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Classification of simple Gelfand–Tsetlin modules of $\mathfrak{sl}(3)$

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We provide a classification and an explicit realization of all simple Gelfand–Tsetlin modules of the complex Lie algebra $\mathfrak{sl}(3)$. The realization of these modules, including those with infinite-dimensional weight spaces, is given via regular and derivative Gelfand–Tsetlin tableaux. Also, we show that all simple Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules can be obtained as subquotients of localized Gelfand–Tsetlin E_{21}-injective modules.

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1. Introduction

One of the most prominent discoveries in representation theory in the 20th century was made by Weyl proving that every finite-dimensional module over a complex semisimple Lie algebra is completely reducible. Further, the irreducible finite-dimensional modules are parameterized by their highest weights and their characters and dimensions can be explicitly written in terms of the celebrated Weyl formulas, see, for example, [40], and the references therein for a detailed exposition.

Unfortunately, Weyl's results do not provide explicit formulas for the action of the generators of the Lie algebras. Such formulas were first discovered in the fundamental works of Gelfand and Tsetlin in 1950. At that time they published two short papers with explicit constructions of tableaux basis of the modules and formulas for the action of the generators of the algebra on the tableaux. These formulas are known today as *Gelfand–Tsetlin formulas*. The formulas are for simple finite-dimensional modules and Lie algebras of type A [31], and Lie algebras of types B and D [32]. It is interesting to note that these works did not include proofs or references to earlier works. In an effort to understand better the results of Gelfand and Tsetlin, mathematicians and physicists developed various tools that lead to formal proofs of Gelfand–Tsetlin formulas. Some of the first works containing such proofs are [1, 2]. In these works, Baird and Biedenharn studied Racah–Wigner calculus on the unitary groups, and discovered a natural relation with the Gelfand–Tsetlin formulas. Independently, in 1962, Zhelobenko introduced lowering operators and suggested that they can be used to prove the Gelfand–Tsetlin formulas for $\mathfrak{gl}(n)$, see [64]. Inspired by the work of Zhelobenko, Nagel and Moshinsky [53], and Pei-Yu [55], obtained yet other proofs of the Gelfand–Tsetlin formulas for $\mathfrak{gl}(n)$.

In the 1970s a series of papers of Gould, Green, and others used the technique of tensor-type identities to obtain further relations with the Gelfand–Tsetlin theory. More precisely, in [38], certain sets of group invariants were used to derive a hierarchy of tensor identities, known as characteristic identities, satisfied by the generators of the general, orthogonal, and symplectic groups. With the aid of characteristic identities, Wigner coefficients formulas and yet another proof for the action of the generators of the algebras of types A, B, and D on finite-dimensional modules were obtained in [34, 35].

The Gelfand–Tsetlin technique providing explicit bases and action of the generators is best suited for the case of $\mathfrak{gl}(n)$. Although in this paper we work with this case only, one should mention that many interesting and deep results are established in the cases of orthogonal and symplectic Lie algebras. One especially powerful technique used to studying these cases is the theory of Yangians. We will not discuss this theory here, and the interested reader is referred to [52] and the references therein.

It is important to note that the Gelfand–Tsetlin theory has strong combinatorics flavor. The representation of the classical groups in tensor product spaces was used to construct bases of the representations of the classical Lie algebras parameterized by standard Young tableaux. The first proof of the formulas presented in [1] relies on calculus of Young patterns. Gelfand–Tsetlin bases were used as prototypes to study combinatorial properties of the supporting graphs of representations of Lie algebras of types B, C, and G_2, see [8].

Undoubtedly, Gelfand–Tsetlin bases are not the only important bases of representations. Finding explicit bases of representations of semisimple Lie algebras has been a central topic of research in the last several decades. Deep results were discovered by De Concini and Kazhdan [5], Kashiwara [43], Littelmann [46, 47], Lusztig [48], among others. We refer the reader to [49] for an overview of the different types of bases. However, explicit formulas for the action of the generators on the corresponding basis elements are known only for bases of Gelfand–Tsetlin type.

The Gelfand–Tsetlin theory for infinite-dimensional modules is a rapidly developing subject. Important categories that include infinite-dimensional modules are highest-weight modules, weight modules, Harish-Chandra modules, among others. Unfortunately, even the classification of the simple objects in some of these categories is still an open problem. One important category involved in this paper, and the Gelfand–Tsetlin theory as a whole, is the category of weight modules. In what follows, we provide some history on the study of this category.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over the complex numbers, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A $\mathfrak{g}$-module M is called a *weight module* if $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$, where $M_\lambda = \{v \in M \mid hv = \lambda(h)v,\ \forall\, h \in \mathfrak{h}\}$. The space M_λ is called a *weight space*, the set $\{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ is called the *weight support* of M, and the dimension of M_λ is called the *weight multiplicity* of λ. If $\mathfrak{h}$ acts locally finitely, but not necessarily diagonally, on M, then we say that M is a *generalized weight module*. It is an easy exercise to show that a simple generalized weight module is a weight module.

A weight module M is *torsion free* provided that all root vectors of $\mathfrak{g}$ act injectively on M. If M is a torsion-free module then all weight multiplicities of M (finite or infinite) are equal. This invariant of M is called the *weight degree* of M. Furthermore, the weight support of a torsion-free module M coincides with a full coset $\lambda + Q$ of $\mathfrak{h}^*/Q$, where Q is the root lattice of $\mathfrak{g}$ and λ is in the weight support of M. On the other hand, a simple weight module may have "full support" without being torsion free, in which case the weight multiplicities are necessarily infinite. The first examples of such modules were given in [16]. Simple modules with full support are called *dense*.

A breakthrough in the theory of weight modules with finite weight multiplicities was made by Fernando [13], in 1990 who reduced the classification of all such simple modules to determining the simple torsion-free modules. He also showed that the only simple Lie algebras admitting torsion-free modules are those of type A or C. The next major breakthrough was made in 2000 by Mathieu [50], who classified and provided a realization of all simple torsion-free modules of finite degree. Previously, the case of degree 1 was worked out in [4]. Important properties of the annihilators of the torsion-free modules were established in [41].

The study of weight modules with infinite multiplicities is still at its initial stage. A result similar to the one of Fernando reduces the classification of all such simple modules to the classification of all simple dense modules of simple Lie algebras. For the classical simple Lie algebras this reduction was obtained in [15] and for all exceptional simple Lie algebras except E_8 in [29]. Finally, in [7], the reduction in all cases, including all important classes of finite-dimensional Lie superalgebras, was completed.

One natural category of weight modules is the category of Gelfand–Tsetlin modules. More precisely, this is the full subcategory of the category of generalized weight modules consisting of modules that admit a generalized eigenbasis for the *Gelfand–Tsetlin subalgebra*, a maximal commutative subalgebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Gelfand–Tsetlin modules were introduced in [9–11] as an attempt to generalize the celebrated tableaux construction of Gelfand–Tsetlin.

Gelfand–Tsetlin subalgebras have applications that extend beyond the study of Gelfand–Tsetlin modules. For example, these subalgebras were related to the solutions of the Euler equation in [14], and to the subalgebras of $U(\mathfrak{g})$ of maximal Gelfand–Kirillov dimension in [59]. Gelfand–Tsetlin subalgebras were studied in [44, 45] in connection with classical mechanics, and also in [36, 37] in connection with general hypergeometric functions on the Lie group $GL(n, \mathbb{C})$.

A general theory of Gelfand–Tsetlin modules for a class of Galois algebras (for a definition, see [27]) was developed in [28]. The results for these Galois algebras can be applied to the universal enveloping algebras of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$, and provide structural properties of the corresponding simple Gelfand–Tsetlin modules. In the *generic* case the characters of the Gelfand–Tsetlin subalgebra parametrize such simple modules. However, in the non-generic case, i.e. in the *singular* case and $n > 2$, we may have more than one isomorphism class of simple Gelfand–Tsetlin modules with a fixed character of the Gelfand–Tsetlin subalgebra. The theory of singular Gelfand–Tsetlin modules was initiated in [21] where 1-singular modules were constructed and studied in detail. Immediately after the construction of the 1-singular modules, there was an abundance of successful attempts to construct simple Gelfand–Tsetlin modules with a given singular character. For more details, we refer the reader to the following papers [12, 21–26, 30, 39, 58, 60, 61, 63]. In particular, a classification of the simple Gelfand–Tsetlin modules was recently announced in [42, 62].

A classification of the simple 1-singular Gelfand–Tsetlin modules was obtained in [23] and leads to the classification of all simple Gelfand–Tsetlin modules of the Lie algebra $\mathfrak{sl}(3)$ (and of $\mathfrak{gl}(3)$). The latter classification is the main purpose of this paper and it is provided via very explicit tableaux construction.

Our classification result relies on various old results on Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules obtained in [3, 15–18], combined with newer results from [19, 57]. We remark that some technical statements in the paper on the properties of

Gelfand–Tsetlin modules can be simplified using the theory recently developed, for example, in [23, 25, 58]. However, for the sake of completeness and for reader's convenience, we opted to keep the original paper containing detailed and explicit proofs.

The structure of the paper is as follows. In Sec. 2, we set up the notation and state basic definitions and results needed in the rest of the paper. In Sec. 3, we prove some general results about the Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$. Section 4 is devoted to the description of certain "easier to study" classes of Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$, namely finite-dimensional modules, generic modules, and 1-singular modules. In Sec. 5, we collect some important definitions and preliminary results that relate Gelfand–Tsetlin modules to their Gelfand–Tsetlin character. In Sec. 6, we prove the main results about existence and uniqueness of simple Gelfand–Tsetlin modules of $\mathfrak{sl}(3)$. The explicit description of all simple Gelfand–Tsetlin modules for $\mathfrak{sl}(3)$ is included in Sec. 7. Finally, in Sec. 8, we study localization functors on the category of Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules and prove that any simple module in this category can be obtain from an E_{21}-injective module using a E_{21}-localization functors.

2. Preliminaries

The ground field will be $\mathbb{C}$. In the first part of the paper, we fix an integer $n \geq 2$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers m such that $m \geq a$. Similarly, we define $\mathbb{Z}_{<a}$, etc. For a Lie algebra $\mathfrak{a}$ by $U(\mathfrak{a})$ we denote the universal enveloping algebra of $\mathfrak{a}$. For a commutative ring R, $\operatorname{Specm} R$ will stand for the set of maximal ideals of R.

By $\mathfrak{gl}(n)$ we denote the general linear Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \,|\, 1 \leq i, j \leq n\}$ — the standard basis of $\mathfrak{gl}(n)$ of elementary matrices. We fix the standard Cartan subalgebra of $\mathfrak{gl}(n)$, the standard triangular decomposition and the corresponding basis of simple roots of $\mathfrak{gl}(n)$. The weights of $\mathfrak{gl}(n)$ will be written as n-tuples $(\lambda_1, \ldots, \lambda_n)$ through the identification $\mathfrak{h}^* \to \mathbb{C}^n$.

The Lie subalgebra $\mathfrak{g} = \mathfrak{sl}(n)$ of $\mathfrak{gl}(n)$ is generated by $\{E_{i,i+1},\ E_{i+1,i} \,|\, 1 \leq i \leq n-1\}$. The standard Cartan subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{h}$, i.e.

$$\mathfrak{h} = \operatorname{Span}\{h_i = E_{ii} - E_{i+1,i+1} \,|\, i = 1, \ldots, n-1\}.$$

Let ϵ_i denote the projection of a $n \times n$ matrix onto its (i,i)th entry. Then a basis of simple roots of the root system Δ of $\mathfrak{g}$ is given by $\pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \,|\, i = 1, \ldots, n-1\}$ and the corresponding positive roots are $\Delta^+ = \{\epsilon_i - \epsilon_j = \alpha_i + \cdots + \alpha_{j-1} \,|\, i < j\}$.

2.1. *Index of notations*

- Section 3.1. c_{mk}; $i(c_{mk})$; Γ; $\gamma_{mk}(l)$; $M(\chi)$; $\operatorname{Supp}_{\mathrm{GT}}(M)$; M_λ; $\operatorname{Supp}(M)$; $\Gamma(\pi)$; $\mathcal{GT}_\pi(n)$.
- Section 3.2. $\mathcal{GT}(n)$.
- Section 4.1. $T(v)$; $T_n(R)$; χ_v.

- Section 4.2. δ^{ij}; Q_n; $\mathcal{B}(T(v))$; $\mathcal{GT}_{T(v)}(n)$.
- Section 4.3. $V(T(v))$; $\tilde{S}_m$; $\Phi_{\ell m}$; ε_{rs}; $e_t^{(+)}(w)$; $e_{t+1}^{(-)}(w)$; $e_{r,s}(w)$; $\Omega(T(w))$; $\Omega^+(T(w))$; $\mathcal{N}(T(w))$; $\mathcal{I}(T(w))$.
- Section 4.4. $\mathcal{F}$; $T_n(\mathbb{C})_{\text{gen}}$; $\mathcal{H}_{ij}^k$; $\Phi_{\ell m}(k,t)$; $\mathcal{H}$; $\overline{\mathcal{H}}$; $\mathcal{F}_{ij}$, $V(T(\bar{v}))$; $\mathcal{B}(T(\bar{v}))$; $\mathcal{V}_{\text{gen}}$; $\mathcal{S}$; $\mathcal{D}^{\bar{v}}$; $\text{Tab}(w)$.
- Section 5. $g_\lambda(x,y)$; $C(\mathfrak{h})$.
- Section 7.3. $\Lambda^+(\text{Tab}(w))$; $\mathcal{A}(\text{Tab}(w))$; $\mathcal{N}(\text{Tab}(w))$; $\mathcal{C}(w)$; $\mathcal{R}(w)$; $\mathcal{N}_{(r,s,p)}(\text{Tab}(w))$; $\mathcal{N}^{(1)}(\text{Tab}(w))$; $\mathcal{N}^{(2)}(\text{Tab}(w))$; $\widehat{\mathcal{A}}(\text{Tab}(w))$; $\widehat{\mathcal{N}}(\text{Tab}(w))$; $\text{Tab}(\mathcal{B})$.
- Section 7.5. $M(\mathcal{B})$; $L(\mathcal{B})$.
- Section 8. $\mathcal{D}_{ij}$; $\mathcal{D}_{ij}^x$; $\mathcal{QD}_{ij}$.
- Section 8.2. $\Theta_x(u)$; Φ_α^x.
- Section 8.3. $L_i^{(Gj)}$; $L_i^{(Cj)}$.

3. Gelfand–Tsetlin Modules of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$

3.1. *Gelfand–Tsetlin modules of* $\mathfrak{gl}(n)$

Let for $m \leq n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \mid i,j = 1,\ldots,m\}$. We have the following chain:

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n\,.$$

It induces the chain $U_1 \subset U_2 \subset \cdots \subset U_n$ for the universal enveloping algebras $U_m = U(\mathfrak{gl}_m)$, $1 \leq m \leq n$. Set $U := U(\mathfrak{gl}_n)$. Let Z_m be the center of U_m. Then Z_m is the polynomial algebra in the m variables $\{c_{mk} \mid k = 1,\ldots,m\}$,

$$c_{mk} = \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}. \tag{1}$$

The (*standard*) Gelfand–Tsetlin subalgebra Γ in U [10] is generated by $\bigcup_{i=1}^n Z_i$.

The algebra Γ is a polynomial algebra in $\frac{n(n+1)}{2}$ variables $\{c_{ij} \mid 1 \leq j \leq i \leq n\}$. For $i = 1,\ldots,n$ denote by S_i the ith symmetric group and set $G = S_n \times \cdots \times S_1$. Let Λ be the polynomial algebra in variables $\{l_{ij} \mid 1 \leq j \leq i \leq n\}$.

Let $\imath : \Gamma \to \Lambda$ be the embedding given by $\imath(c_{mk}) = \gamma_{mk}(l)$ where

$$\gamma_{mk}(l) = \sum_{i=1}^m (l_{mi} + m - 1)^k \prod_{j\neq i}^m \left(1 - \frac{1}{l_{mi} - l_{mj}}\right). \tag{2}$$

The image of $\imath$ coincides with the subalgebra of G-invariant polynomials in Λ which we identify with Γ, see [65] for more details.

Remark 3.1. Note that Γ contains the standard Cartan subalgebra of $\mathfrak{gl}(n)$ spanned by E_{ii}, $i = 1,\ldots,n$. Indeed, $c_{m1} = \sum_{i=1}^m E_{ii}$ for each $1 \leq m \leq n$. Therefore, E_{ii} belong to Γ for each $1 \leq i \leq n$.

Remark 3.2. We should note that the polynomial $\gamma_{mk}(l)$ is symmetric of degree k in variables $l_{m1},\ldots,l_{mm}$, and $\{\gamma_{m1}(l),\ldots,\gamma_{mm}(l)\}$ generate the algebra of S_m-invariant polynomials in the variables $l_{m1},\ldots l_{mm}$ (see [65]).

Example 3.3. The polynomials $\gamma_{mk}(l)$ for $m \leq 4$ are listed as follows:

$$
\begin{aligned}
\gamma_{11}(l) &= l_{11},\\
\gamma_{21}(l) &= (l_{21}+l_{22})+1,\\
\gamma_{22}(l) &= (l_{21}^2+l_{22}^2)+(l_{21}+l_{22}),\\
\gamma_{31}(l) &= (l_{31}+l_{32}+l_{33})+3,\\
\gamma_{32}(l) &= (l_{31}^2+l_{32}^2+l_{33}^2)+2(l_{31}+l_{32}+l_{33})+1,\\
\gamma_{33}(l) &= (l_{31}^3+l_{32}^3+l_{33}^3)+4(l_{31}^2+l_{32}^2+l_{33}^2)-(l_{31}l_{32}+l_{31}l_{33}+l_{32}l_{33})-6\\
&\quad+(l_{31}+l_{32}+l_{33}),\\
\gamma_{41}(l) &= (l_{41}+l_{42}+l_{43}+l_{44})+6,\\
\gamma_{42}(l) &= (l_{41}^2+l_{42}^2+l_{43}^2+l_{44}^2)+3(l_{41}+l_{42}+l_{43}+l_{44})+4,\\
\gamma_{43}(l) &= (l_{41}^3+l_{42}^3+l_{43}^3+l_{44}^3)-(l_{41}l_{42}+l_{41}l_{43}+l_{41}l_{44}+l_{42}l_{43}+l_{42}l_{44}+l_{43}l_{44})\\
&\quad+6(l_{41}^2+l_{42}^2+l_{43}^2+l_{44}^2)+3(l_{41}+l_{42}+l_{43}+l_{44})-19,\\
\gamma_{44}(l) &= (l_{41}^4+l_{42}^4+l_{43}^4+l_{44}^4)+9(l_{41}^3+l_{42}^3+l_{43}^3+l_{44}^3)+21(l_{41}^2+l_{42}^2+l_{43}^2+l_{44}^2)\\
&\quad-(l_{41}^2l_{42}+l_{41}^2l_{43}+l_{41}^2l_{44}+l_{41}l_{42}^2+l_{41}l_{43}^2+l_{41}l_{44}^2+l_{42}^2l_{43}+l_{42}^2l_{44}+l_{42}l_{43}^2)\\
&\quad-(l_{42}l_{44}^2+l_{43}^2l_{44}+l_{43}l_{44}^2)-10(l_{41}l_{42}+l_{41}l_{44}+l_{42}l_{43}+l_{43}l_{44}\\
&\quad+l_{41}l_{43}+l_{42}l_{44})-19(l_{41}+l_{42}+l_{43}+l_{44})-120.
\end{aligned}
$$

Definition 3.4. A finitely generated U-module M is called a *Gelfand–Tsetlin module* (*relative to* Γ) provided that the restriction $M|_\Gamma$ is a direct sum of Γ-modules:

$$M|_\Gamma = \bigoplus_{\mathsf{m}\in\operatorname{Specm}\Gamma} M(\mathsf{m}), \tag{3}$$

where

$$M(\mathsf{m}) = \{v \in M \mid \mathsf{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Definition 3.5. An algebra homomorphism $\chi : \Gamma \to \mathbb{C}$ will be called *Gelfand–Tsetlin character*.

Remark 3.6. For each $\mathbf{m} \in \operatorname{Specm}\Gamma$ we have associated a character $\chi_{\mathbf{m}} : \Gamma \to \Gamma/\mathbf{m} \sim \mathbb{C}$. In the same way, for each nonzero character $\chi : \Gamma \to \mathbb{C}$, $\operatorname{Ker}(\chi)$ is a maximal ideal of Γ. So, we have a natural identification between characters of Γ and elements of $\operatorname{Specm}\Gamma$. So, using Gelfand–Tsetlin characters, a Gelfand–Tsetlin module (with respect to Γ) M can be decomposed as $M = \bigoplus_{\chi\in\Gamma^*} M(\chi)$, where

$$M(\chi) = \{v \in M \,|\, \text{for each } \gamma \in \Gamma,\ \exists\, k \in \mathbb{Z}_{\geq 0} \text{ such that } (\gamma - \chi(\gamma))^k v = 0\}.$$

Definition 3.7. Given a Gelfand–Tsetlin module M, the Gelfand–Tsetlin support of M is the set

$$\mathrm{Supp}_{\mathrm{GT}}(M) := \{\chi \in \Gamma^* \mid M(\chi) \neq 0\}.$$

Definition 3.8. A finitely generated U-module M is called a *weight module* (*relative to* $\mathfrak{h}$) provided that the restriction $M|_{\mathfrak{h}}$ is a direct sum of $\mathfrak{h}$-modules:

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \tag{4}$$

where

$$M_\lambda := \{v \in M \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

The *weight support* (or simply, the *support*) of M is

$$\mathrm{Supp}(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}.$$

Remark 3.9. Any simple Gelfand–Tsetlin module M over $\mathfrak{gl}(n)$ is a weight module with respect to the standard Cartan subalgebra $\mathfrak{h}$ spanned by E_{ii}, $i = 1, \ldots, n$, see Remark 3.1. Moreover, Γ is diagonalizable on any simple finite-dimensional module. On the other hand, a simple weight module need not to be Gelfand–Tsetlin, however, simple weight modules with finite $\mathfrak{h}$-weight multiplicities are Gelfand–Tsetlin. The latter is true since in this case Γ has a common eigenvector in every nonzero weight space. In particular, every highest-weight module or, more general, every module from the category $\mathcal{O}$ is Gelfand–Tsetlin.

The definition of a Gelfand–Tsetlin module depends on the choice of the Gelfand–Tsetlin subalgebra Γ. One can easily define a family of Gelfand–Tsetlin subalgebras of $\mathfrak{gl}(n)$ as follows. Let $\pi = \{\beta_1, \ldots, \beta_{n-1}\}$ be a base of the root system of $\mathfrak{gl}(n)$, where $\beta_k = \epsilon_{i_k} - \epsilon_{i_{k+1}}$, $k = 1, \ldots, n-1$. Let $\mathfrak{gl}_k$ be the subalgebra of $\mathfrak{gl}(n)$ spanned by E_{ij}, $i, j \in \{i_1, \ldots, i_k\}$. In particular, $\mathfrak{gl}_k \simeq \mathfrak{gl}(k)$ and $\beta_1, \ldots, \beta_k$ are the simple roots of $\mathfrak{gl}_k$. Then we have a chain of embeddings

$$\mathfrak{gl}_1 \subset \cdots \subset \mathfrak{gl}_n\,.$$

Let Z_i be the center of $U(\mathfrak{gl}_i)$ and $\Gamma(\pi)$ be the subalgebra generated by Z_i, $i = 1, \ldots, n$. We will call $\Gamma(\pi)$ a *Gelfand–Tsetlin subalgebra associated with* π.

Each subalgebra $\Gamma(\pi)$ gives rise to a category of Gelfand–Tsetlin modules which we denote by $\mathcal{GT}_\pi(n)$. Let π and π' be different bases of the root system. Then π and π' are conjugate under the action of the Weyl group of $\mathfrak{gl}(n)$. Hence, $\Gamma(\pi)$ and $\Gamma(\pi')$ are also conjugate which leads to an equivalence of the categories $\mathcal{GT}_\pi(n)$ and $\mathcal{GT}_{\pi'}(n)$.

Example 3.10. Two bases π and π' may define the same Gelfand–Tsetlin subalgebra. Indeed, take for example the bases $\pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$ and $\pi' = \{\epsilon_2 - \epsilon_1, \epsilon_1 - \epsilon_3\}$ of root system of $\mathfrak{gl}(3)$. Then $\Gamma(\pi) = \Gamma(\pi')$. One easily checks that $\mathfrak{gl}(3)$ has three distinct Gelfand–Tsetlin subalgebras and they are parameterized by the $\mathfrak{gl}_2$-part of the chain.

3.2. *Gelfand–Tsetlin modules of* $\mathfrak{sl}(n)$

Let Γ be a Gelfand–Tsetlin subalgebra of $\mathfrak{gl}(n)$. Consider the natural projection $\tau : \mathfrak{gl}(n) \to \mathfrak{sl}(n)$, $\tau(A) = A - \frac{\mathrm{tr}(A)}{n} I_n$, which extends to an epimorphism $\bar{\tau} : U(\mathfrak{gl}(n)) \to U(\mathfrak{sl}(n))$. Then the image $\bar{\tau}(\Gamma)$ of Γ is called the (standard) Gelfand–Tsetlin subalgebra of $\mathfrak{sl}(n)$. It is a maximal commutative subalgebra of $U(\mathfrak{sl}(n))$ isomorphic to a polynomial ring in $\frac{n(n+1)}{2} - 1$ generators. With a small abuse of notation, by $\mathcal{GT}(n)$, we denote the category of all Gelfand–Tsetlin $\mathfrak{sl}(n)$-modules relative to Γ.

4. Families of Gelfand–Tsetlin Modules for $\mathfrak{gl}(n)$

4.1. *Gelfand–Tsetlin tableaux*

The simple finite-dimensional modules are the first natural examples of Gelfand–Tsetlin modules. In this case, an eigenbasis for the action of the generators of Γ (1) is given by the so-called Gelfand–Tsetlin tableaux, following the original work of Gelfand and Tsetlin. In particular, for every simple finite-dimensional module M, $\dim(M(\chi)) = 1$ whenever $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$. In order to describe the Gelfand–Tsetlin tableaux, we first fix some notation.

Definition 4.1. Fix a vector $v = (v_{ij})_{j \le i}^{n} \in \mathbb{C}^{\frac{n(n+1)}{2}}$.

(i) By $T(v)$ we will denote the following array with complex entries $\{v_{ij}\}$.

$$\begin{array}{ccccccccc}
\boxed{v_{n1}} & & \boxed{v_{n2}} & & \cdots & & \boxed{v_{n,n-1}} & & \boxed{v_{nn}} \\
& \boxed{v_{n-1,1}} & & & \cdots & & & \boxed{v_{n-1,n-1}} & \\
& & \cdots & & \cdots & & \cdots & & \\
& & & \boxed{v_{21}} & & \boxed{v_{22}} & & & \\
& & & & \boxed{v_{11}} & & & &
\end{array}$$

Such an array will be called a Gelfand–Tsetlin *tableau* of height n.

(ii) Throughout the paper, for any ring R, $T_n(R)$ will stand for the space of the Gelfand–Tsetlin tableaux of height n with entries in R. We will identify $T_n(\mathbb{C})$ with the set $\mathbb{C}^{\frac{n(n+1)}{2}}$ in the following way: to each

$$v = (v_{n1}, \dots, v_{nn} | v_{n-1,1}, \dots, v_{n-1,n-1} | \cdots | v_{21}, v_{22} | v_{11}) \in \mathbb{C}^{\frac{n(n+1)}{2}}$$

we associate a tableau $T(v) \in T_n(\mathbb{C})$ as above.

Remark 4.2. There is a natural correspondence between the set Γ^* of characters $\chi : \Gamma \to \mathbb{C}$ and the set of Gelfand–Tsetlin tableaux of height n. In fact, to obtain a Gelfand–Tsetlin tableau $T(l)$ from a character χ we find a solution (l_{ij}) of the system of equations

$$\{\gamma_{mk}(l) = \chi(c_{mk})\}_{1\leq k\leq m\leq n}.$$

Conversely, for every Gelfand–Tsetlin tableau $T(v)$ with entries $\{v_{ij} \mid 1 \leq j \leq i \leq n\}$, we associate $\chi_v \in \Gamma^*$ by defining $\chi_v(c_{mk}) = \gamma_{mk}(v)$. It is clear that each tableau defines such a character uniquely. On the other hand, a tableau is defined by a character up to a permutation of the rows, i.e. an element of $S_n \times S_{n-1} \times \cdots \times S_1$.

4.2. *Gelfand–Tsetlin formulas for finite-dimensional modules*

In this section, we recall the classical result of I. Gelfand and M. Tsetlin.

Definition 4.3. A Gelfand–Tsetlin tableau of height n is called *standard* if $v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n-1$.

Note that, for the sake of convenience, the second condition in the definition above is slightly different from the original condition in [31]. Here is the classical result of Gelfand and Tsetlin [31].

Theorem 4.4. *Let $L(\lambda)$ be the simple finite-dimensional module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exists a basis of $L(\lambda)$ parameterized by the set of all standard tableaux $T(v) = T(v_{ij})$ with fixed top row $v_{nj} = \lambda_j - j + 1$, $j = 1, \ldots, n$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the* Gelfand–Tsetlin formulas:

$$E_{k,k+1}(T(v)) = -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(v_{ki} - v_{k+1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})}\right)T(v + \delta^{ki}),$$

$$E_{k+1,k}(T(v)) = \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(v_{ki} - v_{k-1,j})}{\prod_{j\neq i}^{k}(v_{ki} - v_{kj})}\right)T(v - \delta^{ki}),$$

$$E_{kk}(T(v)) = \left(k - 1 + \sum_{i=1}^{k} v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i}\right)T(v),$$

where $\delta^{ij} \in T_n(\mathbb{Z})$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero. If the new tableau $T(v \pm \delta^{ki})$ is not standard, then the corresponding summand of $E_{k,k+1}(T(v))$ or $E_{k+1,k}(T(v))$ is zero by definition.

We call the formulas above the *Gelfand–Tsetlin formulas* for $\mathfrak{gl}(n)$.

Set $e_{kk}(v) := k - 1 + \sum_{i=1}^{k} v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i}$. We note that $E_{kk}(T(v)) = e_{kk}(v)T(v)$ is well defined for any Gelfand–Tsetlin tableau $T(v)$.

Definition 4.5. Let $T(v)$ be a Gelfand–Tsetlin tableau. Then we call $(e_{11}(v), \ldots, e_{nn}(v))$ (respectively, $(e_{11}(v) - e_{22}(v), \ldots, e_{n-1,n-1}(v) - e_{nn}(v))$) the $\mathfrak{gl}(n)$*-weight* (respectively, the $\mathfrak{sl}(n)$*-weight*) of the tableau $T(v)$.

The formulas for the action of the generators of $\mathfrak{gl}(n)$ in the theorem above imply that the standard tableaux $T(v)$ forms an eigenbasis for the action of the standard Cartan subalgebra $\mathfrak{h}$. The following result shows that such basis is an eigenbasis for the Gelfand–Tsetlin subalgebra. For the proof, see [65].

Theorem 4.6. *Let $L(\lambda)$ be the simple finite-dimensional module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, with basis as described in Theorem* 4.4. *The action of the generators c_{rs} of Γ (see Eq.* (1)) *is given by*

$$c_{rs}(T(v)) = \gamma_{rs}(v)T(v), \tag{5}$$

where $\gamma_{rs}(v)$ are the symmetric polynomials defined in (2).

As a direct consequence of Theorems 4.4 and 4.6, any simple finite-dimensional $\mathfrak{gl}(n)$-module is a Gelfand–Tsetlin module with one-dimensional Gelfand–Tsetlin subspaces.

Remark 4.7. Whenever we refer to finite-dimensional $\mathfrak{sl}(n)$-modules we will use the same vector space and the Gelfand–Tsetlin formulas for generators $E_{r,r+1}$ or $E_{r+1,r}$, for the action of the generators of a Cartan subalgebra $\{h_1, \dots, h_{n-1}\}$ we define $h_i(T(v)) := (E_{ii} - E_{i+1,i+1})(T(v))$. We also fix the action of the central element $E_{11} + \cdots + E_{nn}$ as zero.

Example 4.8. Let us denote by M the simple highest-weight $\mathfrak{gl}(3)$-module with highest weight $(1, 0, -1)$. M is a finite-dimensional module of dimension 8. The tableaux realization guaranteed by Theorem 4.4 consists of a vector space spanned by the set of all standard tableaux of height 3 with top row $(1, -1, -3)$.

$$T_1 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 1 & & -1 & \\ & & 1 & & \end{array} \qquad T_2 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 1 & & -1 & \\ & & 0 & & \end{array}$$

$$T_3 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 1 & & -2 & \\ & & 1 & & \end{array} \qquad T_4 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 1 & & -2 & \\ & & 0 & & \end{array}$$

$$T_5 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 0 & & -2 & \\ & & -1 & & \end{array} \qquad T_6 = \begin{array}{ccccc} 1 & & -1 & & -3 \\ & 0 & & -2 & \\ & & 0 & & \end{array}$$

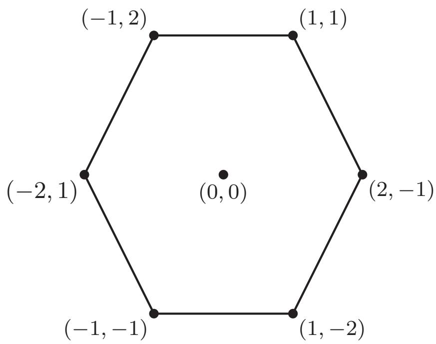
$$T_7 = \begin{array}{ccccc} \boxed{1} & & \boxed{-1} & & \boxed{-3} \\ & \boxed{1} & & \boxed{-2} & \\ & & \boxed{-1} & & \end{array} \qquad T_8 = \begin{array}{ccccc} \boxed{1} & & \boxed{-1} & & \boxed{-3} \\ & \boxed{0} & & \boxed{-1} & \\ & & \boxed{0} & & \end{array}$$

By Theorem 4.4, the simple module M is isomorphic to $\mathrm{Span}_{\mathbb{C}}\{T_i \mid i = 1, \ldots, 8\}$ endowed with the action of $\mathfrak{gl}(3)$ given by the Gelfand–Tsetlin formulas.

Since the action of $E_{11} + E_{22} + E_{33}$ is fixed to be trivial and $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{h_1 = E_{11} - E_{22},\ h_2 = E_{22} - E_{33}\}$, M becomes an $\mathfrak{sl}(3)$-module with weight support

$$\mathrm{Supp}(M) = \{(1,1), (-1,2), (2,-1), (0,0), (-2,1), (1,-2), (-1,-1)\}.$$

Then as an $\mathfrak{sl}(3)$-module, M is isomorphic to $L(1,1)$ (the simple finite-dimensional $\mathfrak{sl}(3)$-module of highest weight $(1,1)$). The following picture shows the weights lattice of the $\mathfrak{sl}(3)$-module M. Note that $M_{(0,0)}$ is two-dimensional with basis $\{T_4,\ T_8\}$.



In particular, the basis elements of $M_{(0,0)}$ cannot be distinguished by the action of the Cartan subalgebra. However, using Γ, the module decomposes as a direct sum of one-dimensional Γ-submodules.

The following theorem will give us information about the dimension of Gelfand–Tsetlin subspaces for simple Gelfand–Tsetlin modules and the possible number of non-isomorphic Gelfand–Tsetlin modules with a given Gelfand–Tsetlin character in its support.

Theorem 4.9 ([28, Theorem 6.1; 54]). *Let $U = U(\mathfrak{gl}(n))$, $\Gamma \subset U$ the Gelfand–Tsetlin subalgebra,* $\mathsf{m} \in \mathrm{Specm}\,\Gamma$. *Set $Q_n = 1!2!\ldots(n-1)!$.*

(i) *For a Gelfand–Tsetlin module M, such that $M(\mathsf{m}) \neq 0$ and M is generated by some $x \in M(\mathsf{m})$ (in particular for a simple module), one has*

$$\dim_{\mathbb{C}} M(\mathsf{m}) \leq Q_n.$$

(ii) *The number of isomorphism classes of simple Gelfand–Tsetlin modules N such that $N(\mathsf{m}) \neq 0$ is always nonzero and does not exceed Q_n.*

The theorem above shows that elements of $\operatorname{Specm}\Gamma$ classify the simple Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules (and, hence, $\mathfrak{sl}(n)$-modules) up to some finiteness and up to an isomorphism of Gelfand–Tsetlin modules which contains two different Gelfand–Tsetlin characters.

In [51], Gelfand–Tsetlin modules with tableaux realization and action given by the Gelfand–Tsetlin formulas are studied, but such modules V satisfy $\dim(V(\chi)) \leq 1$ for all $\chi \in \Gamma^*$. In what follows, we will consider a more general definition of tableaux realization, which will allow to consider certain classes of modules with $\dim(V(\chi))$ greater than 1.

For any Gelfand–Tsetlin tableau $T(v) \in T_n(\mathbb{C})$ we consider the set

$$\mathcal{B}(T(v)) := \{T(v+z) \mid z \in T_n(\mathbb{Z}), z_{nk} = 0,\, 1 \leq k \leq n\}. \tag{6}$$

If an indecomposable Gelfand–Tsetlin module V has a tableaux realization and $T(v)$ is one of the basis tableaux then it has a basis which is a subset of $\mathcal{B}(T(v))$. On the other hand, we might have a module with a basis consisting of a subset of tableaux from $\mathcal{B}(T(v))$ but without a tableaux realization. This may happen, for example, when V has a Gelfand–Tsetlin character of multiplicity more than 1. For this reason, we extend the notion of modules with a tableaux realization.

Definition 4.10. We say that a Gelfand–Tsetlin module M admits a *generalized tableaux realization* with respect to a Gelfand–Tsetlin subalgebra Γ if M has a basis $\mathcal{B}_M$ labeled by a subset of $\mathcal{B}(T(v))$ for some tableau $T(v)$, such that every $m_{T(w)} \in \mathcal{B}_M$, $T(w) \in \mathcal{B}(T(v))$, is a generalized eigenvector of c_{rs} of eigenvalue $\gamma_{rs}(w)$, for all r, s. We will denote by $\mathcal{GT}_{T(v)}(n)$ the full subcategory of the category of Gelfand–Tsetlin modules $\mathcal{GT}(n)$ which consists of modules with a generalized tableaux realization with respect to Γ whose basis contains $T(v)$.

The subcategory $\mathcal{GT}_{T(v)}(n)$ is closed under the operations of taking submodules and quotients. Moreover, as our main result will imply, simple modules of the categories $\mathcal{GT}_{T(v)}(3)$ for all $T(v)$ exhaust all simple Gelfand–Tsetlin modules for $\mathfrak{sl}(3)$.

Conjecture. *Simple modules of the categories* $\mathcal{GT}_{T(v)}(n)$ *for all* $T(v)$ *exhaust all simple Gelfand–Tsetlin modules for* $\mathfrak{sl}(n)$.

Remark 4.11. There are modules in $\mathcal{GT}(n)$ that do not belong to $\mathcal{GT}_{T(v)}(n)$ for any $T(v)$. For example, consider $n = 2$ and a simple weight module $V(\lambda, \gamma)$ with a weight $\lambda \in \mathbb{C}$ in its weight support and on which the Casimir element c_{22} acts as a multiplication by $\gamma \in \mathbb{C}$, where $\gamma \neq (\lambda + k)^2 + 2(\lambda + 2k)$ for any integer k. Then $V(\lambda, \gamma)$ has a non-split self-extension which remains a weight module but on which c_{22} does not act semisimply. This self-extension is an indecomposable Gelfand–Tsetlin module that does not admit a generalized tableaux realization.

Definition 4.12. We will call the subcategory $\mathcal{GT}_{T(v)}(n)$ *the block of* $\mathcal{GT}(n)$ *generated by* $T(v)$.

From now on, whenever n is clear from the context, we will write $\mathcal{GT}_{T(v)}$ instead of $\mathcal{GT}_{T(v)}(n)$.

4.3. *Generic modules*

Observing that the coefficients in the Gelfand–Tsetlin formulas in Theorem 4.4 are rational functions on the entries of the tableaux, Drozd *et al.* [11] extended the Gelfand–Tsetlin construction to more general modules. In the case when all denominators are nonzero for all possible integral shifts, one can use the same formulas and define a new class of infinite-dimensional $\mathfrak{gl}(n)$-modules: *generic* Gelfand–Tsetlin modules (cf. [11, Sec. 2.3]).

Definition 4.13. A Gelfand–Tsetlin tableau $T(v)$ (equivalently, $v \in T_n(\mathbb{C})$) is called *generic* if $v_{ki} - v_{kj} \notin \mathbb{Z}$ for all $1 \leq i \neq j \leq k \leq n-1$. A Gelfand–Tsetlin character χ_v associated to a generic tableau $T(v)$ (see Remark 4.2) will be called a *generic* Gelfand–Tsetlin character.

Recall that $\mathcal{B}(T(v)) = \{T(v+z) \mid z \in T_{n-1}(\mathbb{Z})\}$ for any Gelfand–Tsetlin tableau $T(v)$.

Theorem 4.14 ([11, Sec. 2.3]). *Let $T(v)$ be a generic Gelfand–Tsetlin tableau of height n.*

(i) *The vector space $V(T(v)) = \mathrm{Span}_{\mathbb{C}}\, \mathcal{B}(T(v))$ has a structure of a $\mathfrak{gl}(n)$-module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand–Tsetlin formulas.*
(ii) *The action of the generators of Γ on the basis elements of $\mathcal{B}(T(v))$ is given by* (5).
(iii) *The $\mathfrak{gl}(n)$-module $V(T(v))$ is a Gelfand–Tsetlin module with Gelfand–Tsetlin multiplicities equal to* 1.

The module $V(T(v))$ constructed in Theorem 4.14 will be extensively used in future and will be referred as *the generic Gelfand–Tsetlin module associated to* $T(v)$. In general $V(T(v))$ need not to be simple. Because Γ has simple spectrum on $V(T(v))$ for $T(w)$ in $\mathcal{B}(T(v))$ we may define the *simple U-module in $V(T(v))$ containing* $T(w)$ to be the simple subquotient of $V(T(v))$ containing $T(w)$.

Remark 4.15. By Theorem 4.14(iii), given two different tableaux $T(w)$ and $T(w')$ in $\mathcal{B}(T(v))$, there exists an element of Γ that has different eigenvalues for $T(w)$ and $T(w')$. Whenever we say that Γ "*separates*" *tableaux of $V(T(v))$* we will refer to this property.

4.3.1. *Gelfand–Tsetlin formulas in terms of permutations*

In this section, we will rewrite the Gelfand–Tsetlin formulas in terms of permutations. These formulas will be very useful when verifying certain identities for the action of $\mathfrak{g}$ on $V(T(v))$.

Let $\widetilde{S}_m$ denote the subset of S_m consisting of the transpositions $(1, i)$, $i = 1, \ldots, m$. For $\ell < m$, set $\Phi_{\ell m} = \widetilde{S}_{m-1} \times \cdots \times \widetilde{S}_\ell$. For $\ell > m$, we set $\Phi_{\ell m} = \Phi_{m\ell}$. Finally, we define $\Phi_{\ell\ell} = \{\mathrm{Id}\}$. Every σ in $\Phi_{\ell m}$ will be written as an $|\ell - m|$-tuple of transpositions and by $\sigma[t]$ we will denote the tth component of the tuple.

Remark 4.16. Recall that in order to have well-defined action of $\sigma \in \Phi_{\ell m}$ on $T_{n-1}(\mathbb{C})$, for $w \in T_{n-1}(\mathbb{C})$ and $\sigma \in \Phi_{\ell m}$ on w we set

$$\sigma(w) := (w_{n-1,\sigma^{-1}[n-1](1)}, \ldots, w_{n-1,\sigma^{-1}[n-1](1)} | \cdots | w_{1,\sigma^{-1}[1](1)}).$$

Definition 4.17. Let $1 \leq r < s \leq n$. Define

$$\varepsilon_{rs} := \delta^{r,1} + \delta^{r+1,1} + \cdots + \delta^{s-1,1} \in T_n(\mathbb{Z}).$$

Furthermore, define $\varepsilon_{rr} = 0$ and $\varepsilon_{sr} = -\varepsilon_{rs}$.

Definition 4.18. For each generic vector w and any $1 \leq t \leq n-1$ define

$$e_t^{(+)}(w) := \frac{\prod_{j\neq 1}^{t+1}(w_{t1} - w_{t+1,j})}{\prod_{j\neq 1}^{t}(w_{t1} - w_{tj})}; \quad e_{t+1}^{(-)}(w) := \frac{\prod_{j\neq 1}^{t-1}(w_{t1} - w_{t-1,j})}{\prod_{j\neq 1}^{t}(w_{t1} - w_{tj})};$$

$$e_{k,k+1}(w) := -\frac{\prod_{j=1}^{k+1}(w_{k1} - w_{k+1,j})}{\prod_{j\neq 1}^{k}(w_{k1} - w_{kj})}; \quad e_{k+1,k}(w) := \frac{\prod_{j=1}^{k-1}(w_{k1} - w_{k-1,j})}{\prod_{j\neq 1}^{k}(w_{k1} - w_{kj})}.$$

Lemma 4.19. *For each $m > k$ the action of E_{mk} is given by the expression*:

$$E_{mk}(T(v)) = \sum_{\sigma\in\Phi_{mk}} e_{k+1,k}(\sigma(v)) \left(\prod_{j=k+2}^{m} \left(e_j^{(-)}(\sigma(v))\right)\right) T\left(v + \sum_{i=k}^{m-1} \sigma(\varepsilon_{i+1,i})\right).$$

Proof. The case $k = m+1$ follows from the Gelfand–Tsetlin formulas. The general case follows by induction on $m - k$ using the relation

$$E_{m,k+1}(E_{k+1,k}T(v)) - E_{k+1,k}(E_{m,k+1}T(v)) = E_{m,k}T(v)$$

for any generic vector v. $\square$

Lemma 4.20. *For each $r < s$ the action of E_{rs} is given by the expression*:

$$E_{rs}(T(v)) = \sum_{\sigma\in\Phi_{rs}} \left(\prod_{j=r}^{s-2} e_j^{(+)}(\sigma(v))\right) e_{s-1,s}(\sigma(v)) T\left(v + \sum_{i=r}^{s-1} \sigma(\varepsilon_{i,i+1})\right).$$

Proof. The case $s = r+1$ follows from the Gelfand–Tsetlin formulas. The general case follows by induction in $s - r$ using the relation

$$E_{r,r+1}(E_{r+1,s}T(v)) - E_{r+1,s}(E_{r,r+1}T(v)) = E_{rs}T(v)$$

for any generic vector v. $\square$

Definition 4.21. For each generic vector $w \in T_n(\mathbb{C})$ and any $1 \leq r, s \leq n$ we define

$$e_{rs}(w) := \begin{cases} \left(\prod_{j=r}^{s-2} e_j^{(+)}(w)\right) e_{s-1,s}(w) & \text{if } r < s, \\ e_{s+1,s}(w)\left(\prod_{j=s+2}^{r} e_{j-2}^{(-)}\right) & \text{if } r > s, \\ r-1+\sum_{i=1}^{r} w_{ri} - \sum_{i=1}^{r-1} w_{r-1,i} & \text{if } r = s. \end{cases}$$

Proposition 4.22. *Let $v \in T_n(\mathbb{C})$ be any generic vector and $z \in T_{n-1}(\mathbb{Z})$. The Gelfand–Tsetlin formulas for the U-module $V(T(v))$ can be written as follows*:

$$E_{\ell m}(T(v+z)) = \sum_{\sigma \in \Phi_{\ell m}} e_{\ell m}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{\ell m})).$$

Proof. Follows from Lemmas 4.19 and 4.20 and the fact that $\sum_{i=\ell}^{m-1} \sigma(\varepsilon_{i+1,i}) = \sigma(\varepsilon_{m,\ell})$ when $m > \ell$ and $\sum_{i=m}^{\ell-1} \sigma(\varepsilon_{i,i+1}) = \sigma(\varepsilon_{m,\ell})$ when $m < \ell$. □

Example 4.23. Let us write explicitly the functions $e_{rs}(w)$ from Definition 4.21 and the Gelfand–Tsetlin formulas in Proposition 4.22 in the case of $\mathfrak{gl}(3)$. Let $v \in T_3(\mathbb{C})$ be a generic vector, $z \in T_2(\mathbb{Z})$ and $w = v+z$. Set also τ to be the permutation that interchanges the entries in positions $(2,1)$ and $(2,2)$. Considering

$\varepsilon_{11} = (0,0,0)$	$e_{11}(w) = w_{11}$
$\varepsilon_{22} = (0,0,0)$	$e_{22}(w) = w_{21} + w_{22} - w_{11} + 1$
$\varepsilon_{33} = (0,0,0)$	$e_{33}(w) = w_{31} + w_{32} + w_{33} - w_{21} - w_{22} + 2$
$\varepsilon_{12} = (0,0,1)$	$e_{12}(w) = -(w_{11} - w_{21})(w_{11} - w_{22})$
$\varepsilon_{21} = (0,0,-1)$	$e_{21}(w) = 1$
$\varepsilon_{23} = (1,0,0)$	$e_{23}(w) = -\frac{(w_{21}-w_{31})(w_{21}-w_{32})(w_{21}-w_{33})}{w_{21}-w_{22}}$
$\varepsilon_{32} = (-1,0,0)$	$e_{32}(w) = \frac{w_{21}-w_{11}}{w_{21}-w_{22}}$
$\varepsilon_{31} = (-1,0,-1)$	$e_{31}(w) = \frac{1}{w_{21}-w_{22}}$
$\varepsilon_{13} = (1,0,1)$	$e_{13}(w) = -\frac{(w_{21}-w_{31})(w_{21}-w_{32})(w_{21}-w_{33})(w_{11}-w_{22})}{w_{21}-w_{22}}$

The action of $\mathfrak{gl}(3)$ on any tableau is given by

$$
\begin{aligned}
&E_{11}(T(w)) = e_{11}(w)T(w),\\
&E_{22}(T(w)) = e_{22}(w)T(w),\\
&E_{33}(T(w)) = e_{33}(w)T(w),\\
&E_{12}(T(w)) = e_{12}(w)T(w+\varepsilon_{12}),\\
&E_{21}(T(w)) = e_{21}(w)T(w+\varepsilon_{21}),\\
&E_{32}(T(w)) = e_{32}(w)T(w+\varepsilon_{32}) + e_{32}(\tau(w))T(w+\tau(\varepsilon_{32})),\\
&E_{23}(T(w)) = e_{23}(w)T(w+\varepsilon_{23}) + e_{23}(\tau(w))T(w+\tau(\varepsilon_{23})),\\
&E_{13}(T(w)) = e_{13}(w)T(w+\varepsilon_{13}) + e_{13}(\tau(w))T(w+\tau(\varepsilon_{13})),\\
&E_{31}(T(w)) = e_{31}(w)T(w+\varepsilon_{31}) + e_{31}(\tau(w))T(w+\tau(\varepsilon_{31})).
\end{aligned}
$$

The explicit description of all simple generic modules for $\mathfrak{gl}(3)$ was obtained first in [56]. The classification of simple generic modules $\mathfrak{gl}(n)$ was completed in [20]. Let us discuss briefly the main results in the last classification.

Definition 4.24. Let $T(v)$ be a fixed Gelfand–Tsetlin tableau. For any $T(w) \in \mathcal{B}(T(v))$, and for any $1 < r \leq n$, $1 \leq s \leq r$, and $1 \leq u \leq r-1$ we define

$$
\begin{aligned}
\Omega(T(w)) &:= \{(r,s,u) \,|\, w_{rs} - w_{r-1,u} \in \mathbb{Z}\},\\
\Omega^{+}(T(w)) &:= \{(r,s,u) \,|\, w_{rs} - w_{r-1,u} \in \mathbb{Z}_{\geq 0}\}.
\end{aligned}
$$

A basis for the simple subquotients of $V(T(v))$ is provided in the following theorem.

Theorem 4.25 ([20, Theorems 6.8 and 6.14]). *Let $T(v)$ be a fixed generic Gelfand–Tsetlin tableau and $T(w)$ in $\mathcal{B}(T(v))$.*

(i) *The module $U \cdot T(w)$ has a basis of tableaux*

$$\mathcal{N}(T(w)) = \{T(w') \in \mathcal{B}(T(w)) \,|\, \Omega^{+}(T(w)) \subseteq \Omega^{+}(T(w'))\}.$$

(ii) *The simple module containing $T(w)$ has a basis of tableaux*

$$\mathcal{I}(T(w)) = \{T(w') \in \mathcal{B}(T(w)) \,|\, \Omega^{+}(T(w)) = \Omega^{+}(T(w'))\}.$$

The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{N}(T(w))$ is given by the Gelfand–Tsetlin formulas. The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{I}(T(w))$ is given by the Gelfand–Tsetlin formulas with the convention that all tableau $T(w' \pm \delta^{ki})$ for which $\Omega^{+}(T(w' \pm \delta^{ki})) \neq \Omega^{+}(T(w))$ are omitted in the sums for $E_{k,k+1}(T(w'))$ and $E_{k+1,k}(T(w'))$.

Corollary 4.26. *Let $T(v)$ be a generic Gelfand–Tsetlin tableau. The module $V(T(v))$ is simple if and only if $\Omega(T(v)) = \emptyset$.*

Example 4.27. Consider $a, b, c \in \mathbb{C}$ such that $\{a-b, a-c, b-c\} \cap \mathbb{Z} = \emptyset$ and $v = (a, b, c|a, b+2|a)$, then

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{b+2} & \\ & & \boxed{a} & & \end{array}$$

$\Omega(T(v)) = \{(3,1,1), (3,2,2), (2,1,1)\}$, $\Omega^+(T(v)) = \{(3,1,1), (2,1,1)\}$. So, by Theorem 4.25, the simple subquotient of $V(T(v))$ containing $T(v)$ has a basis

$$\mathcal{I}(T(v)) = \{T(v + (m, n, k)) \,|\, m \leq 0, k \leq m, \text{ and } n > -2\}.$$

Example 4.28 (See also [51, Sec. 4.3]). Let $a_1, \ldots, a_n$ be complex numbers such that $a_i - a_j \notin \mathbb{Z}$ for any $i \neq j$. Denote by $T(v)$ the Gelfand–Tsetlin tableau of height n with entries v_{ij}, such that $v_{rs} = a_s$ for $1 \leq s \leq r \leq n$. The tableau $T(v)$ is a generic Gelfand–Tsetlin tableau and by Theorem 4.25 a basis for a simple $\mathfrak{gl}(n)$-module containing $T(v)$ has a basis

$$\mathcal{I}(T(v)) = \{T(v+z) \,|\, z_{rs} - z_{r-1,s} \in \mathbb{Z}_{\geq 0} \text{ for any } r, s\}.$$

Moreover, we can easily check that $\mathrm{Span}_{\mathbb{C}} \mathcal{I}(T(v))$ is a submodule of $V(T(v))$ isomorphic to the simple Verma module $M(a_1, a_2+1, \ldots, a_n+n-1)$.

Using the description of simple subquotients of $V(T(v))$, we will also be able to describe the Loewy series decomposition for $V(T(v))$. We will use the convention that the first module in the list is the socle of $V(T(v))$.

Theorem 4.29. *Let $T(v)$ be a generic tableau and set $t := |\Omega(T(v))|$. The Loewy series decomposition of the Gelfand–Tsetlin module $V(T(v))$ is given by*

$$D_t, \quad D_{t-1}, \ldots, D_0,$$

where $D_i = \mathrm{Span}_{\mathbb{C}}\{T(w) \in \mathcal{B}(T(v)) \,|\, |\Omega^+(T(w))| = i\}$ and $0 \leq i \leq t$. If $D_i = \emptyset$ for some $1 < i < t$ we omit this term in the Loewy decomposition.

Proof. Let us show first that D_t is a simple submodule of $V(T(v))$. By Theorem 4.25(i), if $|\Omega^+(T(w))| = t$, the module generated by $T(w)$ is simple and hence, equal to $\mathrm{Span}_{\mathbb{C}}\{T(w') \,|\, |\Omega^+(T(w'))| = t\}$. That is, $V(T(v))$ has a unique simple submodule M, namely $M = D_t$.

Set $M_{t+1} := V(T(v))$ and define $M_i := M_{i+1}/D_i$. Note that

$$M_{i+1} = \mathrm{Span}_{\mathbb{C}}\{T(w) \in \mathcal{B}(T(v)) \,|\, |\Omega^+(T(w))| \leq i\}.$$

So, by Theorem 4.25(ii) any element basis of D_i is a basis element of a simple submodule of M_{i+1} and, then D_i is the sum of all simple submodules of M_{i+1}. $\square$

Remark 4.30. We will often apply Theorem 4.29 in the following way. If V_j, $j \in J$, are all non-isomorphic simple subquotients of $V(T(v))$, and $T(w_j) \in V_j$, then the modules Loewy series components of $V(T(v))$ are precisely

$$D_i = D_{i1} \oplus \cdots \oplus D_{i,r_i},$$

where $\{D_{ij}\}_{j=1}^{r_i}$ is the set of all V_j such that $|\Omega^+(w_j)| = i$.

Although Theorem 4.25 gives a nice relation between the category $\mathcal{GT}_{T(v)}$ (see Definition 4.10) and $\Omega(T(v))$, it is not true that $\mathcal{GT}_{T(v)}$ is completely determined by $\Omega(T(v))$ — see the following example.

Example 4.31. Consider the tableaux $T(v)$ and $T(v')$ such that $\Omega(T(v)) = \Omega(T(v'))$ but $\mathcal{GT}_{T(v)}$ is not equivalent to $\mathcal{GT}_{T(v')}$. Take

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{a} \\ & \boxed{a} & & \boxed{y} & \\ & & \boxed{z} & & \end{array} \qquad T(v') = \begin{array}{ccccc} \boxed{a} & & \boxed{a+1} & & \boxed{a+2} \\ & \boxed{a} & & \boxed{y} & \\ & & \boxed{z} & & \end{array}$$

Then $\Omega(T(v)) = \Omega(T(v'))$. The Loewy series of $T(v)$ is D_3, D_0, however, the Loewy series of $T(v')$ is D_3, D_2, D_1, D_0.

4.4. *Singular Gelfand–Tsetlin modules*

The classical constructions of simple finite-dimensional modules and of generic modules presented in Secs. 4.2 and 4.3 have one common feature — an explicit basis parameterized by a set of Gelfand–Tsetlin tableaux. In the finite-dimensional case all the entries of the tableaux $T(v)$ in the basis satisfy $v_{ki} - v_{kj} \in \mathbb{Z}$, while in the generic case they satisfy $v_{ki} - v_{kj} \notin \mathbb{Z}$ for any $k \neq n$. We may consider the standard and generic tableaux as the two extreme cases of the singular Gelfand–Tsetlin tableaux where the latter are defined as follows.

Definition 4.32. A vector $v \in T_n(\mathbb{C})$ will be called *singular* if there exist $1 \leq s < t \leq r \leq n-1$ such that $v_{rs} - v_{rt} \in \mathbb{Z}$. The vector v will be called 1-*singular* if there exist k, i, j with $1 \leq i < j \leq k \leq n-1$ such that $v_{ki} - v_{kj} \in \mathbb{Z}$ and $v_{rs} - v_{rt} \notin \mathbb{Z}$ for all $(r, s, t) \neq (k, i, j)$, $r \neq n$. If v is 1-singular, the tableau $T(v)$ will be called 1-*singular tableau*. A Gelfand–Tsetlin character χ_v associated to a singular (respectively, 1-singular) tableau $T(v)$ (see Remark 4.2) will be called a *singular* (respectively, 1-*singular*) character.

4.4.1. *Construction of* 1-*singular Gelfand–Tsetlin modules*

In [21], an explicit construction of modules with a generalized tableaux realization (see Definition 4.10) associated with any 1-singular Gelfand–Tsetlin tableau was provided. In this section, we provide the main details of this construction.

Set v a vector of variables with $\frac{n(n+1)}{2}$ entries indexed by (r,s) such that $1 \leq s \leq r \leq n$. By $\mathcal{F}$ we will denote the space of rational functions on v_{ij}, $1 \leq j \leq i \leq n$, with poles on the hyperplanes $v_{rs} - v_{rt} = 0$. Note that $V(T(v))$ is defined for all generic v and that $V(T(v)) = V(T(v'))$ whenever $v - \sigma(v') \in T_{n-1}(\mathbb{Z})$ for some $\sigma \in G$. Thus, $V(T(v))$ is defined for elements v in the (generic) complex torus $T = T_n(\mathbb{C})/T_{n-1}(\mathbb{Z})$. Denote by $T_n(\mathbb{C})_{\text{gen}}$ the set of all generic vectors v in $T_n(\mathbb{C})$ such that $V(T(v))$ is simple, equivalently, $v_{rs} - v_{r-1,t} \notin \mathbb{Z}$ for any r,s,t. Until the end of this section we fix (i,j,k) such that $1 \leq i < j \leq k \leq n-1$.

By $\mathcal{H}$ we denote the hyperplane $v_{ki} - v_{kj} = 0$ in $T_n(\mathbb{C})$, also by $\tau \in S_{n-1} \times \cdots \times S_1$ we denote the transposition on the kth row interchanging the ith and jth entries. $\overline{\mathcal{H}}$ stands for the subset of all w in $T_n(\mathbb{C})$ such that $w_{tr} \neq w_{ts}$ for all triples (t,r,s) except for $(t,r,s) = (k,i,j)$. Finally, by $\mathcal{F}_{ij}$ denote the subspace of $\mathcal{F}$ consisting of all functions that are smooth on $\overline{\mathcal{H}}$.

Let us fix $\bar{v}$ in $\mathcal{H}$ such that $\bar{v}_{ki} = \bar{v}_{kj}$ and all other differences $v_{mr} - v_{ms}$ are non-integer. In other words, $\bar{v} \in \mathcal{H}$ and $\bar{v} + T_{n-1}(\mathbb{Z}) \subset \overline{\mathcal{H}}$.

Remark 4.33. For any generic vector w we can choose a representative of the class $w + T_{n-1}(\mathbb{Z})$ of w in $T = T_n(\mathbb{C})_{\text{gen}}/T_{n-1}(\mathbb{Z})$ as "close" as possible to $\bar{v}$ as follows. Let $m_{rs} := \lfloor \text{Re}(\bar{v}_{rs} - w_{rs}) \rfloor$ (the integer part of the real part of $\bar{v}_{rs} - w_{rs}$), m be the vector in $T_{n-1}(\mathbb{Z})$ with components m_{rs}, and $\bar{v}[w] := w + m$. Then $\mathcal{S} = \{\bar{v}[w] \,|\, w \in T_n(\mathbb{C})_{\text{gen}}\}$ is a set of representatives of T.

Our goal is construct a module $V(T(\bar{v}))$ with Gelfand–Tsetlin support $\{\chi_{\bar{v}+m} \,|\, m \in T_{n-1}(\mathbb{Z})\}$. We will refer to this module as the 1-*singular universal tableaux Gelfand–Tsetlin module associated with* $\bar{v}$, or simply as the *universal module*.

We formally introduce the complex vector space $V(T(\bar{v}))$ as the one spanned by vectors $\{T(\bar{v}+z), \mathcal{D}T(\bar{v}+z) \,|\, z \in T_{n-1}(\mathbb{Z})\}$ subject to the relations $T(\bar{v}+z) - T(\bar{v}+\tau(z)) = 0$ and $\mathcal{D}T(\bar{v}+z) + \mathcal{D}T(\bar{v}+\tau(z)) = 0$. We will refer to $T(u)$ as *the regular Gelfand–Tsetlin tableau* associated with u and to $\mathcal{D}T(u)$ as *the derivative Gelfand–Tsetlin tableau* associated with u.

Remark 4.34. Although $\{T(\bar{v}+z), \mathcal{D}T(\bar{v}+z) \,|\, z \in T_{n-1}(\mathbb{Z})\}$ is not a basis, we have the following natural basis of $V(T(\bar{v}))$:

$$\mathcal{B}(T(\bar{v})) = \{T(\bar{v}+z), \mathcal{D}T(\bar{v}+w) \,|\, z_{ki} \leq z_{kj}, w_{ki} > w_{kj}\}.$$

Set $\mathcal{V}_{\text{gen}} = \bigoplus_{v \in S} V(T(v))$ and $\mathcal{V}' = V(T(\bar{v})) \oplus \mathcal{V}_{\text{gen}}$. Then $\mathcal{F} \otimes \mathcal{V}_{\text{gen}}$ is a $\mathfrak{gl}(n)$-module with the trivial action on $\mathcal{F}$. We next define a $\mathfrak{gl}(n)$-module structure on $V(T(\bar{v}))$.

The *evaluation map* $\text{ev}(\bar{v}) : \mathcal{F}_{ij} \otimes \mathcal{V}' \to \mathcal{V}'$ is the linear map defined by $\text{ev}(\bar{v})(fT(v+z)) = f(\bar{v})T(\bar{v}+z)$, $\text{ev}(\bar{v})(f\mathcal{D}T(\bar{v}+z)) = f(\bar{v})\mathcal{D}T(\bar{v}+z)$. Furthermore, $\mathcal{D}^{\bar{v}} : \mathcal{F}_{ij} \otimes V(T(v)) \to V(T(\bar{v}))$ will denote the linear map defined by $\mathcal{D}^{\bar{v}}(fT(v+z)) = \mathcal{D}^{\bar{v}}(f)T(\bar{v}+z) + f(\bar{v})\mathcal{D}T(\bar{v}+z)$, where $\mathcal{D}^{\bar{v}}(f) = \frac{1}{2}\left(\frac{\partial f}{\partial v_{ki}} - \frac{\partial f}{\partial v_{kj}}\right)(\bar{v})$,

$z \in T_{n-1}(\mathbb{Z})$, $f \in \mathcal{F}_{ij}$, and $v \in \mathcal{S}$. We may think of $\mathcal{D}^{\bar{v}}$ as the map

$$\mathcal{D}^{\bar{v}} \otimes \mathrm{ev}(\bar{v}) + \mathrm{ev}(\bar{v}) \otimes \mathcal{D}^{\bar{v}}.$$

This map extends to a linear map $\mathcal{F}_{ij} \otimes \mathcal{V}_{\mathrm{gen}} \to V(T(\bar{v}))$ which we will also denote by $\mathcal{D}^{\bar{v}}$.

Theorem 4.35 ([21, Theorems 4.9 and 5.6]). *$V(T(\bar{v}))$ has structure of Gelfand–Tsetlin module over $\mathfrak{gl}(n)$ with action of the generators of $\mathfrak{gl}(n)$ given by*

$$E_{rs}(T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}((v_{ki}-v_{kj})E_{rs}(T(v+z))), \tag{7}$$

$$E_{rs}(\mathcal{D}(T(\bar{v}+w))) = \mathcal{D}^{\bar{v}}(E_{rs}(T(v+w))), \tag{8}$$

and action of the generators of Γ *given by*

$$c_{rs}(T(\bar{v}+z)) = \mathcal{D}^{\bar{v}}((v_{ki}-v_{kj})c_{rs}(T(v+z))), \tag{9}$$

$$c_{rs}(\mathcal{D}^{\bar{v}}(T(v+w))) = \mathcal{D}^{\bar{v}}(c_{rs}(T(v+w))), \tag{10}$$

where v is a generic vector in the set of representatives $\mathcal{S}$, and $z, w \in T_{n-1}(\mathbb{Z})$ with $w \neq \tau(w)$.

Remark 4.36. In the case of $\mathfrak{gl}(3)$ we can give the following interpretation of the basis elements of the module $V(T(\bar{v}))$. Let $T(v)$ be a generic tableau such that $V(T(v))$ is simple, and let $T(\bar{v})$ be such that $\bar{v}_{21} = \bar{v}_{22}$:

$$T(v) = \begin{array}{ccccc} \boxed{v_{31}} & & \boxed{v_{32}} & & \boxed{v_{33}} \\ & \boxed{v_{21}} & & \boxed{v_{22}} & \\ & & \boxed{v_{11}} & & \end{array} \qquad T(\bar{v}) = \begin{array}{ccccc} \boxed{\bar{v}_{31}} & & \boxed{\bar{v}_{32}} & & \boxed{\bar{v}_{33}} \\ & \boxed{\bar{v}_{21}} & & \boxed{\bar{v}_{22}} & \\ & & \boxed{\bar{v}_{11}} & & \end{array}$$

Then $T(\bar{v}+(m,n,k))$ and $\mathcal{D}T(\bar{v}+(m,n,k))$ can be considered as formal limits in the following way:

$$T(\bar{v}+(m,n,k)) := \lim_{v\to\bar{v}} T(v+(m,n,k)), \tag{11}$$

$$\mathcal{D}T(\bar{v}+(m,n,k)) := \lim_{v\to\bar{v}} \left(\frac{T(v+(m,n,k)) - T(v+(n,m,k))}{v_{21}-v_{22}}\right). \tag{12}$$

One essential property of generic Gelfand–Tsetlin modules described in Theorem 4.14 is that Γ "*separates*" the basis elements of $\mathcal{B}(T(v))$, that is, for any two different tableaux T_1, T_2 in $\mathcal{B}(T(v))$ there exists an element $\gamma \in \Gamma$ such that $\gamma \cdot T_1 = T_1$ and $\gamma \cdot T_2 = 0$ (see Remark 4.15). In the case of 1-singular modules $V(T(\bar{v}))$ this is also true and follows from the fact that no derivative tableau $\mathcal{D}(T(\bar{v}+w))$ is an eigenvector for the action of $c_{k2} \in \Gamma$. A detailed proof can be found in [33, Sec. 5].

Theorem 4.37. *Let $\bar{v}$ be any 1-singular vector and $\mathcal{B}(T(\bar{v}))$ be as before. Then Γ separates the tableaux in the basis $\mathcal{B}(T(\bar{v}))$.*

In the case of $T_3(\mathbb{C})$ (equivalently, Gelfand–Tsetlin tableaux of height 3) every singular vector is a 1-singular vector. Therefore, in the case of $\mathfrak{gl}(3)$ the 1-singular modules exhaust all singular Gelfand–Tsetlin modules.

Example 4.38. The simple Verma $\mathfrak{sl}(3)$-module $M = M(-1,-1)$ admits a tableaux realization as a subquotient of the module $V(T(\bar{v}))$, where $\bar{v}$ is the singular vector $(-1,-1,-1,-1,-1,-1)$. This module contains Gelfand–Tsetlin characters of dimension 2. For example, if χ is the Gelfand–Tsetlin character associated with the tableaux $T(\bar{v}+(-1,0,-1))$ and $\mathcal{D}T(\bar{v}+(0,-1,-1))$, then $\dim(M(\chi)) = 2$ (see Sec. 7.5 (C13) for details).

5. Gelfand–Tsetlin Modules of $\mathfrak{sl}(3)$ and Modules of $C(\mathfrak{h})$

From now on we focus on the case $n = 3$ and $\mathfrak{g} = \mathfrak{sl}(3)$. We fix the standard Gelfand–Tsetlin subalgebra Γ of $\mathfrak{g}$, that is the one corresponding to the chain whose second component is generated by E_{12} and E_{21}. The corresponding category of Gelfand–Tsetlin modules $\mathcal{GT}(3)$ will be denoted simply by $\mathcal{GT}$.

Let $C(\mathfrak{h})$ be the centralizer of the Cartan subalgebra $\mathfrak{h}$ in $U(\mathfrak{g})$, where

$$\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{H_1 := E_{11} - E_{22},\ H_2 := E_{22} - E_{33}\}.$$

In this section, we collect some properties of modules in $\mathcal{GT}$ that are related to the category of modules of $C(\mathfrak{h})$. The results are based on the works [3, 16, 17], but for reader's convenience we provide proofs for some statements.

The following result provides an important relation between the simple $C(\mathfrak{h})$-modules and the simple weight modules (for a proof, see for example, [15]).

Lemma 5.1. *For any simple $C(\mathfrak{h})$-module W there exists a simple weight $\mathfrak{g}$-module M such that $M_\lambda \simeq W$ for some $\lambda \in \mathfrak{h}^*$. Conversely, if M is a simple weight $\mathfrak{g}$-module then M_λ is a simple $C(\mathfrak{h})$-module.*

Denote $A := E_{12}E_{21}$, $B := E_{23}E_{32}$. Recall that the center of $U(\mathfrak{g})$ is generated by c_{32} and c_{33}, see Eq. (1). For convenience we will also use the following generators of the center of $U(\mathfrak{g})$:

$$c_1 = \frac{1}{12}c_{32} \quad \text{and} \quad c_2 = \frac{3}{2}c_{32} - c_{33}.$$

The following two lemmas provide important technical properties of $C(\mathfrak{h})$ and the simple $C(\mathfrak{h})$-modules.

Lemma 5.2 ([3, Lemma 1.1]). *The centralizer $C(\mathfrak{h})$ is an associative algebra generated by H_1, H_2, A, B, c_1, and c_2.*

Lemma 5.3 ([16]). *Let W be a simple $C(\mathfrak{h})$-module, and let $H_i = h_i\mathrm{Id}$, $c_i = \gamma_i\mathrm{Id}$ on W, for some constants h_i, γ_i, $i = 1, 2$. Then the following identities hold on W:*

$$aA = A^2 + AB + BA + ABA - \frac{1}{2}A^2B - \frac{1}{2}BA^2 + rB + \tau I,$$

$$aB = B^2 + AB + BA + BAB - \frac{1}{2}B^2A - \frac{1}{2}AB^2 + r_1B + \tau_1 I,$$

$$\frac{1}{4}(AB - BA)^2 = ABA + BAB + \frac{1}{2}rB^2 + \frac{1}{2}r_1A^2 - \left(\frac{a}{2} - 1\right)(AB + BA) + (\tau + r)B + (\tau_1 + r_1)A + \eta I,$$

where

$$r := \frac{1}{2}(h_1^2 - 2h_1), \quad r_1 := \frac{1}{2}(h_2^2 - 2h_2),$$

$$a := 6\gamma_1 + h_1 + h_2 - \frac{1}{3}h_1^2 - \frac{1}{3}h_2^2 + \frac{1}{6}h_1h_2,$$

$$3p := \frac{1}{9}(h_1 - h_2)^3 - \gamma_2 + 6\gamma_1(h_2 - h_1 + 3) - h_1^2 - h_2^2 - h_1h_2 + 2h_1 - 2h_2,$$

$$\tau := \frac{1}{2}h_1p + h_1h_2, \quad \tau_1 = -\frac{1}{2}ph_2^2 + \frac{1}{3}h_2(18\gamma_1 - h_1^2 - h_2^2 - h_1h_2 + 3h_1),$$

$$\eta := \frac{1}{4}p^2 + \frac{1}{6}p(h_1h_2 + h_1^2 + h_2^2 - 18\gamma_1) + \frac{1}{4}h_1h_2(h_1h_2 + 4 - 2a).$$

Let M be a simple module in $\mathcal{GT}$. In particular, it is a weight module. Consider any $\lambda \in \mathfrak{h}^*$ from the weight support of M. The central elements c_1 and c_2 act on M, and hence on M_λ, as a multiplication by some complex scalars γ_1 and γ_2, respectively. If $\lambda(H_i) = h_i$, then $H_i = h_i\mathrm{Id}$, $i = 1, 2$, on M_λ. Since M is a Gelfand–Tsetlin module, then each component $M(\chi)$ is finite-dimensional. Hence, one can choose a basis of M_λ with respect to which $A|_{M_\lambda}$ has a Jordan canonical form A_λ.

Consider the following polynomial in two variables:

$$g_\lambda(x, y) = (x - y)^2 - 2(x + y) - 2r.$$

Recall $r = \frac{1}{2}(h_1^2 - 2h_1)$. Note that $g_\lambda(x, y)$ depends only on $h_1 = \lambda(H_1)$.

Definition 5.4. Let $\lambda \in \mathfrak{h}^*$.

(i) A sequence $(\mu_i)_{i\in J}$ is λ-*connected* (or, simply, *connected*) if $g_\lambda(\mu_i, \mu_{i+1}) = 0$ for all $i \in J$, for some connected subset J of $\mathbb{Z}$. A subsequence $(\mu_{j'})_{j'\in J'}$ of a connected sequence $(\mu_j)_{j\in J}$ will be called a *connected subsequence* if J' is connected.

(ii) A λ-connected sequence $(\mu_j)_{j\in J}$ with $\mu_i \neq \mu_j$ for any $i \neq j$, is called λ-*connected chain*.

(iii) We will say that a set B is λ-*connected* if the elements of B can be ordered in a λ-connected chain $(\mu_j)_{j\in J}$. We will call the sequence $(\mu_j)_{j\in J}$ *the λ-connected chain associated to B*.

We note that if B is λ-connected, then there are at most two λ-connected chains associated to B.

Lemma 5.5. *Let M be a simple Gelfand–Tsetlin module and $\lambda \in \mathfrak{h}^*$ a weight of M. Then the distinct eigenvalues of A_λ is a λ-connected set.*

Proof. This lemma follows from [3, Lemmas 2.2 and 2.3], but for reader's convenience, a brief proof is written. Let $[A_\lambda] = \mathrm{diag}(A_i)$ be the matrix of A_λ in a fixed Jordan canonical basis of M_λ, where A_i corresponds to the generalized eigenspace with eigenvalue μ_i, and let $[B_\lambda] = (B_{ij})$ be the block matrix of B_λ relative to this basis, for which B_{ii} and A_i are in the same position. If $i \neq j$, looking at the (i,j)th block of the matrix equation corresponding to the first relation in Lemma 5.3, we obtain

$$0 = A_iB_{ij} + B_{ij}A_j + A_iB_{ij}A_j - \frac{1}{2}B_{ij}A_j^2 - \frac{1}{2}A_i^2B_{ij} + rB_{ij}.$$

Applying the above identity to suitable elements of the basis of M_λ, one can see that $B_{ij} = 0$ if $(\mu_i + \mu_j) - \frac{1}{2}(\mu_i - \mu_j)^2 + r \neq 0$ (for details, see the proof of [3, Lemma 2.2]). Since M is simple, then M_λ is a simple $C(\mathfrak{h})$-module. However, if the set of eigenvalues of A is a union of two "disconnected" sets, then one easily can prove that M_λ is a direct sum of two modules, which is a contradiction (for details, see the proof of [3, Lemma 2.3]). □

Until the end of this section we assume that M is a simple Gelfand–Tsetlin module, $\lambda \in \mathrm{Supp}\, M$.

Lemma 5.6. *If $(\mu_j)_{j\in J}$ is a λ-connected sequence with $|J| \geq 3$, then the following recurrence relation holds for any i:*

$$\mu_{i+1} + \mu_{i-1} = 2\mu_i + 2. \tag{13}$$

Proof. As $(\mu_j)_{j\in J}$ is a connected sequence, we have $g_\lambda(\mu_i, \mu_{i-1}) = 0$ and $g_\lambda(\mu_i, \mu_{i+1}) = 0$. By solving both quadratic equations in terms of μ_i we have

$$\{\mu_{i-1}, \mu_{i+1}\} = \{\mu_i + 1 \pm \sqrt{1 + 4\mu_i + h_1^2 - 2h_1}\}$$

therefore, $\mu_{i+1} + \mu_{i-1} = 2\mu_i + 2$. □

Definition 5.7. Let $(\mu_j)_{j\in J}$ be a λ-connected sequence. We say that $(\mu_j)_{j\in J}$ is

(i) *degenerate* if $\mu_i = -\frac{1}{2}r$ for some $i \in J$;
(ii) *critical* if $\mu_i = -1/4 - \frac{1}{2}r$ for some $i \in J$;
(iii) *singular* if $(\mu_j)_{j\in J}$ is a connected subsequence of a degenerate or a critical connected sequence;
(iv) *generic* if it is not singular.

Example 5.8. If $\lambda = (0, 0)$, then the sequence $\{0, 2, 6, 12, \ldots\} = \{n(n+1) \mid n \geq 0\}$ is a degenerate connected chain and $C_i := \{n(n+1) \mid n \geq i\}$ is a connected chain that is not degenerate for each $i > 0$.

Lemma 5.9. *Let $(\mu_j)_{j\in J}$ be a λ-connected sequence.*

(i) *If $(\mu_i)_{j\in J}$ is degenerate, then $\mu_i \in \{n(n+1) - \frac{r}{2} \mid n \geq 0\}$ for all $i \in J$.*
(ii) *If $(\mu_i)_{j\in J}$ is critical, then $\mu_i \in \{n^2 - \frac{1}{4} - \frac{r}{2} \mid n \geq 0\}$ for all $i \in J$.*
(iii) *If $(\mu_i)_{j\in J}$ is generic, then $\mu_i \in \{n^2 + n\sqrt{1 + 4\mu_0 + 2r} + \mu_0 \mid n \in \mathbb{Z}\}$ for all $i \in J$, where $\sqrt{1 + 4\mu_0 + 2r}$ is a fixed solution of $x^2 = 1 + 4\mu_0 + 2r$.*

Proof. By Lemma 5.6, in order to determine a connected sequence, it is enough to know two connected values. In the case of a degenerate sequence we have $\mu_i = -\frac{1}{2}r$ for some i, and hence, $\mu_{i+1} = \mu_i$ or $\mu_{i-1} = \mu_i$. Assume $\mu_{i-1} = \mu_i$ and $i = 0$ for simplicity. Then $\mu_n = n(n+1) - \frac{r}{2}$, $n \in \mathbb{Z}$, give the unique solution of the recursive equation (13). Therefore, all μ_i are in the set $\{n(n+1) - \frac{r}{2} \mid n \geq 0\}$. This proves part (i). For parts (ii) and (iii) we reason in the same manner. Namely, for a critical chain, we use that $\mu_i = -\frac{1}{4} - \frac{1}{2}r$ for some i, hence $\mu_{i+1} = \mu_i + 1$ or $\mu_{i-1} = \mu_i + 1$, while in the generic case, given μ_i in the connected sequence, we have $\mu_{i+1} \in \{\mu_i + 1 \pm \sqrt{1 + 4\mu_i + 2r}\}$. □

The properties of A_λ are described in the following theorem.

Theorem 5.10. *Let M be a simple Gelfand–Tsetlin module. Then the following hold*:

(i) *For every $\lambda \in \operatorname{Supp} M$, every eigenvalue of A_λ has multiplicity at most two.*
(ii) *If $(\mu_i)_{i\in J}$ is a connected chain of the set of distinct eigenvalues of $A_\lambda = A\,|_{M_\lambda}$, then*
   (a) *if the chain $(\mu_i)_{i\in J}$ is generic, then all eigenvalues of A_λ are distinct*;
   (b) *if the chain $(\mu_i)_{i\in J}$ is degenerate, then the chain can be chosen so that $\mu_1 = -\frac{r}{2}$, and if the multiplicity of μ_i equals* 1 *then the multiplicity of μ_{i+1} is also* 1;
   (c) *if the chain $(\mu_i)_{i\in J}$ is critical then the chain can be chosen so that $\mu_1 + 1 = \mu_2$, the multiplicity of μ_1 is* 1, *and if the multiplicity of μ_i equals* 1 *for $i > 1$, then the multiplicity of μ_{i+1} also equals* 1;
   (d) *if the chain $(\mu_i)_{i\in J}$ is singular but not degenerate or critical, then all eigenvalues of A_λ are distinct.*

Proof. The proof of all parts can be found in [15, 17]. The strategy is to apply the relations from Lemma 5.3 to a Jordan form of A_λ. Proofs of parts (b) and (d) can be also found in [3, Theorem 2.7]. □

As a consequence of Theorem 5.10, we have the following statement.

Corollary 5.11. *Let M be a simple weight $\mathfrak{g}$-module. Then for any $\lambda \in \mathfrak{h}^*$ and any $i \neq j$ we have* $\dim(\ker(E_{ij}|_{M_\lambda})) \leq 1$.

Proof. Suppose that $\ker(E_{ij}|_{M_\lambda}) \neq 0$. Consider the Lie subalgebra $\mathfrak{a}$ of $\mathfrak{sl}(3)$ isomorphic to $\mathfrak{sl}(2)$ that is generated by E_{ij} and E_{ji}. Then M is a Gelfand–Tsetlin $\mathfrak{a}$-module with respect to the Gelfand–Tsetlin subalgebra generated by $\mathfrak{h}$, the center of $U(\mathfrak{sl}(3))$, and the center of $U(\mathfrak{a})$. The statement follows from Theorem 5.10. $\square$

Remark 5.12. In what follows, we give an interpretation of the eigenvalues of A_λ in terms of tableaux and the Gelfand–Tsetlin formulas. Set for $i \in \mathbb{Z}$

$$T(v_i) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{x+i} & & \boxed{y-i} & \\ & & \boxed{z} & & \end{array}$$

The set of all tableaux in $\mathcal{B}(T(v_0))$ with fixed $\mathfrak{g}$-weight $\lambda \in \mathfrak{h}^*$ (see Definition 4.5) is $\{T(v_i)\,|\, i \in \mathbb{Z}\}$. Moreover, the Gelfand–Tsetlin formulas imply that $A = E_{12}E_{21}$ acts on $T(v_i)$ as multiplication by $\mu_i := -(x+i-z)(y-i-z)$, and $r = \frac{1}{2}((2z-x-y-1)^2 - (2z-x-y-1))$. We have:

(i) $(\mu_i)_{i\in\mathbb{Z}}$ is degenerate with $\mu_j = -\frac{1}{2}r$ if and only if $x - y \in \{2j-1, 2j+1\}$.
(ii) $(\mu_i)_{i\in\mathbb{Z}}$ is critical with $\mu_j = -\frac{1}{4} - \frac{1}{2}r$ if and only if $x - y = 2j$.

In particular, $(\mu_i)_{i\in\mathbb{Z}}$ is singular if and only if the tableau $T(v_0)$ (respectively, χ_{v_0}) is singular and $\{\mu_i\}$ is generic if and only if $T(v_0)$ (respectively, χ_{v_0}) is generic. Note also that $\chi(A) = -\frac{1}{4} - \frac{1}{2}r$ if and only if $\chi = \chi_w$ with $w_{21} = w_{22}$. Finally, $\chi(A) = -\frac{1}{2}r$ if and only if $\chi = \chi_w$ with $|w_{21} - w_{22}| = 1$.

Definition 5.13. We say that a Gelfand–Tsetlin character χ is *critical* (respectively, *degenerate*) if $\chi = \chi_v$ for some v such that $v_{21} = v_{22}$ (respectively, $|v_{21} - v_{22}| = 1$).

Since $A \in \Gamma$, we can extend the concepts of generic and singular chains to Gelfand–Tsetlin modules.

Definition 5.14. A Gelfand–Tsetlin module M is called *generic* if every Gelfand–Tsetlin character of M is generic. A Gelfand–Tsetlin module M is called *singular* if it has a singular Gelfand–Tsetlin character.

Note that any finite-dimensional module is a singular Gelfand–Tsetlin module, moreover, any 1-singular module as defined in Sec. 4.4 is a singular Gelfand–Tsetlin module. Also, generic modules as defined in Sec. 4.3 are generic Gelfand–Tsetlin modules.

Proposition 5.15. *If a simple Gelfand–Tsetlin* $\mathfrak{g}$*-module* M *is singular, then each Gelfand–Tsetlin character of* M *is singular.*

Proof. The statement follows by a direct computation. Let χ be a singular Gelfand–Tsetlin character of M, $v \in M(\chi)$, $v \neq 0$. If $E_{12}v \neq 0$ then we easily check that $E_{12}v \in M(\chi')$ for a singular χ'. Similar reasoning applies for $E_{21}v$. Suppose now $E_{23}v \neq 0$. Then $E_{23}v \in M(\chi') \oplus M(\chi'')$, where χ' and χ'' are both singular (one of the subspaces can be zero). Moreover, if χ belongs to a critical (respectively, degenerate) connected chain, then χ' and χ'' belong to a degenerate (respectively, critical) connected chain. We reason similarly for E_{32}. $\square$

Definition 5.16. Given a Gelfand–Tsetlin character χ and M a Gelfand–Tsetlin module, we say that M is a *simple extension* of the character χ if M is simple and $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$ (i.e. $M(\chi) \neq 0$).

Lemma 5.17. *Let* M *be a simple Gelfand–Tsetlin module*, $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$ *and* $\lambda = \chi|_{\mathfrak{h}}$. *If there exists a basis* $\{w_i\}_{i\in I}$ *of* M_λ *such that the action of* B_λ *on this basis is completely determined by the eigenvalues of* A_λ *and* χ, *then* M *is the unique simple extension of* χ.

Proof. Under these conditions, the simple $C(\mathfrak{h})$-module M_λ is defined uniquely, moreover, as $M_\lambda^{(1)} \simeq M_\lambda^{(2)}$ implies $M^{(1)} \simeq M^{(2)}$, the uniqueness follows. $\square$

The generic Gelfand–Tsetlin modules are completely determined by any of their characters, as the following result shows.

Theorem 5.18. *If* M *is a generic simple Gelfand–Tsetlin module then for any* $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$ *the subspace* $M(\chi)$ *is one-dimensional and* M *is the unique simple extension of* χ.

Proof. The result can be found in [15, 17], but for the sake of completeness we provide a proof. Let $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$ and let $\lambda = \chi|_{\mathfrak{h}}$ be the associated to χ weight. Let $\mu_0 \in \mathbb{C}$ be an eigenvalue of the operator $A_\lambda = A|_{M_\lambda}$. Then all eigenvalues of A_λ form a connected chain, i.e. belong to a sequence $\mu_i = i^2 + i\sqrt{1+4\mu_0+2r} + \mu_0$, $i \in \mathbb{Z}$ for some choice of the square root (see Lemma 5.9(iii)).

Using relations between A and B we can choose a basis $\{w_i \,|\, i \in \mathbb{Z}\}$ (this set can be finite or bounded from one side or unbounded) of M_λ such that

$$A_\lambda w_i = \mu_i w_i; \quad \text{and} \quad B_\lambda w_i = \begin{cases} w_{i-1} + b_i w_i + d_{i+1} w_{i+1}, & i < 0, \\ w_{-1} + b_0 w_0 + w_1, & i = 0, \\ d_i w_{i-1} + b_i w_i + w_{i+1}, & i > 0 \end{cases}$$

with

$$\begin{aligned}
b_i &:= \frac{a\mu_i - \mu_i^2 - \tau}{2\mu_i + r},\\
d_i &:= \frac{\xi(\mu_{i-1})(3 + \mu_{i-1} - \mu_i) - \theta(\mu_{i-1})\big(\frac{7}{2}\mu_{i-1} - \frac{3}{2}\mu_i + 3 + r\big)}{4(\mu_{i-1} - \mu_i + 1)\big(\mu_{i-1} - \frac{3}{4} + \frac{1}{2}r\big)},\\
\xi(\mu_i) &:= \frac{1}{2}(2\mu_i + r)b_i^2 - (2\mu_i + r)b_i - \frac{1}{2}r_1\mu_i^2 - (r_1 + \tau_1)\mu_i - \eta,\\
\theta(\mu_i) &:= (a - 2\mu_i)b_i - b_i^2 - r_1\mu_i - \tau_1,
\end{aligned}$$

where r, r_1, η, τ, and τ_1 are defined in Lemma 5.3.

Hence, in this case B_λ is completely determined by χ and A_λ. The uniqueness follows from Lemma 5.17. □

Note that in singular cases the subspace $M(\chi)$ can be two-dimensional (see Example 4.38). Also, in these cases for a given $\chi \in \Gamma^*$ there can exist two non-isomorphic simple extensions of χ. Such examples were first constructed in [16].

6. Simple Extensions of Singular Gelfand–Tsetlin Characters

In this section, we provide sufficient conditions for a singular Gelfand–Tsetlin character to admit a unique simple extension.

Theorem 6.1. *If χ is a critical Gelfand–Tsetlin character then χ admits a unique simple extension.*

Proof. By Theorem 4.9, there exist at most two simple modules $M^{(1)}$ and $M^{(2)}$ such that $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M^{(i)})$ for $i = 1, 2$. Assume that we have two such modules and let $\lambda \in \mathrm{Supp}(M^{(1)}) \cap \mathrm{Supp}(M^{(2)})$ such that $\lambda = \chi|_{\mathfrak{h}}$. For $i = 1, 2$, consider the restriction $A_\lambda^{(i)}$ of $A = E_{12}E_{21}$ on $M_\lambda^{(i)}$. Since $M^{(i)}$ are Gelfand–Tsetlin modules, then we can choose bases $\mathcal{B}_1 = \{w_0, \ldots, w_m\}$, $0 \le m \le \infty$ and $\mathcal{B}_2 = \{w'_0, \ldots, w'_k\}$, $0 \le k \le \infty$ of $M_\lambda^{(1)}$ and $M_\lambda^{(2)}$ such that the matrix $[A_\lambda^{(i)}]$ of $A_\lambda^{(i)}$ with respect to $\mathcal{B}_i$ is in a Jordan normal form and each eigenvalue of $A_\lambda^{(i)}$ has algebraic multiplicity at most two.

By Theorem 5.10(ii)(c), the eigenvalue $\chi(A)$ of $A_\lambda^{(1)}$ and $A_\lambda^{(2)}$ has multiplicity 1. Suppose first that all eigenvalues of both $A_\lambda^{(1)}$ and $A_\lambda^{(2)}$ have multiplicity 1. Then they can be ordered in connected chains $\{\mu_n \,|\, 0 \le n \le m\}$ and $\{\mu_m \,|\, 0 \le m \le k\}$ with

$$\mu_0 = \chi(A) = -\frac{1}{4} - \frac{1}{2}r$$

and $\mu_i = i^2 - \frac{1}{4} - \frac{1}{2}r$ for $i \ge 1$ (see Lemma 5.9(ii)).

Applying the relations from Lemma 5.3 we obtain

$$Aw_i = \mu_i w_i, \quad i \geq 0, \quad Bw_i = \begin{cases} b_0 w_0 + w_1, & i = 0, \\ c_i w_{i-1} + b_i w_i + w_{i+1}, & 0 < i \leq m, \end{cases}$$
$$Aw'_i = \mu_i w'_i, \quad i \geq 0, \quad Bw'_i = \begin{cases} b_0 w'_0 + w'_1, & i = 0, \\ c_i w'_{i-1} + b_i w'_i + w'_{i+1}, & 0 < i \leq k, \end{cases} \tag{14}$$

where

$$c_1 = \frac{\theta(\mu_0)}{2},$$
$$c_2 = \begin{cases} \dfrac{\theta(\mu_1)}{(1 + \mu_2 - \mu_1)}, & n = 1, \\ d_2, & n > 1, \end{cases}$$
$$c_i = d_i, \quad i > 2,$$

and

$$2\xi(\mu_0) = \left(\frac{3}{2} + 2\mu_0 + r\right)\theta(\mu_0).$$

If $m < k$, then $c_{m+1} = 0$ implying that $M^{(2)}$ is reducible. Similarly, if $m > k$, then $c_{k+1} = 0$ and $M^{(1)}$ is reducible. It follows that $k = m$. But then formulas (14) define uniquely a simple $C(\mathfrak{h})$-module $M_\lambda^{(1)} \simeq M_\lambda^{(2)}$. Therefore, $M^{(1)} \simeq M^{(2)}$.

Suppose now that the algebraic multiplicity of some μ_i is two in $M_\lambda^{(1)}$. For simplicity assume that

$$[A_\lambda^{(1)}] = \left(\begin{array}{c|cc} \mu_0 & 0 & 0 \\ \hline 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{array}\right); \quad [B_\lambda^{(1)}] = \left(\begin{array}{c|cc} b_{11} & b_{12} & b_{13} \\ \hline b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array}\right),$$

where $[B_\lambda^{(1)}]$ stands for the matrix of $B|_{M_\lambda^{(1)}}$ relative to $\mathcal{B}_1$. Applying the relations from Lemma 5.3 we obtain $b_{32} = 0$. Note that due that $M_\lambda^{(1)}$ is simple as a $C(\mathfrak{h})$-module we have $b_{12} \neq 0$ and $b_{31} \neq 0$. Hence, using row operations, one can change the basis $\mathcal{B}_1$ so that b_{13} becomes 0.

Now, applying the relations from Lemma 5.3 we obtain

$$\begin{cases} b_{22} = b_{33}, \\ \dfrac{1}{2}b_{11} + \dfrac{3}{2}b_{22} = a - 2\mu_0 - 1, \\ \dfrac{1}{2}b_{12}b_{31} = -ab_{22} + b_{22}^2 + 2\mu_1 b_{22} + r_1\mu_1 + \tau_1, \\ \dfrac{3}{2}b_{23} = -1 - b_{11} - 2b_{22}, \\ 2b_{12}b_{21} = -(2\mu_0 + 1 - a)(4\mu_0 + 1) - 3r_1\mu_1 + r_1 - 3\tau_1, \\ \dfrac{1}{2}b_{31}b_{12} = (4\mu_0 + 3)(2\mu_0 + 1 - a) + r_1\mu_1 + \tau_1, \\ (b_{11} + b_{22})b_{31} = 0. \end{cases}$$

By changing the basis if needed, we can assume $b_{12} = 1$. Therefore, the matrix $[B_\lambda^{(1)}]$ is completely determined by χ and the matrix $[A_\lambda^{(1)}]$. We can show that the latter holds for any Jordan normal form $[A_\lambda^{(1)}]$.

Consider now the matrix $[A_\lambda^{(2)}]$. If this Jordan normal form is not equivalent to $[A_\lambda^{(1)}]$, then one of the modules $M_\lambda^{(1)}$ or $M_\lambda^{(2)}$ will not be simple. Indeed, this can be immediately seen from the form of matrices $[B_\lambda^{(1)}]$ and $[B_\lambda^{(2)}]$. On the other hand, if $[A_\lambda^{(1)}] = [A_\lambda^{(2)}]$ then $M_\lambda^{(1)} \simeq M_\lambda^{(2)}$ as $C(\mathfrak{h})$-modules and hence $M^{(1)} \simeq M^{(2)}$. ◻

If χ is a singular Gelfand–Tsetlin character in a critical connected chain and χ is not critical then there might exist two simple extensions of χ (see [16] for examples). On the other hand we have.

Corollary 6.2. *Suppose that χ is a singular character in a critical connected chain and χ is not critical. If $\lambda = \chi|_{\mathfrak{h}}$, then there exists a unique simple extension M of χ with diagonalizable A_λ.*

Proof. Indeed, if A_λ is diagonalizable then B_λ is determined uniquely. As it was shown in the proof of Theorem 6.1, it is sufficient to know one eigenvalue of A_λ to reconstruct the whole A_λ in a simple module. Hence, the statement follows. ◻

Lemma 6.3. *Let M be a simple Gelfand–Tsetlin module, χ a degenerate character of M associated with the weight $\lambda \in \mathfrak{h}^*$. Let $\rho_1 = -\frac{1}{2}r$, and $\rho_2 = 2 - \frac{1}{2}r$ be connected eigenvalues of A_λ with $\chi(A) = \rho_1$. Suppose that both ρ_1 and ρ_2 have multiplicity 2. Then the Gelfand–Tsetlin support of M contains a critical character χ'.*

Proof. Let u_1, u_2, u_3, u_4 be nonzero elements of M such that

$$A_\lambda u_1 = \rho_1 u_1, \quad A_\lambda u_2 = u_1 + \rho_1 u_2,$$

$$A_\lambda u_3 = \rho_2 u_3, \quad A_\lambda u_4 = u_3 + \rho_2 u_4.$$

Suppose that $M_{\lambda+\epsilon_2-\epsilon_3}$ does not contain a critical character. Then the eigenvalues $\{\mu_k, \mu_{k+1}, \ldots, \mu_m\}$, $k > 1$, of $A_{\lambda+\epsilon_2-\epsilon_3}$ form a part of a critical connected chain but without the critical character μ_1. By Theorem 5.10 these eigenvalues are of multiplicity 1. Let v_1, v_2 be eigenvectors of $A_{\lambda+\epsilon_2-\epsilon_3}$. Then we have

$$\begin{cases} E_{23}(u_1) = a_1 v_1, \\ E_{23}(u_2) = a_2 v_1, \\ E_{23}(u_3) = a_3 v_1 + a_4 v_2, \\ E_{23}(u_4) = a_5 v_1 + a_6 v_2. \end{cases}$$

Since u_1, u_2, u_3, and u_4 are linearly independent and their images span at least a two-dimensional space we have that $\dim(\ker(E_{23}\mid_{M_\lambda})) \geq 2$ which is impossible by Corollary 5.11. $\square$

From Lemma 6.3, we immediately have the following.

Corollary 6.4. *Let χ be a degenerate Gelfand–Tsetlin character such that $\mu_1 = \chi(A)$ and $\lambda = \chi|_{\mathfrak{h}}$. If $\{\mu_1, \mu_2\}$ is a λ-connected set, then there exists at most one simple extension of χ such that both μ_1 and μ_2 have multiplicity* 2.

Proof. Indeed, any such simple module M will contain a critical character χ' determined by the condition $E_{23}(M(\chi)) \subset M(\chi') + M(\chi'')$. But, by Theorem 6.1, χ' defines M uniquely. $\square$

Lemma 6.5. *Let M be a simple Gelfand–Tsetlin module such that M is singular but has no critical characters. Then A is diagonalizable on M.*

Proof. Fix $\lambda \in \operatorname{Supp} M$, then the distinct eigenvalues of A_λ form a singular λ-connected chain. If this chain is critical then A_λ is diagonalizable since there is no critical eigenvalue, see Theorem 5.10(ii)(d). Suppose that the chain is degenerate. Then by Theorem 5.10(ii)(b) one can order the distinct eigenvalues of A_λ in the following way: $\{\mu_1, \mu_2, \ldots, \mu_m\}$, where $\mu_1 = -\frac{1}{2}r$, and if the multiplicity of μ_i equals 1 then the multiplicity of μ_{i+1} is also 1. Suppose that M has a character $\widetilde{\chi}$ such that $\widetilde{\chi}(A) = \mu_1$ and μ_1 has multiplicity 2. If μ_2 has multiplicity 2, then by Lemma 6.3 there exists a critical character χ' in the Gelfand–Tsetlin support of every simple extension of $\widetilde{\chi}$, and we obtain a contradiction.

Assume now that μ_1 has multiplicity 2 but μ_2 has multiplicity 1. Consider the weight subspace $M' = M_{\lambda+\epsilon_2-\epsilon_3}$ and $A' = A|_{M'}$. Observe that $M' \neq 0$, since otherwise $\dim(\ker E_{23}|_{M_\lambda}) \geq 2$ which is a contradiction by Corollary 5.11. If A' has no critical eigenvalue, then neither does $A_{\lambda+\epsilon_2-\epsilon_3+k(\epsilon_1-\epsilon_2)}$, for all integer k. In this case all these subspaces $M_{\lambda+\epsilon_2-\epsilon_3+k(\epsilon_1-\epsilon_2)}$, $k \in \mathbb{Z}$ can generate only one eigenvector of A' with eigenvalue μ_1 and, hence, produce only multiplicity 1 eigenvalue μ_1 of A'. But this contradicts to the simplicity of M. Therefore, A' must contain a critical eigenvalue giving a contradiction again. Therefore, A_λ is diagonalizable, which completes the proof. □

The proof of Lemma 6.5 implies also the following statement.

Corollary 6.6. *Let M be a simple Gelfand–Tsetlin module and χ a Gelfand–Tsetlin character of M associated with $\lambda \in \mathfrak{h}^*$ and such that* $\dim(M(\chi)) = 2$. *Then M has a critical character χ' associated with the weight $\lambda + \epsilon_2 - \epsilon_3$.*

Theorem 6.7. *Let M be a simple Gelfand–Tsetlin module and χ be a Gelfand–Tsetlin character such that* $\dim(M(\chi)) = 2$. *Then M is the unique simple extension of χ.*

Proof. Let $\lambda = \chi|_{\mathfrak{h}}$. Since $\dim(M(\chi)) = 2$, the distinct eigenvalues of A_λ form a singular λ-connected chain $\{\mu_1, \ldots, \mu_m\}$, $m \leq \infty$. Moreover, there exists μ_i of multiplicity 2. We proceed with two cases.

Case 1. *The chain $\{\mu_1, \ldots, \mu_m\}$ is critical.* By Theorem 5.10 all distinct eigenvalues can be ordered in the following way: $\{\mu_1, \mu_2, \ldots, \mu_m\}$, where $\mu_1 + 1 = \mu_2$, multiplicity of μ_1 is 1, and if the multiplicity of μ_i equals 1 for $i > 1$ then the multiplicity of μ_{i+1} is also 1. Therefore, the module M has a critical character χ' such that $\chi'(A) = \mu_1$. Thus, every simple extension of χ contains χ' in its Gelfand–Tsetlin support. Applying Theorem 6.1, we conclude that M is unique.

Case 2. *The chain $\{\mu_1, \ldots, \mu_m\}$ is degenerate.* By Theorem 5.10 one can order the distinct eigenvalues of A_λ in the following way: $\{\mu_1, \mu_2, \ldots, \mu_m\}$, where $\mu_1 = -\frac{r}{2}$, and if the multiplicity of μ_i equals 1 then the multiplicity of μ_{i+1} is also 1. Therefore, M has a character $\widetilde{\chi}$ such that $\widetilde{\chi}(A) = \mu_1$ and μ_1 has multiplicity 2. By Corollary 6.6, $A|_{M_{\lambda+\epsilon_2-\epsilon_3}}$ must contain a critical eigenvalue. Thus, M is unique by Theorem 6.1. This completes the proof. □

Theorem 6.8. *Let M be a simple Gelfand–Tsetlin module such that M is singular but has no critical characters. Then for each character $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$, M is the unique simple extension of χ with the property that it has no critical characters.*

Proof. Let as usual $\lambda = \chi|_{\mathfrak{h}}$ and $A_\lambda = A|_{M_\lambda}$. It follows from Lemma 6.5 that A_λ is diagonalizable. We proceed in two steps.

Step I. *Suppose that χ belongs to a critical connected chain.* As M does not have critical characters, $\chi(A)$ belongs to a critical connected chain $\mu_i = i^2 - \frac{1}{4} - \frac{1}{2}r$, $1 \leq n \leq i \leq m \leq \infty$ for some integers n and m. Then as in Theorem 6.1, there exists a basis $\{w_i, n \leq i \leq m\}$ such that

$$Aw_i = \mu_i w_i, \quad Bw_i = \begin{cases} b_n w_n + w_{n+1}, & i = n, \\ d_i w_{i-1} + b_i w_i + w_{i+1}, & i > n, \\ d_m w_{m-1} + b_m w_m, & i = m. \end{cases} \tag{15}$$

Hence, $B_\lambda = B$ is determined uniquely by χ and A_λ. The uniqueness follows from Lemma 5.17.

Step II. *Suppose that* $\chi(A)$ *belongs to a degenerate chain* $\{\mu_1, \mu_2, \ldots\}$ *where* $\mu_n = n(n-1) - \frac{1}{2}r$, $n \geq 1$ (see Lemma 5.9(i)). We proceed considering two cases depending on the connected chain $\{\mu_1, \mu_2, \ldots\}$.

Case 1. *The chain* $\{\mu_1, \mu_2, \ldots\}$ *does not contain a degenerate character*, that is the eigenvalues of A_λ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_s\}$ for some $k > 1$ and some $s \leq \infty$. Applying relations from Lemma 5.3 one can choose a basis $\{w_k, \ldots, w_s\}$ of M_λ such that the matrix of A_λ is diagonal and the matrix of B_λ has a tridiagonal form as in the generic case. Suppose there exists another simple extension W of χ satisfying the conditions of the theorem such that the eigenvalues of $A|_{W_\lambda}$ are $\{\mu_d, \mu_{d+1}, \ldots, \mu_t\}$ for some $d > 1$ and some $t \leq \infty$. If $d = k$ and $s = t$, then the diagonal matrices $[A|_{M_\lambda}] = [A|_{W_\lambda}]$ will give the same matrix of B_λ, hence $M \simeq W$ by Lemma 5.17.

Suppose $d < k$ (note that in this case $k \leq t$). Then applying relations from Lemma 5.3 we obtain that W_λ has a $C(\mathfrak{h})$-submodule U such that the eigenvalues of $A|_U$ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_t\}$. Hence, M has a nontrivial proper submodule $\widetilde{M}$ such that the eigenvalues of $A|_{\widetilde{M}_\lambda}$ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_t\}$. This contradicts the irreducibility of M. The case $d > k$ is treated analogously.

Case 2. *The chain* $\{\mu_1, \mu_2, \ldots\}$ *does contain a degenerate character*, that is the eigenvalues of A_λ are $\{\mu_1, \mu_2, \ldots, \mu_s\}$ for some $s \leq \infty$. Let χ' be the character associated with μ_1. Using relations from Lemma 5.3, we see that $r^2 + 2ar + 4\tau = 0$ and there exists a basis $\{w_i, 1 \leq i \leq s\}$ of M_λ such that

$$Aw_i = \lambda_i w_i, \quad 1 \leq i \leq s, \quad Bw_i = \begin{cases} Tw_1 + w_2, & i = 1, \\ q_i w_{i-1} + b_i w_i + w_{i+1}, & 1 < i < s, \\ q_s w_{s-1} + b_s w_s, & i = s, \end{cases} \tag{16}$$

where

$$q_2 = \frac{1}{3}\left(-\frac{1}{8}r_1 r^2 + \frac{1}{2}(r_1 + \tau_1)r - \eta\right),$$
$$q_i = d_i, \quad i > 2$$

and T is a root of the equation

$$x^2 - (r+a)x + \tau_1 - \frac{1}{8}r_1 r^2 + \frac{1}{2}r\tau_1 - \eta = 0.$$

Let W be another simple extension of χ. Then μ_1 must be an eigenvalue of $A|_{W_\lambda}$, otherwise M is not simple. In fact, the quadratic equation on T shows that there might exist two non-isomorphic simple modules with the same degenerate chain $\{\mu_1, \ldots, \mu_s\}$. We will show that only one such module will satisfy the conditions of the theorem.

The hypothesis that there is no critical characters in all connected chains implies that $E_{23}(M(\chi')) \subset M(\tilde{\chi})$, where $\tilde{\chi}$ is a character such that $\tilde{\chi}(A)$ belongs to a critical connected chain (without critical characters by hypothesis). Also, $E_{23}(W(\chi')) \subset W(\tilde{\chi})$. If both $E_{23}(M(\chi'))$ and $E_{23}(W(\chi'))$ are nonzero then $M(\tilde{\chi}) \neq 0$ and $W(\tilde{\chi}) \neq 0$. We immediately conclude that $M \simeq W$ by Theorem 6.1. Suppose $E_{23}(M(\chi')) = E_{23}(W(\chi')) = 0$. Apply the same arguments for E_{13}. If both $E_{13}(M(\chi'))$ and $E_{13}(W(\chi'))$ are nonzero, then $M \simeq W$ as above. On the other hand, if $E_{13}(M(\chi')) = E_{13}(W(\chi')) = 0$, then M and W are simple quotients of the same generalized Verma module generated by a weight vector v such that $E_{23}v = E_{13}v = 0$. But such generalized Verma module has a unique simple quotient implying $M \simeq W$. Hence, it remains to consider mixed cases. We finish the proof considering three subcases. Recall that a weight module is *pointed* if all its nonzero weight spaces are one-dimensional. All other cases are considered analogously.

Case 2(a). Suppose $E_{23}(M(\chi')), E_{13}(M(\chi')) \neq 0$, *and* $E_{23}(W(\chi')), E_{13}(W(\chi')) = 0$. Then W is a quotient of the generalized Verma module M_1 generated by an element $v \in M_1$ such that $E_{23}v = E_{13}v = 0$. Suppose $E_{32}(W(\chi')) \neq 0$ and $E_{31}(W(\chi')) \neq 0$. If one of $E_{32}(M(\chi'))$ or $E_{31}(M(\chi'))$ is nonzero then we are done. Suppose $E_{32}(M(\chi')) = E_{31}(M(\chi')) = 0$ and thus M is a quotient of generalized Verma module M_2 generated by v' such that $E_{32}v' = E_{31}v' = 0$. In order not to have critical characters both M and W must be pointed modules, that is all weight spaces have dimension 1. Let $\chi'(H_1) = h_1$, $\chi'(H_2) = h_2$. Comparing the values of Casimir elements on M and W we obtain $h_1 = -2h_2$. This condition guarantees that M and W have common degenerate character χ'. Let us find the condition when W is a pointed module. It is sufficient to check when the following system has a nontrivial solution:

$$0 = E_{23}(\alpha E_{32}v + \beta E_{31}E_{12}v) = h_2\alpha v + \beta E_{21}E_{12},$$

$$0 = E_{13}(\alpha E_{32}v + \beta E_{31}E_{12}v) = \alpha E_{12}v + \beta(h_1 + h_2 + 1)E_{12}.$$

Assume that $E_{12}v \neq 0$. Then we have $h_2\alpha - \beta h_1 - \frac{1}{2}\beta r = 0$, and $\alpha + \beta(h_1 + h_2 + 1) = 0$. If $h_1 = 0$, then $M \simeq W \simeq \mathbb{C}$. If $h_1 \neq 0$ then $h_1 = 2$, $h_2 = -1$, and $\chi'(A) = 0$. It follows that $E_{23}v = E_{13}v = E_{21}v = 0$ with $h_1 = 2$, $h_2 = -1$ or $E_{23}v = E_{13}v = E_{12}v = 0$ with $h_1 = -2$, $h_2 = 1$.

Consider first the case $h_1 = -2$, and let $\mu \in \mathfrak{h}^*$ be such that $\mu(H_1) = \mu(H_2) = -1$. Then W_μ is one-dimensional and $E_{12}W_\mu = 0$. Hence, W_μ is a Gelfand–Tsetlin subspace and $A|_{W_\mu} = -\mathrm{Id}$. But, this is a critical value and, thus, W contains critical characters, which is a contradiction.

Suppose now that $h_1 = 2$ and consider $\mu \in \mathfrak{h}^*$ such that $\mu(H_1) = 1$, $\mu(H_2) = -2$. Then W_μ is one-dimensional and $E_{21}W_\mu = 0$. Hence, W_μ is a Gelfand–Tsetlin subspace and $A|_{W_\mu} = 0$. Again, this is a critical value which is a contradiction.

Suppose now that $E_{32}(W(\chi')) = 0$. Then $E_{12}(W(\chi')) = 0$ and W is a highest-weight module of highest weight $\chi'|_H$ and $\chi'(H_2) = h_2 = 0$. Since χ' is degenerate we have $h_1 = 0$ or $h_1 = -2$. In the case $h_1 = 0$, we obtain $M \simeq W \simeq \mathbb{C}$.

Now suppose $h_1 = -2$ and $A|_{W(\chi')} = -2\mathrm{Id}$. This highest-weight module has no critical characters. Since $E_{31}W(\chi') \neq 0$ we have $E_{31}M(\chi') = 0$, otherwise $M \simeq W$ as before. Since $A|_{M(\chi')} = -2\mathrm{Id}$ and all characters have multiplicity 1, we have $E_{12}M(\chi') = 0$. Thus, in addition we have $E_{32}M(\chi') = 0$. Consider a weight $\mu \in \mathfrak{h}^*$ such that $\mu(H_1) = -1$, $\mu(H_2) = 1$. The subspace M_μ is one-dimensional. In fact, this is a critical Gelfand–Tsetlin subspace, since $A|_{M_\mu} = -\mathrm{Id}$. Hence, M does not satisfy the conditions of the theorem and W again is a unique required module.

Case 2(b). *Suppose* $E_{23}(M(\chi')), E_{13}(W(\chi')) \neq 0$, *and* $E_{23}(W(\chi')), E_{13}(M(\chi')) = 0$. Now, we act by E_{32} and E_{31}. Suppose first that $E_{32}(M(\chi')) = 0$ and $E_{31}(W(\chi')) = 0$. Hence, M contains a nonzero vector v such that $E_{13}v = E_{32}v = E_{12}v = 0$. On the other hand, W contains a nonzero vector v' such that $E_{31}v' = E_{23}v' = E_{21}v' = 0$. Moreover, $H_1v = H_1v' = 0$. But, since A is diagonalizable on M and on W, we have $E_{21}v = E_{12}v' = 0$. Thus, $M \simeq W \simeq \mathbb{C}$.

Suppose now $E_{31}(M(\chi')) = 0$ and $E_{32}(W(\chi')) = 0$. Hence, M contains a nonzero vector v such that $E_{13}v = E_{31}v = 0$, and W contains a nonzero vector v' such that $E_{32}v' = E_{23}v' = 0$. We obtain that $H_1v = H_1v' = 0$ and $H_2v = H_2v' = 0$. Moreover, $Av = Av' = 0$. Since A is diagonalizable on M and W we have $E_{12}v = E_{12}v' = 0$ and $E_{21}v = E_{21}v' = 0$. Thus, $M \simeq W \simeq \mathbb{C}$.

Finally, let $E_{31}(M(\chi')) = 0$ and $E_{32}(M(\chi')) = 0$. Hence, M contains a nonzero vector v such that $E_{13}v = E_{31}v = E_{32}v = 0$, implying $E_{12}v = 0$. So, either $H_1v = H_2v = 0$ and $M \simeq W \simeq \mathbb{C}$, or $H_1v = -2$ and $H_2v = 2$. In the latter case W contains a nonzero vector v' such that $E_{23}v' = 0$, $H_1v' = -2v'$, and $Av' = -2v'$. Hence, $E_{12}v' = 0$ and $E_{13}v' = 0$ since A is diagonalizable. We have $c_1v = c_1v'$ implying $H_2v = H_2v' = 0$ which is a contradiction.

Case 2(c). *Suppose* $E_{23}(M(\chi')) \neq 0$, *and* $E_{23}(W(\chi')), E_{13}(M(\chi')), E_{13}(W(\chi')) = 0$. Now, we act by E_{32} and E_{31} on $M(\chi')$ and $W(\chi')$. Without loss of generality we may assume that $E_{32}(M(\chi')) = 0$ and $E_{31}(W(\chi')) = 0$. Therefore, $E_{21}W(\chi') = [E_{23}, E_{31}]W(\chi') = 0$ and $AW(\chi') = 0$. Hence, we have either $H_1w = H_2w = 0$ or $H_1w = 2w$, $H_2w = -2w$ for any $w \in W(\chi')$. In the first case, we obtain $M \simeq W \simeq \mathbb{C}$. Consider the second case. Since $AM(\chi') = 0$, we must have $E_{21}(M(\chi')) = 0$ (otherwise A will not be diagonalizable on M_μ), where $E_{21}(M(\chi')) \subset M_\mu$. Thus, $E_{23}(M(\chi')) = [E_{21}, E_{13}]M(\chi) = 0$, which is a contradiction. □

We next state the main theorem in this section.

Theorem 6.9. *Let M be a simple Gelfand–Tsetlin $\mathfrak{g}$-module and $\chi \in \mathrm{Supp}_{\mathrm{GT}}(M)$. Consider the following conditions*:

(i) χ *is non-critical and* $\dim M(\chi') \leq 1$ *for any* $\chi' \in \Gamma^*$;
(ii) χ *is non-critical and* $\dim(M(\chi)) = 2$;
(iii) χ *is critical*;
(iv) M *has no critical characters.*

If any of (ii)–(iv) *holds, then M is the unique simple extension of χ. If* (i) *holds, then M is the unique simple extension of χ with property* (i), *but χ may have another simple extension with two-dimensional Gelfand–Tsetlin multiplicities.*

Proof. Let χ be a non-critical character of M. Suppose first that M is generic. Then M is the only simple extension of χ by Theorem 5.18. Suppose now that M is singular and satisfies the conditions in (i). Assume first that $\chi(A)$ belongs to a critical chain but $\chi(A)$ itself is non-critical. Then M is the unique simple extension of χ by Corollary 6.2. Assume now that $\chi(A)$ belongs to a degenerate chain $\{\mu_1, \mu_2, \ldots\}$ where $\mu_n = n(n-1) - \frac{1}{2}r$, $n \geq 1$ (see Lemma 5.9(i)). Suppose that $M^{(1)}$ and $M^{(2)}$ are two simple extensions of χ satisfying the conditions of (i). If μ_1 is not an eigenvalue of $A_\lambda^{(i)} := A|_{M_\lambda^{(i)}}$, where $\lambda = \chi|_{\mathfrak{h}}$, then $M^{(1)} \simeq M^{(2)}$ since $A_\lambda^{(i)}$ (and thus $M^{(i)}$) is uniquely determined by $\chi(A)$ in this case. Assume that μ_1 is an eigenvalue of both $A_\lambda^{(i)}$, $i = 1, 2$. Consider the weight $\tilde{\lambda} := \lambda + \epsilon_2 - \epsilon_3$ and the eigenvalues of $A_{\tilde{\lambda}}^{(i)}$. They belong to the same critical chain. If both $A_{\tilde{\lambda}}^{(i)}$ have a common non-critical eigenvalue then $M^{(1)} \simeq M^{(2)}$ by Corollary 6.2. Hence, may assume that $A_{\tilde{\lambda}}^{(i)}$ have distinct eigenvalues. This is only possible if $\dim M_\lambda^{(i)} = 1$ and $\dim M_{\tilde{\lambda}}^{(i)} \leq 1$, $i = 1, 2$. Let $B|_{M_\lambda^{(i)}} = b_i \mathrm{Id}$. Recall that a weight module is torsion free if all root vectors act injectively on the module. Now consider the following cases.

Case 1. $\chi(A) - k\chi(H_1) \neq 0$, $b_i - m\chi(H_2) \neq 0$ *for* $i = 1, 2$ *and* $k, m \in \mathbb{Z}$. In this case both $M^{(1)}$ and $M^{(2)}$ are pointed torsion-free modules. Set $\tilde{\lambda}(H_i) = \tilde{h}_i$, $i = 1, 2$. Then we have $A_{\tilde{\lambda}}^{(1)} = \tilde{\mu}_1 \mathrm{Id}$, $A_{\tilde{\lambda}}^{(2)} = \tilde{\mu}_2 \mathrm{Id}$, $B_\rho^{(1)} = \tilde{b}_1 \mathrm{Id}$, and $B_{\tilde{\lambda}}^{(2)} = \tilde{b}_2 \mathrm{Id}$, where we can assume that $\tilde{\mu}_1 = -\frac{1}{4} - \frac{1}{4}\tilde{h}_1^2 + \tilde{h}_1$ and $\tilde{\mu}_2 = \tilde{\mu}_1 + 1$. Using the second identity in Lemma 5.3 we also have

$$a\tilde{b}_i = 2\tilde{\mu}_i \tilde{b}_i + \tilde{b}_i^2 + r_1 \tilde{\mu}_i + \tau_1, \quad i = 1, 2. \tag{17}$$

Keeping in mind that τ_1 and a depend of $\tilde{h}_1$ and $\tilde{h}_2$, r depends of $\tilde{h}_1$ and b_i, r_1 depend of $\tilde{h}_2$, $(a - 2\tilde{\mu}_1)\tilde{b}_1 - \tilde{b}_1^2 - r_1\tilde{\mu}_1 - \tau_1$ can be express as a polynomial in $\tilde{h}_1$, $\tilde{h}_2$. Let us consider the two-variable polynomial

$$f_i(x, y) = (a(x, y) - 2\mu_i(x))\tilde{b}_i(y) - \tilde{b}_i^2(y) - r_1(y)\mu_i(x) - \tau_1(x, y), \quad i = 1, 2.$$

Then by (17), $f_i(\tilde{h}_1, \tilde{h}_2) = 0$. An easy calculation shows that

$$A^{(1)}_{\tilde{\lambda}+2\epsilon_2-2\epsilon_3} = \tilde{\mu}_2(\tilde{h}_1 - 2)\mathrm{Id}, \quad A^{(2)}_{\tilde{\lambda}+2\epsilon_2-2\epsilon_3} = \tilde{\mu}_1(\tilde{h}_1 - 2)\mathrm{Id},$$

$$B^{(1)}_{\tilde{\lambda}+2\epsilon_2-2\epsilon_3} = \tilde{b}_2(\tilde{h}_2 + 4)\mathrm{Id}, \quad B^{(2)}_{\tilde{\lambda}+2\epsilon_2-2\epsilon_3} = \tilde{b}_1(\tilde{h}_2 + 4)\mathrm{Id}.$$

Then $f_i(\tilde{h}_1 - 2, \tilde{h}_2 + 4) = 0$, $i = 1, 2$. Similarly, using the operator E_{12}^2 we obtain $f_i(\tilde{h}_1 + 4, \tilde{h}_2 - 2) = 0$. Hence, $f_i(\tilde{h}_1 + 6, \tilde{h}_2) = 0$. If we repeat this argument again we will obtain $f_i(\tilde{h}_1 + 12, \tilde{h}_2) = 0$ and so on. Hence, if $g(x) = f_1(x, \tilde{h}_2)$, we have shown that $g(t) = 0$ implies $g(t + 6) = 0$, so g has infinitely many roots, thus $g = 0$, which is impossible.

Case 2. $\chi(A) - k\chi(H_1) = 0$, $b_i - m\chi(H_2) \neq 0$ *for some* $k \in \mathbb{Z}$ *and all* $i = 1, 2$, $m \in \mathbb{Z}$. Without loss of generality we may assume that $\chi(A) = 0$. Since A is diagonalizable on both modules, then this immediately implies that both $M^{(1)}$ and $M^{(2)}$ are quotients of generalized Verma modules (induced from an infinite-dimensional simple $\mathfrak{sl}(2)$-module W) that have the same central character and the same weight support. Since the generalized Harish-Chandra homomorphism defines W uniquely we conclude that $M^{(1)} \simeq M^{(2)}$.

Case 3. $\chi(A) - k\chi(H_1) = 0$, $b_1 - m_1\chi(H_2) = 0$, $b_2 - m_2\chi(H_2) \neq 0$, *for some* $k, m_1 \in \mathbb{Z}$ *and all* $i = 1, 2$, $m_2 \in \mathbb{Z}$. This case is handled in a similar way as Case 2.

Case 4. $\chi(A) - k\chi(H_1) = 0$ *and* $b_i = m_i\chi(H_2)$ *for some integer* k, m_i, $i = 1, 2$. Therefore, as in Case 2, both $M^{(1)}$ and $M^{(2)}$ are quotients of the same generalized Verma module. Hence, $m_1 = m_2$ and $M^{(1)} \simeq M^{(2)}$. This completes the proof of (i).

If χ is a non-critical character such that $\dim(M(\chi)) = 2$, then M is a unique simple extension of χ by Theorem 6.7 implying the result for (ii). The uniqueness of the extension if (iii) holds follows immediately from Theorem 6.1. It remains to consider (iv). Suppose M is generic. Then the uniqueness of simple extension for any character of M again follows from Theorem 5.18. If M is singular but without critical characters, then we apply Theorem 6.8. $\square$

7. Realizations of all Simple Gelfand–Tsetlin Modules for $\mathfrak{sl}(3)$

In this section, we give an explicit realization of all simple Gelfand–Tsetlin modules of $\mathfrak{sl}(3)$. For this purpose, we consider any Gelfand–Tsetlin character $\chi \in \Gamma^*$ and construct a Gelfand–Tsetlin module M such that any simple extension of χ is isomorphic to some subquotient of M (recall that, by Theorem 4.9, the number of non-isomorphic simple extensions is at least one and at most two).

Remark 4.2 provides a natural correspondence between the Gelfand–Tsetlin characters and the Gelfand–Tsetlin tableaux. Hence, given a character χ we can associate a tableau $T(v)$ and the problem of constructing simple extensions of χ is reduced to the problem of finding simple modules with tableaux realization containing $T(v)$ as a basis element. Recall that any Gelfand–Tsetlin tableau $T(v)$ of height 3 is either generic ($v_{21} - v_{22} \notin \mathbb{Z}$) or 1-singular ($v_{21} - v_{22} \in \mathbb{Z}$), and the constructions in Sec. 4 allow us to describe an explicit Gelfand–Tsetlin module $V(T(v))$ for any $T(v)$. This, combined with Theorem 6.9, implies that for the desired classification, it is sufficient to describe all simple subquotients of the modules $V(T(v))$.

7.1. *Structure of generic* $\mathfrak{sl}(3)$*-modules* $V(T(v))$

In this section, we consider all possible generic Gelfand–Tsetlin tableaux $T(v)$ and describe all simple subquotients of the $\mathfrak{sl}(3)$-module $V(T(v))$. The description includes an explicit basis for each simple subquotient, its weight support and its Loewy decomposition. Since $\mathfrak{g} = \mathfrak{sl}(3)$, the action of $E_{11} + E_{22} + E_{33}$ is zero, thus

$w_{31}+w_{32}+w_{33}+3=0$ for any Gelfand–Tsetlin tableaux $T(w)$. We first rewrite Theorem 4.14 in the case of $\mathfrak{sl}(3)$.

Theorem 7.1. *If $T(v)$ is a generic Gelfand–Tsetlin tableau of height* 3, *then the vector space $V(T(v))$ spanned by the set of tableaux $\mathcal{B}(T(v))$ has a structure of a Gelfand–Tsetlin $\mathfrak{sl}(3)$-module with the action of $\mathfrak{sl}(3)$ on $V(T(v))$ given by the Gelfand–Tsetlin formulas*:

$$\begin{aligned}
E_{12}(T(w)) &= -(w_{21}-w_{11})(w_{22}-w_{11})T(w+\delta^{11}),\\
E_{21}(T(w)) &= T(w-\delta^{11}),\\
E_{32}(T(w)) &= \frac{w_{21}-w_{11}}{w_{21}-w_{22}}T(w-\delta^{21}) - \frac{w_{22}-w_{11}}{w_{21}-w_{22}}T(w-\delta^{22}),\\
E_{23}(T(w)) &= \frac{(w_{31}-w_{21})(w_{32}-w_{21})(w_{33}-w_{21})}{w_{21}-w_{22}}T(w+\delta^{21})\\
&\quad - \frac{(w_{31}-w_{22})(w_{32}-w_{22})(w_{33}-w_{22})}{w_{21}-w_{22}}T(w+\delta^{22}),\\
E_{13}(T(w)) &= \frac{(w_{21}-w_{31})(w_{21}-w_{32})(w_{21}-w_{33})(w_{22}-w_{11})}{w_{21}-w_{22}}T(w+\delta^{21}+\delta^{11})\\
&\quad + \frac{(w_{31}-w_{22})(w_{32}-w_{22})(w_{33}-w_{22})(w_{21}-w_{11})}{w_{21}-w_{22}}T(w+\delta^{22}+\delta^{11}),\\
E_{31}(T(w)) &= \frac{1}{w_{21}-w_{22}}T(w-\delta^{21}-\delta^{11}) - \frac{1}{w_{21}-w_{22}}T(w-\delta^{22}-\delta^{11}),\\
H_1(T(w)) &= (2w_{11}-(w_{21}+w_{22}+1))T(w), \qquad (18)\\
H_2(T(w)) &= (2(w_{21}+w_{22}+1)-w_{11})T(w). \qquad (19)
\end{aligned}$$

By Theorem 5.18, a generic character admits a unique simple extension. In order to describe such simple extension, given a tableau $T(w)\in\mathcal{B}(T(v))$ we will describe explicit basis of tableaux for the simple subquotient of $V(T(v))$ that contains $T(w)$.

By (18) and (19), it is clear that $\mathcal{B}(T(v))$ is an eigenbasis for the action of the generators of the Cartan subalgebra $\mathfrak{h}$. In particular, any subquotient of $V(T(v))$ is a weight module. The following proposition describes explicit bases for the weight subspaces of the subquotients of $V(T(v))$.

Proposition 7.2. *Let M be a Gelfand–Tsetlin module with basis of tableaux $\mathcal{B}_M\subset\mathcal{B}(T(v))$ for some generic tableau $T(v)$. If $T(w)\in\mathcal{B}_M$ is a tableau of weight λ, then the weight space M_λ is spanned by the set of tableaux $\{T(w+(i,-i,0))\mid i\in\mathbb{Z}\}\cap\mathcal{B}_M$.*

Proof. As $\mathcal{B}(T(v))=\mathcal{B}(T(w))$, we just need to characterize tableaux of the form $T(w+(m,n,k))$ in $\mathcal{B}_M$ with the same weight λ of $T(w)$. By the Gelfand–Tsetlin formulas we have

$$H_1(T(w+(m,n,k))) = (2(w_{11}+k) - (w_{21}+m+w_{22}+n+1))T(w+(m,n,k)),$$

$$H_2(T(w+(m,n,k))) = (2(w_{21}+m+w_{22}+n+1) - (w_{11}+k))T(w+(m,n,k)).$$

In particular, the weight of $T(w)$ is

$$\lambda = (2w_{11} - (w_{21}+w_{22}+1), 2(w_{21}+w_{22}+1) - w_{11}).$$

Furthermore, a tableau $T(w+(m,n,k))$ in $\mathcal{B}_M$ has weight λ if m,n,k satisfy $m+n = 0$ and $k = 0$. $\square$

In this section, we will use Theorem 4.25 to describe all simple subquotients of the generic $\mathfrak{sl}(3)$-modules $V(T(v))$. Let us recall this result.

Let $T(v)$ be a generic Gelfand–Tsetlin tableau of height 3 and $T(w) \in \mathcal{B}(T(v))$ and $\Omega^+(T(u)) = \{(r,s,t) \mid u_{rs} - u_{r-1,t} \in \mathbb{Z}_{\geq 0}\}$. The complex vector space with basis $\mathcal{N}(T(w)) = \{T(w') \in \mathcal{B}(T(v)) \mid \Omega^+(T(w)) \subseteq \Omega^+(T(w'))\}$ is a submodule of $V(T(v))$ containing $T(w)$. Moreover, $\mathcal{I}(T(w)) := \{T(w') \in \mathcal{B}(T(v)) \mid \Omega^+(T(w)) = \Omega^+(T(w'))\}$ is a basis of the unique simple extension of χ_w, where $\chi_w \in \Gamma^*$ is given by $\chi_w(c_{rs}) = \gamma_{rs}(w)$.

By Theorem 4.25, bases of the subquotients of $V(T(v))$ can be described by subsets of $\mathbb{Z}^3$. In order to describe these bases, we introduce the following notation.

Definition 7.3. Let $T(w)$ be a tableau and $\mathcal{B}$ be a subset of $\mathbb{Z}^3$. Assume that M is a Gelfand–Tsetlin module with basis $\{T(w+(m,n,k)) \mid (m,n,k) \in \mathcal{B}\}$. Then we will denote M by $M(\mathcal{B}, T(w))$, or simply by $M(\mathcal{B})$ if $T(w)$ is fixed. If $M(\mathcal{B})$ is simple, then we will write $L(\mathcal{B})$ for $M(\mathcal{B})$.

Example 7.4. With the notation of Definition 7.3, the simple module from Example 4.27 can be written as follows:

$$L\begin{pmatrix} m \leq 0 \\ k \leq m \\ n > -2 \end{pmatrix} = M\begin{pmatrix} m \leq 0, & \\ k \leq m, & T(v) \\ n > -2, & \end{pmatrix},$$

where $v = (a,b,c,a,b+2,a)$.

7.2. *Realizations of all simple generic Gelfand–Tsetlin* $\mathfrak{sl}(3)$*-modules*

In this section, we describe all simple objects in every generic block $\mathcal{GT}_{T(v)}$ (see Definition 4.12 and Remark 4.2). Such description will include an explicit tableaux basis of each simple subquotient M in $\mathcal{GT}_{T(v)}$ and the weight support of M. For the weight support we will use Proposition 7.2 and the explicit basis. If the weight multiplicities are finite, a picture of the weight support along with the multiplicities is provided. We also present the components of the Loewy series of the universal module $V(T(v))$. A rigorous proof based on Theorem 4.25, Proposition 7.2, and Theorem 4.29 is given for Case $(G6)$ only, however, for all other cases the reasoning is the same.

Until the end of this section we use the following convention. The entries of the Gelfand–Tsetlin tableaux will be integer shifts of some of the complex numbers a, b, c, x, y, z. We also assume that if any two of a, b, c, x, y, z appear in the same row or in consecutive rows of a given tableau, then their difference is not integer.

Remark 7.5. By Theorem 5.18, any generic character has a unique simple extension. In particular, if $T(v)$ is generic, the number of simple subquotients of $V(T(v))$ is equal to the number of simple modules in $\mathcal{GT}_{T(v)}$. This number depends only on $\Omega(T(v))$ (see [20, Theorem 7.6]).

$(G1)$ Consider the following Gelfand–Tsetlin tableau:

$$T(v)= \begin{array}{ccccc} a & & b & & c \\ & x & & y & \\ & & z & & \end{array}$$

The module $V(T(v))$ is simple, and $\mathcal{GT}_{T(v)}$ has unique (up to isomorphism) simple module. This module has infinite-dimensional weight multiplicities.

Module	Basis
L_1	$L(\mathbb{Z}^3)$

$(G2)$ Let $T(v)$ be the tableau:

$$\begin{array}{ccccc} a & & b & & c \\ & x & & y & \\ & & x & & \end{array}$$

I. Simple subquotients.
In this case, the module $V(T(v))$ has two simple subquotients and they have infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L(k \leq m)$
L_2	$L(m < k)$

II. Loewy series.

$$L_1, \quad L_2.$$

$(G3)$ Consider the tableau:

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{y} & \\ & & \boxed{z} & & \end{array}$$

I. Simple subquotients.

In this case, the module $V(T(v))$ has two simple subquotients and they have infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L(m \leq 0)$
L_2	$L(0 < m)$

II. Loewy series.

$$L_1, \quad L_2.$$

$(G4)$ Consider the tableau:

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{y} & \\ & & \boxed{y} & & \end{array}$$

I. Simple subquotients.

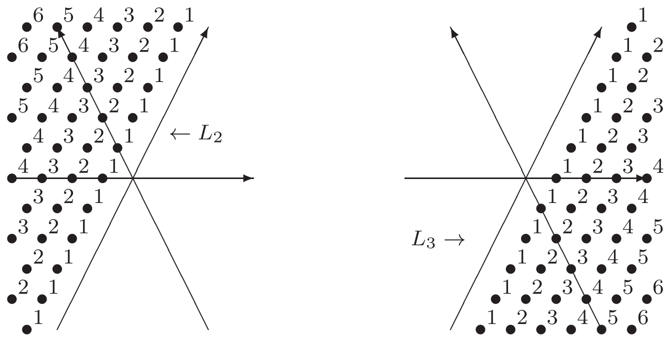
In this case, the module $V(T(v))$ has four simple subquotients. The bases and corresponding weight lattices are given by

(i) Modules with infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ k \leq n \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < m \\ n < k \end{pmatrix}$

(ii) Modules with unbounded finite weight multiplicities:

Module	Basis
L_2	$L\begin{pmatrix} 0 < m \\ k \leq n \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq 0 \\ n < k \end{pmatrix}$



The origin of the picture above corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v)$.

II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4.$$

$(G5)$ Let $T(v)$ be the generic tableau:

a	b	c
a	y	
a		

I. Simple subquotients.

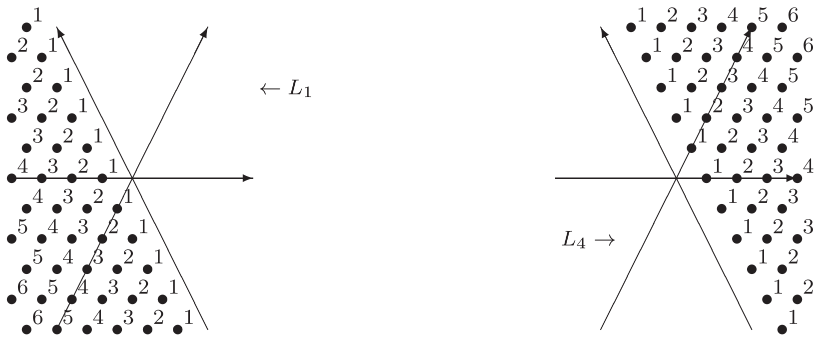
In this case, the module $V(T(v))$ has four simple subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_2	$L\begin{pmatrix} m \leq 0 \\ m < k \end{pmatrix}$
L_3	$L\begin{pmatrix} 0 < m \\ k \leq m \end{pmatrix}$

(ii) Two modules with unbounded finite weight multiplicities. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v + \delta^{11})$:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ k \leq m \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < m \\ m < k \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4.$$

$(G6)$ Consider the tableau:

$$T(v) = \begin{array}{ccc} \boxed{a} & \boxed{b} & \boxed{c} \\ \boxed{a} & \boxed{b} & \\ & \boxed{a} & \end{array}$$

I. Simple subquotients.

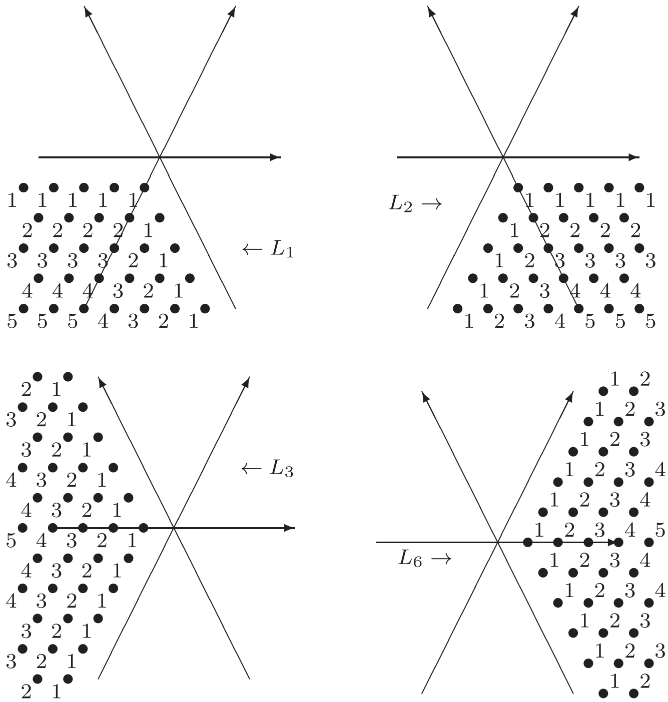
In this case, the module $V(T(v))$ has eight simple subquotients. The bases and corresponding weight lattices are given by

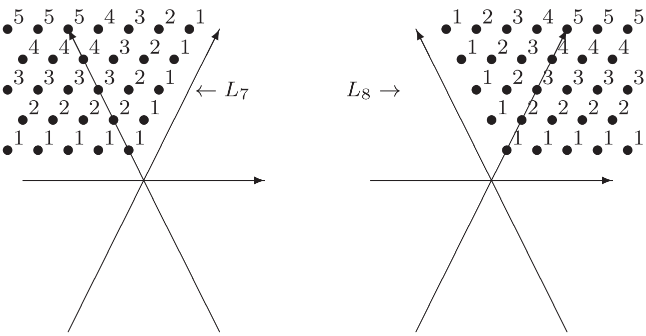
(i) Two modules with infinite-dimensional weight spaces

Module	Basis
L_4	$L\begin{pmatrix} 0 < m \\ n \leq 0 \\ k \leq m \end{pmatrix}$
L_5	$L\begin{pmatrix} m \leq 0 \\ 0 < n \\ m < k \end{pmatrix}$

(ii) Six modules with unbounded finite weight multiplicities. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v + \delta^{11} + \delta^{21})$:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq m \end{pmatrix}$
L_2	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ m < k \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq 0 \\ 0 < n \\ k \leq m \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ n \leq 0 \\ m < k \end{pmatrix}$
L_7	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \leq m \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ 0 < n \\ m < k \end{pmatrix}$





II. Loewy series.

$$L_1, \quad L_2 \oplus L_3 \oplus L_4, \quad L_5 \oplus L_6 \oplus L_7, \quad L_8.$$

Proof. If M denotes the universal module $V(T(v))$, we proceed as follows: first prove that L_1 is a simple submodule of M, then that $M_1 := M/L_1$ has simple submodules isomorphic to $L_2,\ L_3$, and L_4, then prove that $M_2 := M_1/(L_2 \oplus L_3 \oplus L_4)$ has simple submodules isomorphic to $L_5,\ L_6$, and L_7, and finally show that $L_8 = M_2/(L_5 \oplus L_6 \oplus L_7)$ is a simple module.

By Theorem 4.25, we see that, L_1 (respectively, L_2, L_3, L_4, L_5, L_6, L_7, and L_8) is a simple subquotient of $V(T(v))$ containing the tableau $T(v)$ (respectively, $T(v+(0,0,1))$, $T(v+(0,1,0))$, $T(v+(1,0,1))$, $T(v+(0,1,1))$, $T(v+(1,0,2))$, $T(v+(1,1,1))$, and $T(v+(1,1,2))$). To find the layers of the Loewy series decomposition for $V(T(v))$ we apply Theorem 4.29 and Remark 4.30. We describe these layers in four steps.

Step 1. L_1 is a simple submodule of M. By Theorem 4.25(i), the module with basis $\mathcal{N}(T(v)) = \{T(w) \mid \Omega^+(T(v)) \subseteq \Omega^+(T(w))\}$ is a submodule of M, but $\Omega^+(T(v)) = \{(3,1,1),(3,2,2),(2,1,1)\} = \Omega(T(v))$, thus $\mathcal{N}(T(v)) = \mathcal{I}(T(v))$. Hence, L_1 is a simple submodule of M.

Step 2. $M_1 := M/L_1$ has simple submodules isomorphic to $L_2,\ L_3$, and L_4. For these modules we have that

$$\Omega^+(T(v+(1,0,1))) = \{(3,2,2),(2,1,1)\},$$

$$\Omega^+(T(v+(0,0,1))) = \{(3,1,1),(3,2,2)\},$$

$$\Omega^+(T(v+(0,1,0))) = \{(3,1,1),(2,1,1)\}.$$

Hence, by Theorem 4.25(i) the modules with bases $\mathcal{I}(T(v+(1,0,1))) \cup \mathcal{I}(T(v))$, $\mathcal{I}(T(v+(0,0,1))) \cup \mathcal{I}(T(v))$ and $\mathcal{I}(T(v+(0,1,0))) \cup \mathcal{I}(T(v))$ are submodules of M. Therefore, the modules with bases $\mathcal{I}(T(v+(1,0,1)))$; $\mathcal{I}(T(v+(0,0,1)))$, and $\mathcal{I}(T(v+(0,1,0)))$ are submodules of $M_1 := M/L_1$ since L_1 is has basis $\mathcal{I}(T(v))$.

Step 3. $M_2 := M_1/(L_2 \oplus L_3 \oplus L_4)$ has simple submodules isomorphic to $L_5,\ L_6$, and L_7. For these modules we have

$$\Omega^+(T(v+(0,1,1))) = \{(3,1,1)\},$$
$$\Omega^+(T(v+(1,1,1))) = \{(2,1,1)\},$$
$$\Omega^+(T(v+(1,0,2))) = \{(3,2,2)\}.$$

Hence, by Theorem 4.25(i) the modules with bases $\mathcal{I}(T(v+(1,0,1))) \cup \mathcal{I}(T(v))$; $\mathcal{I}(T(v+(0,0,1))) \cup \mathcal{I}(T(v))$ and $\mathcal{I}(T(v+(0,1,0))) \cup \mathcal{I}(T(v))$ are submodules of M. Therefore, the modules with bases $\mathcal{I}(T(v+(1,0,1)))$; $\mathcal{I}(T(v+(0,0,1)))$ and $\mathcal{I}(T(v+(0,1,0)))$ are submodules of $M_2 := M_1/(L_2 \oplus L_3 \oplus L_4)$ because L_1 has a basis $\mathcal{I}(T(v))$.

Step 4. $M_3/(L_5 \oplus L_6 \oplus L_7) \simeq L_8$. In fact, $\Omega^+(T(v+(1,1,2))) = \emptyset$ so the submodule of $V(T(v))$ generated by $T(v+(1,1,2))$ is $V(T(v))$ and the simple subquotient containing $T(v+(1,1,2))$ has the same basis as $M_3/(L_5 \oplus L_6 \oplus L_7)$ so, we have $M_3/(L_5 \oplus L_6 \oplus L_7) \simeq L_8$. $\square$

$(G7)$ Consider the tableau:

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{b} & \\ & & \boxed{z} & & \end{array}$$

I. Simple subquotients.

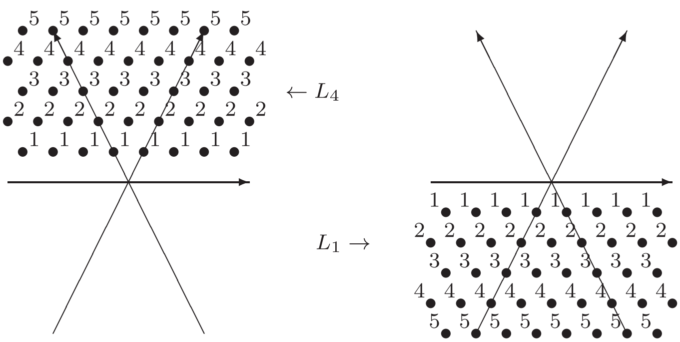
In this case, the module $V(T(v))$ has four simple subquotients.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_2	$L\begin{pmatrix} 0<m \\ n \leq 0 \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq 0 \\ 0<n \end{pmatrix}$

(ii) Two modules with unbounded finite weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \end{pmatrix}$
L_4	$L\begin{pmatrix} 0<m \\ 0<n \end{pmatrix}$



The origin of the picture above corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v + \delta^{11} + \delta^{21})$.

II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4.$$

$(G8)$ Set $t \in \mathbb{Z}_{>0}$ and consider the following Gelfand–Tsetlin tableau:

$$T(v) = \begin{array}{ccc} \boxed{a} & \boxed{a-t} & \boxed{c} \\ & \boxed{a} \quad \boxed{y} & \\ & \boxed{z} & \end{array}$$

I. Simple subquotients.

In this case, the module $V(T(v))$ has three simple subquotients.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L\big(m \leq -t\big)$
L_3	$L\big(0 < m\big)$

(ii) A cuspidal module with t-dimensional weight spaces:

Module	Basis
L_2	$L\big(-t < m \leq 0\big)$

II. Loewy series.

$$L_1, \quad L_2, \quad L_3.$$

$(G9)$ For each $t \in \mathbb{Z}_{>0}$, let consider the following generic tableau:

$$T(v) = \begin{array}{ccc} \boxed{a} & \boxed{a-t} & \boxed{c} \\ & \boxed{a} \quad \boxed{y} & \\ & \boxed{a} & \end{array}$$

I. Simple subquotients.

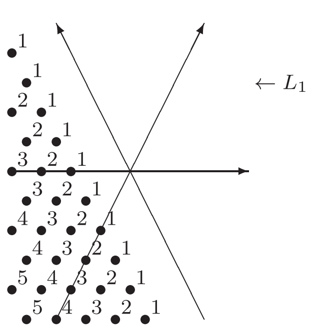
In this case, the module $V(T(v))$ has six simple subquotients. The bases and corresponding weight lattices are given by:

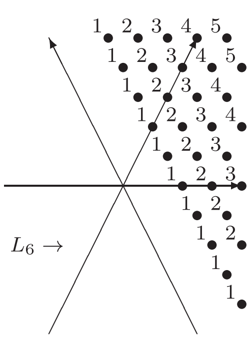
(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_2	$L\begin{pmatrix} m \le -t \\ m < k \end{pmatrix}$
L_5	$L\begin{pmatrix} 0 < m \\ k \le m \end{pmatrix}$

(ii) Two modules with unbounded finite weight multiplicities:

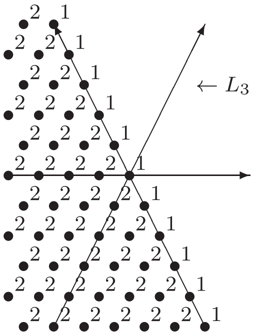
Module	Basis
L_1	$L\begin{pmatrix} m \le -t \\ k \le m \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ m < k \end{pmatrix}$

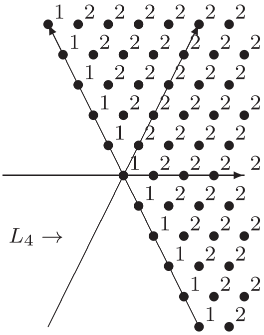




(iii) Two modules with weight multiplicities bounded by t:

Module	Basis
L_3	$L\begin{pmatrix} -t < m \leq 0 \\ k \leq m \end{pmatrix}$
L_4	$L\begin{pmatrix} -t < m \leq 0 \\ m < k \end{pmatrix}$





The pictures above correspond to the case $t = 2$, and the origin is the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v - 2\delta^{21})$.

II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4 \oplus L_5, \quad L_6.$$

$(G10)$ For each $t \in \mathbb{Z}_{>0}$, let $T(v)$ be the following Gelfand–Tsetlin tableau:

a	$a-t$	c
a	y	
y		

I. Simple subquotients.

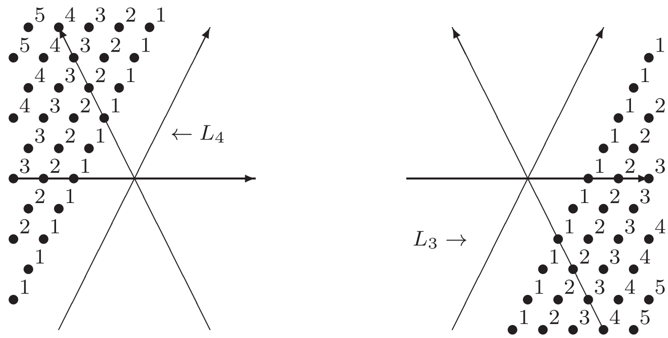
In this case, the module $V(T(v))$ has six simple subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq -t \\ k \leq n \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ n < k \end{pmatrix}$

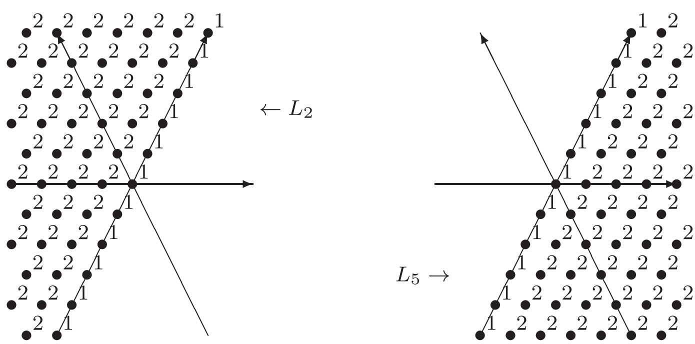
(ii) Two modules with unbounded finite weight multiplicities:

Module	Basis
L_3	$L\begin{pmatrix} m \leq -t \\ n < k \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < m \\ k \leq n \end{pmatrix}$



(iii) Two modules with finite weight multiplicities bounded by t.

Module	Basis
L_2	$L\begin{pmatrix} -t < m \leq 0 \\ k \leq n \end{pmatrix}$
L_5	$L\begin{pmatrix} -t < m \leq 0 \\ n < k \end{pmatrix}$



The pictures above correspond to the case $t = 2$, and the origin is the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v - \delta^{22})$.

II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4 \oplus L_5, \quad L_6.$$

$(G11)$ For any $t \in \mathbb{Z}_{>0}$, let $T(v)$ be the tableau:

$$\begin{array}{|c|c|c|} \hline a & a-t & c \\ \hline \end{array}$$
$$\begin{array}{|c|c|} \hline a & c \\ \hline \end{array}$$
$$\begin{array}{|c|} \hline a \\ \hline \end{array}$$

I. Simple subquotients.

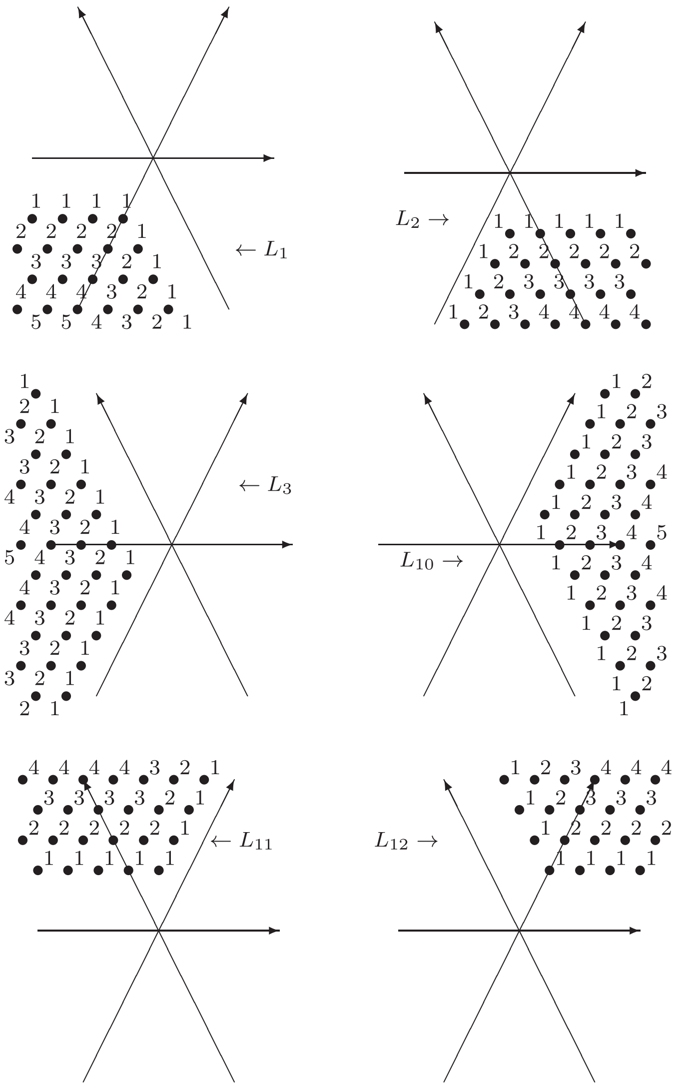
In this case, the module $V(T(v))$ has 12 simple subquotients. We provide pictures of the weight lattice corresponding to the case $t = 2$, and with origin at the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v)$.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_5	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ m < k \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ n \leq 0 \\ k \leq m \end{pmatrix}$

(ii) Six modules with unbounded finite weight multiplicities:

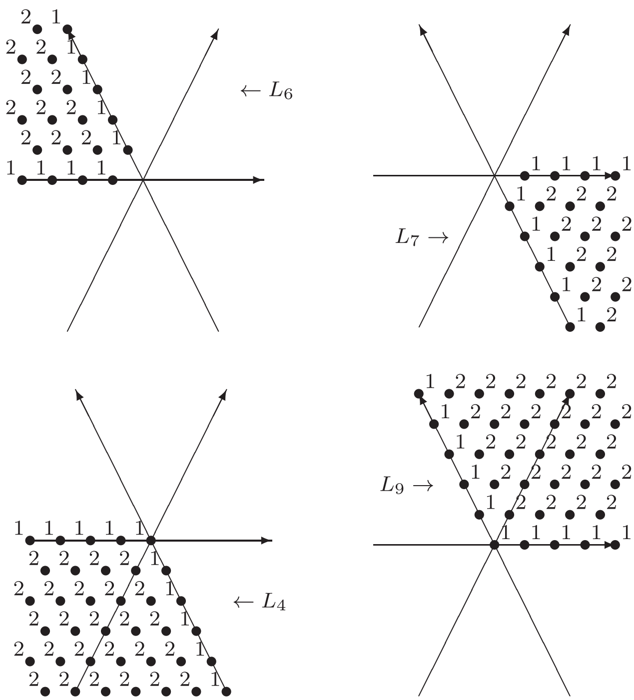
Module	Basis
L_1	$L\begin{pmatrix} m \leq -t \\ n \leq 0 \\ k \leq m \end{pmatrix}$
L_2	$L\begin{pmatrix} m \leq -t \\ n \leq 0 \\ m < k \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ k \leq m \end{pmatrix}$
L_{10}	$L\begin{pmatrix} 0 < m \\ n \leq 0 \\ m < k \end{pmatrix}$
L_{11}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \leq m \end{pmatrix}$
L_{12}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ m < k \end{pmatrix}$



(iii) Four modules with finite weight multiplicities bounded by t:

Module	Basis
L_4	$L\begin{pmatrix} -t < m \leq 0 \\ n \leq 0 \\ k \leq m \end{pmatrix}$
L_6	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ k \leq m \end{pmatrix}$

Module	Basis
L_7	$L\begin{pmatrix} -t < m \leq 0 \\ n \leq 0 \\ m < k \end{pmatrix}$
L_9	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ m < k \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3 \oplus L_4, \quad L_5 \oplus L_6 \oplus L_7 \oplus L_8, \quad L_9 \oplus L_{10} \oplus L_{11}, \quad L_{12}.$$

$(G12)$ Consider $t \in \mathbb{Z}_{>0}$, and $T(v)$ to be the Gelfand–Tsetlin tableau:

$$T(v) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{b-t} \\ & \boxed{a} & & \boxed{b} & \\ & & \boxed{a} & & \end{array}$$

I. Simple subquotients.

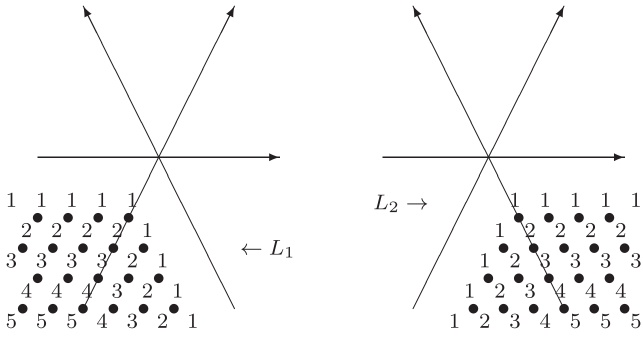
In this case, the module $V(T(v))$ has 12 simple subquotients. We provide pictures of the weight lattice corresponding to the case $t = 2$, and with origin at the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v + \delta^{11})$.

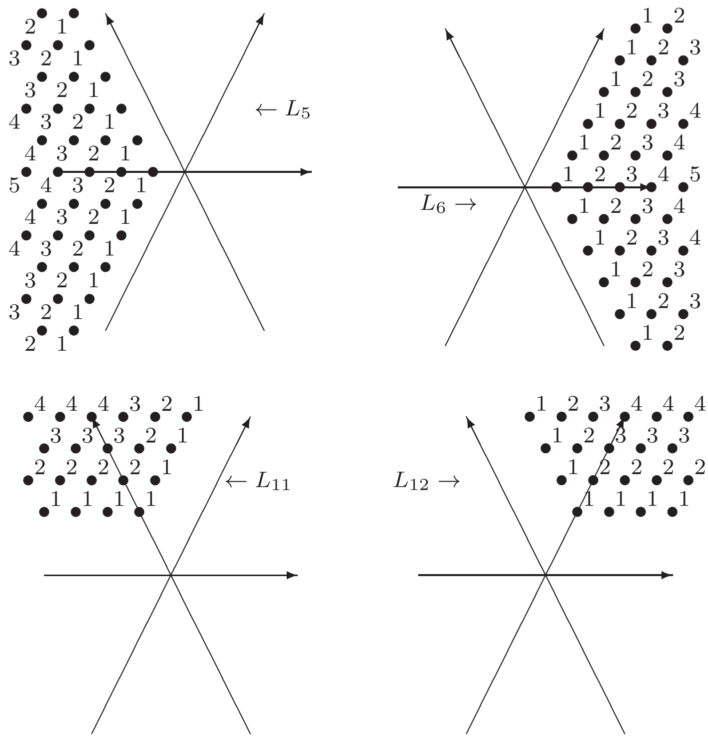
(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_4	$L\begin{pmatrix} 0 < m \\ n \leq -t \\ k \leq m \end{pmatrix}$
L_9	$L\begin{pmatrix} m \leq 0 \\ 0 < n \\ m < k \end{pmatrix}$

(ii) Six modules with unbounded weight multiplicities:

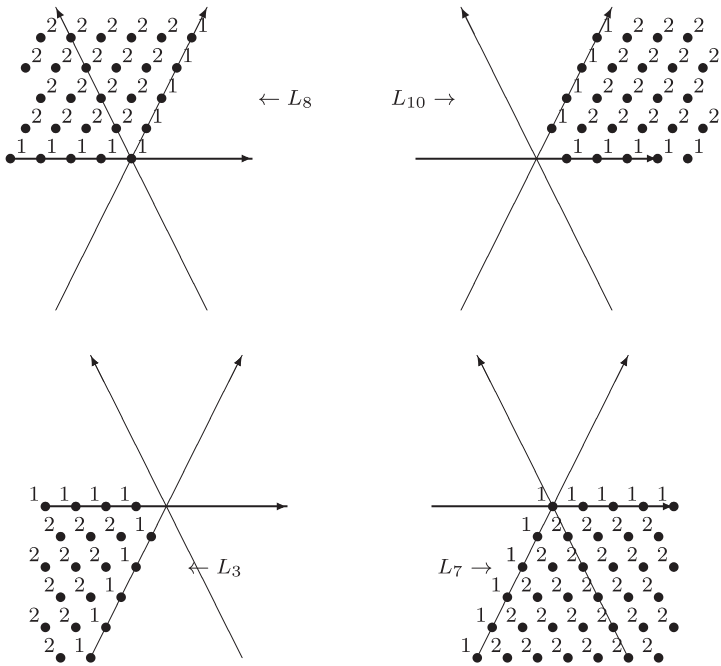
Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq -t \\ k \leq m \end{pmatrix}$
L_2	$L\begin{pmatrix} m \leq 0 \\ n \leq -t \\ m < k \end{pmatrix}$
L_5	$L\begin{pmatrix} m \leq 0 \\ 0 < n \\ k \leq m \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ n \leq -t \\ m < k \end{pmatrix}$
L_{11}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \leq m \end{pmatrix}$
L_{12}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ m < k \end{pmatrix}$





(iii) Four modules with finite weight multiplicities bounded by t:

Module	Basis
L_3	$L\begin{pmatrix} -t < n \leq 0 \\ m \leq 0 \\ k \leq m \end{pmatrix}$
L_7	$L\begin{pmatrix} m \leq 0 \\ -t < n \leq 0 \\ m < k \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ -t < n \leq 0 \\ k \leq m \end{pmatrix}$
L_{10}	$L\begin{pmatrix} 0 < m \\ -t < n \leq 0 \\ m < k \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3 \oplus L_4, \quad L_5 \oplus L_6 \oplus L_7 \oplus L_8, \quad L_9 \oplus L_{10} \oplus L_{11}, \quad L_{12}.$$

$(G13)$ Consider $t \in \mathbb{Z}_{>0}$. Set $T(v)$ to be the following Gelfand–Tsetlin tableau:

$$\begin{array}{ccccc} \boxed{a} & & \boxed{a-t} & & \boxed{c} \\ & \boxed{a} & & \boxed{c} & \\ & & \boxed{z} & & \end{array}$$

I. Simple subquotients.

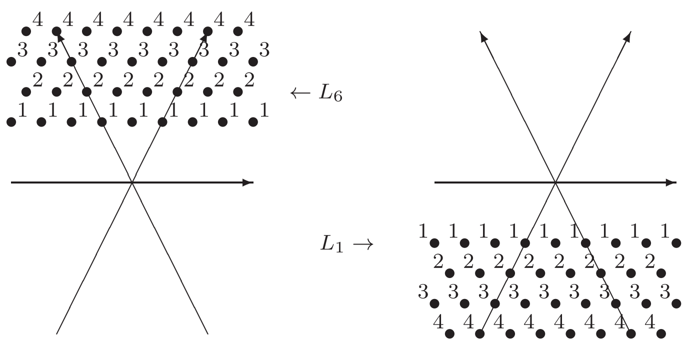
In this case, the module $V(T(v))$ has six simple subquotients. We provide pictures of the weight lattice corresponding to the case $t = 2$, and with origin at the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v + \delta^{11})$.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_3	$L\begin{pmatrix} m \leq -t \\ 0 < n \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < m \\ n \leq 0 \end{pmatrix}$

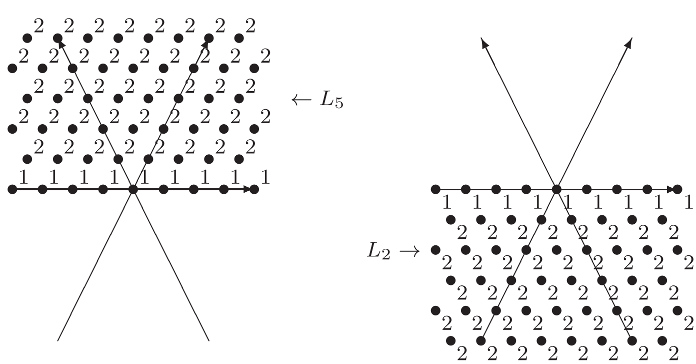
(ii) Two modules with unbounded finite weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \le -t \\ n \le 0 \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ 0 < n \end{pmatrix}$



(iii) Two modules with finite weight multiplicities bounded by t:

Module	Basis
L_2	$L\begin{pmatrix} -t < m \le 0 \\ n \le 0 \end{pmatrix}$
L_5	$L\begin{pmatrix} -t < m \le 0 \\ 0 < n \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4 \oplus L_5, \quad L_6.$$

$(G14)$ Set $t, s \in \mathbb{Z}_{>0}$ with $t < s$ and let $T(v)$ be the following Gelfand–Tsetlin tableau:

$$T(v) = \begin{array}{|c|c|c|} \hline a & a-t & a-s \\ \hline \end{array} \atop {\begin{array}{|c|c|} \hline a & y \\ \hline \end{array} \atop \begin{array}{|c|} \hline a \\ \hline \end{array}}$$

I. Simple subquotients.

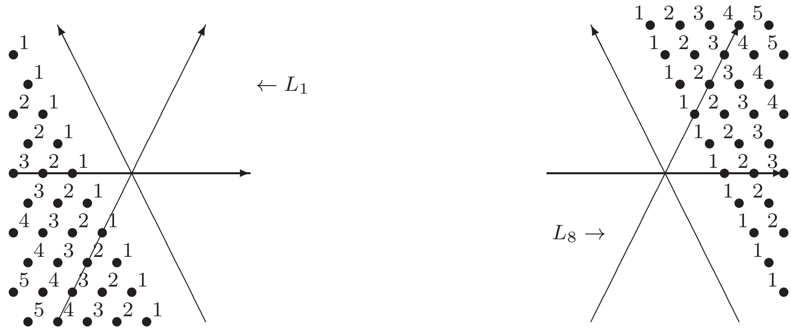
In this case, the module $V(T(v))$ has eight simple subquotients. We provide pictures of the weight lattice corresponding to the case $t = 1$, $s = 2$, and with origin at the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v)$.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_3	$L\begin{pmatrix} m \leq -s \\ m < k \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ k \leq m \end{pmatrix}$

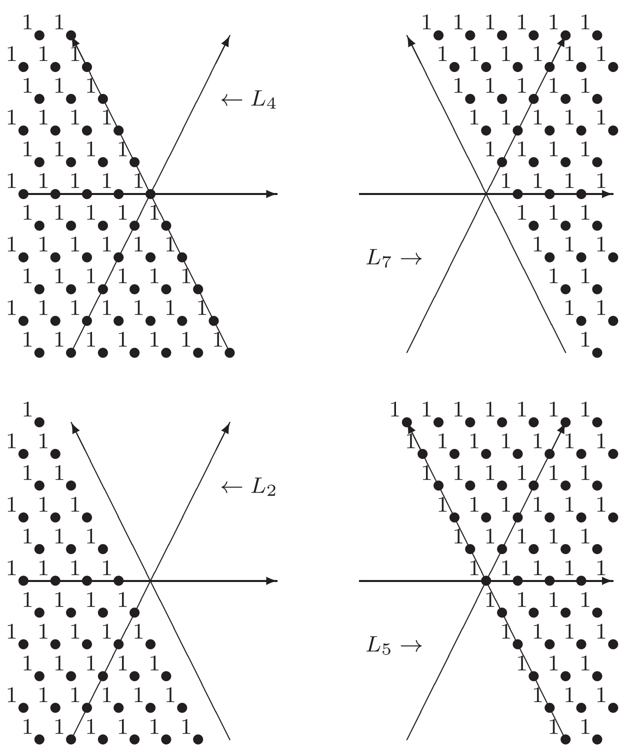
(ii) Two modules with unbounded finite weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq -s \\ k \leq m \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ m < k \end{pmatrix}$



(iii) Four modules with finite weight multiplicities. The modules L_4 and L_7 have multiplicities bounded by t, and L_2, L_5 have multiplicities bounded by $s-t$:

Module	Basis
L_2	$L\begin{pmatrix} -s < m \leq -t \\ k \leq m \end{pmatrix}$
L_4	$L\begin{pmatrix} -t < m \leq 0 \\ k \leq m \end{pmatrix}$
L_5	$L\begin{pmatrix} -s < m \leq -t \\ m < k \end{pmatrix}$
L_7	$L\begin{pmatrix} -t < m \leq 0 \\ m < k \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4 \oplus L_5, \quad L_6 \oplus L_7, \quad L_8.$$

$(G15)$ Set $t, s \in \mathbb{Z}_{>0}$ with $t < s$ and let $T(v)$ be the following tableau:

$$T(v) = \begin{array}{|c|c|c|} \hline a & a-t & a-s \\ \hline \multicolumn{3}{c}{\begin{array}{|c|c|} \hline a & y \\ \hline \end{array}} \\ \multicolumn{3}{c}{\begin{array}{|c|} \hline y \\ \hline \end{array}} \end{array}$$

I. Simple subquotients.

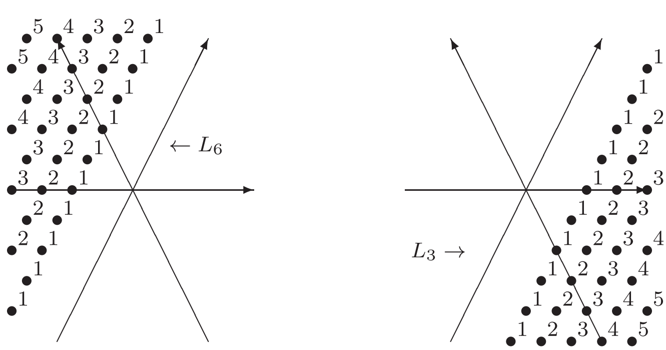
In this case, the module $V(T(v))$ has eight simple subquotients. We provide pictures of the weight lattice corresponding to the case $t = 1$, $s = 2$, and with origin at the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v - \delta^{22})$.

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq -s \\ k \leq n \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ n < k \end{pmatrix}$

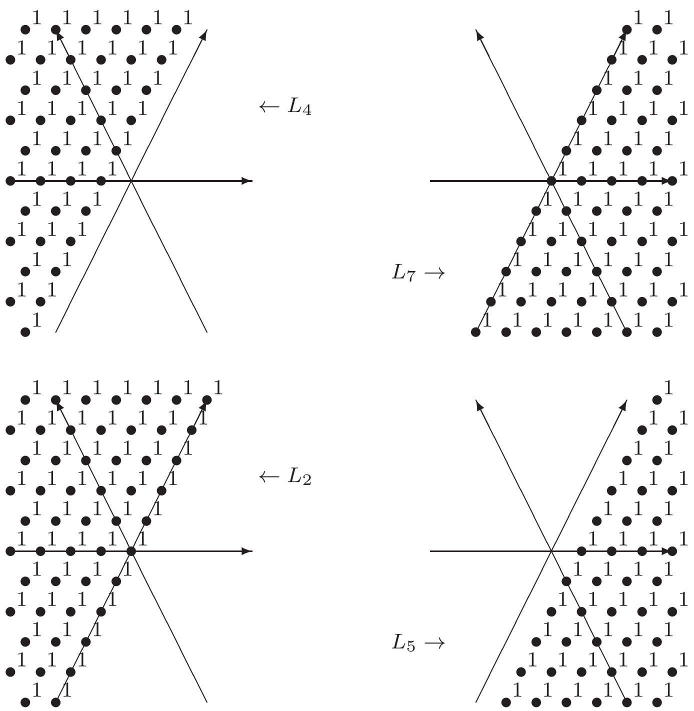
(ii) Two modules with unbounded finite weight multiplicities:

Module	Basis
L_3	$L\begin{pmatrix} m \leq -s \\ n < k \end{pmatrix}$
L_6	$L\begin{pmatrix} 0 < m \\ k \leq n \end{pmatrix}$



(iii) Four modules with bounded weight multiplicities. The modules L_4 and L_7 have multiplicities bounded by t, and L_2, L_5 have multiplicities bounded by $s - t$:

Module	Basis
L_2	$L\begin{pmatrix} -s < m \leq -t \\ k \leq n \end{pmatrix}$
L_4	$L\begin{pmatrix} -t < m \leq 0 \\ k \leq n \end{pmatrix}$
L_5	$L\begin{pmatrix} -s < m \leq -t \\ n < k \end{pmatrix}$
L_7	$L\begin{pmatrix} -t < m \leq 0 \\ n < k \end{pmatrix}$



II. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4 \oplus L_5, \quad L_6 \oplus L_7, \quad L_8.$$

$(G16)$ For any $t, s \in \mathbb{Z}_{>0}$ with $t < s$ let $T(v)$ be the following tableau:

$$T(v) = \begin{array}{|c|c|c|} \hline a & a-t & a-s \\ \hline \end{array}$$
$$\begin{array}{|c|c|} \hline a & y \\ \hline \end{array}$$
$$\begin{array}{|c|} \hline z \\ \hline \end{array}$$

I. Simple subquotients.
In this case, the module $V(T(v))$ has four simple subquotients. The bases and corresponding weight multiplicities are given by

(i) Two modules with infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L\big(m \leq -s\big)$
L_4	$L\big(0 < m\big)$

(ii) Two cuspidal modules; L_2 with weight multiplicities $s-t$ and L_3 with weight multiplicities t:

Module	Basis
L_2	$L\big(-s < m \leq -t\big)$
L_3	$L\big(-t < m \leq 0\big)$

II. Loewy series.

$$L_1, \quad L_2, \quad L_3, \quad L_4.$$

7.3. *Structure of singular $\mathfrak{sl}(3)$-modules $V(T(\bar{v}))$*

In this section, we describe all simple singular Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules. Like in the generic case it is enough to explicitly present all simple subquotients of $V(T(\bar{v}))$ for every 1-singular vector $\bar{v}$.

7.3.1. *Singular Gelfand–Tsetlin formulas*

Recall the construction of the 1-singular $\mathfrak{sl}(3)$-modules $V(T(\bar{v}))$ in Sec. 4.4. We adapt this construction to $\mathfrak{g} = \mathfrak{sl}(3)$. Since the singularity appears in row 2, we fix $\bar{v} \in T_3(\mathbb{C})$ such that $\bar{v}_{21} = \bar{v}_{22}$. Also, we denote by τ the permutation in $S_3 \times S_2 \times S_1$ that interchanges the $(2,1)$th and $(2,2)$th entries and is identity on rows 1 and 3.

Recall that $V(T(\bar{v}))$, as a vector space, is generated by the set of tableaux $\{T(\bar{v}+z), \mathcal{D}T(\bar{v}+z') \mid z, z' \in \mathbb{Z}^3\}$ satisfying the relations $T(\bar{v}+z) - T(\bar{v}+\tau(z)) = 0$ and $\mathcal{D}T(\bar{v}+z) + \mathcal{D}T(\bar{v}+\tau(z)) = 0$. As explained in Remark 4.34,

$$\mathcal{B}(T(\bar{v})) = \{T(\bar{v}+z), \mathcal{D}T(\bar{v}+w) \mid z_{21} \leq z_{22} \text{ and } w_{21} > w_{22}\}$$

is a basis of $V(T(\bar{v}))$.

Definition 7.6. Given $w \in \mathbb{Z}^3$, the *tableau associated to* w with respect to $\mathcal{B}(T(\bar{v}))$ is defined by

$$\mathrm{Tab}(w) := \begin{cases} T(\bar{v}+w) & \text{if } w_{21} \leq w_{22}, \\ \mathcal{D}T(\bar{v}+w) & \text{if } w_{21} > w_{22}. \end{cases}$$

In particular, $\mathcal{B}(T(\bar{v})) = \{\mathrm{Tab}(w) \mid w \in \mathbb{Z}^3\}$.

We next write explicitly the formulas for the action of $\mathfrak{sl}(3)$ on $V(T(\bar{v}))$. We again use the conventions $v = (v_{31}, v_{32}, v_{33}, v_{21}, v_{22}, v_{11})$, $\bar{v} = (a, b, c, x, x, z)$, and $w = (0, 0, 0, m, n, k)$. For any rational function f in $\{v_{ij}\}_{1\leq j\leq i\leq 3}$, $\partial^{\bar{v}}_{v_{ij}}(f)$ will stand for $\frac{\partial f}{\partial v_{ij}}(\bar{v})$.

Recall that by Proposition 4.22, the Gelfand–Tsetlin formulas for generic modules can be written as follows:

$$\begin{aligned} H_i(T(v+z)) &= h_i(v+z)T(v+z), \\ E_{rs}T(v+z) &= \sum_{\sigma\in\Phi_{rs}} e_{rs}(\sigma(v+z))T(v+z+\sigma(\varepsilon_{rs})), \end{aligned}$$

where $h_1(w) = 2w_{11} - (w_{21} + w_{22} + 1)$, $h_2(w) = 2(w_{21} + w_{22} + 1) - w_{11}$, and $e_{rs}(w)$, ε_{rs} are defined in Example 4.23.

Using the relations $T(\bar{v}+z) - T(\bar{v}+\tau(z)) = 0$ and $\mathcal{D}T(\bar{v}+z) + \mathcal{D}T(\bar{v}+\tau(z)) = 0$ (see Sec. 4.4) we can write the above formulas in a simpler form. Set for convenience $\bar{w} = \bar{v} + w$. Then

$$H_i(\mathrm{Tab}(w)) = h_i(\bar{v}+w)\,\mathrm{Tab}(w) \quad \text{for } i = 1, 2, \tag{20}$$

$$E_{21}(\mathrm{Tab}(w)) = \mathrm{Tab}(w+\varepsilon_{21}), \tag{21}$$

$$E_{12}(\mathrm{Tab}(w)) = e_{12}(\bar{v}+w)\mathrm{Tab}(w+\varepsilon_{12}). \tag{22}$$

If $E = E_{ij} \in \{E_{32}, E_{23}, E_{31}, E_{13}\}$, ε denotes the corresponding ε_{ij}, and $\tilde{e}(w) := (v_{21} - v_{22})e(w)$, then the action of E on $\mathrm{Tab}(w)$ is given by

$$2(\tilde{e}(\bar{w})\mathcal{D}T(\bar{w}+\varepsilon) + \mathcal{D}^{\bar{v}}(\tilde{e}(v+w))T(\bar{w}+\varepsilon)) \qquad \text{if } w_{21} = w_{22}, \tag{23}$$

$$e(\bar{w})T(\bar{w}+\varepsilon) + e(\tau(\bar{w}))T(\bar{w}+\tau(\varepsilon)) \qquad \text{if } w_{21} < w_{22}, \tag{24}$$

$$\begin{cases} \mathcal{D}^{\bar{v}}(e(v+w))T(\bar{w}+\varepsilon) + \mathcal{D}^{\bar{v}}(e(\tau(v+w)))T(\bar{w}+\tau(\varepsilon)) \\ +e(\bar{w})\mathcal{D}T(\bar{w}+\varepsilon) + e(\tau(\bar{w}))\mathcal{D}T(\bar{w}+\tau(\varepsilon)) \end{cases} \qquad \text{if } w_{21} > w_{22}. \tag{25}$$

Remark 7.7. Note that the Gelfand–Tsetlin formulas for singular tableaux have the same coefficients as in the classical formulas for tableaux of the same type (see formulas (20)–(25)). More precisely, the action of the generators E_{ij} on a regular tableau is a linear combination of regular tableaux with the same coefficients as those that appear in the classical Gelfand–Tsetlin formulas. On the other hand, the corresponding action on a derivative tableau is a linear combination of both regular and derivative tableaux, and the coefficients of the derivative tableaux are the same as those that appear in the classical formulas.

7.3.2. *Explicit formulas and computations*

Some explicit computations are included in the following example.

Example 7.8. For any complex numbers a and v_{ij}, $1 \leq j \leq i \leq 3$, consider the following Gelfand–Tsetlin tableaux:

$$T(v) = \begin{array}{ccccc} \boxed{v_{31}} & & \boxed{v_{32}} & & \boxed{v_{33}} \\ & \boxed{v_{21}} & & \boxed{v_{22}} & \\ & & \boxed{v_{11}} & & \end{array} \qquad T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{a} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

Consider $w = (0, 1, 0) = \delta^{22}$ and $w' = (1, 0, 0) = \delta^{21}$ in $\mathbb{Z}^3$. Then the following hold:

(i) Using (24), $E_{32}(T(\bar{v} + w)) = \mathcal{D}^{\bar{v}}((v_{21} - v_{22})E_{32}(T(v + w)))$ which equals to

$$\begin{aligned}
&\mathcal{D}^{\bar{v}}\left((v_{21} - v_{22})\left(\frac{v_{21} - v_{11}}{v_{21} - (v_{22} + 1)}T(v + w - \delta^{21})\right.\right.\\
&\qquad\left.\left.+\frac{(v_{22} + 1) - v_{11}}{(v_{22} + 1) - v_{21}}T(v + w - \delta^{22})\right)\right)\\
&\quad= \mathcal{D}^{\bar{v}}\left(\frac{(v_{21} - v_{22})(v_{21} - v_{11})}{v_{21} - (v_{22} + 1)}\right)T(\bar{w} - \delta^{21})\\
&\qquad+ ev(\bar{v})\left(\frac{(v_{21} - v_{22})(v_{21} - v_{11})}{v_{21} - (v_{22} + 1)}\right)\mathcal{D}T(\bar{w} - \delta^{21})\\
&\qquad+ \mathcal{D}^{\bar{v}}\left(\frac{(v_{21} - v_{22})((v_{22} + 1) - v_{11})}{(v_{22} + 1) - v_{21}}\right)T(\bar{v})\\
&\qquad+ ev(\bar{v})\left(\frac{(v_{21} - v_{22})((v_{22} + 1) - v_{11})}{(v_{22} + 1) - v_{21}}\right)\mathcal{D}T(\bar{v})\\
&\quad= ev(\bar{v})\left(\frac{v_{21} - v_{11}}{v_{21} - (v_{22} + 1)}\right)T(\bar{w} - \delta^{21}) + ev(\bar{v})\left(\frac{(v_{21} - v_{22})(v_{21} - v_{11})}{v_{21} - (v_{22} + 1)}\right)\\
&\qquad\times \mathcal{D}T(\bar{w} - \delta^{21}) + ev(\bar{v})\left(\frac{(v_{22} + 1) - v_{11}}{(v_{22} + 1) - v_{21}}\right)T(\bar{v})\\
&\quad= T(\bar{v}).
\end{aligned}$$

(ii) Using (25), $E_{32}(\mathcal{D}T(\bar{v} + w')) = \mathcal{D}^{\bar{v}}(E_{32}(T(v + w')))$ which equals to

$$\begin{aligned}
&\mathcal{D}^{\bar{v}}\left(\frac{v_{21} + 1 - v_{11}}{v_{21} + 1 - v_{22}}T(v + w' - \delta^{21})\right) + \mathcal{D}^{\bar{v}}\left(\frac{v_{22} - v_{11}}{v_{22} - (v_{21} + 1)}T(v + w' - \delta^{22})\right)\\
&\quad= \mathcal{D}^{\bar{v}}\left(\frac{v_{21} + 1 - v_{11}}{v_{21} + 1 - v_{22}}\right)T(\bar{v}) + ev(\bar{v})\left(\frac{v_{21} + 1 - v_{11}}{v_{21} + 1 - v_{22}}\right)\mathcal{D}T(\bar{v})
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{D}^{\bar{v}}\left(\frac{v_{22}-v_{11}}{v_{22}-(v_{21}+1)}\right) T(\bar{v}+\delta^{21}-\delta^{22}) \\
&+ ev(\bar{v})\left(\frac{v_{22}-v_{11}}{v_{22}-(v_{21}+1)}\right) \mathcal{D}T(\bar{v}+\delta^{21}-\delta^{22}) \\
={}& \mathcal{D}^{\bar{v}}\left(\frac{v_{21}+1-v_{11}}{v_{21}+1-v_{22}}\right) T(\bar{v}) + \mathcal{D}^{\bar{v}}\left(\frac{v_{22}-v_{11}}{v_{22}-(v_{21}+1)}\right) T(\bar{v}+\delta^{21}-\delta^{22}) \\
={}& \frac{1}{2}T(\bar{v}) - \frac{1}{2}T(\bar{v}+\delta^{21}-\delta^{22}).
\end{aligned}
$$

The computations made in the last example can easily be applied to the formulas (20)–(25). As a result we have the following set of formulas.

Action of the generators on regular tableaux:

$$
\begin{aligned}
E_{21}(T(\bar{w})) &= T(\bar{w}-\delta^{11}), \\
E_{12}(T(\bar{w})) &= -(x+m-z-k)(x+n-z-k)T(\bar{w}+\delta^{11}). \\
E_{32}(T(\bar{w})) &= \begin{cases} T(\bar{w}-\delta^{21}) + 2(x+m-z-k)\mathcal{D}T(\bar{w}-\delta^{21}) & \text{if } m=n, \\ \dfrac{x+m-z-k}{m-n}T(\bar{w}-\delta^{21}) - \dfrac{x+n-z-k}{m-n}T(\bar{w}-\delta^{22}) & \text{if } m\neq n, \end{cases} \\
E_{23}(T(\bar{w})) &= \begin{cases} \partial^{\bar{v}}_{v_{21}}\left(\displaystyle\prod_{i=1}^{3}(v_{3i}-v_{21}-m)\right) T(\bar{w}+\delta^{22}) & \\ \quad -2(a-x-m)(b-x-m)(c-x-m)\mathcal{D}T(\bar{w}+\delta^{22}) & \text{if } m=n, \\ \dfrac{(a-x-m)(b-x-m)(c-x-m)}{m-n}T(\bar{w}+\delta^{21}) & \\ \quad -\dfrac{(a-x-n)(b-x-n)(c-x-n)}{m-n}T(\bar{w}+\delta^{22}) & \text{if } m\neq n. \end{cases}
\end{aligned}
$$

Action of the generators on derivative tableaux (*recall that we assume* $m>n$):

$$
\begin{aligned}
E_{21}(\mathcal{D}T(\bar{w})) &= \mathcal{D}T(\bar{w}-\delta^{11}), \\
E_{12}(\mathcal{D}T(\bar{w})) &= -(x+m-z-k)(x+n-z-k)\mathcal{D}T(\bar{w}+\delta^{11}) \\
&\quad + \frac{m-n}{2}T(\bar{w}+\delta^{11}), \\
E_{32}(\mathcal{D}T(\bar{w})) &= \frac{x+m-z-k}{m-n}\mathcal{D}T(\bar{w}-\delta^{21}) - \frac{x+n-z-k}{m-n}\mathcal{D}T(\bar{w}-\delta^{22}) \\
&\quad + \frac{1}{2}\left(\frac{1}{m-n} - \frac{2(x+m-z-k)}{(m-n)^2}\right) T(\bar{w}-\delta^{21})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{1}{m-n}-\frac{2(x+n-z-k)}{(n-m)^2}\right)T(\bar{w}-\delta^{22}),\\
E_{23}(\mathcal{D}T(\bar{w})) &= -\frac{(a-x-m)(b-x-m)(c-x-m)}{n-m}\mathcal{D}T(\bar{w}+\delta^{21})\\
&-\frac{(a-x-n)(b-x-n)(c-x-n)}{m-n}\mathcal{D}T(\bar{w}+\delta^{22})\\
&-\partial^{\bar{v}}_{v_{22}}\left(\frac{\prod_{i=1}^{3}(v_{3i}-v_{22}-n)}{m-n}\right)T(\bar{w}+\delta^{22})\\
&-2\frac{(a-x-n)(b-x-n)(c-x-n)}{(m-n)^2}T(\bar{w}+\delta^{22})\\
&-\partial^{\bar{v}}_{v_{21}}\left(\frac{\prod_{i=1}^{3}(v_{3i}-v_{21}-m)}{n-m}\right)T(\bar{w}+\delta^{21})\\
&-2\frac{(a-x-m)(b-x-m)(c-x-m)}{(m-n)^2}T(\bar{w}+\delta^{21}).
\end{aligned}
$$

Lemma 7.9. *The action of* Γ *on* $V(T(\bar{v}))$ *is given by the formulas*:

$$c_{rs}(T(\bar{v}+w)) = \gamma_{rs}(\bar{v}+w)T(\bar{v}+w), \tag{26}$$

$$c_{rs}(\mathcal{D}T(\bar{v}+w)) = \gamma_{rs}(\bar{v}+w)\mathcal{D}T(\bar{v}+w)+\mathcal{D}^{\bar{v}}(\gamma_{rs}(v+w))T(\bar{v}+w). \tag{27}$$

Proof. The identities follow from Theorem 4.35. Indeed,

$$
\begin{aligned}
c_{rs}(T(\bar{v}+z)) &= \mathcal{D}^{\overline{v}}((v_{21}-v_{22})c_{rs}T(v+z))\\
&= \mathcal{D}^{\overline{v}}((v_{21}-v_{22})\gamma_{rs}(v+z)T(v+z))\\
&= \mathcal{D}^{\overline{v}}((v_{21}-v_{22})\gamma_{rs}(v+z))T(\bar{v}+z)\\
&\quad + ev(\bar{v})\left((v_{21}-v_{22})\gamma_{rs}(v+z)\right)\mathcal{D}T(\bar{v}+z)\\
&= \gamma_{rs}(\bar{v}+z)T(\bar{v}+z),\\
c_{rs}(\mathcal{D}T(\bar{v}+z)) &= \mathcal{D}^{\overline{v}}(c_{rs}(T(v+z)))\\
&= \mathcal{D}^{\overline{v}}(\gamma_{rs}(v+z)T(v+z))\\
&= \mathcal{D}^{\overline{v}}(\gamma_{rs}(v+z))T(\bar{v}+z)+ev(\bar{v})\left(\gamma_{rs}(v+z)\right)\mathcal{D}T(\bar{v}+z)\\
&= \mathcal{D}^{\overline{v}}(\gamma_{rs}(v+z))T(\bar{v}+z)+\gamma_{rs}(\bar{v}+z)\mathcal{D}T(\bar{v}+z).
\end{aligned}
$$
∎

7.3.3. *Submodules generated by singular tableaux*

In this section, we obtain an analogous to Theorem 4.25(i) for 1-singular tableaux.

Recall that $\mathcal{B}(T(\bar{v})) = \{\mathrm{Tab}(z)\,|\,z\in\mathbb{Z}^3\}$ is a basis of $V(T(\bar{v}))$.

Definition 7.10. Let $\bar{v}$ be a fixed critical vector, $\mathrm{Tab}(w) \in \mathcal{B}(T(\bar{v}))$, and $\bar{w} = \bar{v}+w$. Define

$$\Lambda^+(\mathrm{Tab}(w)) = \begin{cases} \Omega^+(\mathrm{Tab}(w)) & \text{if } w_{21} \leq w_{22}, \\ \Omega^+(\mathrm{Tab}(\tau(w))) & \text{if } w_{21} > w_{22}. \end{cases}$$

Lemma 7.11. *Assume that* $w_{21} \neq w_{22}$. *Then* $T(\bar{v}+w)$ *belongs to* $U \cdot \mathcal{D}T(\bar{v}+w)$.

Proof. The action of $c_{22} - \gamma_{22}(\bar{v}+w)$ on $\mathcal{D}T(\bar{v}+w)$ is given by the formula (27) and can be easily check that is a nonzero multiple of $(w_{21} - w_{22})T(\bar{v}+w)$. □

Lemma 7.12. *Suppose* $\mathrm{Tab}(w)$ *is a critical tableau. If* $\mathrm{Tab}(w')$ *is a derivative tableau such that* $\Lambda^+(\mathrm{Tab}(w)) \subseteq \Lambda^+(\mathrm{Tab}(w'))$, *then* $\mathrm{Tab}(w') \in U \cdot \mathrm{Tab}(w)$. *In particular, the simple subquotient* M *of* $V(T(\bar{v}))$ *containing* $\mathrm{Tab}(w)$ *satisfies* $\dim(M_{\chi_w}) = 2$.

Proof. The statement follows from Remark 7.7 and formulas (21)–(23). In fact, the numerators of the coefficients of the derivative tableaux appearing in the decomposition of $gT(\bar{v}+w)$ as linear combination of basis elements, are either zero (if g is a product of generators of the form E_{21}, E_{12}, or E_{ii}) or the same as the numerators of the coefficients that appear in the classical Gelfand–Tsetlin formulas. In the latter case $\mathrm{Tab}(w')$ is a derivative tableau, hence we cannot have zero coefficients. Therefore, we can use the same arguments as in the proof of Lemma 4.25(i). □

Definition 7.13. For any tableau $\mathrm{Tab}(w) \in \mathcal{B}(T(\bar{v}))$ define

$$\mathcal{A}(\mathrm{Tab}(w)) := \{\mathrm{Tab}(w') \in \mathcal{B}(T(\bar{v})) \,|\, \Lambda^+(\mathrm{Tab}(w)) \subseteq \Lambda^+(\mathrm{Tab}(w'))\}.$$

By $\mathcal{C}(w)$ we will denote the set of all critical tableaux in $\mathcal{A}(\mathrm{Tab}(w))$ and by $\mathcal{R}(w)$ we will denote the set of all regular tableaux in $\mathcal{A}(\mathrm{Tab}(w))$. Also, set

$$\mathcal{N}(\mathrm{Tab}(w)) = \begin{cases} \mathcal{R}(w) \cup \left(\displaystyle\bigcup_{w' \in \mathcal{C}(w)} \mathcal{A}(\mathrm{Tab}(w')) \right) & \text{if } w_{21} < w_{22}, \\ \mathcal{A}(\mathrm{Tab}(w)) & \text{if } w_{21} \geq w_{22}. \end{cases}$$

Lemma 7.14. *For any tableau* $\mathrm{Tab}(w)$ *we have* $\mathcal{N}(\mathrm{Tab}(w)) \subseteq U \cdot \mathrm{Tab}(w)$.

Proof. The statement follows from Remark 7.7, Lemmas 7.11, and 7.12. More precisely, as the Gelfand–Tsetlin formulas for singular tableaux have the same coefficients as in the classical formulas for tableaux of the same type, we can use the reasoning in [20, Proof of Theorem 6.8] and adapt it to the singular case. □

The following lemma together with Lemma 7.11 gives a sufficient condition in order to have modules with Gelfand–Tsetlin multiplicity 2.

Lemma 7.15. *Suppose that* $\mathrm{Tab}(w)$ *is a regular tableau such that* $\Omega^+(\mathrm{Tab}(w)) = \Omega^+(\mathrm{Tab}(w'))$ *for some critical tableau* $\mathrm{Tab}(w')$, *then* $\mathrm{Tab}(\tau(w)) \in U \cdot \mathrm{Tab}(w)$.

Proof. The statement follows directly from Lemma 7.12. □

Corollary 7.16. *Let* $\mathrm{Tab}(w)$ *be a regular tableau associated to a Gelfand–Tsetlin character* χ. *If* $\{\mathrm{Tab}(w') \mid \Omega^+(\mathrm{Tab}(w')) = \Omega^+(\mathrm{Tab}(w))\}$ *does not contain critical tableaux, then any simple subquotient* N *of* $V(T(\bar{v}))$ *satisfies* $\dim(N_\chi) \leq 1$.

Proof. Since $\mathrm{Tab}(w)$ is regular, $w_{21} < w_{22}$. Then $\mathrm{Tab}(\tau(w))$ is a derivative tableau such that $\mathrm{Tab}(w) \in U \cdot \mathrm{Tab}(\tau(w))$ (see Lemma 7.11). Therefore, it is enough to prove that $\mathrm{Tab}(\tau(w)) \notin U \cdot \mathrm{Tab}(w)$. However, this follows from Theorem 4.35 and the fact that we cannot obtain critical tableaux from $\mathrm{Tab}(w)$ with the same $\Omega^+(\mathrm{Tab}(w))$, in particular we cannot obtain derivative tableaux $\mathrm{Tab}(w')$ such that $\Omega^+(\mathrm{Tab}(w')) = \Omega^+(\mathrm{Tab}(w))$. □

Remark 7.17. By definition of $\mathcal{N}(\mathrm{Tab}(w))$, any $\mathrm{Tab}(w')$ in $\mathcal{N}(\mathrm{Tab}(w))$ satisfies the relation $|\Omega^+(\mathrm{Tab}(w))| \leq |\Omega^+(\mathrm{Tab}(w'))|$. However, it is possible to have $\mathrm{Tab}(w') \in U \cdot \mathrm{Tab}(w)$ with $|\Omega^+(\mathrm{Tab}(w'))| = |\Omega^+(\mathrm{Tab}(w))| - 1$. For instance, consider $\bar{v} = (a, b, c, x, x, x)$ such that $\{a-x, b-x, c-x\} \cap \mathbb{Z} = \emptyset$ and $w = (0, 0, 0)$, then $|\Omega^+(\mathrm{Tab}(w))| = 2$ while $E_{32}\,\mathrm{Tab}(w) = \mathrm{Tab}(w - \delta^{21})$ and $|\Omega^+(\mathrm{Tab}(w - \delta^{21}))| = 1$.

Let us write $\mathrm{Tab}(w') \prec_g \mathrm{Tab}(w)$ if $\mathrm{Tab}(w')$ appears with non-zero coefficient in the decomposition of $g \cdot \mathrm{Tab}(w)$ for some generator $g \in \mathfrak{gl}(n)$.

Lemma 7.18 ([33, Lemma 7.4]). *Suppose that* $\mathrm{Tab}(w') \prec_g \mathrm{Tab}(w)$ *with* $g \in \mathfrak{gl}(n)$ *of the form* $E_{k,k+1}$ *or* $E_{k+1,k}$, *then* $|\Omega^+(\mathrm{Tab}(w'))| \geq |\Omega^+(\mathrm{Tab}(w))| - 1$. *Moreover, the complete list of Gelfand–Tsetlin tableaux* $\mathrm{Tab}(w)$ *and* $\mathrm{Tab}(w')$ *such that* $\mathrm{Tab}(w') \prec_g \mathrm{Tab}(w)$, *and* $|\Omega^+(\mathrm{Tab}(w'))| = |\Omega^+(\mathrm{Tab}(w))| - 1$ *is as follows*:

(i)
$$\mathrm{Tab}(w) = \begin{array}{ccccc} a & & b & & c \\ & x & & x-t & \\ & & x & & \end{array} \qquad \mathrm{Tab}(w') = \begin{array}{ccccc} a & & b & & c \\ & x-t & & x & \\ & & x+1 & & \end{array}$$

for $t \in \mathbb{Z}_{>0}$.

(ii)
$$\mathrm{Tab}(w) = \begin{array}{ccccc} a & & b & & c \\ & x & & x-t & \\ & & x & & \end{array} \qquad \mathrm{Tab}(w') = \begin{array}{ccccc} a & & b & & c \\ & x-t & & x-1 & \\ & & x & & \end{array}$$

for $t \in \mathbb{Z}_{>0}$.

(iii)
$$\mathrm{Tab}(w) = \begin{array}{c} \boxed{a}\,\boxed{b}\,\boxed{c} \\ \boxed{x}\,\boxed{x} \\ \boxed{x} \end{array} \qquad \mathrm{Tab}(w') = \begin{array}{c} \boxed{a}\,\boxed{b}\,\boxed{c} \\ \boxed{x-1}\,\boxed{x} \\ \boxed{x} \end{array}.$$

(iv)
$$\mathrm{Tab}(w) = \begin{array}{c} \boxed{x}\,\boxed{b}\,\boxed{c} \\ \boxed{x}\,\boxed{x-t} \\ \boxed{z} \end{array} \qquad \mathrm{Tab}(w') = \begin{array}{c} \boxed{x}\,\boxed{b}\,\boxed{c} \\ \boxed{x-t}\,\boxed{x+1} \\ \boxed{z} \end{array}$$

for $t \in \mathbb{Z}_{>0}$, $b \neq x$, *and* $c \neq x$.

(v)
$$\mathrm{Tab}(w) = \begin{array}{c} \boxed{x}\,\boxed{b}\,\boxed{c} \\ \boxed{x}\,\boxed{x} \\ \boxed{z} \end{array} \qquad \mathrm{Tab}(w') = \begin{array}{c} \boxed{x}\,\boxed{b}\,\boxed{c} \\ \boxed{x}\,\boxed{x+1} \\ \boxed{z} \end{array}$$

for $b \neq x$ *and* $c \neq x$.

Remark 7.19. For the tableaux in Lemma 7.18(iv) and (v), one may consider x at positions $3i$, $i = 1, 2, 3$, obtaining the same property for Ω^+.

Definition 7.20. We will say that a tableau $\mathrm{Tab}(w)$ is *of type* (I) if it can be written in the form of one of the tableaux $\mathrm{Tab}(w)$ of parts (i), (ii), or (iii) of Lemma 7.18 for some x, a, b, c, t, z. We also say that the tableau is *of type* $(\mathrm{II})_i$ if can be written in the form of one of the tableaux $\mathrm{Tab}(w)$ of parts (iv) or (v) of Lemma 7.18 for some x, b, c, t, z, and x appear in the top row in position $3i$.

Remark 7.21. With the notation of Lemma 7.18 for tableaux of type (I) we have $\Omega^+(\mathrm{Tab}(w')) = \Omega^+(\mathrm{Tab}(w))\backslash\{(2,1,1)\}$ and for tableaux of type $(\mathrm{II})_i$ we have $\Omega^+(\mathrm{Tab}(w')) = \Omega^+(\mathrm{Tab}(w))\backslash\{(3,i,2)\}$.

Definition 7.22. Let $\mathrm{Tab}(w) \in \mathcal{B}(T(\bar{v}))$ and $(r,s,t) \in \Omega^+(\mathrm{Tab}(w))$. Set

$$\mathcal{N}_{(r,s,p)}(\mathrm{Tab}(w)) := \mathcal{N}(\mathrm{Tab}(w)) \cup \mathcal{N}(\mathrm{Tab}(w')),$$

where $\mathrm{Tab}(w')$ is any tableau such that $\Omega^+(\mathrm{Tab}(w))\backslash\{(r,s,p)\} = \Omega^+(\mathrm{Tab}(w'))$. Also, define

$$\mathcal{N}^{(1)}(\mathrm{Tab}(w)) := \begin{cases} \mathcal{N}_{(2,1,1)}(\mathrm{Tab}(w)) & \text{if } \mathrm{Tab}(w) \text{ is of type (I)}, \\ \mathcal{N}(\mathrm{Tab}(w)) & \text{otherwise}, \end{cases} \tag{28}$$

$$\mathcal{N}^{(2)}(\mathrm{Tab}(w)) := \begin{cases} \mathcal{N}_{(3,i,2)}(\mathrm{Tab}(w)) & \text{if } \mathrm{Tab}(w)\text{is of type } (\mathrm{II})_i, \\ \mathcal{N}(\mathrm{Tab}(w)) & \text{otherwise}, \end{cases} \tag{29}$$

$$\widehat{\mathcal{A}}(\mathrm{Tab}(w)) := \mathcal{N}^{(1)}(\mathrm{Tab}(w)) \cup \mathcal{N}^{(2)}(\mathrm{Tab}(w)). \tag{30}$$

Lemma 7.23. *Let* $\mathrm{Tab}(w)$ *be any Gelfand–Tsetlin tableau and* $\mathrm{Tab}(w') \in \mathcal{N}(\mathrm{Tab}(w))$. *We have* $\widehat{\mathcal{A}}(\mathrm{Tab}(w')) \subseteq U \cdot \mathrm{Tab}(w)$.

Proof. As $\mathrm{Tab}(w') \in \mathcal{N}(\mathrm{Tab}(w))$, by Lemmas 7.14, 7.18, and Remark 7.21 we have $\mathcal{N}^{(i)}(\mathrm{Tab}(w')) \subseteq U \cdot \mathrm{Tab}(w)$ for $i = 1, 2$. □

The following theorem summarize the results of this section.

Theorem 7.24. *Let* $\mathrm{Tab}(w) \in \mathcal{B}(T(\bar{v}))$. *The submodule* $U \cdot \mathrm{Tab}(w)$ *has the following basis of tableaux*:

$$\widehat{\mathcal{N}}(\mathrm{Tab}(w)) := \bigcup_{\mathrm{Tab}(w') \in \mathcal{N}(\mathrm{Tab}(w))} \widehat{\mathcal{A}}(\mathrm{Tab}(w')).$$

Proof. The statement follows from Lemmas 7.18 and 7.23. □

Definition 7.25. Let M be a Gelfand–Tsetlin module with basis $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$ for some 1-singular vector $\bar{v}$. We say that $\mathrm{Tab}(w) \in \mathcal{B}_M$ is Ω^+*-maximal in* M if $|\Omega^+(\mathrm{Tab}(w))|$ is maximal for all $\mathrm{Tab}(w)$ in $\mathcal{B}_M$. Also, denote by $U \cdot_M \mathrm{Tab}(w)$ the submodule of M generated by $\mathrm{Tab}(w)$.

The next two corollaries follow from Theorem 7.24 and will be useful when describing the simple subquotients of $V(T(\bar{v}))$.

Corollary 7.26. *Let* M *be a Gelfand–Tsetlin module with basis* $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$ *for some* 1*-singular vector* $\bar{v}$. *If* $\mathrm{Tab}(w) \in \mathcal{B}_M$ *is a regular tableau that is* Ω^+*-maximal in* M, *then* $U \cdot_M \mathrm{Tab}(w)$ *is a simple submodule of* M.

Proof. It is enough to proof that $\mathrm{Tab}(w)$ belongs to $U \cdot_M \mathrm{Tab}(w')$ for any $\mathrm{Tab}(w')$ in $U \cdot_M \mathrm{Tab}(w)$. As $\mathrm{Tab}(w')$ in $U \cdot_M \mathrm{Tab}(w)$ and $\mathrm{Tab}(w)$ is a regular tableau, we have $\Omega^+(\mathrm{Tab}(w)) \subseteq \Omega^+(\mathrm{Tab}(w')) \cup \{(r, s, t)\}$ for some (r, s, t). As $\mathrm{Tab}(w)$ is Ω^+-maximal, we should have $\Omega^+(\mathrm{Tab}(w)) = \Omega^+(\mathrm{Tab}(w')) \cup \{(r, s, t)\}$ for some (r, s, t). Therefore, $U \cdot_M \mathrm{Tab}(w) \subseteq U \cdot_M \mathrm{Tab}(w')$ and, then we have $\mathrm{Tab}(w) \in U \cdot_M \mathrm{Tab}(w')$. □

Corollary 7.27. *Let* M *be a Gelfand–Tsetlin module with basis* $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$ *for some* 1*-singular vector* $\bar{v}$. *If* $\{\mathrm{Tab}(w) \in \mathcal{B}_M \mid \mathrm{Tab}(w)$ *is* Ω^+*-maximal*$\}$ *does not contain regular tableaux, then for any* Ω^+*-maximal tableau* $\mathrm{Tab}(w)$ *the submodule* $U \cdot_M \mathrm{Tab}(w)$ *is a simple submodule of* M.

Proof. The proof is analogous to the proof of Corollary 7.26. □

In order to describe the basis of the simple subquotients of $V(T(\bar{v}))$ we modify Definition 7.3 to singular vectors.

Definition 7.28. Let $\bar{v}$ be any 1-singular vector and $\mathcal{B}$ be a subset of $\mathbb{Z}^3$. By $\mathrm{Tab}(\mathcal{B})$ we will denote the set of tableaux $\{\mathrm{Tab}(m,n,k) \,|\, (m,n,k) \in \mathcal{B}\} \subseteq \mathcal{B}(T(\bar{v}))$. Assume that M is a Gelfand–Tsetlin module with basis $\mathrm{Tab}(\mathcal{B})$. Then we will denote M by $M(\mathcal{B}, \bar{v})$, or simply by $M(\mathcal{B})$ if $\bar{v}$ is fixed. If $M(\mathcal{B})$ is simple, we will write $L(\mathcal{B})$ for $M(\mathcal{B})$.

Example 7.29. Let $\bar{v} = (a,b,c|a,a|z)$. Below we give a basis for the submodule of $V(T(\bar{v}))$ generated by $\mathrm{Tab}(0,0,0)$. In this case $\mathcal{B}(T(\bar{v}))$ does not contain tableaux of type (I), $(\mathrm{II})_2$, or $(\mathrm{II})_3$. However, the set of all tableaux of type $(\mathrm{II})_1$ is $\{\mathrm{Tab}(0,n,k) \,|\, n \in \mathbb{Z}_{\leq 0}\}$. By definition we have

$$\begin{aligned}
\widehat{\mathcal{N}}(\mathrm{Tab}(0,0,0)) &= \widehat{\mathcal{A}}(\mathrm{Tab}(0,0,0)) = \mathcal{A}(\mathrm{Tab}(0,0,0)) \\
&= \{\mathrm{Tab}(m,n,k) \,|\, m \leq 0,\ n \leq 0\}, \\
\widehat{\mathcal{N}}(\mathrm{Tab}(0,-1,0)) &= \widehat{\mathcal{A}}(\mathrm{Tab}(0,-1,0)) = \mathcal{N}^{(2)}(\mathrm{Tab}(0,-1,0)) \\
&= \{\mathrm{Tab}(m,n,k) \,|\, m \leq 0\}.
\end{aligned}$$

Therefore, by Theorem 7.24, the submodule of $V(T(\bar{v}))$ generated by $\mathrm{Tab}(0,0,0)$ has basis:

$$\bigcup_{\mathcal{N}(\mathrm{Tab}(0,0,0))} \widehat{\mathcal{A}}(\mathrm{Tab}(w')) = \widehat{\mathcal{A}}(\mathrm{Tab}(0,0,0)) \cup \widehat{\mathcal{A}}(\mathrm{Tab}(0,-1,0)) = \mathrm{Tab}(m \leq 0).$$

7.4. *The singular block containing* $L(-\rho)$

In this section, we describe all simple subquotients of the module $V(T(\bar{v}))$ with $\bar{v} = (a,a,a|a,a|a)$.

Next, we give an algorithm, which based on Theorem 7.24 and Corollaries 7.26 and 7.27, provides an explicit basis of all simple subquotients of a module M with basis $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$.

Step 1. If there is an Ω^+-maximal regular tableau in $\mathcal{B}_M$, choose any such tableau $\mathrm{Tab}(w)$. By Corollary 7.26, $U \cdot_M \mathrm{Tab}(w)$ is a simple submodule of M.
Step 2. If there are no Ω^+-maximal regular tableaux in $\mathcal{B}_M$, consider any Ω^+-maximal (derivative) tableau $\mathrm{Tab}(w)$. By Corollary 7.27, the module $U \cdot_M \mathrm{Tab}(w)$ will be a simple submodule of M.
Step 3. Using the bases of M and $U \cdot \mathrm{Tab}(w)$ (see Theorem 7.24), we find a basis of $M/(U \cdot_M \mathrm{Tab}(w))$.
Step 4. Start over the procedure with the module $M' := M/(U \cdot_M \mathrm{Tab}(w))$.

Example 7.30. Set $\bar{v} = (a,a,a|a,a|a)$. Below we define explicit bases of all simple subquotients of $V(T(\bar{v}))$.

Note that none of the tableaux in $\mathcal{B}(T(\bar{v}))$ can be of type $(\mathrm{II})_i$, $i = 1, 2, 3$. Therefore, $\mathcal{N}^{(2)}(\mathrm{Tab}(w)) = \mathcal{N}(\mathrm{Tab}(w))$ for any $\mathrm{Tab}(w) \in \mathcal{B}(T(\bar{v}))$. Moreover, the set of all tableaux of type (I) is $\{\mathrm{Tab}(m, n, k) \mid n \leq m = k\}$. Now, we apply Steps 1–4 described above to the module $V(T(\bar{V}))$.

(1) The tableau $\mathrm{Tab}(0, 0, 0)$ is Ω^+-maximal on $V(T(\bar{v}))$. By Corollary 7.26, $U \cdot \mathrm{Tab}(0, 0, 0)$ is a simple submodule of $V(T(\bar{v}))$ and by Theorem 7.24, the submodule $U \cdot \mathrm{Tab}(0, 0, 0)$ has a basis

$$\begin{aligned}\widehat{\mathcal{N}}(\mathrm{Tab}(0,0,0)) &= \widehat{\mathcal{A}}(\mathrm{Tab}(0,0,0)) \\ &= \mathcal{N}^{(1)}(\mathrm{Tab}(0,0,0)) \\ &= \mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{pmatrix}.\end{aligned}$$

Denote this module by L_1, and $M_1 = V(T(\bar{v}))/L_1$.

(2) Now, the derivative tableau $\mathrm{Tab}(0, -2, -1)$ is Ω^+-maximal in M_1. By Theorem 7.24, $U \cdot \mathrm{Tab}(0, -2, -1)$ has a basis

$$\begin{aligned}\widehat{\mathcal{N}}(\mathrm{Tab}(0,-2,-1)) &= \widehat{\mathcal{A}}(\mathrm{Tab}(0,0,1)) \\ &= \mathcal{N}^{(1)}(\mathrm{Tab}(0,-2,-1)) \\ &= \mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \end{pmatrix}.\end{aligned}$$

Moreover, by Corollary 7.27, $U \cdot_{M_1} \mathrm{Tab}(0, -2, -1)$ is a simple submodule of M_1 and has a basis

$$\mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \end{pmatrix} \Big\backslash \mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{pmatrix}$$

$$= \mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ n < k \end{pmatrix}.$$

Denote by L_3 this module and $M_2 = M_1/L_3$.

(3) The tableau $\mathrm{Tab}(0, 1, 0)$ is Ω^+-maximal in M_2 and $U \cdot \mathrm{Tab}(0, 1, 0)$ has a basis $\widehat{\mathcal{A}}(\mathrm{Tab}(0, 1, 0)) \cup \widehat{\mathcal{A}}(\mathrm{Tab}(0, 0, 0))$ which is equal to

$$\mathrm{Tab}\begin{pmatrix} m \leq n \\ m \leq 0 \\ k \leq m \\ k \leq n \end{pmatrix} \bigcup \mathrm{Tab}\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{pmatrix}.$$

Therefore, $U \cdot_{M_2} \mathrm{Tab}(0,1,0)$ has basis

$$\left(\mathrm{Tab}\begin{pmatrix} m \le n \\ m \le 0 \\ k \le m \\ k \le n \end{pmatrix} \bigcup \mathrm{Tab}\begin{pmatrix} m \le 0 \\ n \le 0 \\ k \le n \end{pmatrix}\right) \Big\backslash \left(\mathrm{Tab}\begin{pmatrix} m \le 0 \\ n \le 0 \\ k \le n \end{pmatrix} \bigcup \mathrm{Tab}\begin{pmatrix} m \le 0 \\ n \le 0 \\ n < k \end{pmatrix}\right)$$
$$= \mathrm{Tab}(k \le m \le 0 < n),$$

call this module L_2 and $M_3 = M_2/L_2$.

(4) There are not Ω^+-maximal regular tableaux in M_3, so we choose the derivative tableau $\mathrm{Tab}(1,0,0)$ which is Ω^+-maximal in M_3. By Corollary 7.27, the module $U \cdot_{M_3} \mathrm{Tab}(1,0,0)$ is a simple submodule of M_3 with basis

$$\mathrm{Tab}(k \le n \le 0 < m)$$

call this module L_5 and $M_4 = M_3/L_5$.

(5) The tableau $\mathrm{Tab}(0,1,1)$ is Ω^+-maximal in M_4 and $U \cdot_{M_4} \mathrm{Tab}(0,1,1)$ is a simple submodule of M_4 with basis

$$\mathrm{Tab}\begin{pmatrix} m \le 0 < n \\ m < k \le n \end{pmatrix}$$

call this module L_4 and $M_5 = M_4/L_4$.

(6) The derivative tableau $\mathrm{Tab}(1,0,1)$ is Ω^+-maximal in M_5. Therefore, by Corollary 7.27, $U \cdot_{M_5} \mathrm{Tab}(1,0,1)$ a simple submodule of M_5 with basis

$$\mathrm{Tab}\begin{pmatrix} n \le 0 < m \\ n < k \le m \end{pmatrix}$$

call this module L_7 and $M_6 = M_5/L_7$.

(7) The tableau $\mathrm{Tab}(0,1,2)$ is Ω^+-maximal in M_6 so, $U \cdot_{M_6} \mathrm{Tab}(0,1,2)$ is a simple submodule of M_6 and has a basis

$$\mathrm{Tab}(m \le 0 < n < k)$$

call this module L_6 and $M_7 = M_6/L_6$.

(8) The tableau $\mathrm{Tab}(1,0,2)$ is Ω^+-maximal in M_7 and $U \cdot_{M_7} \mathrm{Tab}(1,0,2)$ is a simple submodule of M_7 with a basis

$$\mathrm{Tab}(n \le 0 < m < k)$$

call this module L_9 and $M_8 = M_7/L_9$.

(9) The tableau $\mathrm{Tab}(1,1,0)$ is Ω^+-maximal in M_8. The module $U\cdot_{M_8}\mathrm{Tab}(1,2,2)$ is a simple submodule of M_8 with a basis

$$\mathrm{Tab}\begin{pmatrix} 0<m \\ 0<n \\ k\leq n \end{pmatrix}$$

call this module L_8 and $M_9 = M_8/L_8$.

(10) The tableau $\mathrm{Tab}(1,1,2)$ is Ω^+-maximal in M_9 and $U\cdot\mathrm{Tab}(1,1,2)$ has a basis $\mathrm{Tab}(\mathbb{Z}^3)$. Therefore, $U\cdot_{M_9}\mathrm{Tab}(1,1,2)$ is has a basis

$$\mathrm{Tab}\begin{pmatrix} 0<m \\ 0<n \\ n<k \end{pmatrix}$$

call this module L_{10}.

Remark 7.31. The reasoning in Example 7.30 can be applied also when finding the Loewy series decomposition of $V(T(\bar v))$. More precisely, the Loewy series decomposition of $V(T(\bar v))$ for $\bar v = (a,a,a|a,a|a)$ is

$$L_1,\quad L_2\oplus L_3,\quad L_4,\quad L_5\oplus L_6,\quad L_7,\quad L_8\oplus L_9,\quad L_{10}.$$

7.5. *Realizations of all simple singular Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules*

In this section, we will describe all simple objects in every block $\mathcal{GT}_{T(v)}$ defined by a singular Gelfand–Tsetlin character χ_v (see Definition 4.12 and Remark 4.2). Such description will include an explicit tableaux basis of each simple subquotient M in $\mathcal{GT}_{T(v)}$ and the weight support of M. For the weight support we will use Proposition 7.32 and the explicit basis to give a description of the weight multiplicities, when the multiplicities are finite, a picture of the weight lattice is provided. We also present the components of the Loewy series of the universal module $V(T(v))$. A rigorous proof based on Theorem 7.24, and Corollaries 7.26 and 7.27 is given for Case ($C13$), see Sec. 7.4, Example 7.30. For all other cases the reasoning is the same.

The simple subquotients will be defined by their corresponding subsets of $\mathbb{Z}^3$, equivalently, by their bases in $\mathcal{B}(T(\bar v))$. We should note that all subsets of $\mathbb{Z}^3$ that define a simple subquotient are defined by a set of inequalities of the form $a\leq b$ or $a<b$ where a,b are elements in the set $\{m,n,k,0,-t,-s\}$.

As we did for the description of generic blocks we will characterize the weight spaces for subquotients of the singular module $V(T(\bar v))$.

Proposition 7.32. *Let M be a singular Gelfand–Tsetlin module with basis of tableaux $\mathcal{B}_M\subseteq\mathcal{B}(T(\bar v))$. If $\mathrm{Tab}(z)\in\mathcal{B}_M$ is a tableau of weight λ, then the weight space M_λ is spanned by the set of tableaux $\{\mathrm{Tab}(z+(i,-i,0))\,|\,i\in\mathbb{Z}\}\cap\mathcal{B}_M$.*

Proof. The action of the generators of $\mathfrak{h}$ in M is given by the same expressions as in the case of generic modules, therefore, we can use the same argument of the proof of Proposition 7.2. $\square$

We now describe the sets $\mathcal{B} \subseteq \mathbb{Z}^3$ that define all simple subquotients of $V(T(\bar{v}))$. For convenience, the modules listed in one row are isomorphic. Recall that τ denote the transposition $(\mathrm{id}, (1,2), \mathrm{id}) \in S_3 \times S_2 \times S_1$. It is worth noting that all isomorphisms between simple subquotients of $V(T(\bar{v}))$ are τ-induced, that is all isomorphisms between simple subquotients are given by $L(\mathcal{B}) \mapsto L(\tau(\mathcal{B}))$.

Remark 7.33. In general it is not true that if $\mathcal{B} \subseteq \mathbb{Z}^3$ defines a subquotient of $V(T(\bar{v}))$ then $\tau(\mathcal{B})$ defines also a subquotient of $V(T(\bar{v}))$.

Remark 7.34. We should note that for singular $\mathfrak{sl}(3)$-modules we may have characters with unique simple extension or with two non-isomorphic simple extensions. In particular, the number of simple subquotients of the singular blocks in general will not coincide with the number of non-isomorphic modules in the block.

Until the end of this section we use the following convention. The entries of the Gelfand–Tsetlin tableaux we will use will be integer shifts of some of the complex numbers a, b, c, x, z. We also assume that if any two of a, b, c, x, z appear in the same row or in consecutive rows of a given tableau, then their difference is not integer. For convenience, in the description of basis of simple subquotients, isomorphic modules are listed in the same row.

($C1$) Consider the Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{x} & & \boxed{x} & \\ & & \boxed{z} & & \end{array}$$

In this case the module $V(T(\bar{v}))$ is simple and all its weight spaces are infinite-dimensional:

Module	Basis
L_1	$L(\mathbb{Z}^3)$

($C2$) Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{x} & & \boxed{x} & \\ & & \boxed{x} & & \end{array}$$

I. Number of simples in the block: 2

II. Simple subquotients.

In this case, the module $V(T(\bar{v}))$ has two simple subquotients and they have infinite-dimensional weight multiplicities:

Module	Basis
L_1	$L(k \leq n)$
L_2	$L(n < k)$

III. Loewy series.

$$L_1, \quad L_2.$$

$(C3)$ Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{z} & & \end{array}$$

I. Number of simples in the block: 2

II. Simple subquotients.

We have two simple subquotients and they have infinite-dimensional weight spaces:

Module	Basis
L_1	$L(m \leq 0)$
L_2	$L(0 < m)$

III. Loewy series.

$$L_1, \quad L_2.$$

$(C4)$ Consider the Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{b} & & \boxed{c} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 5

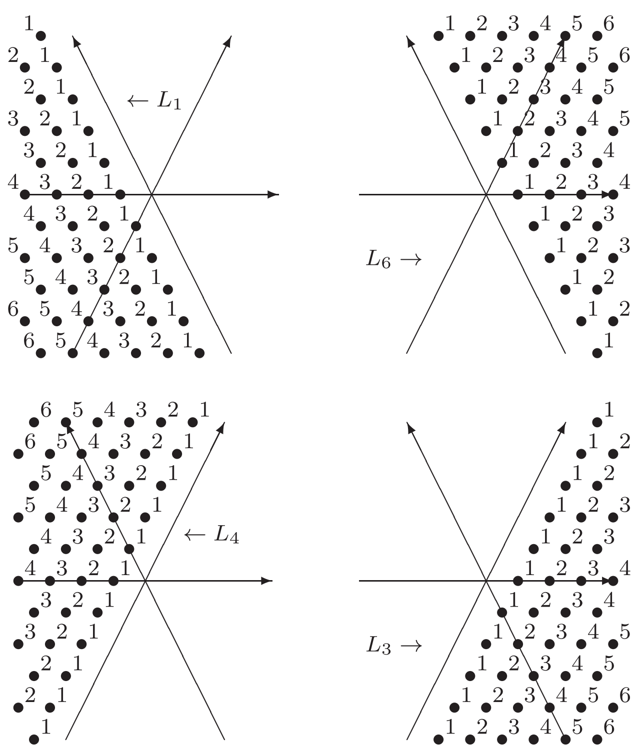
II. Simple subquotients.

In this case, we have six simple subquotients. The origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau

$T(\bar{v} + \delta^{11} + \delta^{21})$.

(i) Modules with unbounded finite weight multiplicities:

Module	Basis
L_1	$L\left(\left\{\begin{matrix} m \leq 0 \\ 0 < n \\ k \leq m \end{matrix}\right\} \bigcup \left\{\begin{matrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{matrix}\right\}\right)$
L_3	$L\begin{pmatrix} m \leq 0 \\ n < k \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < m \\ k \leq n \end{pmatrix}$
L_6	$L\left(\left\{\begin{matrix} 0 < m \\ 0 < n \\ n < k \end{matrix}\right\} \bigcup \left\{\begin{matrix} 0 < m \\ n \leq 0 \\ m < k \end{matrix}\right\}\right)$



(ii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq 0 < n \\ m < k \leq n \end{pmatrix}$	L_5	$L\begin{pmatrix} n \leq 0 < m \\ n < k \leq m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2, \quad L_3 \oplus L_4, \quad L_5, \quad L_6.$$

$(C5)$ For any $t \in \mathbb{Z}_{>0}$, consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a-t} & & \boxed{c} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 11

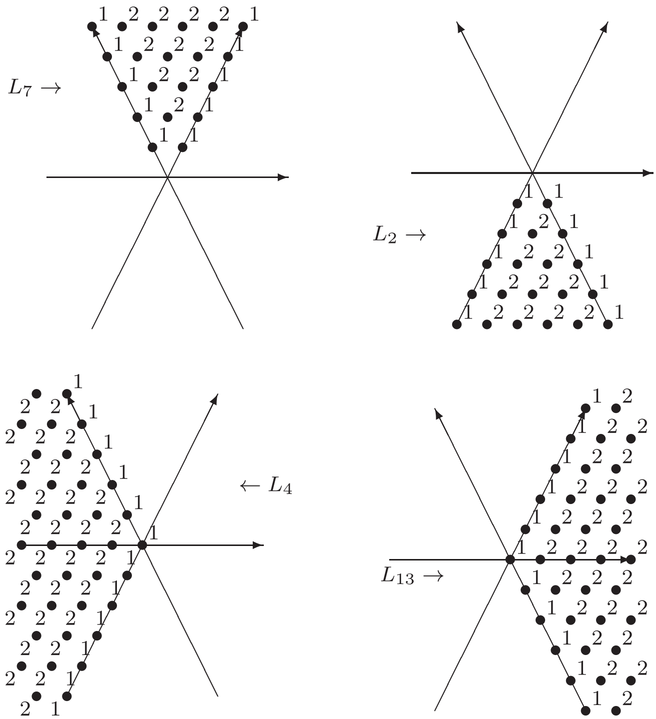
II. Simple subquotients.

We have 16 simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} - \delta^{21})$. The pictures correspond to the case $t = 2$.

(i) Six modules with weight multiplicities bounded by t, two pairs of isomorphic modules and two more modules:

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq -t < n \\ n \leq 0 \\ m < k \leq n \end{pmatrix}$	L_8	$L\begin{pmatrix} n \leq -t < m \\ m \leq 0 \\ n < k \leq m \end{pmatrix}$
L_7	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{15}	$L\begin{pmatrix} -t < n \leq 0 \\ 0 < m \\ n < k \leq m \end{pmatrix}$

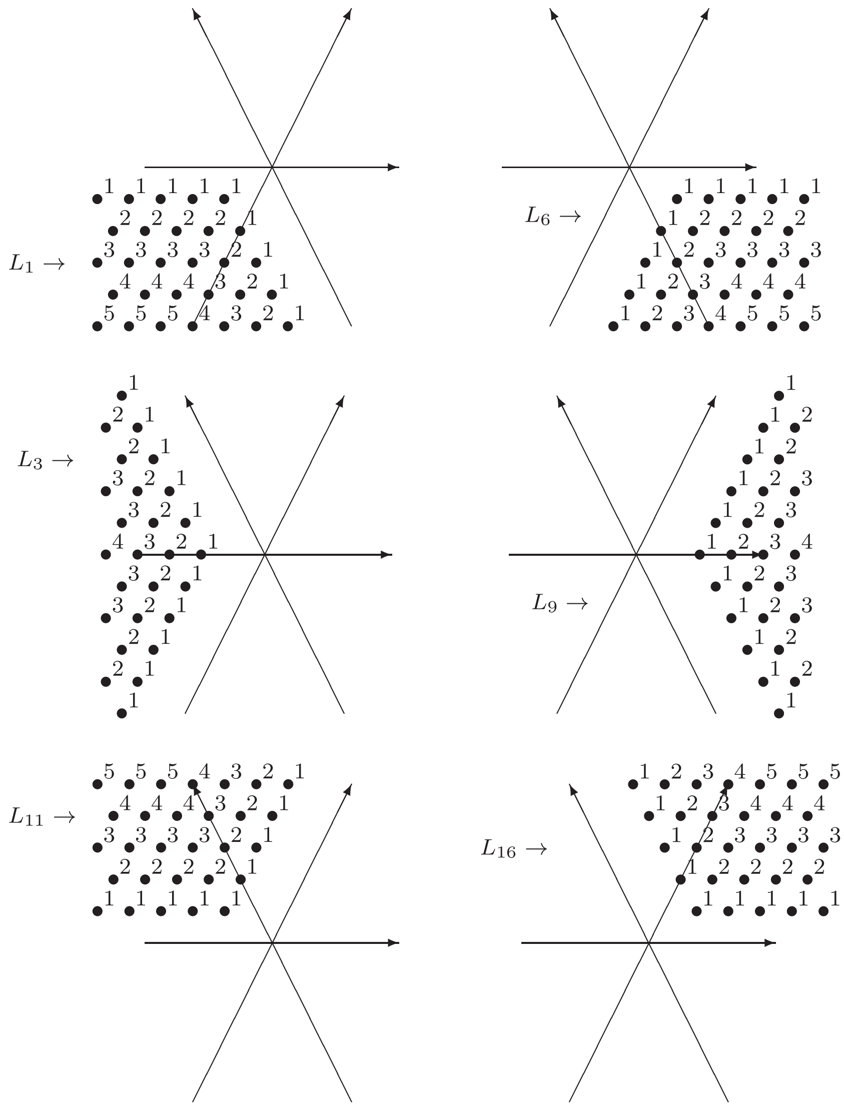
Module	Basis
L_4	$L\left(\left\{\begin{matrix} -t < m \leq 0 \\ 0 < n \\ k \leq m \end{matrix}\right\} \bigcup \left\{\begin{matrix} -t < m \leq 0 \\ n \leq 0 \\ k \leq n \end{matrix}\right\}\right)$
L_{13}	$L\left(\left\{\begin{matrix} m \leq n \\ -t < m \leq 0 \\ n < k \end{matrix}\right\} \bigcup \left\{\begin{matrix} -t < m \leq 0 \\ n \leq -t \\ m < k \end{matrix}\right\} \bigcup \left\{\begin{matrix} m \leq 0 \\ -t < n \\ n < k \end{matrix}\right\}\right)$



(ii) Modules with unbounded weight multiplicities:

Module	Basis	Module	Basis
L_3	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ k \leq m \end{pmatrix}$	L_{10}	$L\begin{pmatrix} n \leq -t \\ 0 < m \\ k \leq n \end{pmatrix}$
L_9	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ n < k \end{pmatrix}$	L_{14}	$L\begin{pmatrix} n \leq -t \\ 0 < m \\ m < k \end{pmatrix}$

Module	Basis
L_1	$L\left(\begin{Bmatrix} m \leq -t < n \leq 0 \\ k \leq m \end{Bmatrix} \bigcup \begin{Bmatrix} m \leq -t \\ n \leq -t \\ k \leq n \end{Bmatrix}\right)$
L_6	$L\left(\begin{Bmatrix} m \leq -t < n \leq 0 \\ n < k \end{Bmatrix} \bigcup \begin{Bmatrix} m \leq -t \\ n \leq -t \\ n < k \end{Bmatrix}\right)$
L_{11}	$L\left(\begin{Bmatrix} m \leq n \\ 0 < m \\ k \leq n \end{Bmatrix} \bigcup \begin{Bmatrix} 0 < m \\ -t < n \leq 0 \\ k \leq n \end{Bmatrix}\right)$
L_{16}	$L\left(\begin{Bmatrix} m \leq n \\ 0 < m \\ n < k \end{Bmatrix} \bigcup \begin{Bmatrix} 0 < m \\ -t < n \leq 0 \\ m < k \end{Bmatrix} \bigcup \begin{Bmatrix} n < m \\ 0 < n \\ n < k \end{Bmatrix}\right)$



(iii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_5	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{12}	$L\begin{pmatrix} n \leq -t \\ 0 < m \\ n < k \leq m \end{pmatrix}$

III. Loewy series.
$L_1,\ L_2 \oplus L_3,\ L_4 \oplus L_5 \oplus L_6,\ L_7 \oplus L_8 \oplus L_9 \oplus L_{10},\ L_{11} \oplus L_{12} \oplus L_{13},$
$L_{14} \oplus L_{15},\ L_{16}.$

$(C6)$ Set $t \in \mathbb{Z}_{>0}$ and consider the following Gelfand–Tsetlin tableau:

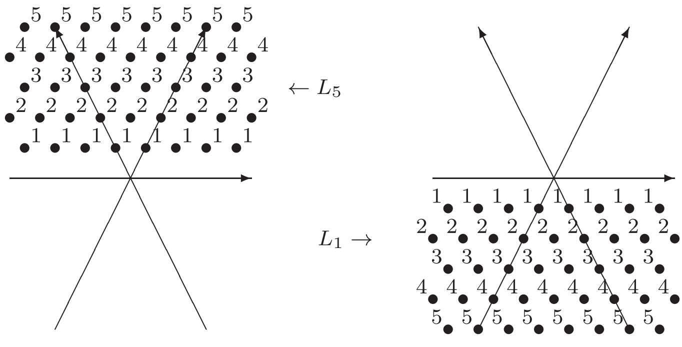
$$T(\bar{v}) = \begin{array}{|c|c|c|} \hline a & a-t & c \\ \hline \end{array}$$
$$\begin{array}{|c|c|} \hline a & a \\ \hline \end{array}$$
$$\begin{array}{|c|} \hline z \\ \hline \end{array}$$

I. Number of simples in the block: 4

II. Simple subquotients.
In this case, we have five simple subquotients. The origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} - \delta^{21})$. We provide the pictures corresponding to the case $t = 2$.

(i) Two modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\left(\left\{\begin{matrix} m \leq n \\ m \leq -t \\ n \leq 0 \end{matrix}\right\} \bigcup \left\{\begin{matrix} n < m \\ m \leq -t \end{matrix}\right\}\right)$
L_5	$L\left(\left\{\begin{matrix} m \leq n \\ 0 < m \end{matrix}\right\} \bigcup \left\{\begin{matrix} n < m \\ 0 < m \\ -t < n \end{matrix}\right\}\right)$



(ii) A cuspidal module with t-dimensional weight multiplicities:

Module	Basis
L_3	$L(-t < m \leq 0)$

(iii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq -t \\ 0 < n \end{pmatrix}$	L_4	$L\begin{pmatrix} n \leq -t \\ 0 < m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2, \quad L_3, \quad L_4, \quad L_5.$$

$(C7)$ Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{c} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 7

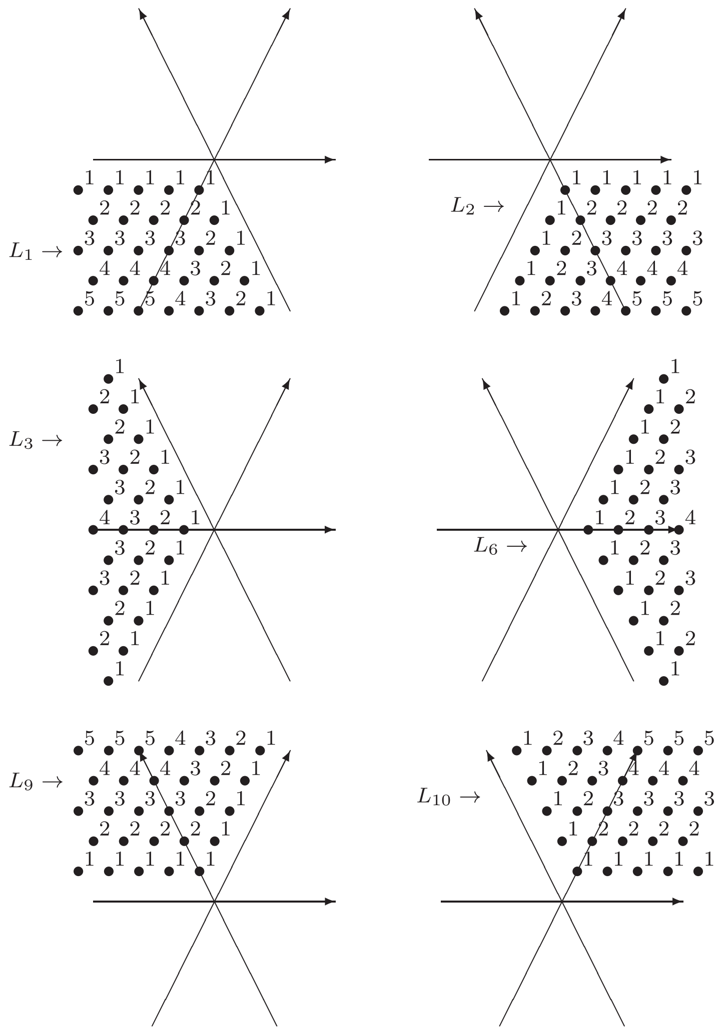
II. Simple subquotients.

The module $V(T(\bar{v}))$ has 10 simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} + \delta^{21} + \delta^{11})$.

(i) Eight modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{pmatrix}$
L_2	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ n < k \end{pmatrix}$
L_9	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \leq n \end{pmatrix}$
L_{10}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ n < k \end{pmatrix}$

Module	Basis	Module	Basis
L_3	$L\big(k \leq m \leq 0 < n\big)$	L_5	$L\big(k \leq n \leq 0 < m\big)$
L_6	$L\big(m \leq 0 < n < k\big)$	L_8	$L\big(n \leq 0 < m < k\big)$



(ii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_4	$L\begin{pmatrix} m \leq 0 < n \\ m < k \leq n \end{pmatrix}$	L_7	$L\begin{pmatrix} n \leq 0 < m \\ n < k \leq m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4, \quad L_5 \oplus L_6, \quad L_7, \quad L_8 \oplus L_9, \quad L_{10}.$$

($C8$) Consider the following Gelfand–Tsetlin tableau:

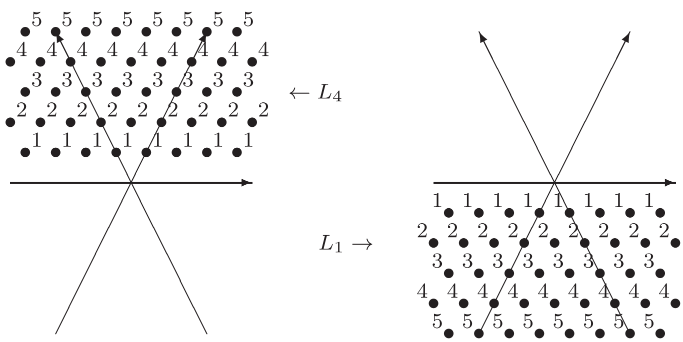
$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{c} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{z} & & \end{array}$$

I. Number of simples in the block: 3

II. Simple subquotients. The module $V(T(\bar{v}))$ has four simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} + \delta^{21} + \delta^{11})$.

(i) Modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < n \\ 0 < m \end{pmatrix}$



(ii) Two isomorphic modules with infinite-dimensional weight multiplicities:

Module	Basis	Module	Basis
L_2	$L(m \leq 0 < n)$	L_3	$L(n \leq 0 < m)$

III. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4.$$

$(C9)$ Let $s, t \in \mathbb{Z}_{>0}$ be such that $t < s$. Consider the following Gelfand–Tsetlin tableau:

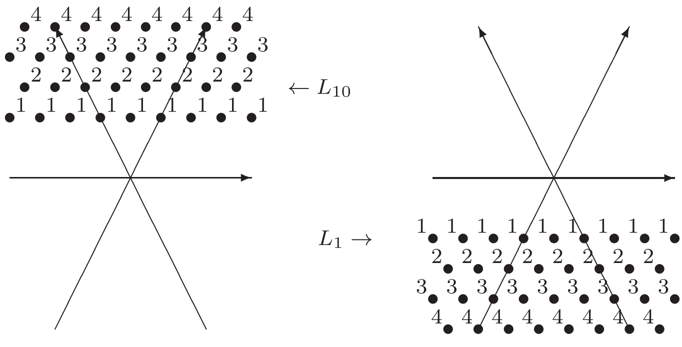
$$T(\bar{v}) = \begin{array}{|c|c|c|}\hline a & a-t & a-s \\ \hline \multicolumn{3}{c}{\begin{array}{|c|c|}\hline a & a \\ \hline\end{array}} \\ \multicolumn{3}{c}{\begin{array}{|c|}\hline z \\ \hline\end{array}}\end{array}$$

I. Number of simples in the block: 7

II. Simple subquotients. The module $V(T(\bar{v}))$ has 10 simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} - \delta^{22})$. The pictures correspond to the case $t = 1$, $s = 2$.

(i) Two modules with unbounded weight multiplicities:

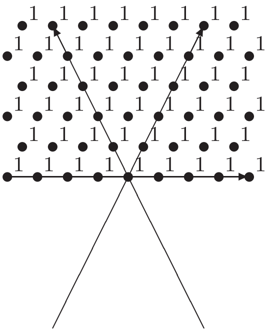
Module	Basis
L_1	$L\left(\left\{\begin{array}{c} m \leq n \\ m \leq -s \\ n \leq -t \end{array}\right\} \bigcup \left\{\begin{array}{c} n < m \\ m \leq -s \\ n \leq -t \end{array}\right\}\right)$
L_{10}	$L\left(\left\{\begin{array}{c} m \leq n \\ 0 < m \end{array}\right\} \bigcup \left\{\begin{array}{c} n < m \\ 0 < m \\ -t < n \end{array}\right\}\right)$

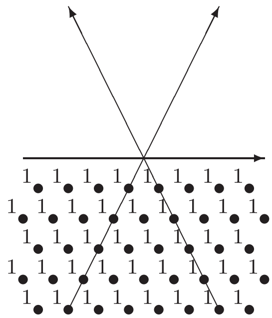


(ii) Three modules with weight multiplicities bounded by t:

Module	Basis	Module	Basis
L_2	$L\left(\begin{array}{c} m \leq -s \\ -t < n \leq 0 \end{array}\right)$	L_6	$L\left(\begin{array}{c} n \leq -s \\ -t < m \leq 0 \end{array}\right)$

Module	Basis
L_8	$L\begin{pmatrix} -t < m \leq 0 \\ -s < n \end{pmatrix}$

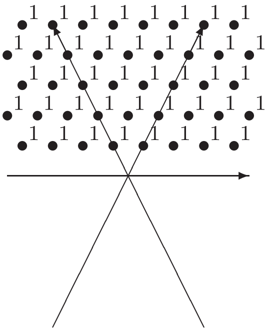


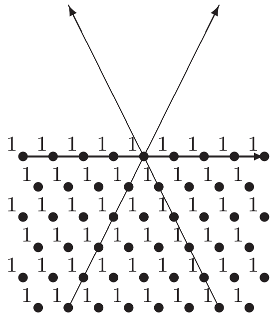


(iii) Three modules with weight multiplicities bounded by $s - t$:

Module	Basis	Module	Basis
L_5	$L\begin{pmatrix} -s < m \leq -t \\ 0 < n \end{pmatrix}$	L_9	$L\begin{pmatrix} -s < n \leq -t \\ 0 < m \end{pmatrix}$

Module	Basis
L_3	$L\left(\left\{\begin{matrix} m \leq n \\ -s < m \leq -t \\ n \leq 0 \end{matrix}\right\} \bigcup \left\{\begin{matrix} -s < m \leq -t \\ n \leq -s \end{matrix}\right\}\right)$





(iv) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_4	$L\begin{pmatrix} m \leq -s \\ 0 < n \end{pmatrix}$	L_7	$L\begin{pmatrix} n \leq -s \\ 0 < m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2, \quad L_3 \oplus L_4, \quad L_5 \oplus L_6, \quad L_7 \oplus L_8, \quad L_9, \quad L_{10}.$$

($C10$) Let $t, s \in \mathbb{Z}_{>0}$ be such that $t < s$. Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a-t} & & \boxed{a-s} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 20

II. Simple subquotients.

The module $V(T(\bar{v}))$ has 32 simple subquotients, two of them are isomorphic to the simple finite-dimensional module with highest weight $\lambda = (t-1, s-t-1)$. Also, there are two isomorphic modules with infinite-dimensional weight spaces. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} - \delta^{22})$. We provide the pictures corresponding to the case $t = 1$, $s = 2$ (i.e. the principal block).

(i) Two isomorphic finite-dimensional modules with weight multiplicities of degree $\min\{t, s-t\}$ and highest weight $\lambda = (t-1, s-t-1)$:

Module	Basis	Module	Basis
L_8	$L\begin{pmatrix} -s < m \leq -t \\ -t < n \leq 0 \\ m < k \leq n \end{pmatrix}$	L_{20}	$L\begin{pmatrix} -s < n \leq -t \\ -t < m \leq 0 \\ n < k \leq m \end{pmatrix}$

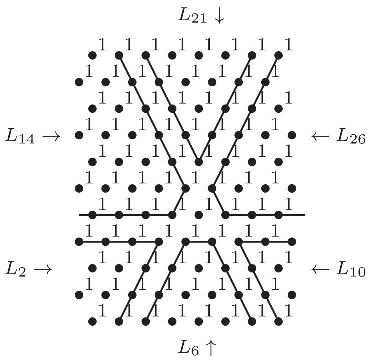
(ii) Twenty weight modules with weight multiplicities bounded by t or $s - t$. There are eight pairs of isomorphic modules and four more modules in the list:

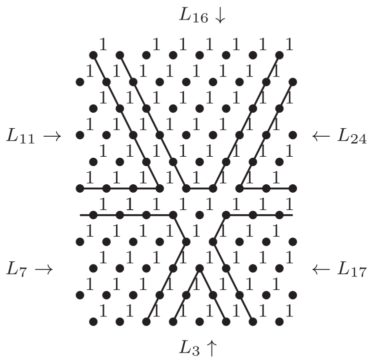
Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq -s \\ -t < n \leq 0 \\ k \leq m \end{pmatrix}$	L_9	$L\begin{pmatrix} n \leq -s \\ -t < m \leq 0 \\ k \leq n \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq -s < n \\ n \leq -t \\ m < k \leq n \end{pmatrix}$	L_{12}	$L\begin{pmatrix} n \leq -s < m \\ m \leq -t \\ n < k \leq m \end{pmatrix}$

Module	Basis	Module	Basis
L_6	$L\begin{pmatrix} m \leq -s \\ -t < n \leq 0 \\ m < k \leq n \end{pmatrix}$	L_{15}	$L\begin{pmatrix} n \leq -s \\ -t < m \leq 0 \\ n < k \leq m \end{pmatrix}$
L_{10}	$L\begin{pmatrix} m \leq -s \\ -t < n \leq 0 \\ n < k \end{pmatrix}$	L_{22}	$L\begin{pmatrix} n \leq -s \\ -t < m \leq 0 \\ m < k \end{pmatrix}$
L_{11}	$L\begin{pmatrix} -s < m \leq -t \\ 0 < n \\ k \leq m \end{pmatrix}$	L_{23}	$L\begin{pmatrix} -s < n \leq -t \\ 0 < m \\ k \leq n \end{pmatrix}$
L_{16}	$L\begin{pmatrix} -s < m \leq -t \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{27}	$L\begin{pmatrix} -s < n \leq -t \\ 0 < m \\ n < k \leq m \end{pmatrix}$
L_{21}	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{30}	$L\begin{pmatrix} -t < n \leq 0 \\ 0 < m \\ n < k \leq m \end{pmatrix}$
L_{24}	$L\begin{pmatrix} -s < m \leq -t \\ 0 < n \\ n < k \end{pmatrix}$	L_{31}	$L\begin{pmatrix} -s < n \leq -t \\ 0 < m \\ m < k \end{pmatrix}$

Module	Basis
L_7	$L\left(\begin{Bmatrix} m \leq n \\ -s < m \leq -t \\ n \leq 0 \\ k \leq m \end{Bmatrix} \cup \begin{Bmatrix} n < m \\ -s < m \leq -t \\ n \leq -t \\ k \leq n \end{Bmatrix}\right)$
L_{14}	$L\left(\begin{Bmatrix} -t < m \leq 0 \\ 0 < n \\ k \leq m \end{Bmatrix} \cup \begin{Bmatrix} m \leq n \\ -t < m \leq 0 \\ k \leq n \leq 0 \end{Bmatrix} \cup \begin{Bmatrix} n < m \\ -t < m \leq 0 \\ -s < n \leq 0 \\ k \leq n \end{Bmatrix}\right)$
L_{17}	$L\left(\begin{Bmatrix} m \leq n \\ -s < m \leq -t \\ n \leq 0 \\ n < k \end{Bmatrix} \cup \begin{Bmatrix} n \leq -s < m \\ m \leq -t \\ m < k \end{Bmatrix} \cup \begin{Bmatrix} -s < n < m \\ m \leq -t \\ m < k \end{Bmatrix}\right)$
L_{26}	$L\left(\begin{Bmatrix} m \leq n \\ -t < m \leq 0 \\ n < k \end{Bmatrix} \cup \begin{Bmatrix} n < m \\ -t < m \leq 0 \\ -s < n \leq 0 \\ m < k \end{Bmatrix}\right)$

The following picture describes the weight support of the above listed modules. Recall that for the modules listed in the left picture, the weight multiplicities are bounded by t, while for the modules in the right picture, the multiplicities are bounded by $s - t$.





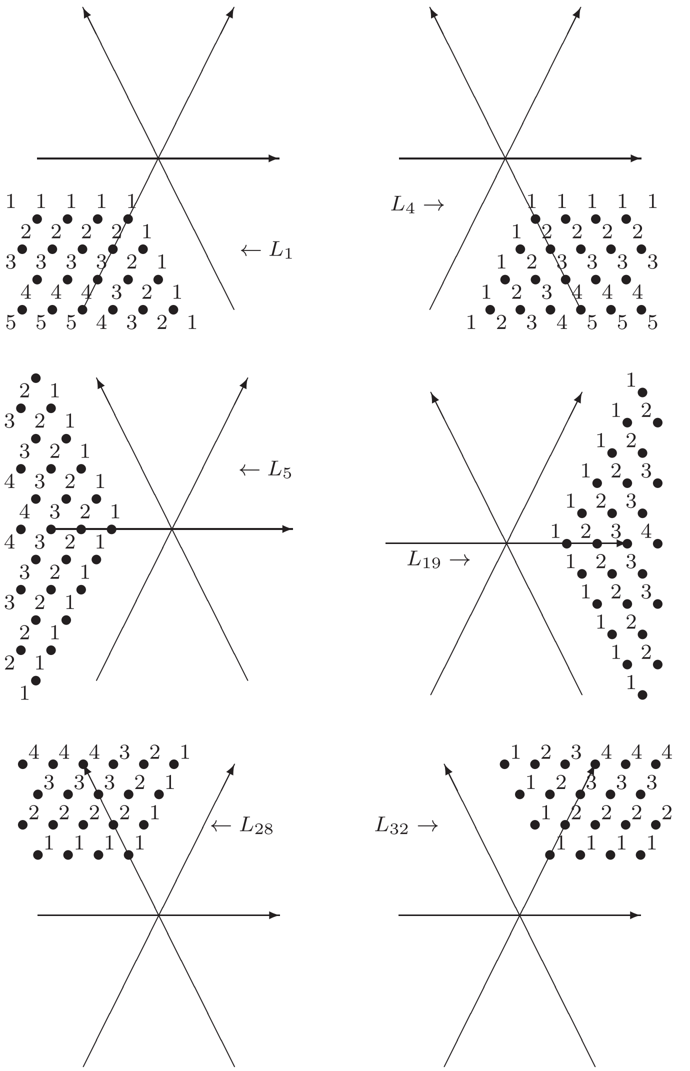
In the pictures above, the point in the middle is the $\mathfrak{sl}(3)$-weight associated to the tableau $T(v-\delta^{22})$ and corresponds to the trivial (i.e. the finite-dimensional) module.

(iii) Eight simple Verma modules (with unbounded set of weight multiplicities). There are two pairs of isomorphic modules and four more modules in the list:

Module	Basis	Module	Basis
L_5	$L\left(\begin{matrix} k\leq m\leq -s \\ 0<n \end{matrix}\right)$	L_{18}	$L\left(\begin{matrix} k\leq n\leq -s \\ 0<m \end{matrix}\right)$
L_{19}	$L\left(\begin{matrix} m\leq -s \\ 0<n<k \end{matrix}\right)$	L_{29}	$L\left(\begin{matrix} n\leq -s \\ 0<m<k \end{matrix}\right)$

Module	Basis
L_1	$L\left(\left\{\begin{matrix} m\leq -s<n \\ n\leq -t \\ k\leq m \end{matrix}\right\}\bigcup\left\{\begin{matrix} m\leq n \\ n\leq -s \\ k\leq n \end{matrix}\right\}\bigcup\left\{\begin{matrix} n<m \\ m\leq -s \\ k\leq n \end{matrix}\right\}\right)$
L_4	$L\left(\left\{\begin{matrix} m\leq n \\ m\leq -s \\ n\leq -t \\ n<k \end{matrix}\right\}\bigcup\left\{\begin{matrix} n<m \\ m\leq -s \\ n\leq -s \\ n<k \end{matrix}\right\}\right)$
L_{28}	$L\left(\left\{\begin{matrix} m\leq n \\ 0<m \\ k\leq n \end{matrix}\right\}\bigcup\left\{\begin{matrix} n<m \\ 0<m \\ -t<n \\ k\leq n \end{matrix}\right\}\right)$
L_{32}	$L\left(\left\{\begin{matrix} m\leq n \\ 0<m \\ n<k \end{matrix}\right\}\bigcup\left\{\begin{matrix} n<m \\ 0<n \\ n<k \end{matrix}\right\}\bigcup\left\{\begin{matrix} 0<m \\ -t<n\leq 0 \\ m<k \end{matrix}\right\}\right)$

The weight supports are listed as follows:



(iv) Two weight modules with infinite weight multiplicities. In this case, we have one pair of isomorphic modules:

Module	Basis	Module	Basis
L_{13}	$L\left(\left\{\begin{matrix} m \leq -s \\ 0 < n \\ m < k \leq n \end{matrix}\right\}\right)$	L_{25}	$L\left(\left\{\begin{matrix} n \leq -s \\ 0 < m \\ n < k \leq m \end{matrix}\right\}\right)$

III. Loewy series.

$L_1,\ L_2 \oplus L_3,\ L_4 \oplus L_5 \oplus L_6 \oplus L_7,\ L_8 \oplus L_9 \oplus L_{10} \oplus L_{11} \oplus L_{12} \oplus L_{13},$
$L_{14} \oplus L_{15} \oplus L_{16} \oplus L_{17} \oplus L_{18} \oplus L_{19},\ L_{20} \oplus L_{21} \oplus L_{22} \oplus L_{23} \oplus L_{24} \oplus L_{25},$
$L_{26} \oplus L_{27} \oplus L_{28} \oplus L_{29},\ L_{30} \oplus L_{31},\ L_{32}.$

$(C11)$ Set $t \in \mathbb{Z}_{>0}$ and consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{a-t} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 13

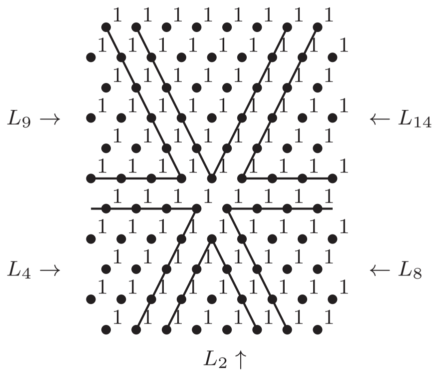
II. Simple subquotients.

The module $V(T(\bar{v}))$ has 20 simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v})$. The pictures correspond to the case $t = 1$.

(i) Six modules with finite-dimensional weight spaces of dimension at most t, given by

Module	Basis
L_4	$L\left(\left\{\begin{matrix} m \leq n \\ -t < m \leq 0 \\ -t < n \leq 0 \\ k \leq m \end{matrix}\right\} \cup \left\{\begin{matrix} -t < m \leq 0 \\ k \leq n \leq -t \end{matrix}\right\}\right)$
L_8	$L\left(\left\{\begin{matrix} m \leq n \\ -t < m \leq 0 \\ -t < n \leq 0 \\ m < k \end{matrix}\right\} \cup \left\{\begin{matrix} -t < m \leq 0 \\ n \leq -t \\ m < k \end{matrix}\right\}\right)$

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq -t < n \\ n \leq 0 \\ m < k \leq n \end{pmatrix}$	L_7	$L\begin{pmatrix} n \leq -t < m \\ m \leq 0 \\ n < k \leq m \end{pmatrix}$
L_9	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ k \leq m \end{pmatrix}$	L_{12}	$L\begin{pmatrix} -t < n \leq 0 \\ 0 < m \\ k \leq n \end{pmatrix}$
L_{10}	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{19}	$L\begin{pmatrix} -t < n \leq 0 \\ 0 < m \\ n < k \leq m \end{pmatrix}$
L_{14}	$L\begin{pmatrix} -t < m \leq 0 \\ 0 < n < k \end{pmatrix}$	L_{17}	$L\begin{pmatrix} -t < n \leq 0 \\ 0 < m < k \end{pmatrix}$

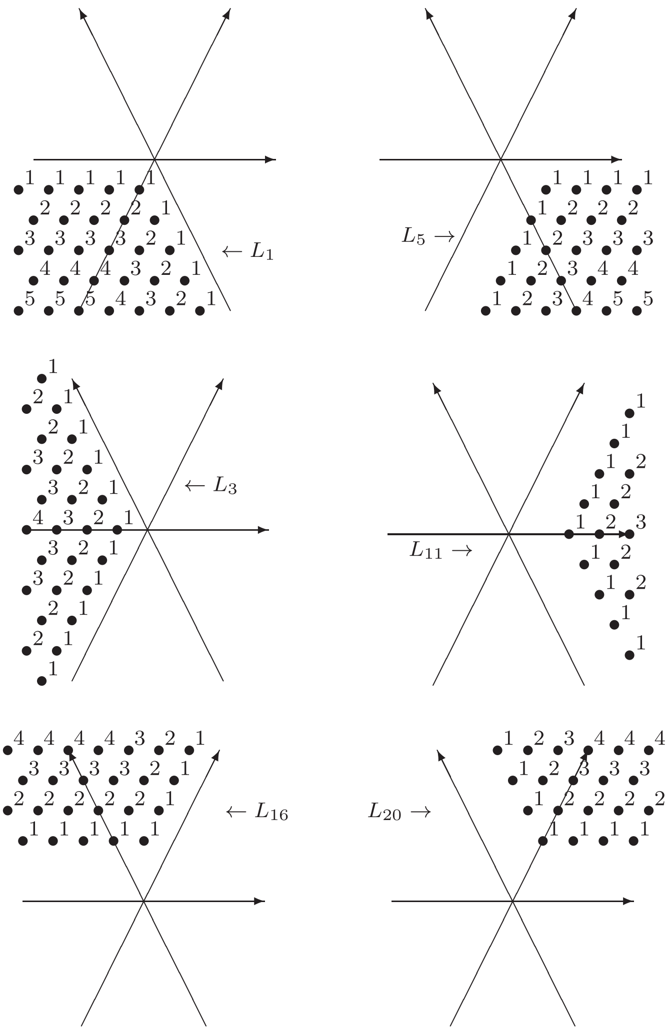


In the pictures above, the highest weight of L_4 corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v})$.

(ii) Six modules with unbounded finite weight multiplicities:

Module	Basis	Module	Basis
L_3	$L\begin{pmatrix} k \le m \le -t \\ 0 < n \end{pmatrix}$	L_{13}	$L\begin{pmatrix} k \le n \le -t \\ 0 < m \end{pmatrix}$
L_{11}	$L\begin{pmatrix} m \le -t \\ 0 < n < k \end{pmatrix}$	L_{18}	$L\begin{pmatrix} n \le -t \\ 0 < m < k \end{pmatrix}$

Module	Basis
L_1	$L\left(\left\{\begin{matrix} m \le n \\ m \le -t \\ -t < n \le 0 \\ k \le m \end{matrix}\right\} \bigcup \left\{\begin{matrix} m \le n \\ n \le -t \\ k \le n \end{matrix}\right\} \bigcup \left\{\begin{matrix} n < m \\ m \le -t \\ k \le n \end{matrix}\right\}\right)$
L_5	$L\left(\left\{\begin{matrix} m \le n \\ m \le -t \\ n \le 0 \\ n < k \end{matrix}\right\} \bigcup \left\{\begin{matrix} n < m \\ m \le -t \\ n < k \end{matrix}\right\}\right)$
L_{16}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \le n \end{pmatrix}$
L_{20}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ n < k \end{pmatrix}$



(iii) Two isomorphic modules with infinite-dimensional weight multiplicities:

Module	Basis	Module	Basis
L_6	$L\begin{pmatrix} m \leq -t \\ 0 < n \\ m < k \leq n \end{pmatrix}$	L_{15}	$L\begin{pmatrix} n \leq -t \\ 0 < m \\ n < k \leq m \end{pmatrix}$

III. Loewy series.

$L_1,\ L_2 \oplus L_3 \oplus L_4,\ L_5 \oplus L_6 \oplus L_7,\ L_8 \oplus L_9 \oplus L_{10} \oplus L_{11} \oplus L_{12} \oplus L_{13},$
$L_{14} \oplus L_{15} \oplus L_{16},\ L_{17} \oplus L_{18} \oplus L_{19},\ L_{20}.$

$(C12)$ For any $t \in \mathbb{Z}_{>0}$ consider the tableau:

$$T(\bar{v}) = \begin{array}{ccc} \boxed{a} & \boxed{a} & \boxed{a-t} \\ & \boxed{a} \quad \boxed{a} & \\ & \boxed{z} & \end{array}$$

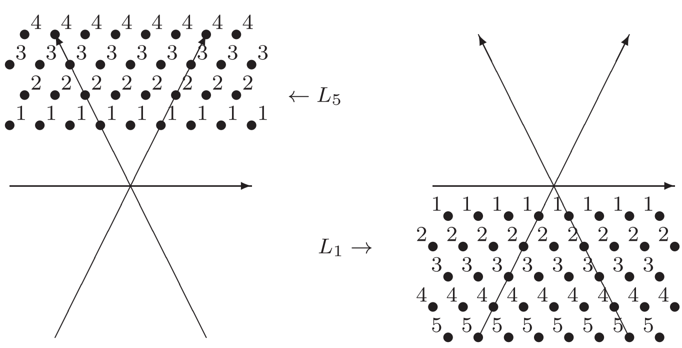
I. Number of simples in the block: 4

II. Simple subquotients.

We have five simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v})$. We provide pictures of the weight lattice corresponding to the case $t = 1$.

(i) Two modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\left(\left\{\begin{matrix} m \le n \\ m \le -t \\ n \le 0 \end{matrix}\right\} \cup \left\{\begin{matrix} n < m \\ m \le -t \end{matrix}\right\}\right)$
L_5	$L\left(\left\{\begin{matrix} m \le n \\ 0 < m \end{matrix}\right\} \cup \left\{\begin{matrix} n < m \\ 0 < m \\ -t < n \end{matrix}\right\}\right)$



(ii) A cuspidal module with t-dimensional weight multiplicities:

Module	Basis
L_3	$L(-t < m \le 0)$

(iii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} m \leq -t \\ 0 < n \end{pmatrix}$	L_4	$L\begin{pmatrix} n \leq -t \\ 0 < m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2, \quad L_3, \quad L_4, \quad L_5.$$

$(C13)$ Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{a} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{a} & & \end{array}$$

I. Number of simples in the block: 7

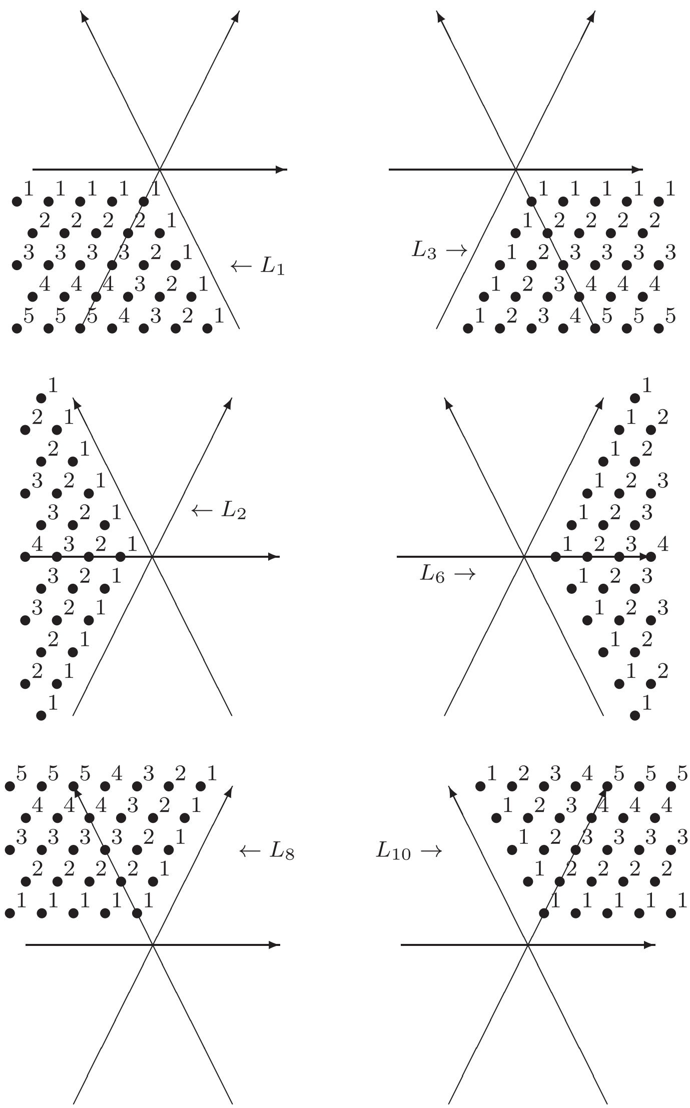
II. Simple subquotients.

The module $V(T(\bar{v}))$ has 10 simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} + \delta^{21} + \delta^{11})$.

(i) Modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ k \leq n \end{pmatrix}$
L_3	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \\ n < k \end{pmatrix}$
L_8	$L\begin{pmatrix} 0 < m \\ 0 < n \\ k \leq n \end{pmatrix}$
L_{10}	$L\begin{pmatrix} 0 < m \\ 0 < n \\ n < k \end{pmatrix}$

Module	Basis	Module	Basis
L_2	$L\begin{pmatrix} k \leq m \leq 0 \\ 0 < n \end{pmatrix}$	L_5	$L\begin{pmatrix} k \leq n \leq 0 \\ 0 < m \end{pmatrix}$
L_6	$L\begin{pmatrix} m \leq 0 < n \\ n < k \end{pmatrix}$	L_9	$L\begin{pmatrix} n \leq 0 < m \\ m < k \end{pmatrix}$



(ii) Two isomorphic modules with infinite-dimensional weight spaces:

Module	Basis	Module	Basis
L_4	$L\begin{pmatrix} m \leq 0 < n \\ m < k \leq n \end{pmatrix}$	L_7	$L\begin{pmatrix} n \leq 0 < m \\ n < k \leq m \end{pmatrix}$

III. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4, \quad L_5 \oplus L_6, \quad L_7, \quad L_8 \oplus L_9, \quad L_{10}.$$

$(C14)$ Consider the following Gelfand–Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{ccccc} \boxed{a} & & \boxed{a} & & \boxed{a} \\ & \boxed{a} & & \boxed{a} & \\ & & \boxed{z} & & \end{array}$$

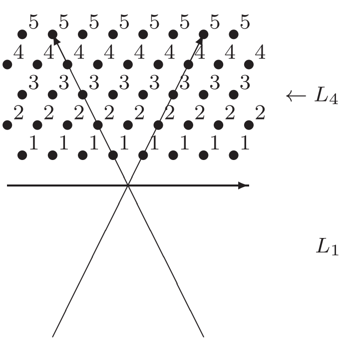
I. Number of simples in the block: 3

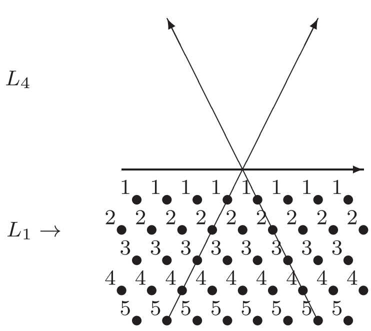
II. Simple subquotients.

The module $V(T(\bar{v}))$ has four simple subquotients. In this case, the origin of the weight lattice corresponds to the $\mathfrak{sl}(3)$-weight associated to the tableau $T(\bar{v} + \delta^{21} + \delta^{11})$.

(i) Modules with unbounded weight multiplicities:

Module	Basis
L_1	$L\begin{pmatrix} m \leq 0 \\ n \leq 0 \end{pmatrix}$
L_4	$L\begin{pmatrix} 0 < n \\ 0 < m \end{pmatrix}$





(ii) Two isomorphic modules with infinite-dimensional weight multiplicities:

Module	Basis	Module	Basis
L_2	$L(m \leq 0 < n)$	L_3	$L(n \leq 0 < m)$

III. Loewy series.

$$L_1, \quad L_2 \oplus L_3, \quad L_4.$$

8. Localization on Gelfand–Tsetlin Modules

8.1. *Localization and twisted localization functors*

We first recall the definition of the localization functor of U-modules. For details, we refer the reader to [6, 50].

For every root $\alpha \in \Delta$ the multiplicative set $\mathbf{F}_\alpha := \{f_\alpha^n \,|\, n \in \mathbb{Z}_{\geq 0}\} \subset U$ satisfies Ore's localizability conditions because $\operatorname{ad} f_\alpha$ acts locally nilpotent on U. By $\mathcal{D}_\alpha U$ we denote the localization of U relative to $\mathbf{F}_\alpha$. For every weight module M, $\mathcal{D}_\alpha M = \mathcal{D}_\alpha U \otimes_U M$ is the *α-localization of M*. If f_α is injective on M, then M can be naturally viewed as a submodule of $\mathcal{D}_\alpha M$. Furthermore, if f_α is injective on M, then it is bijective on M if and only if $\mathcal{D}_\alpha M = M$.

For $x \in \mathbb{C}$ and $u \in \mathcal{D}_\alpha U$ we set

$$\Theta_x(u) := \sum_{i \geq 0} \binom{x}{i} (\operatorname{ad} f_\alpha)^i(u)\, f_\alpha^{-i}, \tag{31}$$

where $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$. Since $\operatorname{ad} f_\alpha$ is locally nilpotent on U_α, the sum above is actually finite. Note that for $x \in \mathbb{Z}$ we have $\Theta_x(u) = f_\alpha^x u f_\alpha^{-x}$. For a $\mathcal{D}_\alpha U$-module M by $\Phi_\alpha^x M$ we denote the $\mathcal{D}_\alpha U$-module M twisted by the action

$$u \cdot v^x := (\Theta_x(u) \cdot v)^x,$$

where $u \in \mathcal{D}_\alpha U$, $v \in M$, and v^x stands for the element v considered as an element of $\Phi_\alpha^x M$. Since $v^n = f_\alpha^{-n} \cdot v$ whenever $n \in \mathbb{Z}$ it is convenient to set $f_\alpha^x \cdot v := v^{-x}$ in $\Phi_\alpha^{-x} M$ for $x \in \mathbb{C}$.

In what follows we set $\mathcal{D}_\alpha^x M := \Phi_\alpha^x(\mathcal{D}_\alpha M)$ and refer to it as a *twisted localization of M*. One easily check that if M is a weight $\mathfrak{g}$-module, then $\mathcal{D}_\alpha^x M$ is a weight module as well, in particular, $v^x \in M_{\lambda + x\alpha}$ whenever $v \in M_\lambda$. Furthermore, one easily verifies the following proposition.

Proposition 8.1. *Let α be a root and $x \in \mathbb{C}$.*

(i) *$\mathcal{D}_\alpha^x$ is an exact functor from the category of U-modules to the category of $\mathcal{D}_\alpha U$-modules.*

(ii) *If* $M \subset N$ *are* U*-modules such that* M *is* f_α*-injective and* N *is* f_α*-bijective, then* $N = \mathcal{D}_\alpha M$*.*

In the case when f_α acts injectively on M, set $\mathcal{QD}_\alpha M := \mathcal{D}_\alpha M/M$. Also, if $\alpha = \varepsilon_i - \varepsilon_j$ we will write $\mathcal{D}_{ij}$, $\mathcal{D}_{ij}^x$, and $\mathcal{QD}_{ij}$ for $\mathcal{D}_\alpha$, $\mathcal{D}_\alpha^x$, and $\mathcal{QD}_\alpha$, respectively.

8.2. *Localization functors in the case of* $\mathfrak{sl}(3)$

From now on, we consider $U = U(\mathfrak{sl}(3))$. In this section, we study the relation between the tableaux bases of a module and its localized module.

Our goal is to apply localization functors to Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules and realize all simple Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules as subquotients of twisted localized modules.

With this in mind, our first step is to obtain conditions on the bases of the modules that guarantee injectivity or surjectivity of the operator f_α. For simplicity, we will work with $\alpha = \varepsilon_1 - \varepsilon_2$, hence $f_\alpha = E_{21}$.

8.2.1. *Injectivity and surjectivity of the operator* E_{21}

In this section, we assume that V is the generic module $V(T(v))$, or the singular module $V(T(\bar{v}))$. By $\mathcal{B}$ we denote the lattice of tableaux $\mathcal{B}(T(v))$ (or $\mathcal{B}(T(\bar{v}))$). Also, the tableaux basis of a Gelfand–Tsetlin module M that is a subquotient of V will be denoted by $\mathcal{B}_M \subseteq \mathcal{B}$.

Remark 8.2. Since V is a weight module, every subquotient M of V is a weight module. Hence, in order to check injectivity or surjectivity of E_{21} on M, it is enough to check those properties on weight spaces of M. Also, recall that for a weight $\lambda = (\lambda_1, \lambda_2)$ in the weight support of M, $E_{21}(M_\lambda) \subseteq M_{(\lambda_1-2,\lambda_2+1)}$.

To unify the notation, in the case of a generic tableau $T(v)$ it will be convenient to write $\mathrm{Tab}(w) := T(v+w)$. Then, the action of E_{21} on $\mathrm{Tab}(w)$ (generic or singular) is given by the formula:

$$E_{21}(\mathrm{Tab}(w)) = \mathrm{Tab}(w - \delta^{11}). \tag{32}$$

From Propositions 7.2 and 7.32, if $\mathrm{Tab}(w) \in \mathcal{B}_M$ is a weight vector of weight λ, then the weight space M_λ is spanned by $\{\mathrm{Tab}(w + (i, -i, 0)) \,|\, i \in \mathbb{Z}\} \cap \mathcal{B}_M$. If w_i denotes the vector $w + (i, -i, 0) \in \mathbb{Z}^3$, any element of M_λ will be of the form $u = \sum_{i \in I} a_i T(w_i)$ for some finite subset I of $\mathbb{Z}$.

Lemma 8.3. *The operator* E_{21} *acts injectively on* M *if and only if* $\mathrm{Tab}(w) \in \mathcal{B}_M$ *implies* $\mathrm{Tab}(w - \delta^{11}) \in \mathcal{B}_M$*.*

Proof. Suppose first that there exists $\mathrm{Tab}(w) \in \mathcal{B}_M$ such that $\mathrm{Tab}(w - \delta^{11}) \notin \mathcal{B}_M$, then $E_{21}(\mathrm{Tab}(w)) = \mathrm{Tab}(w - \delta^{11}) = 0$ (on M), which implies that E_{21} is not injective. On the other hand, suppose $u := \sum_{i\in I} a_i \operatorname{Tab}(w_i) \in M_\lambda$ with $\operatorname{Tab}(w_i) \in \mathcal{B}_M$ for any i. If u is such that $0 = E_{21}(u) = \sum_{i\in I} a_i \operatorname{Tab}(w_i - \delta^{11})$, then $a_i T(w_i - \delta^{11}) = 0$ for any $i \in I$. Since by hypothesis $T(w_i - \delta^{11}) \in \mathcal{B}_M$, we have $a_i = 0$ for any $i \in I$. $\square$

Lemma 8.4. *The operator E_{21} acts surjectively on M if and only if $T(w) \in \mathcal{B}_M$ implies $T(w + \delta^{11}) \in \mathcal{B}_M$.*

Proof. Any element of $M_{(\lambda_1-2,\lambda_2+1)}$ is of the form $u' = \sum_{i\in I} a_i \operatorname{Tab}(w_i - \delta^{11})$ and a direct computation using (32) shows that $E_{21}(\sum_{i\in I} a_i \operatorname{Tab}(w_i)) = u'$. $\square$

8.2.2. *Twisted localization with respect to E_{21}*

Recall that for $x \in \mathbb{C}$ and $u \in \mathcal{D}_{12}U$ we have

$$\Theta_x(u) := \sum_{i\geq 0} \binom{x}{i} (\operatorname{ad} E_{21})^i(u)\, E_{21}^{-i}. \tag{33}$$

Lemma 8.5. *Let $\{c_{ij}\}_{1\leq j\leq i\leq 3}$ be the generators of Γ defined in (1). Then*

$$\Theta_x(c_{ij}) = \begin{cases} c_{ij} & \text{if } (i,j) \neq (1,1), \\ c_{11} + x & \text{if } (i,j) = (1,1). \end{cases}$$

Proof. We first note that if u commutes with E_{21}, then $\Theta_x(u) = u$. Since the generators $\{c_{ij}\}_{2\leq j\leq i\leq 3}$ commute with E_{21}, the first part of the lemma is proven. For the second part, we use that $c_{11} = E_{11}$ and $(\operatorname{ad} E_{21})^2(E_{11}) = 0$. $\square$

As an immediate consequence of Lemma 8.5 we have the following corollary that will be frequently applied.

Corollary 8.6. *Let M be any Gelfand–Tsetlin module on which E_{21} acts injectively.*

(i) *The twisted localized module $\mathcal{D}_{12}^x M$ is also a Gelfand–Tsetlin module.*
(ii) *If $v \in M$ has Gelfand–Tsetlin character $\chi = (a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33})$, then $v^x \in \mathcal{D}_{12}^x M$ has Gelfand–Tsetlin character $\tilde{\chi} = (a_{11} + x, a_{21}, a_{22}, a_{31}, a_{32}, a_{33})$.*

Next, for a Gelfand–Tsetlin module M with tableaux basis $\mathcal{B}_M$ and injective action of E_{21}, we explicitly describe the tableaux basis of $\mathcal{D}_{12}^x M$. For this, we introduce some notation.

For $\mathcal{B} \subseteq \mathbb{Z}^3$, denote by $\mathcal{B} + \delta^{11}$ the region $\{(m,n,k) \mid (m,n,k-1) \in \mathcal{B}\}$. Set $\mathcal{B} + t\delta^{11} = (\mathcal{B} + (t-1)\delta^{11}) + \delta^{11}$ for $t \in \mathbb{N}$ and $\mathcal{B} + \mathbb{N}\delta^{11} = \bigcup_{t=0}^{\infty} (\mathcal{B} + t\delta^{11})$.

Recall Definition 7.3 for $L(\mathcal{B}; v)$.

Proposition 8.7. *Let $\mathcal{B} \subset \mathbb{Z}^3$ and $L(\mathcal{B}) = L(\mathcal{B}; v)$ be a simple Gelfand–Tsetlin module. Assume that E_{21} acts injectively on $L(\mathcal{B}; v)$. Then $\mathcal{D}_{12}^x L(\mathcal{B}; v) \simeq M(\mathcal{B} +$*

$\mathbb{N}\delta^{11};\ v+x\delta^{11})$ and $\mathcal{QD}_{12}^{x}L(\mathcal{B};v)\simeq M((\mathcal{B}+\mathbb{N}\delta^{11})\backslash\mathcal{B};\ v+x\delta^{11})$. In particular, if x is an integer we have $\mathcal{D}_{12}L(\mathcal{B})\simeq M(\mathcal{B}+\mathbb{N}\delta^{11})$ and $\mathcal{QD}L(\mathcal{B})\simeq M((\mathcal{B}+\mathbb{N}\delta^{11})\backslash\mathcal{B})$.

Corollary 8.8. *Let M be a simple module in $\mathcal{GT}$, generated by a tableau* $\mathrm{Tab}(w)$ *and such that E_{21} acts injectively on M. Then for any $x\in\mathbb{C}$, $\mathcal{D}_{12}^{x}M$ has a subquotient isomorphic to a simple $\mathfrak{sl}(3)$-module in $\mathcal{GT}$ generated by the tableau* $\mathrm{Tab}(w+x\delta^{11})$.

8.3. *Simple Gelfand–Tsetlin modules and localization functors*

In this section, we will describe the simple Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules via localization functors and subquotients starting with some simple E_{21}-injective Gelfand–Tsetlin module. In order to give such description we rely on Lemmas 8.3, 8.4, and Proposition 8.7. In fact, in order to use Proposition 8.7 we have to check if the corresponding module defined in such region is E_{21}-bijective.

For convenience, we denote by $L_i^{(Gj)}$ the simple module L_i in the jth generic block from the list in Sec. 7.2, and by $L_i^{(Cj)}$ the simple module L_i in the jth singular block in the list in Sec. 7.5. For example, $L_1^{(G2)}$ stands for the module $L(\{k\le m\};T(a,b,c|x,y|x))$.

Below we list of all simple modules which are E_{21}-injective.

(i) Simple E_{21}-injective generic modules:

$L_1^{(G2)}$	$L_1^{(G4)}$	$L_2^{(G4)}$	$L_1^{(G5)}$	$L_3^{(G5)}$	$L_1^{(G6)}$
$L_3^{(G6)}$	$L_4^{(G6)}$	$L_7^{(G6)}$	$L_1^{(G9)}$	$L_3^{(G9)}$	$L_5^{(G9)}$
$L_1^{(G10)}$	$L_2^{(G10)}$	$L_4^{(G10)}$	$L_1^{(G11)}$	$L_3^{(G11)}$	$L_4^{(G11)}$
$L_6^{(G11)}$	$L_8^{(G11)}$	$L_{11}^{(G11)}$	$L_1^{(G12)}$	$L_3^{(G12)}$	$L_4^{(G12)}$
$L_5^{(G12)}$	$L_8^{(G12)}$	$L_{11}^{(G12)}$	$L_1^{(G14)}$	$L_2^{(G14)}$	$L_4^{(G14)}$
$L_6^{(G14)}$	$L_1^{(G15)}$	$L_2^{(G15)}$	$L_4^{(G15)}$	$L_6^{(G15)}$	

(ii) Simple E_{21}-injective singular modules:

$L_1^{(C2)}$	$L_1^{(C4)}$	$L_4^{(C4)}$	$L_1^{(C5)}$	$L_3^{(C5)}$	$L_4^{(C5)}$	$L_{10}^{(C5)}$	$L_{11}^{(C5)}$
$L_1^{(C7)}$	$L_3^{(C7)}$	$L_5^{(C7)}$	$L_9^{(C7)}$	$L_1^{(C10)}$	$L_2^{(C10)}$	$L_5^{(C10)}$	$L_7^{(C10)}$
$L_9^{(C10)}$	$L_{11}^{(C10)}$	$L_{14}^{(C10)}$	$L_{23}^{(C10)}$	$L_{28}^{(C10)}$	$L_1^{(C11)}$	$L_3^{(C11)}$	$L_4^{(C11)}$
$L_9^{(C11)}$	$L_{12}^{(C11)}$	$L_{13}^{(C11)}$	$L_{16}^{(C11)}$	$L_1^{(C13)}$	$L_2^{(C13)}$	$L_5^{(C13)}$	$L_8^{(C13)}$

Finally, we apply (twisted) localization functors on the modules above and obtain all simple modules in the block as shown in the following tables. Set

$\mathcal{QD}^x := \mathcal{QD}^x_{12}$, and $\mathcal{D}^x = \mathcal{D}^x_{12}$.

Block	Module	Subquotient of E_{21}-localization
$G1$	$L_1^{(G1)}$	$\mathcal{D}^{(z-x)}(L_1^{(G2)})$
$G2$	$L_2^{(G2)}$	$\mathcal{QD}(L_1^{(G2)})$
$G3$	$L_1^{(G3)}$	$\mathcal{D}^{(y-z)}(L_1^{(G4)}) \simeq \mathcal{D}^{(z-a)}(L_1^{(G5)})$
	$L_2^{(G3)}$	$\mathcal{D}^{(y-z)}(L_2^{(G4)}) \simeq \mathcal{D}^{(z-a)}(L_3^{(G5)})$
$G4$	$L_3^{(G4)}$	$\mathcal{QD}(L_1^{(G4)})$
	$L_4^{(G4)}$	$\mathcal{QD}(L_2^{(G4)})$
$G5$	$L_2^{(G5)}$	$\mathcal{QD}(L_1^{(G5)})$
	$L_4^{(G5)}$	$\mathcal{QD}(L_3^{(G5)})$
$G6$	$L_2^{(G6)}$	$\mathcal{QD}(L_1^{(G6)})$
	$L_5^{(G6)}$	$\mathcal{QD}(L_3^{(G6)})$
	$L_6^{(G6)}$	$\mathcal{QD}(L_4^{(G6)})$
	$L_8^{(G6)}$	$\mathcal{QD}(L_7^{(G6)})$
$G7$	$L_1^{(G7)}$	$\mathcal{D}^{(z-a)}(L_1^{(G6)})$
	$L_2^{(G7)}$	$\mathcal{D}^{(z-a)}(L_4^{(G6)})$
	$L_3^{(G7)}$	$\mathcal{D}^{(z-a)}(L_3^{(G6)})$
	$L_4^{(G7)}$	$\mathcal{D}^{(z-a)}(L_7^{(G6)})$
$G8$	$L_1^{(G8)}$	$\mathcal{D}^{(z-a)}(L_1^{(G9)}) \simeq \mathcal{D}^{(y-z)}(L_1^{(G10)})$
	$L_2^{(G8)}$	$\mathcal{D}^{(z-a)}(L_3^{(G9)}) \simeq \mathcal{D}^{(y-z)}(L_2^{(G10)})$
	$L_3^{(G8)}$	$\mathcal{D}^{(z-a)}(L_5^{(G9)}) \simeq \mathcal{D}^{(y-z)}(L_4^{(G10)})$
$G9$	$L_2^{(G9)}$	$\mathcal{QD}(L_1^{(G9)})$
	$L_4^{(G9)}$	$\mathcal{QD}(L_3^{(G9)})$
	$L_6^{(G9)}$	$\mathcal{QD}(L_5^{(G9)})$
$G10$	$L_3^{(G10)}$	$\mathcal{QD}(L_1^{(G10)})$
	$L_5^{(G10)}$	$\mathcal{QD}(L_2^{(G10)})$
	$L_6^{(G10)}$	$\mathcal{QD}(L_4^{(G10)})$

Block	Module	Subquotient of E_{21}-localization
$G11$	$L_2^{(G11)}$	$\mathcal{QD}(L_1^{(G11)})$
	$L_5^{(G11)}$	$\mathcal{QD}(L_3^{(G11)})$
	$L_7^{(G11)}$	$\mathcal{QD}(L_4^{(G11)})$
	$L_9^{(G11)}$	$\mathcal{QD}(L_6^{(G11)})$
	$L_{10}^{(G11)}$	$\mathcal{QD}(L_8^{(G11)})$
	$L_{12}^{(G11)}$	$\mathcal{QD}(L_{11}^{(G11)})$
$G12$	$L_2^{(G12)}$	$\mathcal{QD}(L_1^{(G12)})$
	$L_6^{(G12)}$	$\mathcal{QD}(L_4^{(G12)})$
	$L_7^{(G12)}$	$\mathcal{QD}(L_3^{(G12)})$
	$L_9^{(G12)}$	$\mathcal{QD}(L_5^{(G12)})$
	$L_{10}^{(G12)}$	$\mathcal{QD}(L_8^{(G12)})$
	$L_{12}^{(G12)}$	$\mathcal{QD}(L_{11}^{(G12)})$
$G13$	$L_1^{(G13)}$	$\mathcal{D}^{(z-a)}(L_1^{(G11)})$
	$L_2^{(G13)}$	$\mathcal{D}^{(z-a)}(L_4^{(G11)})$
	$L_3^{(G13)}$	$\mathcal{D}^{(z-a)}(L_3^{(G11)})$
	$L_4^{(G13)}$	$\mathcal{D}^{(z-a)}(L_8^{(G11)})$
	$L_5^{(G13)}$	$\mathcal{D}^{(z-a)}(L_6^{(G11)})$
	$L_6^{(G13)}$	$\mathcal{D}^{(z-a)}(L_{11}^{(G11)})$
$G14$	$L_3^{(G14)}$	$\mathcal{QD}(L_1^{(G14)})$
	$L_5^{(G14)}$	$\mathcal{QD}(L_2^{(G14)})$
	$L_7^{(G14)}$	$\mathcal{QD}(L_4^{(G14)})$
	$L_8^{(G14)}$	$\mathcal{QD}(L_6^{(G14)})$
$G15$	$L_3^{(G15)}$	$\mathcal{QD}(L_1^{(G15)})$
	$L_5^{(G15)}$	$\mathcal{QD}(L_2^{(G15)})$
	$L_7^{(G15)}$	$\mathcal{QD}(L_4^{(G15)})$
	$L_8^{(G15)}$	$\mathcal{QD}(L_6^{(G12)})$
$G16$	$L_1^{(G16)}$	$\mathcal{D}^{(z-a)}(L_1^{(G14)}) \simeq \mathcal{D}^{(y-z)}(L_1^{(G15)})$
	$L_2^{(G16)}$	$\mathcal{D}^{(z-a)}(L_2^{(G14)}) \simeq \mathcal{D}^{(y-z)}(L_2^{(G15)})$
	$L_3^{(G16)}$	$\mathcal{D}^{(z-a)}(L_4^{(G14)}) \simeq \mathcal{D}^{(y-z)}(L_4^{(G15)})$
	$L_4^{(G16)}$	$\mathcal{D}^{(z-a)}(L_6^{(G14)}) \simeq \mathcal{D}^{(y-z)}(L_6^{(G15)})$

Block	Module	Subquotient of E_{21}-localization
$C1$	$L_1^{(C1)}$	$\mathcal{QD}^{(z-x)}(L_1^{(C2)})$
$C2$	$L_2^{(C2)}$	$\mathcal{QD}(L_1^{(C2)})$
$C3$	$L_1^{(C3)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C4)})$
	$L_2^{(C3)}$	$\mathcal{QD}^{(z-a)}(L_4^{(C4)})$
$C4$	$L_2^{(C4)} \simeq L_5^{(C4)}$	$\mathrm{soc}(\mathcal{QD}(L_1^{(C4)}) \simeq \mathrm{soc}(\mathcal{QD}(L_4^{(C4)}))$
	$L_3^{(C4)}$	$\mathcal{QD}(L_1^{(C4)})/L_2^{(C4)}$
	$L_6^{(C4)}$	$\mathcal{QD}(L_4^{(C4)})/L_5^{(C4)}$
$C5$	$L_2^{(C5)} \simeq L_8^{(C5)}$	$\mathrm{soc}(\mathcal{QD}(L_1^{(C5)})) \simeq \mathrm{soc}(\mathcal{QD}(L_4^{(C5)}))/L_7^{(C5)}$
	$L_5^{(C5)} \simeq L_{12}^{(C5)}$	$\mathrm{soc}(\mathcal{QD}(L_3^{(C5)})) \simeq \mathrm{soc}(\mathcal{QD}(L_{10}^{(C5)}))$
	$L_6^{(C5)}$	$\mathcal{QD}(L_1^{(C5)})/L_2^{(C5)}$
	$L_7^{(C5)} \simeq L_{15}^{(C5)}$	$\mathrm{soc}(\mathcal{QD}(L_4^{(C5)}))/L_8^{(C5)} \simeq \mathrm{soc}(\mathcal{QD}(L_{11}^{(C5)}))$
	$L_9^{(C5)} \simeq L_{14}^{(C5)}$	$\mathcal{QD}(L_3^{(C5)})/L_5^{(C5)} \simeq \mathcal{QD}(L_{10}^{(C5)})/L_{12}^{(C5)}$
	$L_{13}^{(C5)}$	$\mathcal{QD}(L_4^{(C5)})/(L_7^{(C5)} \oplus L_8^{(C5)})$
	$L_{16}^{(C5)}$	$\mathcal{QD}(L_{11}^{(C5)})$
$C6$	$L_1^{(C6)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C5)})$
	$L_2^{(C6)} \simeq L_4^{(C6)}$	$\mathcal{QD}^{(z-a)}(L_3^{(C5)}) \simeq \mathcal{QD}^{(z-a)}(L_{10}^{(C5)})$
	$L_3^{(C6)}$	$\mathcal{QD}^{(z-a)}(L_4^{(C5)})$
	$L_5^{(C6)}$	$\mathcal{QD}^{(z-a)}(L_{11}^{(C5)})$
$C7$	$L_2^{(C7)}$	$\mathcal{QD}(L_1^{(C7)})$
	$L_4^{(C7)} \simeq L_7^{(C7)}$	$\mathrm{soc}(\mathcal{QD}(L_3^{(C7)})) \simeq \mathrm{soc}(\mathcal{QD}(L_5^{(C7)}))$
	$L_6^{(C7)} \simeq L_8^{(C7)}$	$\mathcal{QD}(L_3^{(C7)})/L_4^{(C7)} \simeq \mathcal{QD}(L_5^{(C7)})/L_7^{(C7)}$
	$L_{10}^{(C7)}$	$\mathcal{QD}(L_9^{(C7)})$
$C8$	$L_1^{(C8)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C7)})$
	$L_2^{(C8)} \simeq L_3^{(C8)}$	$\mathcal{QD}^{(z-a)}(L_3^{(C7)}) \simeq \mathcal{QD}^{(z-a)}(L_5^{(C7)})$
	$L_4^{(C8)}$	$\mathcal{QD}^{(z-a)}(L_9^{(C7)})$
$C9$	$L_1^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C10)})$
	$L_2^{(C9)} \simeq L_6^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_2^{(C10)}) \simeq \mathcal{QD}^{(z-a)}(L_9^{(C10)})$
	$L_3^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_7^{(C10)})$
	$L_4^{(C9)} \simeq L_7^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_2^{(C10)}) \simeq \mathcal{QD}^{(z-a)}(L_9^{(C10)})$
	$L_5^{(C9)} \simeq L_9^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_2^{(C10)}) \simeq \mathcal{QD}^{(z-a)}(L_{23}^{(C10)})$
	$L_8^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_{14}^{(C10)})$
	$L_{10}^{(C9)}$	$\mathcal{QD}^{(z-a)}(L_{28}^{(C10)})$

Block	Module	Subquotient of E_{21}-localization
$C10$	$L_3^{(C10)} \simeq L_{12}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_7^{(C10)}/L_8^{(C10)}) \simeq \mathrm{soc}(\mathcal{QD}(L_1^{(C10)}))$
	$L_4^{(C10)}$	$\mathcal{QD}(L_1^{(C10)})/L_3^{(C10)}$
	$L_6^{(C10)} \simeq L_{15}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_2^{(C10)}) \simeq \mathrm{soc}(\mathcal{QD}(L_9^{(C10)}))$
	$L_8^{(C10)} \simeq L_{20}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_7^{(C10)})/L_3^{(C10)}) \simeq \mathrm{soc}(\mathcal{QD}(L_{14}^{(C10)})/L_{21}^{(C10)})$
	$L_{10}^{(C10)}$	$\mathcal{QD}(L_2^{(C10)})/L_6^{(C10)}$
	$L_{13}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_5^{(C10)}))$
	$L_{16}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_{11}^{(C10)}))$
	$L_{17}^{(C10)}$	$\mathcal{QD}(L_7^{(C10)})/(L_8^{(C10)} \oplus L_{12}^{(C10)})$
	$L_{19}^{(C10)}$	$\mathcal{QD}(L_5^{(C10)})/L_{13}^{(C10)}$
	$L_{21}^{(C10)} \simeq L_{30}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_{14}^{(C10)})/L_{20}^{(C10)}) \simeq \mathrm{soc}(\mathcal{QD}(L_{28}^{(C10)}))$
	$L_{22}^{(C10)}$	$\mathcal{QD}(L_9^{(C10)})/L_{15}^{(C10)}$
	$L_{24}^{(C10)}$	$\mathcal{QD}(L_{11}^{(C10)})/L_{16}^{(C10)}$
	$L_{25}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_{18}^{(C10)}))$
	$L_{26}^{(C10)}$	$\mathcal{QD}(L_{14}^{(C10)})/(L_{20}^{(C10)} \oplus L_{21}^{(C10)})$
	$L_{27}^{(C10)}$	$\mathrm{soc}(\mathcal{QD}(L_{23}^{(C10)}))$
	$L_{29}^{(C10)}$	$\mathcal{QD}(L_{18}^{(C10)})/L_{25}^{(C10)}$
	$L_{31}^{(C10)}$	$\mathcal{QD}(L_{23}^{(C10)})/L_{27}^{(C10)}$
	$L_{32}^{(C10)}$	$\mathcal{QD}(L_{28}^{(C10)})/L_{21}^{(C10)}$
$C11$	$L_2^{(C11)} \simeq L_7^{(C11)}$	$\mathrm{soc}(\mathcal{QD}(L_1^{(C11)})) \simeq \mathrm{soc}(\mathcal{QD}(L_4^{(C11)}))$
	$L_5^{(C11)}$	$\mathcal{QD}(L_1^{(C11)})/L_2^{(C^{11})}$
	$L_6^{(C11)} \simeq L_{15}^{(C11)}$	$\mathrm{soc}(\mathcal{QD}(L_3^{(C11)})) \simeq \mathrm{soc}(\mathcal{QD}(L_{13}^{(C11)}))$
	$L_8^{(C11)}$	$\mathcal{QD}(L_4^{(C11)})/L_7^{(C11)}$
	$L_{10}^{(C11)} \simeq L_{19}^{(C11)}$	$\mathrm{soc}(\mathcal{QD}(L_9^{(C11)})) \simeq \mathrm{soc}(\mathcal{QD}(L_{12}^{(C11)}))$
	$L_{11}^{(C11)} \simeq L_{18}^{(C11)}$	$\mathcal{QD}(L_3^{(C11)})/L_6^{(C^{11})} \simeq \mathcal{QD}(L_{13}^{(C11)})/L_{15}^{(C^{11})}$
	$L_{14}^{(C11)} \simeq L_{17}^{(C11)}$	$\mathcal{QD}(L_9^{(C11)})/L_{10}^{(C^{11})} \simeq \mathcal{QD}(L_{12}^{(C11)})/L_{19}^{(C^{11})}$
	$L_{20}^{(C11)}$	$\mathcal{QD}(L_{16}^{(C11)})$
$C12$	$L_1^{(C12)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C11)})$
	$L_2^{(C12)} \simeq L_4^{(C12)}$	$\mathcal{QD}_{12}^{(z-a)}(L_3^{(C11)}) \simeq \mathcal{QD}^{(z-a)}(L_{13}^{(C11)})$
	$L_3^{(C12)}$	$\mathcal{QD}^{(z-a)}(L_3^{(C11)}) \simeq \mathcal{QD}^{(z-a)}(L_{13}^{(C11)})$
	$L_5^{(C12)}$	$\mathcal{QD}^{(z-a)}(L_{16}^{(C11)})$

Block	Module	Subquotient of E_{21}-localization
$C13$	$L_3^{(C13)}$	$\mathcal{QD}(L_1^{(C13)})$
	$L_4^{(C13)} \simeq L_7^{(C13)}$	$\mathrm{soc}(\mathcal{QD}(L_2^{(C13)})) \simeq \mathrm{soc}(\mathcal{QD}(L_5^{(C13)}))$
	$L_6^{(C13)} \simeq L_9^{(C13)}$	$\mathcal{QD}(L_2^{(C13)})/L_4^{(C13)} \simeq \mathcal{QD}(L_5^{(C13)})/L_7^{(C13)}$
	$L_{10}^{(C13)}$	$\mathcal{QD}(L_8^{(C13)})$
$C14$	$L_1^{(C14)}$	$\mathcal{QD}^{(z-a)}(L_1^{(C13)})$
	$L_2^{(C14)} \simeq L_3^{(C14)}$	$\mathcal{QD}^{(z-a)}(L_2^{(C13)}) \simeq \mathcal{QD}^{(z-a)}(L_5^{(C13)})$
	$L_4^{(C14)}$	$\mathcal{QD}^{(z-a)}(L_8^{(C13)})$

Corollary 8.9. *Every simple Gelfand–Tsetlin module can be obtained via a composition of a twisted localization functor and taking a subquotient from a simple E_{21}-injective Gelfand–Tsetlin module.*

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