients of $d\sigma_1, \ldots, d\sigma_n$ with respect to $db_1, \ldots, db_n$ is equal to

$$\prod_{1 \leq i < j \leq n} (b_i - b_j)$$

which, by lemma 2.3, is strictly positive for t big enough. It follows now that $dI_1, \ldots, dI_n$ are linearly independent in a neighborhood of $(p_1(+\infty), \ldots, p_n(+\infty))$, then everywhere in the phase space, since the flow of a vector field is a diffeomorphism. The functions $\sigma_1, \ldots, \sigma_n$ as functions of $p_1, \ldots, p_n$ are in involution . Since $I_1, \ldots, I_n$, as functions of $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are first integrals of the original Hamiltonian system, then the Poisson bracket (I_i, I_j) is also an integral. But (I_i, I_j) tends to $(\sigma_i, \sigma_j) = 0$ along any solution, as $t \to +\infty$. Then, $(I_i, I_j) = 0$ everywhere. □

Example 2.4: The n-particle system on the line, with the inverse square potential considered by Calogero and Marchioro.

Consider n particles on the line with coordinates $q_1, \ldots, q_n$ and define $U = \sum_{k<\ell} (q_k - q_\ell)^{-2}$, $k, \ell = 1, \ldots, n$, as their potential, so that the equations of motion are given by

$$\ddot{q}_k = -\frac{\partial U}{\partial q_k} = 2 \sum_{j \neq k} (q_k - q_j)^{-3}, \quad k = 1, \ldots, n.$$

As observed by Moser in [M-2], this system possesses n integrals, independent and in involution, which are polynomials in $\dot{q}_k$ and $(q_k - q_\ell)^{-2}$, a fact that can be derived below following the same lines as used for the Toda lattice system. As before, matrices L and B are introduced and the differential equation $\dot{L} = BL - LB$ is considered, which can be transformed in the above equations of motion. In order to do that, set

$$z_{k\ell} = \begin{cases} (q_k - q_\ell)^{-1} & \text{for } k \neq \ell \\ 0 & \text{for } k = \ell \end{cases}$$

and define the matrices

$$Z_\alpha = (z_{k\ell}^\alpha) \quad \text{for } \alpha = 1,2,$$

$$Y = \operatorname{diag}(y_1,\ldots,y_n),$$

$$D_\alpha = \operatorname{diag}\left(\sum_{j=1}^{n} z_{kj}^\alpha\right) \quad \text{for } \alpha = 2,3.$$

Defining $L = Y + iZ_1$, $B = iD_2 - iZ_2$, one sees that L is Hermitian, that is, $L = L^*$ (its conjugate transpose) and B is skew-Hermitian, that is, $B = -B^*$. The equations of motion in terms of the variables $z_{k\ell}$ and $y_k = -\dot{q}_k$ can be written as

$$\dot{y}_k = -\ddot{q}_k = -2 \sum_{j=1}^{n} z_{kj}^3,$$

$$\dot{z}_{k\ell} = z_{k\ell}^2 (y_k - y_\ell), \quad k,\ell = 1,\ldots,n.$$

This last system is redundant, since only the (n-1) variables $z_{k,k+1}$ are independent, the other being determined by the relations $z_{k\ell}^{-1} = z_{kr}^{-1} + z_{r\ell}^{-1}$ if k,ℓ,r distinct and $z_{k\ell} + z_{\ell k} = 0$. It is easy to check that

$$[B,L] = BL - LB = i[Y,Z_2] - [D_2,Z_1] + [Z_2,Z_1]$$

and a straightforward computation shows that $\dot{L} = [B,L]$ leads to the equations of motion in the variables $z_{k\ell}$ and y_k. This implies, by a similar argument used in the lemma 2.2, that the coeficients I_k of the characteristic polynomial

$$\det(\lambda I - L) = \lambda^n + I_1\lambda^{n-1} + \dots + I_n$$

are integrals of the motion.

The next lemma proves that, as t goes to $+\infty$ or $-\infty$, that the matrix L(t) tends to a real diagonal matrix with distinct eigenvalues. The proof is found in [M-2] and it is presented here for a sake of completeness.

Lemma 2.4: <u>The elements</u> $y_1(t),\dots,y_n(t)$ <u>and</u> $z_{k\ell}(t)$, $k,\ell=1,\dots,n$, <u>are bounded functions</u>, $\lim_{t\to\pm\infty} y_k(t) = y_k(\pm\infty)$ <u>exist</u>, $\lim_{t\to\pm\infty} z_{k\ell}(t) = 0$ <u>and</u>

$$+\infty > y_1(+\infty) > \dots > y_n(+\infty) > -\infty ,$$

$$-\infty < y_1(-\infty) < \dots, < y_n(-\infty) < +\infty .$$

Proof. The matrix $L(t)$ is defined for all $t \in \mathbb{R}$ by Lemma 2.1 . Moreover, $L(t)$ satisfies the equation $L(t) = Q^*L(0)Q$ where $Q(t)$ is an unitary matrix defined by $\dot{Q} = -QB$, $Q(0) = I$. Therefore, $L(t)$ and so its components are bounded functions of $t \in \mathbb{R}$. Without loss of generality, the particles can be labelled according to the order $q_1 < \ldots < q_n$ at $t=0$ and then for all $t \in \mathbb{R}$, by the conservation of energy. The next step is to prove that there exist and are finite the limits $\lim_{t\to\pm\infty} \dot{q}_k(t)$, $k=1,\ldots,n$. From the equations of motion it follows that for any $T > 0$ one has

$$\frac{1}{2}\left[\dot{q}_n-\dot{q}_1\right](T)-\frac{1}{2}\left[\dot{q}_n-\dot{q}_1\right](-T) = \int_{-T}^{T}\left(\sum_{j<n}(q_n-q_j)^{-3}+\sum_{j>1}(q_j-q_1)^{-3}\right)dt.$$

Since the left-hand side is bounded and the right-hand side is an increasing function of T, one sees that $\int_{-\infty}^{+\infty}(q_k-q_\ell)^{-3}dt < \infty$ for $k>\ell=1$ and for $\ell<k=n$, and by induction for all pair (k,ℓ) with $k>\ell$. From

$$\dot{q}_k(T) - \dot{q}_k(0) = 2\sum_{j\neq k}\int_0^T (q_k - q_j)^{-3}dt$$

it follows that $\lim_{t\to\pm\infty} \dot{q}_k(t)$ exists and is finite. The limits of the velocities satisfy

$$\dot{q}_1(+\infty) \leq \ldots \leq \dot{q}_n(+\infty) \quad \text{and}$$

$$\dot{q}_1(-\infty) \geq \ldots \geq \dot{q}_n(-\infty).$$

In fact, if $f(t) = q_{k+1}(t) - q_k(t)$ and assuming , by contradiction, that

$\lim_{t\to+\infty} \dot f(t)=b<0$, one sees that $f(t)-f(t_0) < \frac{b}{2}(t-t_0)$ for all t and t_0 sufficiently big, $t>t_0$, which implies that $f(t) = q_{k+1}(t)-q_k(t)$ is negative for t big enough, in contradiction with the fact that $q_{k+1} > q_k$. The above inequalities for the limit velocities are, in fact, strict inequalities. To prove this, let $\varphi(t) = q_n-q_1 > 0$. Then $\frac{1}{2}\ddot\varphi(t) \geq 2(q_n - q_1)^{-3} > 0$, so that $\dot\varphi$ is increasing and $\dot\varphi(+\infty) = \dot q_n(+\infty) - \dot q_1(+\infty) \geq 0$. If, by contradiction, $\dot\varphi(+\infty) = 0$, then $\dot\varphi(t) < 0$ for all $t \in \mathbb{R}$ so that $\varphi(t)$ is a decreasing positive function; therefore $\varphi(t)$ is bounded. But, then, $\frac{1}{2}\ddot\varphi(t) \geq (\max\varphi)^{-3}>0$; hence integrating twice one gets

$$\varphi(t) \geq \varphi(0) + \dot\varphi(0)t + (\max\varphi)^{-3}\, t^2/2$$

which shows that $\varphi(t)$ is unbounded, which is a contradiction. Therefore, $\dot q_n(+\infty) > \dot q_1(+\infty)$. This and the inequalities $\dot q_1(+\infty) \leq \ldots \leq \dot q_n(+\infty)$ imply the existence of an s such that $\dot q_s(+\infty) < \dot q_{s+1}(+\infty)$. It will be shown now that $\dot q_1(+\infty) < \dot q_s(+\infty)$. First of all, observe that, setting $a = \dot q_{s+1}(+\infty) - \dot q_s(+\infty)$, then

$$\frac{a}{2}(t-t_0) \leq \Big(q_{s+1}(t)-q_s(t)\Big)-\Big(q_{s+1}(t_0)-q_s(t_0)\Big) \leq \frac{3a}{2}(t-t_0)$$

for all t and t_0 big enough. Therefore, for $j > s$:

$$q_j - q_s = \Big(q_j - q_{j-1}\Big) + \ldots + \Big(q_{s+1} - q_s\Big) \geq q_{s+1}-q_s$$

implies that

$$0 \le (q_j - q_s)^{-1} \le (q_{s+1} - q_s)^{-1} = O(t^{-1}).$$

Now, $\frac{1}{2}\frac{d^2}{dt^2}(q_s - q_1) = \sum_{j<s}(q_s - q_j)^{-3} - O(t^{-3}) + \sum_{j>1}(q_j - q_1)^{-3}$ $\ge 2(q_s - q_1)^{-3} - O(t^{-3})$. Calling $\psi(t) = q_s - q_1 + \frac{A}{t}$, A being a positive constant, one verifies that

$$\ddot{\psi}(t) = \ddot{q}_s - \ddot{q}_1 + \frac{2A}{t^3} \ge 4(q_s - q_1)^{-3} + \frac{2A}{t^3} - O(t^{-3}),$$

so that

$\ddot{\psi} \ge 4(q_s - q_1)^{-3} > 0$ if t is big enough and A is chosen suitably. Thus $\dot{\psi}$ is increasing and $\dot{\psi}(+\infty) > 0$ since $\dot{\psi}(+\infty)=0$ leads to a contradiction. Since $\dot{\psi}(+\infty) = \dot{q}_s(+\infty) - \dot{q}_1(+\infty)$ it follows that $\dot{q}_1(+\infty) < \dot{q}_s(+\infty)$. Analogously, one proves that $\dot{q}_{s+1}(+\infty) < \dot{q}_n(+\infty)$ and, by induction,

$$-\infty < \dot{q}_1(+\infty) < \dots < \dot{q}_n(+\infty) < +\infty,$$

and, analogously,

$$+\infty > \dot{q}_1(-\infty) > \dots > \dot{q}_n(-\infty) > -\infty .$$

Finally, since $z_{k\ell} = (q_k - q_\ell)^{-1} = O(t^{-1})$ as $|t|\to+\infty$, one obtains

$\lim_{t\to\pm\infty} z_{k\ell}(t) = 0.$ □

The next theorem proves the integrability of the mechanical system with the potential considered by Calogero and Marchioro.

Theorem 2.2: The Hamiltonian system

$$\dot{q}_k = p_k \ , \ \dot{p}_k = -\frac{\partial U}{\partial q_k} \ , \quad k=1,\dots,n \ ,$$

where $U = \sum_{k<\ell} (q_k - q_\ell)^{-2}$, is an integrable system.

Proof. Using Lemma 2.4 and following the same ideas as in Theorem 2.1, one can prove that the integrals $I_1,\dots,I_n$ are real functions of $p_1,\dots,p_n$ and $q_1,\dots,q_n$ and they are independent and in involution. □

Remarks. These integrals are rational functions of the coordinates $p_1,\dots,p_n$ and $q_1,\dots,q_n$ since they are polynomials in $z_{k\ell}$ and $p_1,\dots,p_n$, and the $z_{k\ell} = (q_k - q_\ell)^{-1}$ for $k \neq \ell$ and $z_{kk} = 0$.

§ 3 - SYMPLECTIC ACTIONS OF A GROUP. MOMENT MAP AND REDUCTION OF THE PHASE SPACE

Let X_H be a Hamiltonian vector field of Hamiltonian function H defined on a symplectic manifold (M^{2n},ω); assume that the flow ϕ_H^t is an one-parameter group of symplectic diffeomorphisms. Given a first integral F, that is $(F,H) = 0$, F is constant along the integral curves of ϕ_H^t and H is constant along the integral curves of the flow ϕ_F^t of X_F; assume also X_F be complete. If one considers the restriction of H to a level surface $F^{-1}(c)$ which is a submanifold of dimension 2n-1 when c is a regular value of F, one sees that H is constant on the integrals curves of ϕ_F^t lying on $F^{-1}(c)$. Then $F^{-1}(c)/\phi_F^t$, the manifold of the orbits of ϕ_F^t on $F^{-1}(c)$, has dimension 2n-2 and H induces a function $\tilde{H}$ which is well defined in the quotient. Moreover, any integral curve of X_H on $F^{-1}(c)$ is projected on an integral curve of the vector field $\tilde{X}_H$ obtained by projection of X_H. Roughly speaking, if it is known a first integral F it is possible to reduce X_H to another system $\tilde{X}_H$ with 2n-2 dimensions.

As in the Arnold-Liouville's theorem, if one considers k first integrals $F_1,\ldots,F_k$ in involution and independent everywhere, it is possible to define $F = (F_1,\ldots,F_k):M \rightarrow \mathbb{R}^k$ and an action of the Abelian group $\mathbb{R}^k$ on the level surface $F^{-1}(c)$, using the fact that the flows ϕ_j^t of X_{F_i} commute and assuming also that they are complete. The action is the map $A : \mathbb{R}^k \times M \to M$ given by $A(t,p) = A((t_1,\ldots,t_k),p) = \phi_1^t \circ \cdots \circ \phi_k^t(p)$ for all $p \in M$ and all $t=(t_1,\ldots,t_k) \in \mathbb{R}^k$. It is clear that $A(t+s,p) = A(t,A(s,p))$. One can prove that the quotient $F^{-1}(c)/\mathbb{R}^k$, the set of all orbits of $\mathbb{R}^k$, is a symplectic manifold of dimension $2n-2k$ and that the function $\tilde{H}$, which is well defined in the quotient, is the Hamiltonian function of the Hamiltonian vector field $\tilde{X}_H$, projection of X_H under the derivative of the quotient map. The above method is called the Jacobi-Liouville method of reduction.

It will be also shown that there exist other kinds of group actions, including noncommutative actions; a good example studied below in example 3.2 is the noncommutative action of the rotation group SO(3) on $\mathbb{R}^3 \times \mathbb{R}^3$ which can be used to reduce a Hamiltonian system invariant under that action.

Example 3.1: Let $H(p,q), (p,q) \in \mathbb{R}^{2n}$, be a Hamiltonian function invariant under the translation $(p,q) \mapsto (p,q + s(1,\ldots,1))$, that is, $H(p,q + s(1,\ldots,1)) = H(p,q)$ for all $s \in \mathbb{R}$, the differentiating with respect to s at $s = 0$, it results $\sum_{k=1}^{n} \frac{\partial H}{\partial q_k}(p,q) = -\sum_{k=1}^{n} \dot{p}_k = 0$. This shows that $F = p_1 + \ldots + p_k$ is a first integral, called the <u>linear momentum</u>. Consider now the following change of coordinates:

$$\xi_k = q_k - q_n \quad , \quad \xi_n = q_n$$

$$\eta_k = p_k \quad , \quad \eta_n = \sum_{i=1}^{n} p_i \quad , \quad k=1,\ldots,n.$$

It defines a canonical transformation since

$$\sum_{i=1}^{n} d\eta_i \wedge d\xi_i = \sum_{i=1}^{n} dp_i \wedge dq_i \ .$$

Let $\Gamma(\eta,\xi)$ be defined by $\Gamma(\eta,\xi) = H(p,q)$. The function Γ does not depend on ξ_n since $\frac{\partial\Gamma}{\partial\xi_n} = -\dot{\eta}_n$ and η_n is an integral. The system restricted to any level surface $\eta_n = c$ becomes $\dot{\xi}_k = \frac{\partial\Gamma}{\partial\eta_k}$, $\dot{\eta}_k = -\frac{\partial\Gamma}{\partial\xi_k}$, $k=1,\ldots,n-1$, with $(n-1)$ degrees of freedom. Knowing how to solve it, it is then possible to integrate the remaining scalar equation $\dot{\xi}_n = \frac{\partial\Gamma}{\partial\eta_n} = \phi(t,c)$. This is, essentially, the Jacobi-Liouville method of reduction in the case of one first integral.

Example 3.2: Let $H(p,q) = \frac{1}{2}|p|^2 + V(|q|)$, $(p,q) \in \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. This function is invariant under the orthogonal group SO(3), since $H(Rp,Rq) = H(p,q)$ for all $R \in SO(3)$. Using the Lie algebra $g=so(3)$ of all skew-symmetric operators, one obtains that $\exp sA \in SO(3)$ for all $s \in \mathbb{R}$ and $A \in g$. Then $H((\exp sA)p,(\exp sA)q) = H(p,q)$, which, after differentiating with respect to s at $s = 0$, gives

$$\langle \frac{\partial H}{\partial p}, A q \rangle + \langle \frac{\partial H}{\partial q}, A p \rangle = 0 \ .$$

Thus defining $\hat{J}_A$ by $\hat{J}_A(p,q) = \langle p, A q \rangle$, $\hat{J}_A$ is an integral for any $A \in g = so(3)$. It is known that to each $A \in so(3)$ there corresponds an unique $a \in \mathbb{R}^3$ such that $Ax = a \wedge x$, for all $x \in \mathbb{R}^3$.

Therefore $\hat{J}_A(p,q) = \hat{J}_a(p,q) = \langle p, a \wedge q \rangle = \langle a, p \wedge q \rangle$. If (e_1,e_2,e_3) is the canonical basis of $\mathbb{R}^3$, one has $q \wedge p = \langle q \wedge p, e_1 \rangle e_1 + \langle q \wedge p, e_2 \rangle e_2 + \langle q \wedge p, e_3 \rangle e_3$. The map $(p,q) \mapsto q \wedge p$ is a vector valued first integral called the <u>angular momentum</u>. Therefore, if $q = (q_1,q_2,q_3)$ and $p = (p_1,p_2,p_3)$, the components $F_1 = q_2p_3 - q_3p_2$, $F_2 = q_3p_1 - q_1p_3$, $F_3 = q_1p_2 - q_2p_1$ of $q \wedge p$ are integrals of motion and one may expect to reduce the phase space $\mathbb{R}^6$ by six dimensions. But, in this case these integrals are not in involution, since $(F_1,F_2) = -F_3$. In order to see that the reduced space has two dimensions, one fixes a value $c = q \wedge p$ of the angular momentum and assume $c \neq 0$. One may choose the coordinates such that $c = \lambda e_3$, $\lambda > 0$, $e_3 = (0,0,1)$. Then $q_2p_3 - q_3p_2 = 0$, $q_1p_3 - q_3p_1 = 0$ and since $\lambda = (q_1p_2 - q_2p_1) > 0$ it results that $p_3 = q_3 = 0$, that means, the problem is reduced to $\mathbb{R}^2 \times \mathbb{R}^2$ with the quadratic integral $\lambda = q_1p_2 - q_2p_1$. The new phase space $\mathbb{R}^2 \times \mathbb{R}^2$ is invariant under the rotations of SO(2). The reduced Hamiltonian function $\tilde{H} = H|_{\mathbb{R}^2 \times \mathbb{R}^2}$ is obtained by making $p_3 = q_3 = 0$ and then

$$\tilde{H} = \frac{1}{2}(p_1^2 + p_2^2) + V(r), \quad r = \sqrt{q_1^2 + q_2^2}.$$

It is also obvious that $\tilde{H}$ is invariant under SO(2). Following a method discovered by Jacobi, one introduces polar coordinates $q_1 = r\cos\phi$, $q_2 = r\sin\phi$ and, for the conjugate variables one chooses the special functions $p_r = p_1\cos\phi + p_2\sin\phi$ and $p_\phi = r(-p_1\sin\phi + p_2\cos\phi)$. Since $p_\phi = q_1p_2 - q_2p_1 = \lambda$, p_ϕ is an integral and a straightforward computation shows that the coordinate system (p_r, p_ϕ, r, ϕ) is canonical since $\tilde{\omega} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 =$ $= dp_r \wedge dr + dp_\phi \wedge d\phi$. The Hamiltonian function $\tilde{H}$ in these coordinates is given by

$$\tilde{H} = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2} p_\phi^2\right) + V(r) = \frac{1}{2}\left(p_r^2 + \frac{\lambda^2}{r^2}\right) + V(r)$$

so that the system reduces to

$$\dot{p}_r = -\frac{\partial \tilde{H}}{\partial r} = -\frac{\lambda^2}{r^3} + \frac{dV}{dr}(r)$$

$$\dot{r} = \frac{\partial \tilde{H}}{\partial p_r} = p_r$$

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial p_\phi} = \frac{1}{r^2} p_\phi = \frac{\lambda}{r^2} .$$

The two first equations define a system in two dimensions; using the conservation of the energy, one can solve the scalar equation $\dot{r} = \pm \sqrt{2c - \frac{\lambda^2}{r^2} - V(r)}$ to find $r(t)$ and then integrate $\dot{\phi} = \frac{\lambda}{r^2(t)}$.

The Moment Map

Let (M,ω) be a symplectic manifold such that $\omega = d\theta$, that is, besides to be closed and nondegenerate, ω is an exact two-form. In this case $(M,d\theta)$ is said to be an exact symplectic manifold. As an example one has the cotangent bundle $M = T^*Q$ of a manifold Q (see example 1.2).

An <u>action of a Lie group</u> G <u>on a manifold</u> M is a smooth map $\phi : G \times M \to M$ such that, for any $g \in G$, the map $\phi_g : M \to M$ defined by $\phi_g(p) = \phi(g,p)$ is a diffeomorphism, $\phi_e = id(M)$, e being the identity of G, and $\phi_g \circ \phi_h = \phi_{gh}$ for all $g,h \in G$. The <u>orbit</u> O(p) of a point $p \in M$ is the set

$$O(p) = \{\phi_g(p) \mid g \in G\} .$$

One also defines the maps $\phi_p : G \to M$ by $\phi_p(g) = \phi(g,p)$. A <u>symplectic action of a Lie group</u> G <u>on a symplectic manifold</u> (M,ω) is an action $\phi : G \times M \to M$ such that ϕ_g is a symplectic map, i.e., $\phi_g^*\omega = \omega$ for all $g \in G$. If, moreover, $\omega = d\theta$ and $\phi_g^*\theta = \theta$ for all $g \in G$, one has an <u>exact symplectic group action</u>.

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and let $\phi : G \times M \to M$ be a symplectic group action of G on a symplectic manifold (M,ω). Given $a \in \mathfrak{g}$, $g(t) = \exp t a \in G$ for all $t \in \mathbb{R}$, then $\phi_{g(t)}$ is a symplectic diffeomorphism. Let X_a be the vector field defined on M by $X_a(p) = \frac{d}{dt} \phi_{g(t)}(p)\big|_{t=0}$. The vector field X_a is called the <u>infinitesimal generator of the action corresponding to</u> $a \in \mathfrak{g}$. The map $a \mapsto X_a$ is linear since $X_a(p) = \frac{d}{dt} \phi(g(t),p)\big|_{t=0} = \frac{d}{dt} \phi_p(g(t))\big|_{t=0} = d\phi_p(e)a$. If, moreover, the symplectic group action is exact, then the vector field X_a is a Hamiltonian vector field with Hamiltonian function $\hat{J}_a = \theta(X_a)$, as proved below. If β is a n-exterior form on M, recall that $X_a \lrcorner \beta$ denotes the (n-1)-exterior form given by $(X_a \lrcorner \beta)(v_1,\ldots,v_{n-1}) = \beta(X_a,v_1,\ldots,v_{n-1})$. By the well known formula $L_{X_a}\theta = d(X_a \lrcorner\, \theta) + X_a \lrcorner\, d\theta$, (see [A-M]) and since $\phi^*_{g(t)}\theta=\theta$ implies $L_{X_a}\theta = 0$, one obtains $d(X_a \lrcorner\, \theta) = -X_a \lrcorner\, d\theta$ that is $d(\theta(X_a)) = -X_a \lrcorner\, d\theta$ which proves the assertion. Since the map $a \in \mathfrak{g} \mapsto \hat{J}_a = \theta(X_a)$ is linear, <u>the moment map</u> J <u>associated to the exact symplectic group action</u> can be defined as $J : M \to \mathfrak{g}^*$, by the formula $(Jp)a = \hat{J}_a(p)$, for all $p \in M$ and $a \in \mathfrak{g}$. It can be shown that the linear map $a \mapsto X_a$, considered above, has the property $X_{[a,b]} = [X_a, X_b]$. Then the linear map $a \mapsto \hat{J}_a$ is a homomorphism of Lie algebras, that is, $\hat{J}_{[a,b]} = (\hat{J}_a, \hat{J}_b)$.

The next theorem shows that the moment map $J:M \to \mathfrak{g}^*$ has the universal property of being a vector valued first integral for any Hamiltonian system X_H such that the Hamiltonian function H is invariant under group action.

Theorem 3.1: <u>Let</u> ϕ <u>be an exact symplectic group action of a Lie group</u> G <u>on an exact symplectic manifold</u> $(M, d\theta)$ <u>and let</u> $H:M \to \mathbb{R}$ <u>be any smooth Hamiltonian function invariant under the action, that is</u>, $H \circ \phi_g = H$ <u>for all</u> $g \in G$. <u>Then the flow</u> ϕ^t_H <u>of the Hamiltonian vector field</u> X_H <u>leaves invariant the moment map</u> $J : M \to \mathfrak{g}^*$, i.e., $J \circ \phi^t_H = J$, <u>or equivalently</u>, <u>the function</u> $\hat{J}_a$ <u>is a first integral of</u> X_H <u>for all</u> $a \in \mathfrak{g}$.

Proof. Since $H(p) = H(\phi_g(p))$, take $g = g(t) = \exp t\, a$, $a \in \mathfrak{g}$, and compute the derivative with respect to t at t=0 of $H(p)=H(\phi_{\exp t a}(p))$.

Thus, one obtains

$$0 = dH(X_a(p)) = d\theta(X_a(p),X_H(p)) = (H,\hat{J}_a)(p) ,$$

which shows that $\hat{J}_a$ is a first integral of X_H, for any $a \in g$. Finally, $\left[(J \circ \phi_H^t)p\right]a = \left[J(\phi_H^t(p))\right]a = \hat{J}_a(\phi_H^t(p)) = \hat{J}_a(p) = (Jp)a$. □

At this point it is interesting to go back to example 3.2, where the group $G = SO(3)$ defines an exact symplectic group action on the exact symplectic manifold $(M,d\theta) = (\mathbb{R}^3 \times \mathbb{R}^3, d\theta)$, $\theta = \sum_{i=1}^{3} p_i\, dq_i\tau$; the group action is given by $\phi_R(p,q) = (Rp,Rq)$ for all $R \in SO(3)$. In fact, it is clear that $\left[d\phi_R(p,q)\right](Y,X) = (RY,RX)$ and $\theta_{(p,q)}(Y,X) = \sum_{i=1}^{3} p_i\, dq_i\tau\,(Y,X) = \sum_{i=1}^{3} p_i\, dq_i(X)$. Then $\theta_{(p,q)}(Y,X) = \sum_{i=1}^{3} p_i X_i = \langle p,X \rangle$.

Finally

$$\theta_{(Rp,Rq)}(RY,RX) = \langle Rp,RX \rangle = \langle p,X \rangle = \theta_{(p,q)}(Y,X).$$

Recall also that $\hat{J}_a(p,q) = \langle a,p \wedge q \rangle$, for all $a \in g = so(3) \cong \mathbb{R}^3$. The moment map $J:M \to g^*$ is given by $J(p,q) = \langle \cdot , p \wedge q \rangle$. By the last theorem, any Hamiltonian function $H : \mathbb{R}^6 \to \mathbb{R}$ invariant under the action of SO(3) (in particular, that one considered in example 3.2) defines a Hamiltonian system for which the functions $\hat{J}_a$ are first integrals; in particular $(p,q) \mapsto p \wedge q$ is a vector valued first integral for such systems.

Any action of a Lie group G on a vector space V is called a <u>representation</u>. The <u>adjoint representation of G</u> on its Lie algebra g is the action $Ad : G \times g \to g$ defined by $Ad(g,a) = \left[dL_g \circ dR_{g^{-1}}(e)\right](a)$ where dL_g and $dR_{g^{-1}}$ denote, respectively, the derivatives of the left and right translations

$$x \in G \mapsto gx \in G \quad \text{and} \quad x \in G \mapsto xg^{-1} \in G.$$

One also introduces the notation $Ad(g)a = Ad(g,a)$. Since $Ad(g) : g \to g$ is the derivative $d\psi_g(e) : g \to g$ of the map

$$\psi_g : x \in G \mapsto gxg^{-1} = L_g R_{g^{-1}}(x) \in G$$

at $e \in G$, one obtains $Ad(gh) = Ad(g) \circ Ad(h)$.

The <u>coadjoint representation of G</u> on g^*, $Ad^* : G \times g^* \to g^*$, is given by the maps $Ad^*(g) : g^* \to g^*$, $g \in G$, defined by the formula $[Ad^*(g)](\mu)a \stackrel{def}{=} \mu[Ad(g^{-1})a]$ for all $\mu \in g^*$ and all $a \in g$. With this definition one obtains $Ad^*(gh) = Ad^*(g) \circ Ad^*(h)$.

If one fixes now $\mu \in g^*$ one obtains the <u>orbit</u> $O(\mu)$ of the coadjoint representation, which is the set $O(\mu) = \{Ad^*(g)\mu \mid g \in G\}$. It is a basic result due to Kirillov and Kostant that $O(\mu)$ has a canonical symplectic structure.

Example 3.3: Isospectral matrices.

Let G be the Lie group of all the $n\times n$ non singular real matrices and let T be an element of G. Then $Ad(T)A = T\,A\,T^{-1}$ for all $A \in g$. The well known inner product

$$(A,B) \in g \times g \to \langle A,B \rangle \overset{def}{=} \text{trace } A.{}^{T}B$$

identifies the Lie algebra g of all $n \times n$ real matrices with its dual g^{*} by

$$A \in g \mapsto \nu \in g^{*},\ \nu(\cdot) = \langle \cdot, A \rangle .$$

For a fixed $\mu \in g$, the orbit $O(\mu)$ of the coadjoint representation is identified with the set of all matrices which are similar to the matrix M given by the identification $\mu(\cdot) = \langle \cdot, M \rangle$. In fact, $O(\mu) = \{X | Ad^{*}(g)\mu(\cdot) = \langle \cdot, X \rangle,\ g \in G\}$ and $Ad^{*}(g)\mu(\cdot) = \langle \cdot, X \rangle$ = $\mu(Ad(g^{-1})(\cdot)) = \mu(g^{-1}(\cdot)g) = \langle g^{-1}(\cdot)g, M \rangle$ = trace $g^{-1}(\cdot)g\ {}^{T}M$ = trace $g^{-1}(\cdot)g\ {}^{T}M\,g^{-1}\ g$ = trace $(\cdot)g\ {}^{T}M\,g^{-1} = \langle \cdot, {}^{T}(g^{-1})M\,g \rangle$ then $X = {}^{T}(g^{-1})M\ {}^{T}g$ and X is similar to M. If μ is such that M has distinct eigenvalues then $O(\mu)$ consists of the so called <u>isospectral matrices</u>. A symplectic structure will be defined on the manifold $O(\mu) = \{TMT^{-1} | T \in G\}$. The point M belongs to $O(\mu)$ and the tangent space of $O(\mu)$ at the point M is the set $\{[A,M] | A \in g\}$. To see this, one considers a generic curve $T(t)$ in G such that $T(0) = I$

and $\dot{T}(0) = A$; then $T(t)\,M\,T(t)^{-1}$ is a generic curve on $O(\mu)$, passing through M at t=0. Since $\left.\frac{d}{dt}\,T(t)^{-1}\right|_{t=0} = -\dot{T}(0) = -A$ one obtains

$$\left.\frac{d}{dt}\,(T(t)\,M\,T(t)^{-1})\right|_{t=0} = AM - MA = [A,M].$$

This description of the tangent space of $O(\mu)$ at M can be used to show that the dimension of $O(\mu)$ is n^2-n. The form ω on $O(\mu)$ is given by $\omega([A,M],[B,M]) \overset{def}{=} \operatorname{trace}([A,B].M)$. One verifies that this form is nondegenerate, skew-symmetric and closed, thus defining a symplectic structure on $O(\mu)$.

Consider again an exact symplectic group action ϕ and its corresponding moment map $J : M \to g^*$. The moment map J is said to be <u>equivariant</u> with respect to a pair of functions $f : M \to M$ and $h : g^* \to g^*$ if $J \circ f = h \circ J$, that is, if the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\ J\ } & g^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ M & \xrightarrow{\ J\ } & g^* \end{array}$$

Theorem 3.2: <u>Given any exact symplectic group action</u> $\phi : G \times M \to M$ <u>of a Lie group</u> G <u>on an exact symplectic manifold</u> $(M, d\theta)$, <u>the corresponding moment map</u> J <u>is equivariant with respect to</u> $\phi_g : M \to M$ <u>and to</u> $Ad^*(g) : g^* \to g^*$, for all $g \in G$.

Proof. To be equivariant means $J \circ \phi_g = [Ad^*(g)] \circ J$, for all $g \in G$, that is, $[J(\phi_g(p))]a$ has to be equal to $[(Ad^*(g))Jp]a$ for all $p \in M$ and $a \in g$. Let X_a be the infinitesimal generator of the action corresponding to $a \in g$. But $[J(\phi_g(p))]a = \hat{J}_a(\phi_g(p)) = [\theta(X_a)]\phi_g(p) = \theta_{\phi_g(p)}(X_a(\phi_g(p)))$. On the other hand one has $[(Ad^*(g))Jp]a = Jp[Ad(g^{-1})a] = \hat{J}_{Ad(g^{-1})a}(p) = \left[\theta\left(X_{Ad(g^{-1})a}\right)\right](p) = \theta_p\left[d\phi_p(e)(Ad(g^{-1})a)\right]$.

The fact that $\exp t\, a$ is the one parameter group of $a \in g$, implies that $g^{-1}(\exp t\, a)g$ is the one parameter group of $Ad(g^{-1})a$; in fact, the curve $g^{-1}(\exp t\, a)g$ passes through $e \in G$ for $t = 0$ and its derivative at $t=0$ is $\frac{d}{dt}(g^{-1}(\exp t\, a)g)\Big|_{t=0} = \frac{d}{dt}(L_{g^{-1}} \cdot R_g)(\exp t\, a)\Big|_{t=0} = dL_{g^{-1}} \circ dR_g\left(\frac{d}{dt}\exp t\, a\right)\Big|_{t=0} = dL_{g^{-1}} \circ dR_g(a) = Ad(g^{-1})a$. Consequently $[(Ad^*(g)Jp)]a = \theta_p\left[\left(\frac{d}{dt}\phi_{\exp t\, Ad(g^{-1})a}(p)\right)_{t=0}\right] = \theta_p\left[\left(\frac{d}{dt}\phi_{g^{-1}(\exp t\, a)g}(p)\right)_{t=0}\right] = \theta_p\left[\left(\frac{d}{dt}\left(\phi_{g^{-1}} \circ \phi_{\exp t\, a} \circ \phi_g(p)\right)\right)_{t=0}\right] = \theta_p\left[d\phi_{g^{-1}}\left(\frac{d}{dt}\phi_{\exp t\, a}(\phi_g(p))\right)_{t=0}\right] = \theta_{\phi_g(p)}\left(X_a(\phi_g(p)\right)$, the last equality being true because $\phi_g^*\theta = \theta$.

The proof is then complete. □

<u>The Reduced Phase Space</u>

Given an exact symplectic group action $G \times M \to M$ on an exact symplectic manifold $(M, d\theta)$, let us consider the corresponding moment map $J : M \to g^*$. Fix $\mu \in g^*$ and consider the set

$$J^{-1}(\mu) = \{p \in M | J(p) = \mu\}.$$

In the example 3.2, G is SO(3) and $J^{-1}(\mu)$ consists of those states for which the angular momentum vector μ is fixed. By the equivariant property, the isotropy subgroup

$$G_\mu \overset{\text{def}}{=} \{g \in G | Ad^*(g)\mu = \mu\}$$

acts on the set $J^{-1}(\mu)$, which in the example consists of all rotations leaving fixed the angular momentum vector μ. Again, in example 3.2, if $\mu \neq 0$, G_μ is a one-dimensional rotation subgroup SO(2). To eliminate this angle one considers the quotient set $J^{-1}(\mu)/G_\mu$, which corresponds to the elimination of the "ignorable" angle of rotation. Under appropriate assumptions, for example (see [A], Apendice 5), if μ is a regular value of J, if the isotropy group G_μ is compact and acts on $J^{-1}(\mu)$ without fixed points, it is possible to construct the bundle

$$\Pi : J^{-1}(\mu) \to \tilde{M} = J^{-1}(\mu)/G_\mu ,$$

and $\tilde{M}$ is called the <u>reduced phase space</u>; $\tilde{M}$ is a symplectic manifold and its structure is given by a two form $\tilde{\omega}$ which is defined as follows: if $j : J^{-1}(\mu) \to M$ is the injection map then $j^*\omega$ is the pullback of $\omega = d\theta$ to $J^{-1}(\mu)$. This form $j^*\omega$ is invariant under G_μ and therefore there exists $\tilde{\omega}$ in the quotient, such that $\Pi^*\tilde{\omega}=j^*\omega$. Moreover, if H is any Hamiltonian function invariant under the action ϕ_g, the <u>reduced Hamiltonian flow</u> $X_{\tilde{H}}$ on $(\tilde{M},\tilde{\omega})$ is defined by $d\tilde{H} = -X_{\tilde{H}} \rfloor \tilde{\omega}$, where the Hamiltonian function $\tilde{H}$ is such that $\tilde{H} \circ \Pi = H \circ j$, that is, the restriction of H to $J^{-1}(\mu)$ is invariant under G_μ and defines $\tilde{H}$ in the quotient (see [A] Apendice 5 and [A-M]).

§ 4 - PERSISTENCE OF TORI. THE THEOREM K.A.M. (Kolmogorov, Arnold and Moser)

Assume it given an integrable system of Hamiltonian function H_0, with the hypothesis of the Theorem 1.2 of Arnold and Liouville. Assume also that in a neighborhood of the Torus M_λ, diffeomorphic to $B \times T^n$, B being a ball in $\mathbb{R}^n$, there are action-angle canonical coordinates (I,φ) and X_{H_0} is given in $B \times T^n$ by

$$\frac{dI}{dt} = 0 \ , \ \frac{d\varphi}{dt} = \nu_0(I) \ ,$$

so that $H_0 = H_0(I)$ and $\nu_0(I) = \frac{\partial H}{\partial I}(I)$.

If, moreover, $\det \dfrac{\partial^2 H_0(I)}{\partial I^2} = \det \dfrac{\partial \nu_0(I)}{\partial I} \neq 0$, the system X_{H_0} is called non-degenerate. In this case the frequencies $(\nu_0^1,\ldots,\nu_0^n)$ are non-resonant, that is, $\langle \nu_0, k \rangle \neq 0$ for all sequences $k \neq 0$ of integers, and the orbits are dense in each invariant torus I=cte, which is then the closure of the orbits. This means that if the frequencies are non resonant, the tori do not depend on the choice of the action-angle coordinates.

Denote (I,φ) by (p,q) and consider a perturbation on the non-degenerate Hamiltonian by

$$H = H_0(p) + H_1(p,q)$$

with $H_1(p,q+2\pi) = H_1(p,q)$, where $H_1(p,q)$ is small in a sense that will be precised below. The equations for the perturbed Hamiltonian are

$$\dot{q} = \nu_0(p) + \frac{\partial H_1}{\partial p} , \ \dot{p} = - \frac{\partial H_1}{\partial q} .$$

All the data in the statement of the next theorem are supposed to be analytic and X_{H_0} non-degenerate. If one selects a non-resonant frequency $\nu = \nu^*$, the equations on the invariant nonperturbed torus $T_0(\nu^*)$ are characterized by $p = p^*$ where $\nu^* = \nu_0(p^*)$.

Theorem 4.1: (K.A.M.)

Given $K > 0$ and for almost all non-resonant ν^* (except for a set of Lebesque measure zero), there exist $\varepsilon > 0$ and a map $p = p(Q)$, $q = q(Q)$ from an abstract torus $T = \{Q | \text{mod } 2\pi\}$ into $B \times T^n$ such that according to the perturbed system one has $\dot{Q} = \nu^*$ and $|p(Q) - p^*| < K$, $|q(Q) - Q| < K$, provided that $|H_1| < \varepsilon = \varepsilon(K, \nu^*, H_0)$.

Proof. (see [A-A]).

The meaning of the Theorem K.A.M. is that there exists an invariant torus $T(\nu^*)$ of the perturbed system, close to the torus $T_0(\nu^*)$, for almost all ν^* (except for a set of measure zero of frequencies). The union of all tori $T(\nu^*)$ is a set with positive measure in the phase space and its complement has measure which tends to zero has $|H_1| \to 0$.

The theorem K.A.M. was presented by Kolmogorov, proved by Arnold in the analytic case and by Moser in the C^k case, k=333, when n=2. Today, one finds other versions for this remarkable result, even C^k versions ($k > 2n$, etc.).

The behavior of the trajectories which are in the complement of the invariant tori set is not completely known. If n=2, the phase space has dimension four, and then two tori T^2 separate in two connected components a surface of energy of dimension three. If one trajectory starts between two tori, it always remains between these tori. But for $n > 2$ the tori T^n do not separate a hypersurface of energy (dim.2n-1) and apparently a trajectory could start close to two tori at t=0 and for big times it could go faraway from them; this phenomenon is called the difusion of Arnold and it is an important field of research now days.

REFERENCES

[A] Arnold V. - "Méthodes mathématiques de la Mécanique Classique", Editions Mir, 1976.

[A-A] Arnold V., Avez A. - "Ergodic Problems of Classical Mechanics", Benjamin 1968.

[A-M] Abraham R., Marsden J. - "Foundations of Mechanics", Benjamin 1978.

[F] Flaschka H. - "The Toda lattice II. Existence of integrals". Physical Review B vol. 9 nº 4, 1974 (p.1924).

[H] Hénon M. - "Integrals of the Toda lattice". Physical Review B vol. 9 nº 4, 1974 (p.1921).

[M-1] Moser J. - "Finitely many mass points on the line under the influence of an exponential potential" - An integrable system -Lect. Notes in Physics, Springer 1975.

[M-2] Moser J. - "Three Integrable Hamiltonian Systems Connected with Isospectral Deformations" - Advances in Math 16, 197-220 (1975).

[M-3] Moser J. - "Various aspects of Integrable Hamiltonian Systems" - CIME Lectures Bressanone Italy, June 1978.

"RELATÓRIO TÉCNICO"
DEPARTAMENTO DE MATEMÁTICA APLICADA
TÍTULOS PUBLICADOS

RT-MAP-7701 - Ivan de Queiroz Barros
On equivalence and reducibility of Generating Matrices of RK-Procedures - Agosto 1977

RT-MAP-7702 - V.W. Setzer
A Note on a Recursive Top-Down Analizer of N.Wirth - Dezembro 1977

RT-MAP-7703 - Ivan de Queiroz Barros
Introdução a Aproximação Ótima - Dezembro 1977

RT-MAP-7704 - V.W. Setzer, M.M. Sanches
A linguagem "LEAL" para Ensino básico de Computação - Dezembro 1977

RT-MAP-7801 - Ivan de Queiroz Barros
Proof of two Lemmas of interest in connection with discretization of Ordinary Differential Equations - Janeiro 1978

RT-MAP-7802 - Silvio Ursic, Cyro Patarra
Exact solution of Systems of Linear Equations with Iterative Methods
Fevereiro 1978

RT-MAP-7803 - Martin Grötschel, Yoshiko Wakabayashi
Hypohamiltonian Digraphs - Março 1978

RT-MAP-7804 - Martin Grötschel, Yoshiko Wakabayashi
Hypotraceable Digraphs - Maio 1978

RT-MAP-7805 - W. Hesse, V.W. Setzer
The Line-Justifier: an example of program development by transformations
Junho 1978

RT-MAP-7806 - Ivan de Queiroz Barros
Discretização
Capitulo I - Tópicos Introdutórios
Capitulo II - Discretização
Julho 1978

RT-MAP-7807 - Ivan de Queiroz Barros
(Γ',Γ) - Estabilidade e Métodos Preditores-Corretores - Setembro 1978

RT-MAP-7808 - Ivan de Queiroz Barros
Discretização
Capitulo III - Métodos de passo progressivo para Eq. Dif. Ord. com condições iniciais - Setembro 1978

RT-MAP-7809 - V.W. Setzer
Program development by transformations applied to relational Data-Base queries - Novembro 1978

RT-MAP-7810 - Nguiffo B. Boyom, Paulo Boulos
Homogeneity of Cartan-Killing spheres and singularities of vector fields - Novembro 1978

RT-MAP-7811 - D.T. Fernandes e C. Patarra
Sistemas Lineares Esparsos, um Método Exato de Solução - Novembro 1978

RT-MAP-7812 - V.W. Setzer e G. Bressan
Desenvolvimento de Programas por Transformações: uma Comparação entre dois Metodos - Novembro 1978

RT-MAP-7813 - Ivan de Queiroz Barros
Variação do Passo na Discretização de Eq. Dif. Ord. com Condições Iniciais - Novembro 1978

RT-MAP-7814 - Martin Grötschel e Yoshiko Wakabayashi
On the Complexity of the Monotone Asymetric Travelling Salesman Polytope I: HIPOHAMILTONIAN FACETS - Dezembro 1978

RT-MAP-7815 - Ana F. Humes e E.I. Jury
Stability of Multidimensional Discrete Systems: State-Space Representation Approach - Dezembro 1978

RT-MAP-7901 - Martin Grötschel, Yoshiko Wakabayashi
On the complexity of the Monotone Asymetric Travelling Salesman Polytope II: HYPOTRACEABLE FACETS - Fevereiro 1979

RT-MAP-7902 - M.M. Sanches e V.W. Setzer
A portabilidade do Compilador para a Linguagem LEAL - Junho 1979

RT-MAP-7903 - Martin Grötschel, Carsten Thomassen, Yoshiko Wakabayashi
Hypotraceable Digraphs - Julho 1979

RT-MAP-7904 - N'Guiffo B. Boyom
Translations non triviales dans les groupes (transitifs) des transformations affines - Novembro 1979

RT-MAP-8001 - Ângelo Barone Netto
Extremos detectáveis por jatos - Junho 1980

RT-MAP-8002 - Ivan de Queiroz Barros
Medida e Integração
Cap. I - Medida e Integração Abstrata - Julho 1980

RT-MAP-8003 - Routo Terada
Fast Algorithms for NP-Hard Problems which are Optimal or Near-Optimal with Probability one - Setembro 1980

RT-MAP-8004 - V.W. Setzer e R. Lapyda
Uma Metodologia de Projeto de Bancos de Dados para o Sistema ADABAS
Setembro 1980

RT-MAP-8005 - Imre Simon
On Brzozowski's Problem: $(1 \cup A)^m = A^*$ - Outubro 1980

RT-MAP-8006 - Ivan de Queiroz Barros
Medida e Integração
Cap. II - Espaços Lp - Outubro 1980

RT-MAP-8101 - Luzia Kazuko Yoshida e Gabriel Richard Bitran
Um algoritmo para Problemas de Programação Vetorial com Variáveis Zero-Um - Fevereiro 1981

RT-MAP-8102 - Ivan de Queiros Barros
Medida e Integração
Cap. III - Medidas em Espaços Topológicos - Março 1981

RT-MAP-8103 - V.W. Setzer, R. Lapyda
Design of Data Models for the ADABAS System using the Entity-Relationship Approach - Abril 1981

RT-MAP-8104 - Iván de Queiroz Barros
Medida e Integração
Cap. IV - Medida e Integração Vetoriais - Abril 1981

RT-MAP-8105 - U.S.R. Murty
Projective Geometries and Their Truncations - Maio 1981

RT-MAP-8106 - V.W. Setzer, R. Lapyda
Projeto de Bancos de Dados, Usando Modelos Conceituais
Este relatorio Tecnico complementa o RT-MAP-8103. Ambos substituem o RT-MAP-8004 ampliando os conceitos ali expostos. - Junho 1981

RT-MAP-8107 - Maria Angela Gurgel, Yoshiko Wakabayashi
Embedding of Trees - August 1981

RT-MAP-8108 - Ivan de Queiroz Barros
Mecânica Analítica Clássica - Outubro 1981

RT-MAP-8109 - Ivan de Queiroz Barros
Equações Integrais de Fredholm no Espaço das Funções A-Uniformemente Contínuas - Novembro 1981

RT-MAP-8110 - Ivan de Queiroz Barros
Dois Teoremas sobre Equações Integrais de Fredholm - Novembro 1981

RT-MAP-8201 - Siang Wun Song
On a High-Performance VLSI Solution to Database Problems - Janeiro 1982

RT-MAP-8202 - Maria Angela Gurgel, Yoshiko Wakabayashi
A Result on Hamilton-Connected Graphs - Junho 1982

RT-MAP-8203 - Jörg Blatter, Larry Schumaker
The Set of Continuous Selections of a Metric Projection in C(X) - Outubro 1981

RT-MAP-8204 - Jörg Blatter, Larry Schumaker
Continuous Selections and Maximal Alternators for Spline Approximation - Dezembro 1981

RT-MAP-8205 - Arnaldo Mandel
Topology of Oriented Matroids - Junho 1982

RT-MAP-8206 - Erich J. Neuhold
Database Management Systems: A General Introduction - Novembro 1982

RT-MAP-8207 - Béla Bollobás
The Evolution of Random Graphs - Novembro 1982

RT-MAP-8208 - V.W. Setzer
Um Grafo Sintático para a Linguagem PL/M-80 - Novembro 1982

RT-MAP-8209 - Jayme Luiz Szwarcfiter
A Sufficient Condition for Hamilton Cycles - Novembro 1982

RT-MAP-8301 - W.M. Oliva
Stability of Morse-Smale Maps - Janeiro 1983

RT-MAP-8302 - Belá Bollobás, Istvan Simon
Repeated Random Insertion into a Priority Queue - Fevereiro 1983

RT-MAP-8303 - V.W. Setzer, P.C.D. Freitas e B.C.A. Cunha
Um Banco de Dados de Medicamentos - Julho 1983

RT-MAP-8304 - Ivan de Queiroz Barros
O Teorema de Stokes em Variedades Celuláveis - Julho 1983

RT-MAP-8305 - Arnaldo Mandel
The 1-Skeleton of Polytopes, oriented Matroids and some other lattices - Julho 1983

RT-MAP-8306 - Arnaldo Mandel
Alguns Problemas de Enumeração em Geometria - Agosto 1983

RT-MAP-8307 - Siang Wun Song
Complexidade de E/S e Projetos Optimais de Dispositivos para Ordenação - Agosto 1983

RT-MAP-8401-A - Dirceu Douglas Salvetti
Procedimentos para Cálculos com Splines
Parte A - Resumos Teoricos - Janeiro 1984
RT-MAP-8401-B
Parte B - Descrição de Procedimentos - Janeiro 1984
RT-MAP-8401-C
Parte C - Listagem de Testes - Janeiro 1984

RT-MAP-8402 - V.W. Setzer
Manifesto contra o uso de computadores no Ensino de 1º Grau - Abril 1984

RT-MAP-8403 - G. Fusco e W.M. Oliva
On Mechanical Systems with Non-Holonomic Constraints: Some Aspects of the General Theory and Results for the Dissipative Case - Julho 1984

RT-MAP-8404 - Imre Simon
A Factorization of Infinite Words - Setembro 1984 - São Paulo - IME-USP
7 pg.

RT-MAP-8405 - Imre Simon
The Subword Structure of a Free Monoid - Setembro 1984 - São Paulo - IME-USP
6 pg.

RT-MAP-8406 - Jairo Z. Gonçalves e Arnaldo Mandel
Are There Free Groups in Division Rings? - Setembro 1984 - São Paulo - IME-USP
25 pg.

RT-MAP-8407 - Paulo Feofiloff and D.H. Younger
Vertex-Constrained Transversals in a Bipartite Graph - Novembro 1984
São Paulo - IME-USP - 18 pg.

RT-MAP-8408 - Paulo Feofiloff
Disjoint Transversals of Directed Coboundaries - Novembro 1984
Sao Paulo - IME-USP - 126 pg.

RT-MAP-8409 - Paulo Feofiloff e D.H. Younger
Directed cut transversal packing for source-sink connected graphs -
São Paulo - IME-USP - 16 pg. - Novembro 1984

RT-MAP-8410 - Gaetano Zampieri e Ângelo Barone Netto
Attractive Central Forces May Yield Liapunov Instability - Dezembro 1984
São Paulo - IME-USP - 8 pg.

RT-MAP-8501 - Siang Wun Song
Disposições Compactas de Árvores no Plano - Maio 1985
São Paulo - IME-USP - 11 pg.

RT-MAP-8502 - Paulo Feofiloff
Transversais de Cortes Orientados em Grafos Bipartidos - Julho 1985
Sao Paulo - IME-USP - 11 pg.

RT-MAP-8503 - Paulo Domingos Cordaro
On the Range of the Lewy Complexy - Outubro 1985
Sao Paulo - IME-USP - 113 pg.

RT-MAP-8504 - Christian Choffrut
Free Partially Commutative Monoids - Setembro 1985
São Paulo - IME-USP - 110 pg.

RT-MAP-8505 - Valdemar W. Setzer
Manifesto Against the use of Computers in Elementary Education - Outubro 1985
Sao Paulo - IME-USP - 40 pg.

RT-MAP-8506 - Ivan Kupka and Waldyr Muniz Oliva
Generic Properties and Structural Estability of Dissipative Mechanical Systems - Novembro 1985
São Paulo - IME-USP - 32 pg.

RT-MAP-8601 - Gaetano Zampieri
Determining and Construting Isochronous Centers - Abril 1986
Sao Paulo - IME-USP - 11 pg.

RT-MAP-8602 - G. Fusco e W.M. Oliva
Jacobi Matrices and Transversality - Abril 1986
Sao Paulo - IME-USP - 25 pg.

RT-MAP-8603 - Gaetano Zampieri
Il Teorema di A.E. Nother per finiti gradi di libertà e per i Campi -
Maio 1986 Sao Paulo - IME-USP - 18 pg.

RT-MAP-8604 - Gaetano Zampieri
Stabilità Dell'Equilibrio Per $\ddot{x}+xf(x)=0$, $\ddot{y}+yf(x)=0$, $f\in C^{\infty}$.
Sap Paulo - IME-USP - 16 pg.

TÍTULOS PUBLICADOS

RT-MAP-8605 - Angelo Barone Netto e Mauro de Oliveira Cesar
Nonconservative Positional Systems - Stability - Junho 1986
São Paulo - IME-USP - 14 pg.

RT-MAP-8606 - Júlio Michael Stern
Fatoração L - U e Aplicações - Agosto 1986
São Paulo - IME-USP - 105 pg.

RT-MAP-8607 - Afonso Galvão Ferreira
O Problema do Dobramento Optimal de PLAs - Agosto 1986
São Paulo - IME-USP - 73 pg.

RT-MAP-8608 - Gaetano Zampieri
Liapunov Stability for Some Central Forces - Novembro 1986
São Paulo - IME-USP - 17 pg.

RT-MAP-8701 - A. Bergamasco, P. Cordaro and J. Hounie
Global Properties of a Class of Vector Fields in the Plane - Fevereiro 1987
São Paulo - IME-USP - 37 pg.

RT-MAP-8702 - P. Cordaro and J. Hounie
Local Solvability in C_c^∞ of Over-Determined Systems of Vector Fields -
Fevereiro - São Paulo - IME-USP - 32 pg.

RT-MAP-8703 - Imre Simon
The Nondeterministic Complexity of a Finite Automaton - Fevereiro 1987
São Paulo - IME-USP - 20 pg.

RT-MAP-8704 - Imre Simon
Infinite Words and a Theorem of Hindman - Abril 1987
São Paulo - IME-USP - 8 pg.

RT-MAP-8705 - Sônia Regina Leite Garcia
Relations Between Critical Points of f and its "Radial Derivate" -
Junho 1987 - IME-USP - 38 pg.

RT-MAP-8706 - Gaetano Zampieri
On the Periodic Oscillations of $\ddot{x} = g(x)$. - Abril 1987
São Paulo - IME-USP - 25 pg.

RT-MAP-8707 - Imre Simon
Factorization Forests of Finite Height - Agosto 1987
São Paulo - IME-USP - 36 pg.

RT-MAP-8708 - Mauro de O. César e Gaetano Zampieri
On Liapunov Stability for $\ddot{x} + xf(x)=0$, $\ddot{y} + yu(x)=0$ - Dezembro 1987
São Paulo - IME-USP - 19 pg.

RT-MAP-8709 - Routo Terada
Um Código Criptográfico para Segurança em Transmissão e Base de Dados -
Março- São Paulo - IME-USP - 31 pg.

RT-MAP-8710 - [illegible] Oliveira and M.S.A. Castilla
Topics on Hamiltonian Systems - Dezembro 1987
São Paulo - IME-USP - 49 pg.

RT-MAP-8801 - C. Fusco and W.M. Oliva
A Perron Theorem for the Existence of Invariant Subspaces - Abril 1988
São Paulo - IME-USP - 36 pg.