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The Contact Process on a Finite Set, III: The Critical Case

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Abstract. We show that if σ_N is the time that the contact process on $\{1,...N\}$ first hits the empty set then for $\lambda = \lambda_c$, the critical value for the contact process on $\mathbb{Z}$, $\sigma_N/N \to \infty$ and $\sigma_N/N^4 \to 0$ in probability as $N \to \infty$. The keys to the proof are a new renormalized bond construction and lower bounds for the fluctuations of the right edge. As a consequence of the result we get bounds on some critical exponents. We also study the analogous problem for bond percolation in $\{1,...N\} \times \mathbb{Z}$ and investigate the limit distribution of $\sigma_N/E\sigma_N$.

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1. Introduction

In this paper we continue the study of the contact process on a large finite set. The reader will find the motivation for such questions and relations to previous work discussed in the introductions of Durrett and Liu (1988) and Durrett and Schonmann (1988). For motivations coming from the physical problem of modeling metastability, the reader is referred to Cassandro, Galves, Olivieri, and Vares (1984) and to Schonmann (1985). For examples of other systems which have been studied on a finite set, see Lebowitz and Schonmann (1987), Cox (1987), and Cox and Greven (1987) and (1988).

We begin by describing the model under consideration. For more details or facts that we cite without reference, see Griffeath (1981), Chapter VI of Liggett (1985), or Chapter 4 of Durrett (1988). The contact process is a Markov process with state space the subsets of $\mathbb{Z}$, and transition probabilities that satisfy

$$P(x \notin \xi_{t+s} \mid \xi_t) \sim s \qquad \text{if } x \in \xi_t \tag{1.1a}$$

$$P(x \in \xi_{t+s} \mid \xi_t) \sim \lambda s |\xi_t \cap \{x-1,x+1\}| \qquad \text{if } x \notin \xi_t \tag{1.1b}$$

as $s \to 0$, where $f(s) \sim g(s)$ means $f(s)/g(s) \to 1$ as $s \to 0$. If we think of the sites in ξ_t as occupied by particles, then the dynamics can be described as: "particles die at rate one and are born at vacant sites at rate λ times the number of occupied neighbors." It is by now well known that there is a unique Markov process with the properties given above and there are several ways to construct it. We will introduce one of these (the graphical representation) in Section 2.

We will use $\{ \xi_t^A , t \geq 0 \}$ to denote the contact process with $\xi_0^A = A \subset \mathbb{Z}$. For simplicity, we write ξ_t^x for $\xi_t^{\{x\}}$ and use similar abbreviations below. Let $\tau^A = \inf\{ t \geq 0 : \xi_t^A = \phi \}$, where $\inf \phi = \infty$. Let $\rho(\lambda) = P(\xi_t^0 \neq \phi \text{ for all } t \geq 0) = P(\tau^0 = \infty)$, and let $\lambda_c = \inf \{ \lambda : \rho(\lambda) > 0 \}$. It is known that $1 < \lambda_c \leq 2$ and that $\rho(\lambda)$ is continuous on (λ_c, ∞), but

it is an open question whether $\rho(\lambda_c) = 0$. While we are still not able to settle the last question in this paper, our results can be used to prove some new facts about the critical contact process on $\mathbb{Z}$.

The contact process on $\{1,...N\}$ has transition probabilities given by (1.1) for $x \in \{1,...N\}$. We denote by ζ_t^N the process starting from $\{1,...N\}$, and let $\sigma_N = \inf\{ t \geq 0 : \zeta_t^N = \phi \}$. Since ζ_t^N is a Markov chain on a finite set, $P(\sigma_N < \infty) = 1$ for all λ. Differences between the λ's appear when we let $N \to \infty$. In Durrett and Liu (1988) and Durrett and Schonmann (1988) the following results were proved:

(1.2) If $\lambda < \lambda_c$ then there is a constant $\gamma_1(\lambda) \in (0,\infty)$ so that

$$\sigma_N/(\log N) \to 1/\gamma_1(\lambda) \text{ in probability as } N \to \infty.$$

(1.3) If $\lambda > \lambda_c$ then there is a constant $\gamma_2(\lambda) \in (0,\infty)$ so that

$$(\log \sigma_N)/N \to \gamma_2(\lambda) \text{ in probability as } N \to \infty.$$

The constants $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ may be defined by

$$\gamma_1(\lambda) = -\lim_{n\to\infty} \frac{1}{n} \log P(\xi_n^0 \neq \phi), \tag{1.4}$$

$$\gamma_2(\lambda) = -\lim_{n\to\infty} \frac{1}{n} \log P(\tau^{\{1,...N\}} < \infty). \tag{1.5}$$

In Durrett, Schonmann, and Tanaka (1988) we argue that $L_{\|}(\lambda) = 1/\gamma_1(\lambda)$ and $L_{\perp}(\lambda) = 1/\gamma_2(\lambda)$ are reasonable definitions for the temporal and spatial correlation lengths used in nonrigorous studies of the contact process and oriented percolation (see e.g., Grassberger and de la Torre (1979) and Kinzel and Yeomans (1981)). We will have more to say about these quantities below.

(1.2) and (1.3) tell us that in the subcritical case σ_N grows logarithmically with N, and in the supercritical case σ_N grows exponentially with N. From these two results the reader can probably guess that in the critical case σ_N grows like a power of N. Indeed, we will show:

(1.6) Theorem If $\lambda = \lambda_c$ and $a,b \in (0,\infty)$ then $P(aN \le \sigma_N \le bN^4) \to 1$ as $N \to \infty$.

The lower bound is essentially due to Griffeath (1981). He got a weaker result (see p.179 of his paper) because it was not known at that time that the edge speed α (defined in the proof of (3.1) below) was 0 at the critical value. Our main contribution is to prove the upper bound. The keys to the proof are the following results concerning the right edge $r_t = \sup \xi_t^{(-\infty,0]}$ and the survival time τ^0.

(1.7) Let $v > 0$. If $t^{-v}E(r_t^2) \to \infty$ as $t \to \infty$, then $\sigma_N/N^{2/v} \to 0$ in probability as $N \to \infty$.

(1.8) For any $\lambda > 0$ there is a constant $C > 0$ so that $Var(r_t) \ge C\,[t]\,P(\tau^0 > t)$, where $[t]$ = the greatest integer $\le t$.

(1.9) If $\lambda = \lambda_c$ then $t^{1/2} P(\tau^0 > t) \to \infty$ as $t \to \infty$.

The last result is (8) in Section 4b of Durrett (1988). When plugged into (1.8) it shows $t^{-1/2} Var(r_t) \to \infty$, which with (1.7) gives the upper bound in the theorem.

We believe that nothing is lost in (1.7). The other two results are not the best possible results. The nonrigorous studies quoted above suggest $P(\tau^0 > t) \approx t^{-.161}$. As for (1.8), which generalizes a result of Galves and Presutti (1987), we believe that when $\lambda = \lambda_c$ $E(r_t^2)/t^{1+\epsilon} \to \infty$ for some $\epsilon > 0$, and hence that the correct power of N is less than 2. The next result makes it clear that 4 is far from the right answer:

(1.10) If $\lambda = \lambda_c$ and $\theta > 2.5$ then $\limsup_{N\to\infty} P(\sigma_N < N^\theta) = 1$.

Proof: On $\{\sigma_N \geq N^\theta\}$, some point of the form $(x,N/2)$ with $1 \leq x \leq N$ must have paths in the graphical representation connecting it to $\mathbb{Z} \times 0$ and to $\mathbb{Z} \times \{N\}$. (This is the key to the proof of (1.9).) So if (1.10) is false

$$\liminf_{N\to\infty} N\, P(\tau^0 \geq N^\theta/2)^2 > 0$$

Changing variables $t = N^\theta/2$ and using (1.8) shows (1.7) is true when $v < 1-(1/2\theta)$, but this is a contradiction unless $\theta \leq 2/(1-(1/2\theta))$, i.e., $\theta \leq 2.5$. To get the last conclusion observe that the left hand side is increasing, the right hand side is decreasing on $(1/2,\infty)$, and they are equal when $\theta = 2.5$.

We believe that for finite range growth models on $\mathbb{Z}$ (i.e. translation invariant attractive systems in which ϕ is a trap), σ_N/N^2 is tight. In support of this conjecture we observe that (i) if we consider the biased voter model on a finite set (as in Durrett and Liu (1988)) then σ_N/N^2 has a limiting distribution, and (ii) if the contact process survives at the critical value then (1.8) and a modification of (1.7) show that σ_N/N^2 is tight. For the contact process, scaling theory predicts (V.Privman, private communication) that the right power is $\nu_\parallel/\nu_\perp \approx 1.74/1.10 = 1.58$, where the ν's are critical exponents defined by

$$L_\parallel(\lambda) \approx |\lambda-\lambda_c|^{-\nu_\parallel} \qquad L_\perp(\lambda) \approx |\lambda-\lambda_c|^{-\nu_\perp}.$$

Here $L_\parallel(\lambda)$ and $L_\perp(\lambda)$ are the correlation lengths defined above and $f(\lambda) \approx |\lambda-\lambda_c|^{-\nu}$ means $\log f(\lambda) / \log|\lambda-\lambda_c| \to -\nu$ as $\lambda \to \lambda_c$. (To complete the picture here we would have to define $L_\parallel(\lambda)$ for $\lambda > \lambda_c$ and $L_\perp(\lambda)$ for $\lambda < \lambda_c$. We will give the definition of the second quantity in Section 5. In Durrett, Schonmann, and Tanaka (1988) other definitions are given and their relationships are discussed.)

(1.9) gives a lower bound on the survival time for the critical contact process. Combining this with (1.6) and (1.10) gives a lower bound on its spatial spread.

(1.11) Let $R^0 = \sup(\bigcup_{t \geq 0} \xi_t^0)$. If $\lambda = \lambda_c$ then as $r \to \infty$

$$r^2 (\log r)^{1/2} P(R^0 > r) \to \infty, \text{ and}$$

$$\limsup_{r \to \infty} r^{1.25+\epsilon} P(R^0 > r) \to \infty \text{ for any } \epsilon > 0.$$

When our results are combined with an idea from Chayes, Chayes, Fisher, and Spencer (1986), who proved the analogue of (1.12) for ordinary percolation and other systems, we can get bounds on the correlation lengths defined above. In all the results below ϵ is an arbitrary positive number.

(1.12) As $\lambda \uparrow \lambda_c$ $\qquad \liminf L_{\|}(\lambda) / (\lambda - \lambda_c)^{-1+\epsilon} > 0.$

(1.12) is proved by using the lower bound in (1.6). If we could improve the lower bound to $N^{1+\delta}$ then we would show that $\nu_{\|} > 1$, its "mean field" value. As we remarked above, the right power is supposed to be $\nu_{\|} = 1.74$.

Using the lower bound in (1.6) and (1.10) gives

(1.13) As $\lambda \downarrow \lambda_c$ $\qquad \liminf L_{\perp}(\lambda) / (\lambda - \lambda_c)^{-(2/5)+\epsilon} > 0,$

and $\qquad \limsup L_{\perp}(\lambda) / (\lambda - \lambda_c)^{-(4/7)+\epsilon} > 0.$

The second result shows that $\nu_{\perp}$ does not take its mean field value 1/2, but is still far from the right answer $\nu_{\perp} = 1.10$. Using (1.11) we can get results for $L_{\perp}$ in the subcritical regime. Our result is worse than in the supercritical case although it is implicit in the definition ($L_{\perp}(\lambda) \approx (\lambda - \lambda_c)^{-\nu_{\perp}}$) that the exponent should not depend upon the direction in which we approach λ_c.

(1.14) As $\lambda \uparrow \lambda_c$ $\qquad \liminf L_{\perp}(\lambda) / (\lambda_c - \lambda)^{-(2/9)+\epsilon} > 0,$

and $\quad \limsup L_\perp(\lambda)/(\lambda_c-\lambda)^{-(1/3)+\epsilon} > 0.$

Having seen three of the four possible combinations in $\{ \lambda > \lambda_c , \lambda < \lambda_c \} \times \{ \| , \perp \}$, the reader should be wondering what we know about $L_\|(\lambda)$ as $\lambda \downarrow \lambda_c$. It follows easily from the definitions in Durrett, Schonmann, and Tanaka (1988) that $L_\|(\lambda) \geq L_\perp(\lambda)$. Combining the last observation with (1.13) gives bounds for $\nu_\|$ but does not come close to beating the mean field value.

(1.7) is proved in Section 3, and (1.8) in Section 4. The main ideas of the proof of the second result are due to Galves and Presutti (1987) but our proof is simpler and extends the result to the critical case. (1.11)–(1.14) are proved in Section 5. The astute reader will have noticed that we have not mentioned Section 2. The title below should indicate what we study there.

2. Limiting Behavior of $\sigma_N/E\sigma_N$

From the proofs of (1.2) and (1.3) one can easily get the corresponding statements about expected values:

(2.1) If $\lambda < \lambda_c$ then $E\sigma_N/(\log N) \to 1/\gamma_1(\lambda)$ as $N \to \infty$.

(2.2) If $\lambda > \lambda_c$ then $\log(E\sigma_N)/ N \to \gamma_2(\lambda)$ as $N \to \infty$.

From (1.2) and (2.1) it follows that

(2.3) $$\sigma_N/E\sigma_N \to 1 \text{ in probability for } \lambda < \lambda_c.$$

We also know that

(2.4) $$P(\sigma_N/E\sigma_N > t) \to e^{-t} \text{ for } \lambda > \lambda_c.$$

This result was first proved by Cassandro, Galves, Olivieri, and Vares (1984) for large λ, and extended to $\lambda > \lambda_c$ by Schonmann (1985). A simple proof is given in Durrett and Schonmann (1988).

Comparing (2.3) and (2.4) we see that if we let χ be the limit of $\sigma_N/E\sigma_N$ then χ is deterministic in the subcritical case, and is unpredictable in the supercritical case, that is, χ has the lack of memory property

(2.5) $$P(\chi > t+s \mid \chi > s) = P(\chi > t).$$

In the critical case we expect that the limiting distribution is something in between these

two extremes. To be precise, we expect χ to be random but the $=$ in (2.5) will be replaced by $<$.

To support the speculation in the last paragraph we will now describe a related result for ordinary (i.e. not oriented) bond percolation. As explained in Durrett and Schonmann (1988), there is a similarity between results for the contact process on a finite set and the results in Grimmett (1981) for sponge crossings in two dimensional bond percolation. To describe the connection, consider bond percolation in $[1,N] \times [0,\infty)$ and let

$$\bar{\sigma}_N = \sup\{\, \ell \geq 0 : \text{there is a path of open bonds from } [1,N] \times \{0\} \text{ to } [1,N] \times \{\ell\} \text{ inside } [1,N] \times [0,\infty) \,\}.$$

Grimmett (1981) showed that

(2.6) If $p < 1/2$ then $\bar{\sigma}_N / (\log N) \to 1/\gamma(p)$ in probability.

(2.7) If $p > 1/2$ then $(\log \bar{\sigma}_N) / N \to \gamma(p)$ in probability.

Here $\gamma(p)$ is a positive constant which in the subcritical case can be defined as

$$-\lim_{n\to\infty} \frac{1}{n} \log P(\, (0,0) \to (n,0) \,),$$

where $x \to y$ stands for "there is an open path from x to y." To define $\gamma(p)$ for $p > 1/2$ we set $\gamma(p) = \gamma(1-p)$. By analogy with the contact process, the reader should guess:

(2.8) If $p < 1/2$ then $\bar{\sigma}_N / E\bar{\sigma}_N \to 1$ in probability.

(2.9) If $p > 1/2$ then $P(\, \bar{\sigma}_N / E\bar{\sigma}_N > t \,) \to e^{-t}$.

The first conclusion is an easy consequence of Grimmett's proof. The second is proved in Durrett and Schonmann (1988). To investigate the critical case $p = 1/2$ we observe that by the self–duality of bond percolation (see Kesten (1982) or Durrett (1988) Section 6a)

$$P(\bar{\sigma}_N \geq N) = 1/2. \tag{2.10}$$

It is easy to see that

$$P(\bar{\sigma}_N \geq L+K) \leq P(\bar{\sigma}_N \geq L)P(\bar{\sigma}_N \geq K), \tag{2.11}$$

since to cross $[1,N]\times[0,L+K]$ there must be crossings of $[1,N]\times[0,L]$ and $[1,N]\times[L,L+K]$ and the last two events are independent. From the last observation it follows that $P(\bar{\sigma}_N \geq kN) \leq P(\bar{\sigma}_N \geq N)^k$, so $E\bar{\sigma}_N / N$ is bounded and $\bar{\sigma}_N/N$ is tight. With a little more work one can show

(2.12) Theorem. No subsequential limit of $\bar{\sigma}_N/N$ is degenerate or exponential.

Proof: We use two well known properties of sponge crossings (see (2) and (3) in Section 6a of Durrett (1988)).

$$P(\bar{\sigma}_N \geq 3N/2) \geq (1 - (1-P(\bar{\sigma}_N \geq N))^{1/2})^3 \tag{2.13}$$

$$P(\bar{\sigma}_N \geq kN) \geq P(\bar{\sigma}_N \geq (k+1)N/2)^3 \text{ for } k \geq 1 \tag{2.14}$$

The first inequality and (2.10) shows that no limit is degenerate. Combining (2.13) and (2.14) one gets easily

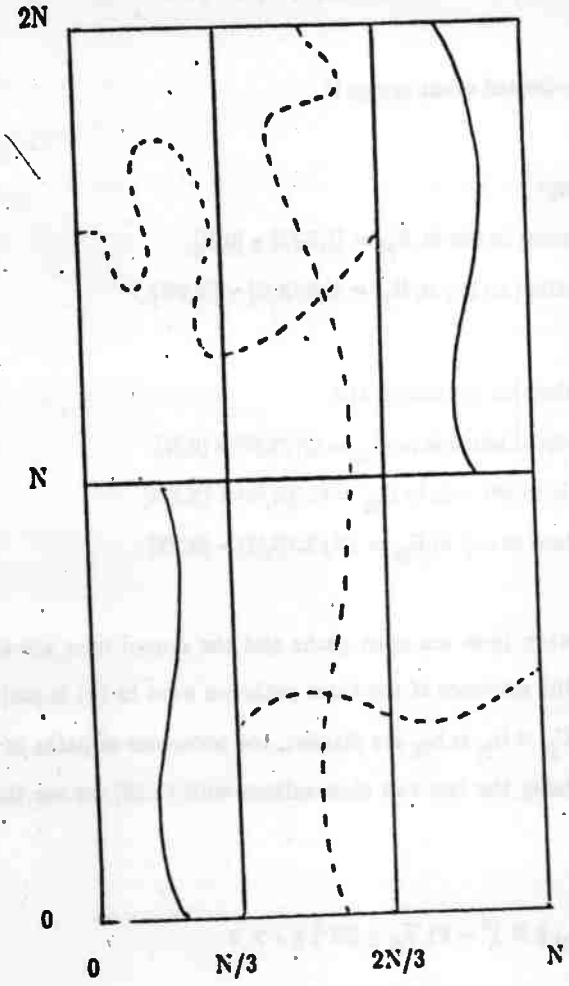

Figure 1

(2.15) $P($ crossings of $[1,N] \times [0,N]$ and $[1,N] \times [N,2N]$ exist but there is no crossing of $[1,N] \times [0,2N]$ $) \geq \epsilon > 0.$

To prove (2.15), notice that the desired event occurs if

(a) there are open crossings

from bottom to top in $A_N = [1,N/3] \times [0,N]$,

and from bottom to top in $B_N = [2N/3,N] \times [N,2N]$,

and

(b) there are closed crossings on the dual graph

from right side to left side in $C_N = (N/3,N) \times [0,N]$,

from right side to left side in $D_N = (0,2N/3) \times [N,2N]$,

and from bottom to top in $E_N = (N/3,2N/3) \times [0,2N]$.

See Figure 1. There the solid wavy lines are open paths and the dotted ones are closed. Harris' inequality implies that the existence of the three paths we want in (b) is positively correlated. Since A_N, B_N, and $C_N \cup D_N \cup E_N$ are disjoint, the occurence of paths in those regions are independent. Combining the last two observations with (2.15) we see that for large N

$$P(\bar{\sigma}_N \geq N)^2 - P(\bar{\sigma}_N \geq 2N) \geq \epsilon > 0$$

so no limits are exponential.

3. Proof of (1.7)

The first thing we have to do is to introduce the usual construction of the contact process. (See the sources cited in the introduction for more details.) To each $x \in \mathbb{Z}$ we associate three independent Poisson processes with rates 1, λ, and λ respectively. Let $\{ T_n^{x,k} , n \geq 1 \}$ $k = 1,2,3$ be the arrival times for these processes. For each $x \in \mathbb{Z}$ and $n \geq 1$, we write a δ at each point $(x,T_n^{x,1})$ and draw arrows from $(x,T_n^{x,2})$ to $(x+1,T_n^{x,2})$ and from $(x,T_n^{x,3})$ to $(x-1,T_n^{x,3})$. The effect of a δ is to kill a particle (if one is present), while the arrows cause a birth to occur if they point from an occupied site to one which is vacant.

To formalize the intuition we say there is a path from (x,s) to (y,t) if it is possible to go from (x,s) to (y,t) by a path which goes up and across arrows in the direction of their orientation without crossing any δ's. Using the "percolation structure" introduced above we can define all the processes we are interested in

$$\xi_t^{(A,s)} = \{ y : \text{for some } x \in A \text{ there is a path from } (x,s) \text{ to } (y,t) \}$$

$$\xi_t^A = \xi_t^{(A,0)}$$

$$\zeta_t^N = \{ y : \text{for some } x \in \{1,\ldots N\} \text{ there is a path from } (x,s) \text{ to } (y,t) \text{ inside } [1,\ldots N] \times [0,t] \}$$

We will now prove the lower bound in (1.8):

(3.1) If $\lambda = \lambda_c$ and $a < \infty$ then $P(\sigma_N < aN) \to 0$ as $N \to \infty$.

Proof: We begin by recalling some facts about the right edge, $r_t = \sup \xi_t^{(-\infty,0]}$. It is known that $r_t/t \to \alpha(\lambda)$ a.s. as $t \to \infty$, where $\alpha(\lambda) = -\infty$ if $\lambda < \lambda_c$, $\alpha(\lambda_c) = 0$, and $\alpha(\cdot)$ is strictly

increasing and continuous on $[\lambda_c,\infty)$. Let $A = \{ x \in \mathbb{Z} : x \le 2N/3 \}$ and $r_t^A = \sup \xi_t^A$. Then r_t^A has the same distribution as $r_t + [2N/3]$. Let $G = \{ r_t^A \in (N/3,N)$ for all $t \in [0,aN] \}$. Since $\alpha(\lambda_c) = 0$, it follows from the limiting behavior of the right edge recalled above that $P(G) \to 1$ as $N \to \infty$.

On G there is a path from $(-\infty,2N/3] \times \{0\}$ to $[N/3,\infty) \times \{aN\}$ which does not cross the vertical line $\{N\} \times [0,aN]$. To finish the proof we have to argue that with high probability the path does not touch $\{0\} \times [0,aN]$. To do this, we observe that by the symmetry of the Poisson process with respect to time reversal (i.e. the self–duality of the contact process), the probability of having a path touch $\{0\} \times [0,aN]$ and end up in $[N/3,\infty) \times \{aN\}$ is the same as the probability of a path from $[N/3,\infty) \times \{0\}$ to $\{0\} \times [0,aN]$, which by the left–right symmetry of the model is more unlikely than G^c. Combining the last observation with results in the last paragraph we have shown

$$P(\sigma_N > aN) \ge 1 - 2P(G^c)$$

and the proof is complete.

Remark. Larry Gray invented the trick used in the second paragraph of the proof above to simplify the renormalized bond construction of Durrett and Griffeath (1983).

We turn now to the proof of the upper bound on σ_N:

(3.2) If $\lambda = \lambda_c$ and $b > 0$ then $P(\sigma_N > bN^4) \to 0$ as $N \to \infty$.

Our argument is divided into several steps. In the first one we use a "renormalized bond construction" which has its roots in the work of Russo (1981), and which has appeared in various forms in a number of papers. Here we introduce yet another variation on the

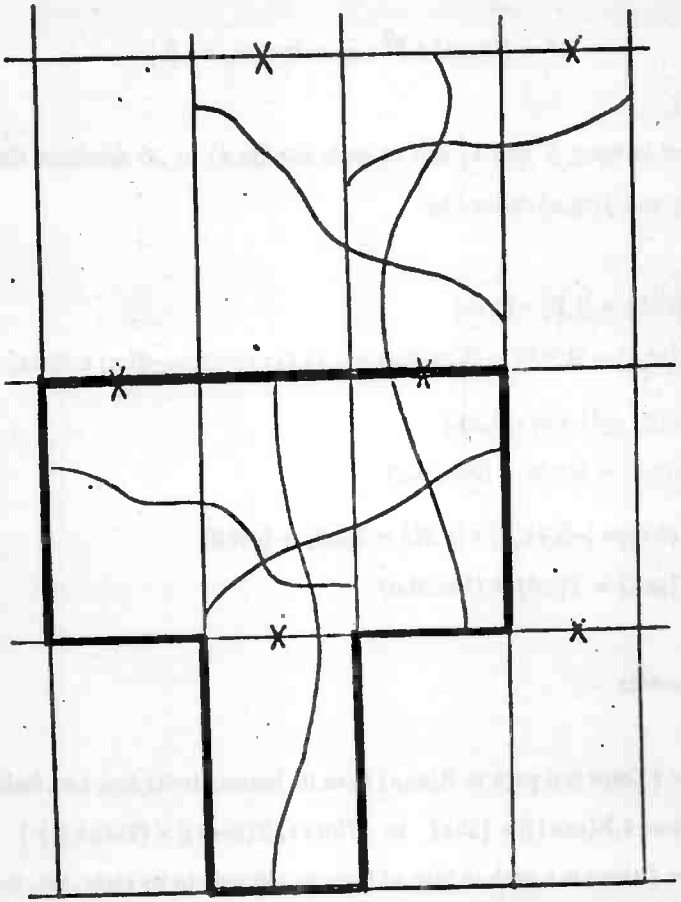

Figure 2

theme. As in most treatments the renormalized lattice is

$$\mathscr{L} = \{ (m,n) \in \mathbb{Z}^2 : m+n \text{ is even}, n \geq 0 \}$$

Fix two positive integers N and L, and to each site (m,n) in $\mathscr{L}$ associate the rectangles R(m,n), S(m,n), and T(m,n) defined by

$$R(0,0) = [1,N] \times [0,2L]$$
$$R(m,n) = R(0,0) + (Nm,2Ln) = \{ (x,t) : (x-Nm,t-2Ln) \in R(0,0) \}$$

$$S(0,0) = [1,2N] \times [L,2L]$$
$$S(m,n) = S(0,0) + (Nm,2Ln)$$

$$T(0,0) = [-N+1,N] \times [L,2L] = S(0,0) + (-N,0)$$
$$T(m,n) = T(0,0) + (Nm,2Ln)$$

We also define events

$F(m,n) = \{$ there is a path in R(m,n) from its bottom to its top, i.e., from
$[Nm+1,N(m+1)] \times \{2Ln\}$ to $[Nm+1, N(m+1)] \times \{2L(n+1)\} \}$

$G(m,n) = \{$ there is a path in S(m,n) from its left side to its right, i.e., from
$\{Nm+1\} \times [L(2n+1),L(2n+2)]$ to $\{N(m+2)\} \times [L(2n+1),L(2n+2)] \}$

$H(m,n) = \{$ there is a path in T(m,n) from its left side to its right, i.e., from
$\{N(m-1)+1\} \times [L(2n+1),L(2n+2)]$ to $\{N(m+1)\} \times [L(2n+1),L(2n+2)] \}$

We will write R for R(0,0), F for F(0,0), etc. Figure 2 may help explain the definitions. In this picture a bold line surrounds the T-shaped region $R \cup S \cup T$. Points of the form $(N/2,L) + (Nm,Ln)$ with $(m,n) \in \mathscr{L}$ are indicated by crosses. The wavy lines are paths in

the graphical representation of the contact process. In this picture F, G, H, F(1,1), G(1,1), and H(1,1) occur.

We will say that the renormalized site $(m,n) \in \mathcal{L}$ is open and set $\eta(m,n) = 1$ if F(m,n), G(m,n), and H(m,n) all happen, otherwise we say that (m,n) is closed and set $\eta(m,n) = 0$. It is easy to see that the random variables $\eta(m,n)$ are 1–dependent, that is, if we let $\|(m,n)\| = (|m|+|n|)/2$ and if (m_1,n_1), ... (m_k,n_k) are points with $\| (m_i,n_i) - (m_j,n_j) \| > 1$ for $i \neq j$, then $\eta(m_1,n_1)$, ... $\eta(m_k,n_k)$ are independent.

By translation invariance $P(\ \eta(m,n) = 1\ )$ is independent of (m,n). Denote this probability by $\Pi(N,L,\lambda)$. We will say that "(oriented) percolation occurs in the η system starting from (0,0)" if there is an infinite sequence of open sites $(0,0) = (m_0,n_0)$, (m_1,n_1), ... with $n_k = k$ and $|m_{k+1}-m_k| = 1$ for $k \geq 0$; and we will let $\theta(N,L,\lambda)$ denote the probability of that event. A result in Section 10 of Durrett (1984) implies

(3.3) If $\Pi(N,L,\lambda) > 1 - 3^{-36}$ then $\theta(N,L,\lambda) > 0$.

The critical relationship between the renormalized and original process is

(3.4) If percolation occurs in the η system starting from (0,0) then $\tau^{\{1,2,...N\}} = \infty$ in the contact process.

Using (3.3), (3.4), and an argument of Russo (1981) we get

(3.5) Lemma. For any N and L, $\Pi(N,L,\lambda_c) \leq 1 - 3^{-36}$.

Proof: Suppose for some N and L that $\Pi(N,L,\lambda_c) > 1-3^{-36}$. $\{\ \eta(0,0) = 1\ \}$ is defined in terms of the finite region $R \cup S \cup T$, so $\lambda \to \Pi(N,L,\lambda)$ is continuous and there is a $\lambda_0 < \lambda_c$

with $\Pi(N,L,\lambda_0) > 1-3^{-36}$. But then by (3.3) and (3.4), $\rho(\lambda_0) > 0$ contradicting the definition of λ_c.

$P(F) = P(\sigma_N \geq L)$ is the event we are interested in. To control G and H we use:

(3.6) Suppose $\lambda = \lambda_c$ and $v > 0$. If $t^{-v}E(r_t^2) \to \infty$ as $t \to \infty$, then $\sigma_N/N^{2/v} \to 0$ in probability as $N \to \infty$.

The proof will be easy once we show:

(3.7) For any $\delta > 0$ $\limsup_{n\to\infty} P(\sigma_N > 2\delta N^{2/v}) \leq 1 - 3^{-38}$.

Proof: Suppose (3.7) is false, and let $A = 1 - 3^{-37}$. Then for some sequence N_i

$$f(N_i) \equiv P(F(N_i,L_i)) > A, \tag{3.8}$$

where $L_i = \delta N_i^{2/v}$. From left–right symmetry

$$g(N_i) \equiv P(G(N_i,L_i)) = P(H(N_i,L_i))$$

Set theory and symmetry tell us that $1 - P(F \cap G \cap H) \leq 1-P(F) + 2(1-P(G))$, or rearranging

$$g(N_i) \leq (1/2)(2 - f(N_i) + \Pi(N_i,L_i,\lambda)).$$

Hence from (3.5) and (3.8) we get

(3.9) $$g(N_i) \le (1/2)\,(\,1 + 3^{-37} + 1 - 3^{-36}\,) = 1 - 3^{-37} \equiv B$$

(Yes B = A, but for future clarity we ignore this accident.) Now if $r(L_i) \ge 2N_i\ell$ then each of the rectangles $[(2N_ik)+1,2N_i(k+1)] \times [0,L_i]$ $k=0,1,\ldots\ell-1$ must be crossed from left to right by paths. So (3.9) implies $P(\, r(L_i) \ge 2N_i\ell\,) \le B^\ell$, and it follows that

(3.10) $$E(\, r(L_i)^2\,;\, r(L_i) \ge 0\,) = \sum_{n=1}^{\infty} (2n-1)\, P(\, r(L_i) \ge n\,)$$

$$\le \sum_{m=1}^{\infty} 2N_im \cdot N_i \cdot P(\, r(L_i) \ge 2N_i(m-1)\,) \le C_1\, N_i^2$$

where $C_1 = \sum_{m=1}^{\infty} 2mB^{m-1}$.

On the other hand if $r(L_i) \le -N_i\ell$, then each of the rectangles $[-N_i(k+1)+1,-N_ik] \times [0,L_i]$ cannot be crossed from bottom to top by a path. So from (3.8), $P(\, r(L_i) \le N_i\ell\,) \le (1-A)^\ell$ and repeating the computation in (3.10) shows

(3.11) $$E(\, r(L_i)^2\,;\, r(L_i) < 0\,) \le C_2\, N_i^2\,,$$

where $C_2 = \sum_{m=1}^{\infty} 2m(1-A)^{m-1}$. Combining (3.10) and (3.11) gives

$$L_i^{-v}\, E(\, r(L_i)^2\,) \le (C_1+C_2)\, N_i^2\, L_i^{-v} = (C_1+C_2)\, \delta^{-v} < \infty,$$

contradicting the hypothesis of the lemma.

Proof of (3.6): Let $b > 0$ and K be a positive integer.

$$P(\, \sigma_N > bN^{2/v}\,) \le P(\, \sigma_N > (b/K)N^{2/v}\,)^K.$$

Using (3.7) now with $2\delta = b/K$ shows

$$\limsup_{N\to\infty} P(\sigma_N > bN^{2/v}) \le (1 - 3^{-38})^K,$$

which proves (3.6) since b and K are arbitrary.

At this point we have proved the result called (1.7) in the introduction. (1.8) is proved in the next section, and (1.9) is proved in Durrett (1988) so we are done with the proof of the upper bound.

4. Galves and Presutti (1987) revisited

In this section we will prove (1.8). The proof is based on the argument of Galves and Presutti (1987), but uses a countable partition of the sample space instead of what they call a "measurable partition." This modification allows us to give a simple proof of the crucial conditional independence property that they state without proof. We give the proof here only for $r_t = \sup \xi_t^{(-\infty,0]}$ but the argument works whenever the initial configuration is in $\{ \eta : 0 \in \eta, \eta \cap (0,\infty) = \phi, |\eta| = \infty \}$

Let $\xi_t^{(x,s)}$ denote the contact process starting with x occupied at time s, and let

$$r_t^{(x,s)} = \sup \xi_t^{(x,s)},$$
$$\tau(x,s) = \inf \{ t \geq s ; \xi_t^{(x,s)} = \phi \}.$$

For $n = 0,1,2,\ldots$ and $s \geq n+1$ let

$$A(n) = \{ \text{there is no } \delta \text{ at } r_n \text{ from time n to n+1} \}$$
$$B(n,s) = \{ \tau(r_{n+1},n+1) \geq s \}$$
$$C(n,s) = A(n) \cap B(n,s)$$
$$D(u,v) = \bigcap_{n=u}^{v-1} C(n,v)^c.$$

Let $T_0 = -1$ and for $i \geq 0$ let

$$T_{i+1} = \inf \{ n : T_i + 1 \leq n \leq [t]-1 , C(n,t) \text{ occurs} \}$$

where $\inf \phi = \infty$. Let $N = \sup \{ i : T_i < \infty \}$. For what follows it is convenient to redefine $T_{N+1} = t$ and to define random variables on $\{ N \geq i \}$ by

$$S_i(\omega) = r(T_{i+1}) - r(T_i+1) \text{ for } i \geq 0$$
$$\Delta_i(\omega) = r(T_i+1) - r(T_i) \text{ for } i \geq 1.$$

Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the graphical representation is defined, and let Π be the partition of Ω defined by considering two outcomes ω_1 and ω_2 to be in the same atom if and only if

(a) $T_i(\omega_1) = T_i(\omega_2)$ for all i, in particular $N(\omega_1) = N(\omega_2)$, and
(b) $S_i(\omega_1) = S_i(\omega_2)$ for $0 \leq i \leq N(\omega_1)$.

It is clear that

$$r_t = \sum_{i=0}^{N} S_i + \sum_{i=1}^{N} \Delta_i. \tag{4.1}$$

We will now show that conditioned on Π, the random variables Δ_i are independent and have the same distribution as $(r_i^0 \mid \tau^0 > 1)$ where $r_1^0 = \sup \xi_1^0$. The key observation is that if A(n) occurs but B(n,t) does not, then we must wait until at least $\tau(r_{n+1},n+1)$ to get the next T_i, since until that time the right edge is part of a process which will die out. If we let

$$x_i = r(t_i+1)$$
$$F_i = D(t_i+1,t_{i+1}) \cap \{ \tau(x_i,t_i+1) \geq t_{i+1}, r^{(x_i,t_i+1)}(t_{i+1}) = s_i \}$$
$$G_i = A(t_i) \cap \{ r^{(x_i,t_i)}(t_i+1) = \delta_i \}$$

then a little thought reveals

(4.2) $$P(N_t = n,\ T_i = t_i\ 1 \le i \le n,\ S_j = s_j\ 0 \le j \le n,\ \Delta_k = \delta_k\ 1 \le k \le n)$$
$$= P(F_0 \cap G_1 \cap F_1 \dots \cap G_n \cap F_n)$$

and the events in the right hand side are independent since they depend on the graphical representation in disjoint regions. The conditional probability

(4.3) $$P(\Delta_k = \delta_k\ 1 \le k \le n \mid N = n,\ T_i = t_i\ 1 \le i \le n,\ S_j = s_j\ 0 \le j \le n)$$
$$= \frac{P(F_0 \cap G_1 \cap F_1 \dots \cap G_n \cap F_n)}{P(F_0 \cap A_1 \cap F_1 \dots \cap A_n \cap F_n)}$$
$$= \prod_{k=1}^{n} P(G_k|A_k) = \prod_{k=1}^{n} P(r_1^0 = \delta_k \mid A_0)$$

This completes the proof of the claim about the conditional independence of the Δ_k. The rest of the argument is easy (and a slight improvement of the calculation on p.1143 of Galves and Presutti). By (4.1)

(4.4) $$E((r_t - Er_t)^2 \mid \Pi) \ge E((r_t - E(r_t|\Pi))^2 \mid \Pi)$$
$$= E(\{\sum_{i=1}^{N} \Delta_i - E(\Delta_i|\Pi)\}^2 \mid \Pi)$$
$$= E(\sum_{i=1}^{N} \{\Delta_i - E(\Delta_i|\Pi)\}^2 \mid \Pi) = CN$$

where $C > 0$ is the variance of r_1^0 given A(0). Taking expected values now gives

(4.5) $$Var(r_t) \ge C\,EN = C \sum_{i=0}^{[t]-1} P(C(i,t)) = C \sum_{i=0}^{[t]-1} e^{-1} P(\tau(x_i,i+1) > t)$$

which proves the desired result since $P(\tau(x_i,i+1) > t) \ge P(\tau^0 > t)$.

5. Bounds on Correlation Lengths

The key to the developments below is the following result, which is the analogue for Poisson processes of a result of Chayes, Chayes, Fisher, and Spencer (1986) for indpendent Bernoulli variables.

(5.1) Consider independent Poisson processes $T^1, \ldots T^M$ with rate λ and $T^0, \ldots T^{-N}$ with rate 1. If A is an event which is determined by the arrivals in [0,t] then

$$\frac{\partial}{\partial\lambda} P(A) \le (Mt/\lambda)^{1/2}.$$

The proof is a little messy and is postponed to the end of the section. We will now demonstrate (1.12), (1.13), (1.11), and (1.14) in that order. As in the introduction the ϵ's which appear in the statements of (5.2)–(5.5) are arbitrary positive numbers. C_1 and C_2 are positive finite constants whose values are unimportant.

(5.2) As $\lambda \uparrow \lambda_c$ $\qquad$ $\liminf L_{\|}(\lambda)/ (\lambda_c-\lambda)^{-1+\epsilon} > 0.$

Proof: Let $G_N = \{ \sigma_N \ge N^\alpha \}$ (1.6) implies that when $\alpha = 1$

(a) $$P_{cr}(G_N) \to 1,$$

where the subscript on P indicates we are considering $\lambda = \lambda_c$. When $\lambda < \lambda_c$ a result from Durrett (1984) gives

(b) $$P_\lambda(G_N) \le N P(\tau^0 \ge N^\alpha) \le N \exp(-N^\alpha / L_{\|}(\lambda)).$$

where P_λ indicates we are considering the contact process with parameter λ. Using (5.1) now we have

(c) $$P_\lambda(G_N) \geq P_{cr}(G_N) - C_1(\lambda_c - \lambda) N^{(1+\alpha)/2}$$

for $\lambda > \lambda_c/2$. Let $\delta = \lambda_c - \lambda$ and

$$N = (1/3C_1\delta)^{2/(1+\alpha)}.$$

Using (a), (b), and (c) now gives that for this choice of N

(d) $$1/3 \leq N \exp(-N^\alpha / L_\parallel(\lambda))$$

when $\delta = \lambda_c - \lambda$ is small. Rearranging the last inequality gives

(e) $$L_\parallel(\lambda) \geq N^\alpha / \log(3N)$$

which implies that for small δ

(f) $$L_\parallel(\lambda) \geq C_2 \delta^{-2\alpha/(1+\alpha)} / \log(\delta^{-1}).$$

Setting $\alpha = 1$ now gives the desired result.

(5.3) As $\lambda \downarrow \lambda_c$

$$\liminf L_\perp(\lambda) / (\lambda - \lambda_c)^{-(2/5)+\epsilon} > 0,$$

and

$$\limsup L_\perp(\lambda) / (\lambda - \lambda_c)^{-(4/7)+\epsilon} > 0.$$

Proof: Let $G_N = \{ \sigma_N \le N^\alpha \}$ (1.6) implies that when $\alpha = 4$

(a) $$P_{cr}(G_N) \to 1.$$

When $\lambda > \lambda_c$, combining Lemma 1 with the proof of Lemma 4 in Section 3 of Durrett and Schonmann (1988) gives

(b) $$P_\lambda(G_N) \le e^2 N^{2\alpha} \exp(-N/ L_\perp(\lambda)).$$

(Lemma 1 implies that the probability of a "dual path" from x on the right side of $(1/2,N+1/2) \times [0,N^\alpha]$ to y on the left side is smaller than $\exp(-N/L_\perp(\lambda))$. Integrating over the possible x and y and using (*) in the proof of Lemma 4 now gives (b).) Using (5.1) now we have

(c) $$P_\lambda(G_N) \ge P_{cr}(G_N) - C_1 (\lambda-\lambda_c) N^{(1+\alpha)/2}$$

for $\lambda > \lambda_c$. Let $\delta = \lambda-\lambda_c$ and

$$N = (1/3C_1\delta)^{2/(1+\alpha)}.$$

Using (a), (b), and (c) now gives that for this choice of N

(d) $$1/3 \le eN^\alpha \exp(-N/L_\perp(\lambda))$$

when $\delta = \lambda_c-\lambda$ is small. Rearranging the last inequality gives

(e) $$L_\perp(\lambda) \ge N/\log(3e^2N^{2\alpha})$$

which implies that for small δ

$$L_{\perp}(\lambda) \geq C_2\, \delta^{-2/(1+\alpha)} / \log(\delta^{-1}). \tag{f}$$

Setting $\alpha = 4$ gives the first result. To prove the second, notice that when $\alpha < 2.5$ we have limsup $P_{cr}(G_N) = 1$ by (1.10), and apply the argument above to a sequence $N_i \to \infty$ for which $P_{cr}(G_{N_i}) \to 1$.

(5.4) Let $R^0 = \sup(\bigcup_{t\geq 0} \xi_t^0)$. If $\lambda = \lambda_c$ then as $r \to \infty$

$$r^2 (\log r)^{1/2}\, P(R^0 > r) \to \infty$$

and

$$\limsup_{r\to\infty} r^{1.25+\epsilon}\, P(R^0 > r) = \infty.$$

Proof. Let $\alpha = 4$. Now

$$P(\tau^0 \geq r^\alpha \log r) \leq P(\sigma_{2r+1} > r^\alpha \log r) + 2\, P(R^0 > r,\, \sigma_{2r+1} \leq r^\alpha \log r),$$

and (1.6) implies

$$P(\sigma_{2r+1} \geq r^\alpha \log r) \leq P(\sigma_{2r+1} \geq r^\alpha)^{[\log r]} \leq (e^{-10})^{[\log r]}$$

for large r. Combining the last two equations gives

$$P(R^0 > r,\, \sigma_{2r+1} \leq r^\alpha \log r) \geq P(\sigma_{2r+1} > r^\alpha \log r) - r^{-9}$$

for large r, so it follows from (1.9) that

$$\liminf_{r\to\infty} r^{\alpha/2} (\log r)^{1/2} P(R^0 > r) \geq \liminf_{r\to\infty} (r^\alpha \log r)^{1/2} P(\tau^0 \geq r^\alpha \log r) = \infty.$$

To prove the second conclusion we repeat the proof with $\alpha > 2.5$ and use (1.10) in place of (1.6).

(5.5) If $\lambda < \lambda_c$ then

$$\liminf L_\perp(\lambda) / (\lambda_c - \lambda)^{-(2/9)+\epsilon} > 0,$$

and

$$\limsup L_\perp(\lambda) / (\lambda_c - \lambda)^{-(1/3)+\epsilon} > 0.$$

Proof: Let $G_N = \{ R^0 > N , \tau^0 \leq N^\alpha \log N \}$ From the proof of (5.3) we see that when $\alpha = 4$

(a) $$(N^\alpha \log N)^{1/2} P_{cr}(G_N) \to \infty.$$

Our next step is to observe that when $\lambda < \lambda_c$

(b) $$P_\lambda(G_N) \leq P_\lambda(R^0 > N) \leq \exp(- N/L_\perp(\lambda)).$$

The second inequality is trivial since $L_\perp(\lambda)$ is defined in Durrett, Schonmann, and Tanaka (1988) to be

$$\lim \{ \tfrac{-1}{n} \log P(R^0 > n) \} = \inf \{ \tfrac{-1}{n} \log P(R^0 > n) \}.$$

Using (5.1) now we have

(c) $$P_\lambda(G_N) \geq P_{cr}(G_N) - C_1(\lambda-\lambda_c)(N^{1+\alpha}\log N)^{1/2}$$

for $\lambda > \lambda_c/2$. Let $\delta = \lambda_c - \lambda$, $\epsilon > 0$, and pick

$$N = \delta^{-(1-\epsilon)/(.5+\alpha)}$$

so that when δ is small $(N^{.5+\alpha}\log N) \leq \delta^{-1}$ and hence

$$\delta(N^{1+\alpha}\log N)^{1/2} \leq (N^{\alpha}\log N)^{-1/2}$$

Using (a), (b) and (c) now gives that for this choice of N

(d) $$(N^{\alpha}\log N)^{-1/2} \leq \exp(-N/L_\perp(\lambda))$$

when $\delta = \lambda_c - \lambda$ is small. Rearranging the last inequality gives

(e) $$L_\perp(\lambda) \geq 2N/\log(N^{\alpha}\log N),$$

or since ϵ is arbitrary

(f) $$L_\perp(\lambda) / \delta^{-\beta} \to \infty \quad \text{for all } \beta < 1/(.5+\alpha)$$

Setting $\alpha = 4$ gives the first result. Letting $\alpha < 2.5$ and modifying the proof slightly as in (5.3) gives the second conclusion.

Proof of (5.1): We begin by proving the result when there is one rate λ Poisson process $\{ T_n , n \geq 1 \}$ and no rate 1 processes. Let $N_t = \sup\{ k : T_k \leq t \}$. If $k \geq 1$ and $0 = t_0 < t_1$

$< t_2 \ldots < t_k < t$ then with an obvious abuse of notation we can write

$$P(N_t = k, T_1 = t_1, \ldots T_k = t_k) = \{\prod_{i=1}^{k} \lambda \exp(-\lambda(t_i - t_{i-1}))\} \exp(-\lambda(t - t_k)) = \lambda^k e^{-\lambda t}.$$

If A is any event concerning the Poisson process in [0,t] and $k \geq 1$ then $A \cap \{N_t = k\} = \{(T_1, T_2, \ldots T_k) \in B_k\}$ for some $B_k \subset \{(t_1, \ldots t_k) : 0 < t_1 < \ldots < t_k\}$, and we have

$$P(A \cap \{N_t = k\}) = \lambda^k e^{-\lambda t} |B_k|$$

where $|B_k|$ denotes the k dimensional Lebesgue measure of B_k. The last formula also holds when $k = 0$ if we consider $|B_0| = 1$ or 0 according as $\{N_t = 0\} \subset A$ or $\subset A^c$. Differentiating now gives

$$\frac{\partial}{\partial \lambda} P(A \cap \{N_t = k\}) = \left[\frac{k}{\lambda} - t\right] P(A \cap \{N_t = k\}).$$

The right hand side is smaller than $|(k/\lambda) - t| P(N_t = k)$, and hence the sum on k converges uniformly on compact subsets of $(0,\infty)$. From this it follows easily (see p.1034 of Durrett (1984) for details) that the derivative of the sum is the sum of the derivatives, and hence

$$\left|\frac{\partial}{\partial \lambda} P(A)\right| \leq E |(N_t/\lambda) - t|.$$

To bound the right hand side we observe

$$E|N_t - \lambda t| \leq (E|N_t - \lambda t|^2)^{1/2} = (\lambda t)^{1/2}$$

so

$$\left|\frac{\partial}{\partial\lambda} P(A)\right| \le (t/\lambda)^{1/2}.$$

The proof for the general case is almost the same. We begin by observing that

$$P(N_t^m = k(m), T_1^m = t_1^m, \ldots T_{k(m)}^m = t_{k(m)}^m \; -N \le m \le M) = \lambda^\kappa e^{-\lambda Mt} e^{-(N+1)t}$$

where $\kappa = k(1) + \ldots + k(m)$. Now if A is any event involving the Poisson processes in [0,t] then

$$P(A \cap \{ \vec{N}_t = \vec{k} \}) = c(\vec{k}) \lambda^\kappa e^{-\lambda Mt} e^{-(N+1)t}$$

where $\rightarrow$ indicates a vector indexed by $-N, \ldots M$, and $c(\vec{k})$ is a constant which only depends on $\vec{k}$. Differentiating with respect to λ, and summing over $\vec{k}$ gives

$$\left|\frac{\partial}{\partial\lambda} P(A)\right| \le E\left| \sum_{m=1}^{M} (N_t^m/\lambda) - Mt \right| \le (tM/\lambda)^{1/2},$$

proving the desired result.

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