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Differentiability with respect to delays for a neutral differential-difference equation

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1 Introduction

This work is concerned with the smoothness properties with respect to delays of the solution of a neutral differential-delay equation. By a neutral differential-delay equation we mean an equation in which $\dot{x}(t)$ is expressed as a function of present and past values of x, and past values of $\dot{x}$. A very important point in the study of these equations is the choice of an appropriate topological space for the solutions. The basic normed linear space in this paper is $W^{1,\infty}([a,b],\mathbf{R}^n)$, the set of absolutely continuous functions taking the interval $[a,b]$ into $\mathbf{R}^n$, with essentially bounded derivatives, endowed with the norm $\|\phi\|^{1,1}_{[a,b]} = |\phi(a)| + \int_a^b |\dot{\phi}(s)|\, ds$. We refer to Driver [1] (see also Melvin [5]) for useful remarks and examples motivating such a choice. Notice that in this linear space we can also define the (complete) norm $\|\phi\|^{1,\infty}_{[a,b]} = |\phi(a)| + \text{ess sup}_{\theta\in[a,b]} |\dot{\phi}(\theta)|$. If $[a,b] = [-r,0]$, we will denote $W^{1,\infty} = W^{1,\infty}([-r,0],\mathbf{R}^n)$, and denote its norms simply by $\|\phi\|^{1,1}$ and $\|\phi\|^{1,\infty}$.

We now formulate our problem in a more precise way. Let $r \geq 0$, $a \geq 0$ be given real numbers, $\tau, \sigma \in [0,r]$, $\sigma > 0$, $\phi \in W^{1,\infty}$. If $F\colon \mathbf{R}^{3n} \to \mathbf{R}^n$ is Lipschitz continuous, then the initial value problem

$$\begin{cases} \dot{x}(t) = F(x(t), x(t-\tau), \dot{x}(t-\sigma)) & \text{if} \quad t > 0 \\ x(t) = \phi(t) & \text{if} \quad -r \leq t \leq 0\,. \end{cases} \tag{1}$$

has a unique solution $x(\phi,\tau,\sigma)$, defined on a maximal interval $[-r,b)$, for some $b > 0$

(see, for example, [1] or [5]). We are concerned with the smoothness properties of $x(\phi, \tau, \sigma)$.

A basic result in this paper is the following formulation of the uniform contraction principle, whose proof is essentially contained in [3] and [4]. In the statement of the theorem and in everything that follows (unless otherwise stated), the expressions open, continuous, etc, should be understood with respect to the norm $\|\cdot\|^{1,1}$.

Theorem 1.1 *Suppose Y is a normed space, Λ is a topological space, $V \subset Y$ and $U \subset W^{1,\infty}(([a,b],\mathbf{R}^n)$, are open subsets, and $T: \bar{U} \times V \times \Lambda \to E$ satisfy:*

(a) For each $(y,\lambda) \in V \times \Lambda$ there exists a unique $x = g(y,\lambda) \in E$ such that $g(y,\lambda) = T(g(y,\lambda), y, \lambda)$.

(b) $T(x,y,\lambda)$ is a contraction in x uniformly with respect to $(y,\lambda) \in V \times \Lambda$.

(c) For each $x \in g(V \times \Lambda)$, $T(x,\cdot,\cdot)$ is continuous in $V \times \Lambda$.

(d) For each $\lambda \in \Lambda$, $T(\cdot,\cdot,\lambda)$ is C^k in $\bar{U} \times V$.

(e) There exists $q \in [0,1)$ such that, for each $(x,y,\lambda) \in \bar{U} \times V \times \Lambda$, we have $\|D_xT(x,y,\lambda)h\|^{1,1}_{[a,b]} \le q\|h\|^{1,1}_{[a,b]}$ and $\|D_xT(x,y,\lambda)h\|^{1,\infty}_{[a,b]} \le q\|h\|^{1,\infty}_{[a,b]}$ for all $h \in W^{1,\infty}(([a,b],\mathbf{R}^n)$.

Then g is a continuous map on $V \times \Lambda$ and, for each $\lambda \in \Lambda$, the partial map $g(\cdot,\lambda)$ is C^k in V.

2 Differentiability of the Solutions.

The following notation will be used in this section: for each $r > 0$, $\alpha > 0$, $\beta > 0$, denote

$$W^{1,\infty} := W^{1,\infty}([-r,0],\mathbf{R}^n), \quad \text{and} \quad \|\phi\|^{1,1} := |\phi(-r)| + \int_{-r}^{0} |\dot{\phi}(s)|\, ds;$$

$$W^{1,\infty}_{\alpha,0} := \{\phi \in W^{1,\infty}([-r,\alpha],\mathbf{R}^n); \phi(t) \equiv 0 \quad \text{on } [-r,0]\} \quad \text{with the norms}$$

$$\|\phi\|^{1,1}_{\alpha} := \int_0^{\alpha} |\dot{\phi}(s)|\, ds \quad \text{and} \quad \|\phi\|^{1,\infty}_{\alpha} := \text{ess sup}_{s\in[0,\alpha]} |\dot{\phi}(s)| \text{ for } \phi \in W^{1,\infty}_{\alpha,0};$$

$$A(\alpha,\beta) := \{\phi \in W^{1,\infty}_{\alpha,0};\, \|\phi\|^{1,\infty}_{\alpha} \le \beta\};$$

$$B(\alpha,\beta) := \{\phi \in W^{1,\infty}_{\alpha,0};\, \|\phi\|^{1,1}_{\alpha} < \beta\};$$

$$B_\beta := \{\psi \in W^{1,\infty};\, \|\psi\|^{1,1} < \beta\}.$$

For each $\alpha > 0$ and $\phi \in W^{1,\infty}$ we define the element $\tilde{\phi} \in W^{1,\infty}([-r,\alpha],\mathbf{R}^n)$ by $\tilde{\phi}(t) := \phi(t)$, if $-r \le t \le 0$, and $\tilde{\phi}(t) := \phi(0)$, if $0 \le t \le \alpha$.

It is easy to see that $W^{1,\infty}_{\alpha,0}$ is a closed subspace of $W^{1,\infty}([-r,\alpha],\mathbf{R}^n)$ and that for any $\alpha,\beta > 0$ we have $A(\alpha,\beta) \subset B(\alpha,\alpha\beta)$.

It is easy to see that a function $x(t)$ is a solution of (1) if and only if $x(t) = \tilde{\phi}(t) + z(t)$ and $z(t)$ satisfies

$$z(t) = \begin{cases} 0, & \text{if } -r \le t \le 0; \\ \displaystyle\int_0^t F(u(s),v(s),w(s)))\,ds & \text{if } t \ge 0. \end{cases}$$

where $u(s) = \phi(0) + z(s)$, $v(s) = \tilde{\phi}(s-\tau) + z(s-\tau)$, $w(s) = \dot{\tilde{\phi}}(s-\sigma) + \dot{z}(s-\sigma)$.

We first remark that if F is nonlinear in its third variable, then the solution $x(\phi,\tau,\sigma)$ may be discontinuous, as the next example (from [1]) shows.

Consider the equation

$$\dot{x}(t) = \dot{x}^2(t-1)$$

It is easy to see that the solution corresponding to the initial function $\phi \equiv 0$ is $x(0)(t) \equiv 0$. Now, taking ψ defined by

$$\psi(\theta) = \begin{cases} \frac{1}{\varepsilon}(\theta+1) & \text{if } \theta \in [-1,-1+\varepsilon^2] \\ \varepsilon & \text{if } \theta \in [-1+\varepsilon^2, 0]. \end{cases}$$

we have

$$x(\psi)(t) = \begin{cases} \varepsilon + \frac{t}{\varepsilon^2} & \text{if } 0 \le t \le \varepsilon^2 \\ \varepsilon + 1 & \text{if } \varepsilon^2 \le t \le 1 \end{cases}$$

so that $\|\phi-\psi\|^{1,1} = \varepsilon$, which can be made arbitrarily small, while $\|x(\psi)-x(\phi)\|_1^{1,1} = 1$, which shows that x does not vary continuously with ϕ.

We remark here that a continuous dependence result for a very general class of functions F has been obtained by Melvin [7] in terms of the so called bw_1^*-topology. However, that topology is not convenient for our purposes, since we need to work in a normed space in order to make use of the uniform contraction principle. Thus, we follow Driver [1] and assume that F is linear in its third variable, that is we assume that F is given by $F(x,y,z) = f(x,y) + g(x,y)z$, where $f: \mathbf{R}^{2n} \to \mathbf{R}^n$ and $g: \mathbf{R}^{2n} \to \mathbf{R}^{n^2}$ are locally Lipschitz continuous. Thus, we will henceforth be considering the initial value problem

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-\tau)) + g(x(t), x(t-\tau))\dot{x}(t-\sigma) & \text{if } t > 0 \\ x(t) = \phi(t) & \text{if } -r \le t \le 0 \end{cases} \tag{2}$$

and analyze the dependence with respect to (ϕ, τ, σ) of the unique solution $x(\phi, \tau, \sigma)$ of (2).

For each $\alpha > 0$, $z \in W_{\alpha,0}^{1,\infty}$, $\phi \in W^{1,\infty}$, $\tau \in [0, r]$, $\sigma \in (0, r]$, we define $S(z, \phi, \tau, \sigma)$ and $G(z, \phi, \tau, \sigma)$ by

$$\begin{cases} S(z, \phi, \tau)(t) = 0 & \text{if } -r \le t \le 0 \\ S(z, \phi, \tau)(t) = \int_0^t f(u(s), v(s))\, ds & \text{if } 0 \le t \le \alpha \end{cases} \tag{3}$$

$$\begin{cases} G(z, \phi, \tau, \sigma)(t) = 0 & \text{if } t \in [-r, 0] \\ G(z, \phi, \tau, \sigma)(t) = \int_0^t g(u(s), v(s))\dot{\phi}(s-\sigma)\, ds & \text{if } 0 \le t \le \alpha \end{cases} \tag{4}$$

Then, a function x is a solution of (2) if and only if $x = \tilde{\phi} + z$ and $z = S(z, \phi, \tau) + G(z, \phi, \tau, \sigma)$.

For Eq. (2) we have the following result on continuous dependence of the solutions. We include the proof in order to introduce the notation to be used later.

Theorem 2.1 *Suppose $f: \mathbf{R}^{2n} \to \mathbf{R}^n$ and $g: \mathbf{R}^{2n} \to \mathbf{R}^{n^2}$ are locally Lipschitz continuous. Then the solution $x(\phi, \tau, \sigma)(t)$ is continuous in (ϕ, τ, σ), for t in each compact interval $[-r, \alpha] \subset [-r, b)$.*

Proof. It suffices to prove the Theorem for t in a small interval $[0,\alpha]$. If $B \subset W^{1,\infty}$ is any bounded set, take an open set $O \subset \mathbf{R}^{2n}$ such that $\phi \in B$ implies $\phi(0) \times \phi([-r,0]) \subset O$. Then, choose $L = L(O) > 0$ such that

$$\begin{aligned} &|f(x,y)|, |g(x,y)| \leq L \\ &|f(x,y) - f(x',y')| \leq L(|x-x'| + |y-y'|) \\ &|g(x,y) - g(x',y')| \leq L(|x-x'| + |y-y'|) \end{aligned} \tag{5}$$

Now we choose $K > 0, \alpha > 0$ such that $2L(\alpha + \int_0^\alpha |\dot{\phi}(s-\tau)| ds) < 1$ and such that for any $\xi \in B(\alpha, K)$ we have $(\phi(0) + \xi(s), \tilde{\phi}(s-\tau) + \xi(s-\tau)) \in O$ for $t \in [0,\alpha]$. Then, for $z, w \in B(\alpha, K)$, $\phi, \psi \in B$, $\tau, \rho \in [0,r]$, $\sigma, \nu \in (0,r]$ we have

$$\begin{aligned} &\|S(z,\phi,\tau) + G(z,\phi,\tau,\sigma) - S(w,\phi,\tau) - G(w,\phi,\tau,\sigma)\|_\alpha^{1,1} = \\ &= \int_0^\alpha |f(u(s),v(s)) - f(\phi(0) + w(s), \tilde{\phi}(s-\tau) + w(s-\tau))|\, ds + \\ &\qquad + \int_0^\alpha |g(u(s),v(s)) - g(\phi(0) + w(s), \tilde{\phi}(s-\tau) + w(s-\tau))||\dot{\phi}(s-\sigma)|\, ds \\ &\leq 2L(\alpha + \int_0^\alpha |\dot{\phi}(s-\sigma)|\, ds)\|z - w\|_\alpha^{1,1} \end{aligned}$$

$$\begin{aligned} &\|S(z,\phi,\tau) + G(z,\phi,\tau,\sigma) - S(z,\psi,\rho) - G(z,\psi,\rho,\nu)\|_\alpha^{1,1} = \\ &= \int_0^\alpha |f(u(s),v(s)) - f(\psi(0) + z(s), \tilde{\psi}(s-\rho) + z(s-\rho))|\, ds + \\ &\qquad + \int_0^\alpha |g(u(s),v(s)) - g(\psi(0) + z(s), \tilde{\psi}(s-\rho) + z(s-\rho))||\dot{\phi}(s-\sigma)|\, ds + \\ &\qquad + \int_0^\alpha |g(\psi(0) + z(s), \tilde{\psi}(s-\rho) + z(s-\rho))||\dot{\phi}(s-\sigma) - \dot{\psi}(s-\nu)|\, ds \\ &\leq L(2(\alpha + \|\phi\|^{1,1})\|\phi - \psi\|^{1,1} + (2 + K(\phi) + K(z))\|\phi\|^{1,1}|\tau - \rho|) + \\ &\qquad + L\int_0^\alpha |\dot{\phi}(s-\sigma) - \dot{\phi}(s-\nu)|\, ds \end{aligned}$$

Thus, $S(z,\phi,\tau) + G(z,\phi,\tau,\sigma)$ is a contraction in z uniformly with respect to (ϕ,τ,σ), and is continuous in (ϕ,τ,σ). Then, Theorem 1 implies that $z(\phi,\tau,\sigma)$ is continuous.

We now analyze the differentiability of the solution $x(\phi,\tau,\sigma)$ of (2). First, we note that the dependence of x with respect to σ is not somewhat complicated. The next example shows that, even for a very simple equation, the solution is not differentiable with respect to σ.

Consider the equation

$$\dot{x}(t) = \dot{x}(t-\sigma)$$

with $\sigma \in (0,1]$. Choose $\sigma_0 \in (0,1)$ and fix the following initial function

$$\phi(\theta) = \begin{cases} 0 & \text{if } -1 \le \theta \le \sigma_0 \\ \theta + \sigma_0 & \text{if } -\sigma_0 \le \theta \le 0 \end{cases}$$

Denote $x(\sigma) = x(\phi, \sigma)$, and fix $\alpha = \sigma_0$. It is easy to see that for $\sigma \le \sigma_0$, we have $x(\sigma)(t) = t + \sigma_0$, so that $D_\sigma^- x(\sigma_0) = 0$.

On the other hand, for any $\xi > 0$, $x(\sigma_0 + \xi)(t)$ is given by

$$x(\sigma_0+\xi)(t) = \begin{cases} \sigma_0 & \text{if } 0 \le t \le \xi \\ t + \sigma_0 - \xi & \text{if } \xi \le t \le \alpha \end{cases}$$

Thus we have

$$\frac{x(\sigma_0+\xi)(t) - x(\sigma_0)(t)}{\xi} = \frac{1}{\xi}\begin{cases} t & \text{if } 0 \le t \le \xi \\ \xi & \text{if } \xi \le t \le \alpha \end{cases}$$

whence

$$\frac{1}{\xi}|||[x(\sigma_0+\xi) - x(\sigma_0)]|||_\alpha^{1,1} = \frac{1}{\xi}\int_0^\xi dt = 1.$$

Then it follows that $D_\sigma x(\sigma_0)$ does not exist.

Let us also use this equation to show that the derivative $D_\phi x(\phi, \tau, \sigma)$ can be discontinuous in σ. Indeed, a simple computation shows that $D_\phi x(\phi, \sigma)h(t) = \dot{h}(t-\sigma)$ and, though this function is continuous in σ, the continuity is not uniform in h, for $\|h\|_\alpha^{1,1} \le 1$. Let, for $0 < \sigma < 1/2$, h be defined by

$$h(\theta) = \begin{cases} 0 & \text{if } -1 \le \theta \le -1/2 \\ \dfrac{1}{1-2\sigma}(\theta + 1/2) & \text{if } -1/2 \le \theta \le -\sigma \\ -\dfrac{1}{1-2\sigma}(\theta + 2\sigma - 1/2) & \text{if } -\sigma \le \theta \le 1/2 - 2\sigma \\ 0 & \text{if } 1/2 - 2\sigma \le \theta \le 0. \end{cases}$$

Then it follows that

$$
\begin{aligned}
&\|D_\phi x(0,\sigma)h - D_\phi x(0,1/2)h\|_\alpha^{1,1} \geq \\
&\geq \int_0^{1/2-\sigma} |\dot{h}(s-\sigma) - \dot{h}(s-1/2)|\, ds = \\
&= (1/2-\sigma)\frac{2}{1-2\sigma} \\
&= 1
\end{aligned}
$$

which implies that $\|D_\phi x(0,\sigma) - D_\phi x(0,\frac{1}{2})\|_\alpha^1 \geq 2$ (here, $\|\cdot\|_\alpha^1$ denotes the operator norm)

Based on this example, we will search smoothness with respect to (ϕ,τ), of the solution $x(\phi,\tau,\sigma)$ of (2).

We now analyze the differentiability properties of the operators G and S defined by (3) e (4), respectively. We only write down the computations for the operator G. The corresponding computations for the operator S are analogous and thus are omited. Since we are interested in the differentiability for $\tau > 0$, we assume that $\tau \in [\delta, r]$, for some $\delta > 0$. This being the case, we can choose $0 < \alpha < \delta$, so that, for $0 \leq t \leq \alpha$, we have $G(z,\phi,\tau,\sigma)(t) = \int_0^t g(\phi(0)+z(s),\phi(s-\tau))\dot{\phi}(s-\tau)\,ds$. To shorten notation, we denote $u(s) = \phi(0)+z(s)$, $v(s) = \phi(s-\tau)$.

Lemma 2.1 *Suppose $f\colon \mathbf{R}^{2n} \to \mathbf{R}^n$ and $g\colon \mathbf{R}^{2n} \to \mathbf{R}^{n^2}$ are continuously differentiable functions. Then S is continuously differentiable with respect to (z,ϕ,τ). For each fixed σ, $G(z,\phi,\tau,\sigma)$ is differentiable with respect to (z,ϕ,τ). Furthermore, if g does not depend upon the second variable, then G is continuously differentiable in (z,ϕ,τ).*

Proof. The candidate to $D_z G(z,\phi,\tau,\sigma)$ is obviously, the operator given by $D_z G(z,\phi,\tau,\sigma)h(t) = \int_0^t D_x g(\phi(0)+z(s),\phi(s-\tau))h(s)\dot{\phi}(s-\sigma)\,ds$, for $0 \leq t \leq \alpha$. We have

$$
\begin{aligned}
&\|G(z+h,\phi,\tau,\sigma)-G(z,\phi,\tau,\sigma)-D_zG(z,\phi,\tau,\sigma)h\|_\alpha^{1,1}\\
&\quad=\int_0^\alpha |g(u(s)+h(s),v(s))-g(u(s),v(s))-D_xg(u(s),v(s))h(s)||\dot\phi(s-\sigma)|\,ds\\
&\quad\le\int_0^\alpha \sup_{0\le\lambda\le1}|D_xg(u(s)+\lambda h(s),v(s))-D_xg(u(s),v(s))||h(s)||\dot\phi(s-\sigma)|\,ds\\
&\quad\le L\int_0^\alpha |h(s)|^2|\dot\phi(s-\sigma)|\,ds\\
&\quad\le L(\|h\|_\alpha^{1,1})^2\int_0^\alpha |\dot\phi(s-\sigma)|\,ds\\
&\quad=O((\|h\|_\alpha^{1,1})^2),\qquad \text{as } \|h\|_\alpha^{1,1}\to 0
\end{aligned}
$$

which shows that $D_zGz,\phi,\tau,\sigma)$ exists. We now show that $D_zG(z,\phi,\tau,\sigma)$ is continuous.

$$
\begin{aligned}
&\|D_zG(z,\phi,\tau,\sigma)h-D_zG(w,\psi,\rho,\nu)h\|_\alpha^{1,1}=\\
&\quad=\int_0^\alpha |D_xg(u(s),v(s))h(s)\dot\phi(s-\sigma)-D_xg(\psi(0)+w(s),\psi(s-\rho))h(s)\dot\psi(s-\nu)|ds\\
&\quad\le\int_0^\alpha |D_xg(\phi(0)+z(s),\phi(s-\tau))||\dot\phi(s-\sigma)-\dot\psi(s-\nu)|\,ds\\
&\qquad+\int_0^\alpha |D_xg(\phi(0)+z(s),\phi(s-\tau))-D_xg(\psi(0)+w(s),\psi(s-\rho))||\dot\psi(s-\nu)|\,ds\\
&\quad\le L\int_0^\alpha |\dot\phi(s-\sigma)-\dot\psi(s-\nu)|\,ds+L\|\psi-\phi\|^{1,1}+L|\phi(0)-\psi(0)|\int_0^\alpha |\dot\psi(s-\nu)|\,ds+\\
&\qquad+L\int_0^\alpha |z(s)-w(s)||\dot\psi(s-\nu)|\,ds+L\|\phi-\psi\|^{1,1}\int_0^\alpha |\dot\psi(s-\nu)|\,ds+\\
&\qquad+L\int_0^\alpha |\psi(s-\tau)-\psi(s-\rho)||\dot\psi(s-\nu)|\,ds\\
&\quad\le L\Big(\int_0^\alpha |\dot\phi(s-\sigma)-\dot\psi(s-\nu)|\,ds+\|\phi-\psi\|^{1,1}+2\|\phi-\psi\|^{1,1}\|\psi\|^{1,1}+\\
&\qquad+\|z-w\|_\alpha^{1,1}\|\psi\|^{1,1}+(\|\psi\|^{1,1})^2|\tau-\rho|\Big)
\end{aligned}
$$

and hence $D_zG(z,\phi,\tau,\sigma)$ is continuous.

Now we show that $D_\phi G(z,\phi,\tau,\sigma)$ exists and is continuous. The computations are similar to the ones made above, except for a lengthier notation. The candidate to $D_\phi G(z,\phi,\tau,\sigma)$ is the operator given by $D_\phi G(z,\phi,\tau,\sigma)\eta(t)=\int_0^t((D_xg(u(s),v(s))\eta(0)+D_yg(u(s),v(s))\eta(s-\tau))\dot\phi(s-\sigma)+g(u(s),v(s))\dot\eta(s-\sigma))\,ds$, for $0\le t\le\alpha$

$$
\begin{aligned}
&\|G(z,\phi+\eta,\tau,\sigma)-G(z,\phi,\tau,\sigma)-D_\phi G(z,\phi,\tau,\sigma)\eta\|_\alpha^{1,1}=\\
&=\int_0^\alpha |g(u(s)+\eta(0),v(s)+\eta(s-\tau))-g(u(s),v(s))-D_xg(u(s),v(s))\eta(0)-\\
&\qquad -D_yg(u(s),v(s))\eta(s-\tau)||\dot\phi(s-\sigma)|\,ds+\\
&\quad+\int_0^\alpha |g(u(s)+\eta(0),v(s)+\eta(s-\tau))-g(u(s),v(s))|\dot\eta(s-\tau)\,ds\\
&\le\int_0^\alpha \sup_{0\le\lambda\le1}(|D_xg(u(s)+\lambda\eta(0),v(s)+\lambda\eta(s-\tau))-D_xg(u(s),v(s))|\eta(0)|+\\
&\quad+|D_yg(u(s)+\lambda\eta(0),v(s)+\lambda\eta(s-\tau))-D_yg(u(s),v(s))|\eta(s-\tau)|)|\dot\phi(s-\sigma)|\,ds\\
&\quad+\int_0^\alpha |g(u(s)+\eta(0),v(s)+\eta(s-\tau))-g(u(s),v(s))|\dot\eta(s-\tau)\,ds\\
&\le L\int_0^\alpha (|\eta(0)|+|\eta(s-\tau)|)^2|\dot\phi(s-\tau)|\,ds\\
&\quad+L\int_0^\alpha |\eta(s-\tau)|(|\eta(0)|+|\eta(s-\tau)|)\,ds\\
&\le L[4(\|\eta\|_\alpha^{1,1})^2\int_0^\alpha |\dot\phi(s-\sigma)|\,ds+2(\|\eta\|_\alpha^{1,1})^2]\\
&\le 2L(\|\eta\|_\alpha^{1,1})^2(1+2\|\phi\|^{1,1})\\
&=O(\|\eta\|_\alpha^{1,1})^2,\qquad \text{as }\ \|\eta\|_\alpha^{1,1}\to 0
\end{aligned}
$$

which shows the differentiability of G with respect to ϕ. The continuity of $D_\phi G$ is a consequence of the following inequality

$$
\begin{aligned}
&\|D_\phi G(z,\phi,\tau,\sigma)\eta - D_\phi G(w,\psi,\rho,\sigma)\eta\|_\alpha^{1,1} \leq \\
&\leq \int_0^\alpha |D_x g(u(s),v(s))\dot\phi(s-\sigma)\eta(0) - D_x g(\psi(0)+w(s),\psi(s-\rho))\dot\psi(s-\sigma)\eta(0)|\,ds \\
&\quad + \int_0^\alpha |D_y g(u(s),v(s))\eta(s-\tau))\dot\phi(s-\sigma) - D_y g(\psi(0)+w(s),\psi(s-\rho))\cdot \\
&\qquad \cdot\eta(s-\rho))\dot\psi(s-\sigma)|\,ds \\
&\quad + \int_0^\alpha |g(u(s),v(s)) - g(\psi(0)+w(s),\psi(s-\rho))||\dot\eta(s-\sigma))|\,ds \\
&\leq \int_0^\alpha |D_x g(u(s),v(s)) - D_x g(\psi(0)+w(s),\psi(s-\rho))||\dot\phi(s-\sigma)\eta(0)|\,ds \\
&\quad + \int_0^\alpha |D_x g(\psi(0)+w(s),\psi(s-\rho))||\eta(0)||\dot\phi(s-\sigma) - \dot\psi(s-\sigma)|\,ds \\
&\quad + \int_0^\alpha |D_y g(u(s),v(s)) - D_y g(\psi(0)+w(s),\psi(s-\rho))\eta(s-\tau)\dot\phi(s-\sigma)|\,ds \\
&\quad + \int_0^\alpha |D_y g(\psi(0)+w(s),\psi(s-\rho))||\dot\phi(s-\sigma)\eta(s-\tau) - \dot\psi(s-\sigma)\eta(s-\rho)|\,ds \\
&\quad + \int_0^\alpha |g(u(s),v(s)) - g(\psi(0)+w(s),\psi(s-\rho))||\dot\eta(s-\sigma))|\,ds \\
&\leq 2L\int_0^\alpha (|\phi(0)-\psi(0)| + |z(s)-w(s)| + |\phi(s-\tau)-\psi(s-\rho)|)|\dot\phi(s-\sigma)|\,ds \\
&\quad + 2L\|\phi-\psi\|^{1,1} + L\int_0^\alpha |\dot\phi(s-\tau)||\eta(s-\tau)-\eta(s-\rho)|\,ds \\
&\quad + L\int_0^\alpha (|\phi(0)-\psi(0)| + |z(s)-w(s)| + |\phi(s-\tau)-\psi(s-\rho)|)|\dot\eta(s-\sigma)|\,ds \\
&\leq L(4(\|\phi\|^{1,1}+1)\|\phi-\psi\|^{1,1} + (2\|\phi\|^{1,1}+1)\|z-w\|_\alpha^{1,1} + 2\|\phi\|^{1,1}(\|\phi\|^{1,1}+1)|\tau-\rho|)
\end{aligned}
$$

We now show that $D_\tau G$ exists and is the operator given by $D_\tau G(z,\phi,\tau,\sigma)(t) = -\int_0^t D_y G(u(s),v(s))\dot\phi(s-\tau)\dot\phi(s-\sigma)\,ds$. First, we note that, if we define, for $\xi > 0$, $\Delta\phi(s) := \phi(s-\tau-\xi) - \phi(s-\tau)$ and $\omega_\xi(s) := \dot\phi(s-\tau) + \Delta\phi(s)/\xi$, then we have $|\omega_\xi(s)| \leq 2K(\phi)$ and $\omega_\xi(s) \to 0$, almost everywhere in s, as $\xi \to 0$ and that $|\Delta\phi(s)| \leq K(\phi)|\xi|$, in which $K(\phi)$ is a Lipschitz constant of ϕ. It follows that

$$
\begin{aligned}
&\|G(z,\phi,\tau+\xi,\sigma)-G(z,\phi,\tau,\sigma)-D_\tau G(z,\phi,\tau,\sigma)\xi\|_\alpha^{1,1}= \\
&=\int_0^\alpha |g(u(s),\phi(s-\tau-\xi))-g(u(s),\phi(s-\tau))+ \\
&\qquad +D_yg(u(s),\phi(s-\tau))\dot\phi(s-\tau)\dot\phi(s-\sigma)\xi|\,ds \\
&\le \int_0^\alpha |g(u(s),\phi(s-\tau-\xi))-g(u(s),\phi(s-\tau))+ \\
&\qquad +D_yg(u(s),\phi(s-\tau))\Delta\phi(s)\dot\phi(s-\sigma)|\,ds+ \\
&\quad +|\xi|\int_0^\alpha |D_yg(u(s),\phi(s-\tau))||\omega_\xi(s)||\dot\phi(s-\sigma)|\,ds \\
&\le \int_0^\alpha |\Delta\phi(s)|\sup_{0\le\lambda\le 1}|D_yg(u(s),v(s)+\lambda\Delta\phi(s))-g(u(s),v(s))||\dot\phi(s-\sigma)|\,ds \\
&\quad +L|\xi|K(\phi)\int_0^\alpha |\omega_\xi(s)|\,ds \\
&\le LK(\phi)(\int_0^\alpha |\Delta\phi(s)|^2\,ds+\int_0^\alpha |\omega_\xi(s)|)\,ds \\
&\le LK(\phi)^3|\xi|^2\alpha+LK(\phi)|\xi|\int_0^\alpha |\omega_\xi(s)|\,ds
\end{aligned}
$$

The Lebesgue Dominated Convergence implies that $\int_0^\alpha |\omega_\xi(s)|\,ds \to 0$, as $\xi\to 0$, and therefore we have

$$\|G(z,\phi,\tau+\xi,\sigma)-G(z,\phi,\tau,\sigma)-D_\tau G(z,\phi,\tau,\sigma)\xi\|_\alpha^{1,1}=o(|\xi|),\ \text{as } |\xi|\to 0$$

For each fixed ϕ , $D_\tau G(z,\phi,\tau,\sigma)$ is continuous in (z,τ,σ)

$$
\begin{aligned}
&\|D_\tau G(z,\phi,\tau,\sigma)-D_\tau G(w,\phi,\rho,\nu)\|_\alpha^{1,1}\le \\
&\le \int_0^\alpha |D_yg(u(s),\phi(s-\tau))||\dot\phi(s-\tau)\dot\phi(s-\sigma)-\dot\phi(s-\rho)\dot\phi(s-\nu)|\,ds+ \\
&\quad +\int_0^\alpha |D_yg(u(s),\phi(s-\tau))-D_yg(\phi(0)+w(s),\phi(s-,\rho))||\dot\phi(s-\rho)\dot\phi(s-\nu)|\,ds \\
&\le LK(\phi)(\int_0^\alpha |\dot\phi(s-\sigma)-\dot\phi(s-\nu)|\,ds+\int_0^\alpha |\dot\phi(s-\tau)-\dot\phi(s-\rho)|\,ds+ \\
&\quad +\alpha K(\phi)(\|z-w\|_\alpha^{1,1}+K(\phi)|\tau-\rho|))
\end{aligned}
$$

which shows the continuity claimed.

The computations made above imply that, for each fixed σ, $G(z,\phi,\tau,\sigma)$ is differentiable in (z,ϕ,τ). For the case that $g(x,y)\equiv g(x)$, then, obviously, G does not depends on τ and we have $D_\tau G$ continuous. Thus it follows that G is continuously differentiable.

Lemma 2.1 and the Uniform Contraction Principle imply the following results

Theorem 2.2 *Suppose the functions $f: \mathbf{R}^{2n} \to \mathbf{R}^n$ and $g: \mathbf{R}^{2n} \to \mathbf{R}^{n^2}$ are C^k, $k \geq 1$, and that the $k-th$ order derivatives k, $D^k f$, $D^k g$, are locally Lipschitz continuous. Then, the unique solution $x(\phi, \tau, \sigma)(t)$ of (2) is continuous in (ϕ, τ, σ); and, for each fixed $\sigma^0 > 0$, $x(\phi, \tau, \sigma)$ is differentiable in (ϕ, τ), for t on each compact subinterval of $[-r, b)$. Furthermore, for each fixed (τ^0, σ^0), with $\tau^0 > 0$, $\sigma^0 > 0$, $x(\phi, \tau, \sigma)(t)$ is C^k in ϕ.*

If $g(x, y)$ is not independent of y, then the derivative may be discontinuous, as the next example shows. Consider the equation

$$\dot{x}(t) = x(t - \tau)\dot{x}(t - 1/2)$$

and, for any $0 \leq \tau < 1/2$, take the initial function

$$\phi^\tau(\theta) = \begin{cases} 0, & \text{if } -1 \leq \theta \leq -1/2; \\ (\theta + 1/2)/(1/2 - \tau), & \text{if } -1/2 \leq \theta \leq -\tau; \\ 1, & \text{if } -\tau \leq \theta \leq 0; \end{cases}$$

Then, we have, for $0 \leq \rho \leq \tau < 1/2$, $\|x(\phi^\tau, \tau) - x(\phi^\rho, \rho)\|^{1,1}_{[0,1/2-\rho]} = 2(\tau - \rho)/(1/2 - \rho)$, which shows that $x(\phi, \tau)$ is not locally Lipschitz continuous in (ϕ, τ).

However, if $g(x, y) \equiv g(x)$, that is, g does not depend on the second variable, then the conclusion of the above theorem may be improved to the following result.

Theorem 2.3 *Suppose the functions $f: \mathbf{R}^{2n} \to \mathbf{R}^n$ are $g: \mathbf{R}^n \to \mathbf{R}^{n^2}$ C^k, $k \geq 1$, and that the $k - th$ order derivatives $D^k f$, $D^k g$, are locally Lipschitz continuous functions. Then, the unique solution $x(\phi, \tau, \sigma)(t)$ of*

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau)) + g(x(t))\dot{x}(t - \sigma) & \text{if } t > 0 \\ x(t) = \phi(t) & \text{if } -r \leq t \leq 0 \end{cases} \tag{6}$$

is continuous in (ϕ, τ, σ); and, for each fixed $\sigma^0 > 0$, $x(\phi, \tau, \sigma)$ is continuously differentiable in (ϕ, τ), for t on each compact subinterval of $[-r, b)$. Furthermore, for each fixed (τ^0, σ^0), with $\tau^0 > 0$, $\sigma^0 > 0$, $x(\phi, \tau, \sigma)(t)$ is C^k in ϕ, and if ϕ^0 is any fixed function $W^{k,\infty}$ (that is, the $k - th$ order derivative of ϕ, $D^k\phi$, belongs to L^∞), then $x(\phi^0, \tau, \sigma^0)$ is C^k in τ.

For the case $g(x) \equiv 0$, the previous theorem gives one of the main results of [4] (actually, we present it slightly modified, with some minor change in the statement).

Corollary 2.1 *Suppose the function* $f: \mathbf{R}^{2n} \to \mathbf{R}^n$ *is* C^k, $k \geq 1$, *and its* $k-th$ *order derivative* $D^k f$, *is locally Lipschitz continuous. Then, the unique solution* $x(\phi, \tau, \sigma)(t)$ *of*

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-\tau)) & se\ t > 0 \\ x(t) = \phi(t) & se\ -r \leq t \leq 0 \end{cases} \tag{7}$$

is continuous in (ϕ, τ); $x(\phi, \tau)$ *is continuously differentiable in* (ϕ, τ), *for* t *in each compact subinterval of* $[-r, b)$. *Furthermore, for each fixed* $\tau^0 > 0$, $x(\phi, \tau, \sigma)(t)$ *is* C^k *in* ϕ, *and if* ϕ^0 *is any fixed in* $W^{k,\infty}$ *(that is, the* $k-th$ *order derivative of* ϕ, $D^k\phi$, *is in* L^∞*), then* $x(\phi^0, \tau)$ *is* C^k *in* τ.

3 Some Remarks and Extensions.

The results of Theorems 3 and 4 above remain valid for equations with finitely many delays, that is, equations of the form

$$\dot{x}(t) = f(x(t), X(t-\tau)) + H(x(t), X(t-\tau) \cdot \dot{X}(t-\sigma)$$

in which

$$X(t-\tau) = (x(t-\tau_1), \ldots, x(t-\tau_p))$$

$$H(x(t), X(t-\tau)) \cdot \dot{X}(t-\sigma) = \textstyle\sum_{j=1}^{s} g_j(x(t), x(t-\tau_1), \ldots, x(t-\tau_p))\dot{x}(t-\sigma_j)$$

Let $(\tau_1^0, \ldots, \tau_p^0)$, $(\sigma_1^0, \ldots, \sigma_s^0)$ be fixed. For each j, $1 \leq j \leq p$, the proof of Lemma 2.1 applies to the operators

$$S(z, \phi, \tau_1^0, \ldots, \tau_{j-1}^0, \tau_j, \tau_j^0 + 1, \ldots, \tau_p^0)$$

$$G(z, \phi, \tau_1^0, \ldots, \tau_{j-1}^0, \tau_j, \tau_j^0 + 1, \ldots, \tau_p^0, \sigma_1^0, \ldots, \sigma_s^0).$$

For example, in the case of Theorem 4, we have that, for each fixed $(\sigma_1^0, \ldots, \sigma_s^0)$, $x(\phi, \tau_1, \ldots, \tau_p, \sigma_1^0, \ldots, \sigma_s^0)$ is continuously differentiable in $(\phi, \tau^1, \ldots, \tau_p)$; for each fixed $(\tau_1^0, \ldots, \tau_p^0)$, it is C^k in ϕ, for each fixed ϕ^0 in $W^{k,\infty}$, $x(\phi^0, \tau^1, \ldots, \tau_p, \sigma_1^0, \ldots, \sigma_s^0)$ is C^k in $(\tau_1^0, \ldots, \tau_p^0)$.

The computations made above also apply to the case in which the delays are functions of t. In order to simplify notation, we describe this result for the equation

$$\dot{x}(t) = f(x(t), x(t - \tau(t))) + g(x(t))\dot{x}(t - \sigma(t))$$

where $\tau(t)$ e $\sigma(t)$ are continuously differentiable functions of t. Consider, for $a > 0$, the Banach space $C([0,a], \mathbf{R})$. The set $V = \{\tau, \sigma \in C([0,a], \mathbf{R}); \tau(t), \sigma(t) \in (0,a)$, for $t \in [0,a]\}$ is open in $C([0,a], \mathbf{R})$. The computations of Lemma 2.1 remain valid here, and we can prove that, for $\tau, \sigma^0 \in V$ (σ^0 fixed), $\phi \in W^{1,\infty}$, the solution $x(\phi, \tau, \sigma^0)$ of the above equation is continuously differentiable in (ϕ, τ); if $\tau^0 \in V$ is fixed, then $x(\phi, \tau^0, \sigma^0)$ is C^k is ϕ; and if ϕ is any fixed function in $W^{k,\infty}$, then $x(\phi^0, \tau, \sigma^0)$ is C^k in τ.

For the retarded equation (7), we also remark that, the solutions of (7) become smoother at each interval of length r. Thus, if a solution of (7) is defined and is bounded in $\mathbf{R}$, then it is as smooth in t as f is. So, the initial function of (7) for such a solution is C^k, if f is C^k, and therefore, the solution will be C^k in τ. In the case f is dissipative, Equation (7) has an attractor. If f is analytic, then the elements of the attractor are analytic functions of t. Then we can conclude that $x(\phi, \tau)$ is analytic in (ϕ, τ). This property is no longer true for the neutral type equation (2), since the solutions of this equation, in general, do not become smoother as t increases (the examples given above show this).

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