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Finite Branched Coverings in a Generalized Inverse Mapping Theorem

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Resumo

Seja $U \subset \mathbb{R}^n$ um conjunto aberto e $f : U \to \mathbb{R}^n$ uma aplicação aberta contínua tal que, para todo $y \in f(U)$, $f^{-1}(y)$ é um conjunto discreto. Dado $x \in U$, existem um conjunto aberto conexo arbitrariamente pequeno V, um inteiro $\ell \geq 1$ e um recobrimento de ℓ folhas $f|_V : V \to f(V)$ tais que: **(a)** $\overline{V}$ e $f(\overline{V})$ são vizinhanças de x e $f(x)$, respectivamente, **(b)** $f|_{\overline{V}} : \overline{V} \to f(\overline{V})$ é uma aplicação própria e **(c)** para todo $y \in f(U)$, $\#(f^{-1}(y) \cap \overline{V}) \leq \ell$. Além disto, se $|\deg(f, x)| \equiv 1$, então f é localmente homeomorfa. Nós discutimos condições sob as quais aplicações localmente homeomorfas de uma variedade são homeomorfismos globais.

FINITE BRANCHED COVERINGS IN A GENERALIZED INVERSE MAPPING THEOREM

CARLOS BIASI AND CARLOS GUTIERREZ

ABSTRACT. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be an open continuous map such that, for all $y \in f(U)$, $f^{-1}(y)$ is a discrete set. Given $x \in U$, there exist an arbitrarily small open connected set V, an integer $\ell \geq 1$ and an ℓ-sheeted covering $f|_V : V \to f(V)$ such that: **(a)** $\overline{V}$ and $f(\overline{V})$ are neighborhoods of x and $f(x)$, respectively, **(b)** $f|_{\overline{V}} : \overline{V} \to f(\overline{V})$ is a proper map, and **(c)** for all $y \in f(U)$, $\#(f^{-1}(y) \cap \overline{V}) \leq \ell$. Moreover, if $|\deg(f,x)| \equiv 1$, then f is locally homeomorphic. We discuss conditions under which locally homeomorphic maps of a manifold are global homeomorphisms.

1. Introduction

The local inversion of maps is one of the most important subjects in Topology and Analysis. With respect to this we have the fundamental Inverse Mapping Theorem for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ which requires regularity and continuity of the derivative, of the map, around the given point. We shall now mention some results that use weaker assumptions. F. H. Clarke [8], using the concept of generalized gradient, proves the Inverse Mapping Theorem under a condition that need the map be differentiable. In [14], S. Radulescu and M. Radulescu, state some Inverse Mapping Theorems for mappings in Banach spaces, where the derivative is regular, around the given point, but may no vary continuously. Our result generalizes, in the finite dimensional case, the Inverse Mapping Theorem of [14]. Some articles somehow related to our work are those of [4], [7], [10], [11], [13], [16].

We say that $f : X \to Y$ is discrete if for all $y \in f(X)$, $f^{-1}(y)$ is a discrete set. The main result of this article generalize the Inverse Mapping Theorem as follows

Theorem 3.1 Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be an open, discrete, continuous map. Given $x \in U$, there exist an arbitrarily

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small open connected set V, an integer $\ell \geq 1$ and an ℓ–sheeted covering $f|_V : V \to f(V)$ such that:

(a) $\overline{V}$ and $f(\overline{V})$ are neighborhoods of x and $f(x)$, respectively,
(b) $f|_{\overline{V}} : \overline{V} \to f(\overline{V})$ is a proper map, and
(c) for all $y \in f(U)$, $\#(f^{-1}(y) \cap \overline{V}) \leq \ell$.

As a consequence of Theorem 3.1, we will obtain

Corollary Let $U \subset I\!\!R^n$ be an open set and let $f : U \to I\!\!R^n$ be a discrete, continuous, open map. If $|\deg(f, x)| \equiv 1$, then f is locally homeomorphic.

In this article the term differentiable will refer to maps which need not be C^1; also, the considered homologies and cohomologies will be that of Cěch having $I\!\!K \in \{\mathbb{Z}, \mathbb{Z}_2\}$ as the coefficient group.

A particular case of corollary above is when f is differentiavel and, for all $x \in U$, the derivative $f'(x)$ is non–singular.

Section 2 and 3 are devoted to prove our main result. The sections 4 – 6 are dedicated to obtain conditions under which locally homeomorphic maps of a manifold are global homeomorphisms. In Section 6, we provide an alternative proof of a Černavskii's Theorem (Černavskii [5], [6], Väisälä [15]).

Before continuing, we wish to thank Benar Swaiter, for very helpful conversations

2. Preliminary results

Let Y be a metric space. Given $A \subset B \subset Y$, we shall denote by $I_B(A)$ (resp. $C_B(A)$) the interior (resp. closure) in B of A.

Proposition 2.1. *Let $f : X \to Y$ be a continuous surjective open map between metric spaces X and Y. Suppose that $B \subset Y$ is a Baire space and that, for all $y \in B$, $\sharp f^{-1}(y)$ is finite. Then,*

$$\bigcup_{n=1}^{\infty} I_B(E_n(B))$$

is dense in B, where

$$E_n(B) = \{y \in B : \sharp f^{-1}(y) = n\}$$

Proof.
Let $F = C_B\big(\bigcup_{n=1}^{\infty} I_B(E_n(B))\big)$ and suppose by contradiction that

$$(*) \qquad U = B \backslash F \neq \emptyset.$$

Given $n \in \mathbb{N}$, denote by

$$O_n(U) = \{y \in U : \sharp f^{-1}(y) \geq n\}$$

As U is open in B and f is an open map, for all n, $O_n(U)$ is an open subset of both U and B. Certainly, $O_1(U) = U$ and – as U is a Baire space (open subsets of Baire spaces are Baire spaces too) – there exists $j \geq 1$ such that

$$U = C_U(O_1(U)) = C_U(O_2(U)) = \cdots = C_U(O_j(U))$$

but $C_U(O_{j+1}(U))$ is properly contained in U. Therefore,

$$I_U(E_j(U)) \neq \emptyset$$

where $E_j(U) = \{y \in U : \sharp f^{-1}(y) = j\}$. As U is open in B

$$\emptyset \neq I_U(E_j(U)) = I_B(E_j(U))$$

However, this is not possible because

$$I_B(E_j(U)) \subset I_B(E_j(B)),\ I_U(E_j(U)) \subset U$$

and

$$I_B(E_j(B)) \cap U = \emptyset$$

This gives a contradiction with (*) and proves the proposition. ∎

In the following, given a topological space W, we shall denote by $W^{\infty} = W \cup \{\infty\}$ the Alexandroff compactification of W. If X is a subspace of W, then we shall suppose that $X^{\infty} \subset W^{\infty}$.

Lemma 2.2. *Let W be a connected topological n-manifold ($n > 0$) and X be a closed subset of W. If $\emptyset \subsetneqq X \subsetneqq W$, then $H^n(X^{\infty}) = 0$, where the considered cohomology is that of Čech with group coefficient $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Z}_2\}$.*

Proof: Considering the cohomology exact sequence of the triple $(W^{\infty}, X^{\infty}, \infty)$, we get

$$(a1) \qquad H^n(W^{\infty}, X^{\infty}) \to H^n(W^{\infty}, \infty) \to H^n(X^{\infty}, \infty) \to 0.$$

Now observe that as W is connected and $W \setminus X \neq \emptyset$,

$$(a2) \qquad H_0(W \setminus X) \xrightarrow{i*} H_0(W) \cong \mathbb{K}$$

is onto, where $i : W \setminus X \to W$ is the inclusion.

As $W^\infty \backslash X^\infty = W \backslash X$ and $W^\infty \backslash \{\infty\} = W$, we may use Alexandroff-Čech Duality to obtain

$$H^n(W^\infty, X^\infty) \cong H^n(W^\infty \setminus X^\infty) \cong H_0(W \setminus X)$$

and

$$(a3) \qquad H^n(W^\infty, \infty) \cong H^n(W^\infty \setminus \{\infty\}) \cong H_0(W)$$

Using this and (a2), the exact sequence (a1), up to isomorphisms, can be rewritten as

$$(a4) \qquad H_0(W \setminus X) \xrightarrow{i*} H_0(W) \cong I\!\!K \to H^n(X^\infty, \infty) \to 0.$$

Hence, as i^* is onto, we obtain that $0 = H^n(X^\infty, \infty) = H^n(X^\infty)$, which implies the Lemma.

Proposition 2.3. *Let W be a connected topological n-manifold ($n > 0$) and Y be a topological space. Let $\emptyset \subsetneqq A \subsetneqq X \subsetneqq W$ and $\emptyset \subsetneqq B \subsetneqq Y$ consist of closed subsets of W and Y, respectively. Suppose that $X \setminus A$ and $Y \setminus B$ are topological n-manifolds and $Y \setminus B$ is connected. If $f : (X, A) \to (Y, B)$ is a proper continuous map such that $f_{|A} : A \to B$ is a homeomorphism. Then $X \setminus A$ is connected.*

Proof: In this proof, the considered homologies and cohomologies will be that of Cěch having $\mathbb{Z}_2$ as the coefficient group.

Denote by $\tilde{f} : (X^\infty, A^\infty) \to (Y^\infty, B^\infty)$ the continuous extension of f which takes ∞ to ∞. Using Lemma 2.2 ($H^n(X^\infty) \cong 0$), consider the following diagram

$$(b1) \qquad \begin{array}{ccccc} H^{n-1}(B^\infty) & \longrightarrow & H^n(Y^\infty, B^\infty) & \longrightarrow & H^n(Y^\infty) \\ \downarrow \tilde{f}^* & & \downarrow \tilde{f}^* & & \downarrow \tilde{f}^* \\ H^{n-1}(A^\infty) & \longrightarrow & H^n(X^\infty, A^\infty) & \longrightarrow & H^n(X^\infty) \cong 0 \end{array}$$

where rows are exact and, by assumption, $\tilde{f}^* : H^{n-1}(B^\infty) \to H^{n-1}(A^\infty)$ is an isomormphism.

Since $X^\infty \setminus A^\infty = X \setminus A \neq \emptyset$ and $Y \setminus B = Y^\infty \setminus B^\infty$ is connected,

$$(b2) \qquad \begin{array}{l} H^n(Y^\infty, B^\infty) = H_0(Y \setminus B) = \mathbb{Z}_2 \\ H^n(X^\infty, A^\infty) = H_0(X \setminus A) \neq 0 \end{array}$$

In this way we obtain, from (b1) and (b2), that $H_0(X \setminus A) = H^n(X^\infty, A^\infty) = \mathbb{Z}_2$, which implies that $X \setminus A$ is connected. ■

Proposition 2.4. *Let X and Y be locally compact, connected, metric spaces. Let A (resp. B) be a closed subset of X (resp. of Y). Suppose that $Y \setminus B$ and $X \setminus A$ are connected, oriented, topological n-manifolds.*

Also, assume that $f\colon (X,A) \to (Y,B)$ *is a continuous proper map,* $f^{-1}(B) = A$, $f|_A\colon A \to B$ *is a homeomorphism and* $f\colon X \to Y$ *is surjective. If* $H^n(X^\infty) = 0$, *where the coefficient group is* $\mathbb{Z}$, *then* $|\deg(f|_{X\setminus A})| = 1$.

Proof:

Observe that as $f\colon X \setminus A \to Y \setminus B$ is a continuous proper map, $\deg(f|_{X\setminus A}) = s$ is well defined. Also,

(a1) $H^n_C(X \setminus A) \cong H^n(X^\infty, A^\infty) \cong H_0(X \setminus A) \cong \mathbb{Z}$,
(a2) $H^n_C(Y \setminus B) \cong H^n(Y^\infty, B^\infty) \cong \mathbb{Z}$.

Moreover, by using the assumption that $H^n(X^\infty) = 0$, we have that

(b) $f^*\colon H^n(Y^\infty, B^\infty) \to H^n(X^\infty, A^\infty)$ is an isomorphism.

Now, if α be a generator of $H^n_C((Y \setminus B)$, then $f^*(\alpha) = \deg(f|_{X\setminus A})\,\alpha = s\,\alpha$, where $f^*\colon H^n_C((Y\setminus B) \to H^n_C(X\setminus A)$. Therefore, by (a1), (a2) and (b), $f^*\colon H^n_C((Y \setminus B) \to H^n_C(X \setminus A)$ is an isomorphism. This implies that $|s| = 1$ ■

3. Main Theorem

In the following theorem, if W is a subset of $\mathbb{R}^n$, $\overline{W}$ will denote the closure of W in $\mathbb{R}^n$ and $\partial W = \overline{W} \setminus W$.

Theorem 3.1. *Let* $U \subset \mathbb{R}^n$ *be an open set and let* $f: U \to \mathbb{R}^n$ *be an open, discrete, continuous map. Let* B_f *be the set of* $x \in U$ *such that* f *is not locally homeomorphic at* x. *Then, given* $x \in B_f$, *there exists* $\epsilon > 0$ *such that if* B *is a ball, centered at* $f(x)$, *of radius* $< \epsilon$, *then* $B \subset f(U)$ *and there exists a connected open neighborhood* $Z \subset U$, *with* $f(Z) = B$, *satisfying the following: if* $h = f|_Z$ *then* $h: Z \to B$ *is a proper map,* $B \setminus h(Z \cap B_f)$ *is a connected set which is open and dense in* B, *and*

$$h : Z \setminus (h^{-1}(h(Z \cap B_f)) \to B \setminus h(Z \cap B_f)$$

is an ℓ*-sheeted covering map with* $\ell \geq 2$ *and such that, for all* $y \in h(Z \cap B_f)$, $\#h^{-1}(y) \leq \ell$.

To prove of this result we shall use the following

Lemma 3.2. *The following is satisfied:*

(a) *given* $x \in U$, *there exists an arbitrarily small connected neighborhood* V *of* x *such that* $V \subset \overline{V} \subset U$ *and*

$$g : V \to g(V)$$

is a proper map, where $g = f|_V$;

(b) *let* $y \in g(V)$ *and let* $\{x_1, x_2, \cdots, x_k\} = g^{-1}(y)$*; there exists* $\epsilon > 0$ *such that if* $0 < r < \epsilon$ *and* B *denotes the open ball centered at* y*, having radius* r*, then* $g^{-1}(B)$ *is the finite union of pairwise disjoint open sets* $V_1, V_2, \cdots, V_k$*, such that, for all* i*,* $x_i \in V_i$*,* $g(V_i) = B$ *and*

$$g|_{V_i} : V_i \to B$$

is a proper map.

Proof: Let W be a small disc centered at x such that $\overline{W} \subset U$. As f is a open mapping, $f(\partial W)$ is a closed set having empty interior. As f is discrete we way take W so that $f(x) \notin f(\partial W)$. Let V be the connected component of $W \setminus (f^{-1}(f(\partial W)))$ containning x. Since $W \setminus (f^{-1}(f(\partial W)))$ is open, V is an open connected neighborhood of x and $f(V) \subset f(W) \setminus f(\partial W)$ is an open connected neighborhood of $f(x)$. As $f(\partial V) \subset f(\partial W)$, $f(\partial V) \cap f(V) = \emptyset$. This proves (a).

Take pairwise disjoint connected open sets $W_1, W_2, \cdots, W_k$, such that, for all i, W_i is a neighborhood of x_i. Let B be any ball centered at y such that $\overline{B} \subset \cap_{i=1}^{k} g(W_i)$. Let V_i be the (open) connected component of $g^{-1}(B)$ which contains x_i. As g is a proper map, $g^{-1}(\overline{B})$ is a compact set. Moreover if Z is an open connected component of $g^{-1}(B)$, then $g(Z)$ is an open connected subset of B such that $g(\partial Z) \subset \partial B$; this implies that $g(Z) = B$ and $g(\partial Z) = \partial B$. Therefore, for some i, $Z = V_i$. This proves (b). ∎

Lemma 3.3. *Suppose that* $B \subset f(U)$ *is an open ball,* V *is an open connected set and* $f|_V : V \to f(V) = B$ *is a proper map. Then* $B \setminus f(V \cap B_f)$ *is a connected set.*

Proof: Let E be a connected component of $B \setminus f(V \cap B_f)$. Denote by $G_2(E)$ the set of points $x \in f(V \cap B_f)$ such that, for some open ball D, $x \in D \in B$ and $(D \setminus f(V \cap B_f)) \subset E$. As E is an open connected set, $E \cup G_2(E)$ is also a open connected set. Let G_2 be the union of all sets $G_2(E)$ such that E is a connected component of $B \setminus f(V \cap B_f)$. Let $G_1 = f(V \cap B_f) \setminus G_2$. We may conclude that

(a) if C is a connected component of $B \setminus G_1$ then there exists exactly one connected component E of $B \setminus f(V \cap B_f)$ such that $C = E \cup G_2(E)$.

Suppose, by contradiction, that

(b) $B \setminus f(V \cap B_f)$ is not connected.

This and (a) imply that

(c) $B \setminus G_1$ is not connected.

By (b) of Lemma 3.2, as f is an open map and by Proposition 2.1 applied to the Baire Space $G_1 \subset B$ (with respect to the map $f|_V V \to B$), we obtain that

(d) there exists an open ball $Z \subset B$, with $\overline{Z} \subset B$, such that $Z \cap G_1 \neq \emptyset$ and if $Y_1, Y_2, \cdots, Y_t$ are the connected components of $(f|_V)^{-1}(Z)$ then, for all $i \in \{1, 2, \cdots, t\}$,

$$h_i : h_i^{-1}(Z \cap G_1) \to Z \cap G_1$$

is a homeomorphism, where $h_i = f|_{Y_i}$.

Let C be a connected component of $Z \setminus G_1$ and let $i \in \{1, 2, \cdots, t\}$. Let denote by $F_Z(C)$ and $\partial_Z(C)$ the closure and frontier, respectively, of C in Z; also, let denote by $F_{Y_i}(h_i^{-1}(C))$ and $\partial_{Y_i}(h_i^{-1}(C))$ the closure and frontier, respectively, of $h_i^{-1}(C)$ in Y_i.

As Y_i is connected,

$$\emptyset \subsetneq h_i^{-1}(F_Z(C)) = F_{Y_i}(h_i^{-1}(C)) \subsetneq Y_i,$$

and $h_i^{-1}(F_Z(C))$ is a closed subset of the open set Y_i, we may apply Lemma 2.2 to obtain that

(e) $H^n(h_i^{-1}(F_Z(C))) = 0$, where the considered cohomology is that of Čech with group coefficient $\mathbb{Z}$.

Also, as Y_i is connected (see (d)), by Proposition 2.3 applied to the map

$$h_i : (F_{Y_i}(h_i^{-1}(C)), \partial_{Y_i}(h_i^{-1}(C))) \to (F_Z(C)), \partial_Z(C))),$$

we obtain that

(f) $h_i^{-1}(C)$ is connected.

Therefore, by (e) and by Proposition 2.4 applied to the map

$$h_i : (F_{Y_i}(h_i^{-1}(C)), \partial_{Y_i}(h_i^{-1}(C))) \to (F_Z(C)), \partial_Z(C))),$$

it follows that h_i takes homeomorphically $h_i^{-1}(C)$ onto C. Since h_i is an open map and

$$\# h_i^{-1} : Z \to \mathbb{N}$$

is a lower semi-continuous map, we conclude that h_i takes homeomorphically $F_{Y_i}(h_i^{-1}(C))$ onto $F_Z(C))$.

As $F_Z(C)$ is the closure of an arbitrary connected component C of $Z \setminus G_1$ we obtain that, for all $i \in \{1, 2, \cdots, t\}$,

(g) h_i takes homeomorphically Y_i onto Z.

This implies that $Z \cap G_1 = \emptyset$. This contradiction with (d) proves that $B \setminus f(V \cap B_f)$ is connected. ∎

Proof of Theorem 3.1:

The existence of B and Z such that $h : Z \to B$ is a proper map, follows from (b) of Lemma 3.2.

Since $B \setminus h(Z \cap B_f)$ is connected (see Lemma 3.3),

$$\#h^{-1} : B \setminus h(Z \cap B_f) \to I\!\!N$$

is a constant map equal to $\ell \in I\!\!N$. Moreover, as $\#h^{-1} : B \to I\!\!N$ is a lower semicontinuous map, we conclude that for all $y \in h(Z \cap B_f)$, $\#h^{-1}(y) \leq \ell$. If ℓ was equal to 1, we would obtain, by using that h is open, that $h : Z \to B$ is a homeomorphism, contradicting the assumption that $x \in B_f$. Therefore $\ell \geq 2$. This proves (b) and the theorem ∎

As a consequence of Theorem 3.1, we obtain

Corollary 3.4. *Let $U \subset I\!\!R^n$ be an open set and let $f : U \to I\!\!R^n$ be a degree one, open, discrete, continuous map. Then f is locally homeomorphic.*

Also as a corollary of Theorem 3.1 we obtain the following Local Inverse Mapping Theorem

Theorem 3.5. *Let $U \subset I\!\!R^n$ be an open set and let $f : U \to I\!\!R^n$ be differentiable (not necessarily C^1) without critical points. Then f is a local homeomorphism.*

This result allow us to extend a classical Darboux Theorem as follows

Theorem 3.6. *Let $U \subset I\!\!R^n$ be an open connected set and let $f : U \to I\!\!R^n$ be differentiable (not necessarily C^1). Given $\alpha \in I\!\!R$ e $x_0, x_1 \in U$ such that $\det(Df(x_0) - \alpha I) < 0$ and $\det(Df(x_1) - \alpha I) > 0$, Then, there exists $x_2 \in U$ such that $\det(Df(x_2) - \alpha I) = 0$ (i.e. α is an eigenvalue of $Df(x_2)$.*

Proof: Let I denote the identity map of $I\!\!R^n$. We may apply Theorem 3.5 to the map $g(x) = f(x) - \alpha x$ to obtain that the sign of $\det(Dg(x) - \alpha I$ is locally constant. Therefore, this theorem follows by observing that U is pathwise connected. ∎

Remark 3.7. *The theorem 3.1 is also true if f is replaced by a map between either generalized manifolds (for definitions, see* [1], [2]*) or topological manifolds.*

4. A global Inverse Mapping Theorem

Theorem 4.1. *Let M and N be smooth riemannian connected manifolds of the same finite dimension n, where M is complete. Let $f : M \to$*

N be a differentiable map (which may or may not be of class C^1) without critical points. Let $x_0 \in M$; given a non-negative real number r, denote by

$$\Gamma(r) = \sup\{||(f'(x))^{-1}|| : d(x, x_0) \leq r\},$$

where $d(\cdot, \cdot)$ denotes the intrinsic metric of M. Suppose that

$$\int_0^\infty \frac{du}{\Gamma(u)} = \infty. \tag{4.1}$$

Then

(a) *f is a covering map (in particular, f is onto);*
(b) *if N is simply connected, f is a (global) homeomorphism.*

The proof of this result will be completed after some preparatory lemmas. First observe that the condition 4.1 does not depend on the particular $x_0 \in M$ chosen.

The following lemma follows inmediately from Theorem 3.5 and canonical arguments.

Lemma 4.2. *Let U, V be open subsets of $I\!R^n$. Suppose that $f \colon U \to V$ is a differentiable homeomorphism without critical points. Then $f^{-1} \colon V \to U$ is also differentiable homeomorphism without critical points.*

Under these conditions, we may obtain the Implicit Function Theorem by standard arguments:

Theorem 4.3. *Let $U \subset I\!R^n$ and $V \subset I\!R^m$ be open sets and let $f : U \times V \to I\!R^n$ be a differentiable map. Suppose that for all $(x, y) \in U \times V$, $\frac{\partial f}{dx}(x, y) : I\!R^n \to I\!R^n$ is non-singular. Given $(x_0, y_0) \in U \times V$ there exists an open neighboorhood V_1 of y_0 and a differentiable map $h : V_1 \to U$ such that $h(y_0) = x_0$ and $f(h(y), y) = f(x_0, y_0)$, for all $y \in V_1$.*

Lemma 4.4. *Let $x \in M$ and $\beta \colon [0, b] \to N$ be differentiable path such that $\beta(0) = f(x)$. Then, there exists a unique continuous path $\alpha \colon [0, b] \to M$ such that $\alpha(0) = x$ and $f \circ \alpha = \beta$. Moreover, α is differentiable.*

Proof: By Theorem 3.5, f is a local homeomorphism. Therefore, we may find a continuous injective path $\alpha \colon [0, b] \to M$ such that $\alpha(0) = x$ and $f \circ \alpha = \beta$. Certainly α is uniquely defined. It follows from Lemma 4.2 that α is differentiable.

Given $x \in M$, we may define the strictly increasing continuous function $\sigma_x \colon [0, \infty) \to [0, \infty)$ given by

$$\sigma_x(s) = \int_{d(x_0,x)}^{d(x_0,x)+s} \frac{du}{\Gamma(u)}.$$

It follows from the condition 4.1 that σ_x is a homeomorphism. Under this condition we have

Lemma 4.5. *Let $x \in M$ and $\beta\colon [0,b] \to N$ be differentiable path such that $\beta(0) = f(x)$. Let $\alpha\colon [0,b] \to M$ be the only differentiable path such that $\alpha(0) = x$ and $f \circ \alpha = \beta$. Then*

$$\sigma_x^{-1}\left(\int_0^b ||\beta'(u)||du\right) \geq \int_0^b ||\alpha'(u)||du$$

Proof: It follows that $\beta'(t) = Df(\alpha(t)) \cdot \alpha'(t)$ and so $Df(\alpha(t))^{-1} \cdot \beta'(t) = \alpha'(t)$. Hence

$$||Df(\alpha(t))^{-1}|| \cdot ||\beta'(t)|| = ||\alpha'(t)||.$$

Given $t \in [0,b]$, let

$$s(t) := \int_0^b ||\alpha'(t)||dt \geq d(\alpha(0), \alpha(t)) = d(x, \alpha(t)).$$

Then we obtain that

$$d(x_0, \alpha(t)) \leq d(x_0, x) + d(x, \alpha(t)) = d(x_0, x) + s(t)$$

which implies that,

$$\Gamma(s(t) + d(x_0, x))||\beta'(t)|| \geq ||\alpha'(t),$$

and so

$$||\beta'(t)|| \geq \frac{||\alpha'(t)}{\Gamma(s(t) + d(x_0, x))}.$$

In this way, we obtain that

$$\int_0^b ||\beta'(t)||dt \geq \int_0^b \frac{||\alpha'(t)dt}{\Gamma(s(t) + d(x_0, x))}.$$

By considering, in the right integral, the following change of variables $v = s(t) + d(x_0, x)$, where $dv = s'(t)dt = ||\alpha'(t)||dt$, we obtain that

$$\int_0^b ||\beta'(t)||dt \geq \int_{d(x_0,x)}^{s(b)+d(x_0,x)} \frac{dv}{\Gamma(v)} = \sigma_x(s(b)) = \sigma_x\left(\int_0^b ||\alpha'(t)||dt\right),$$

which proves the lemma. ∎

Proof of Theorem 4.1.

It follows from Lemma 4.4 and the fact that M is complete that f is a covering map. This proves (a). When N is simply connected, then M is also simply connected and f is a homeomorphism. ∎

5. Other results

In respect to the natural question on which results about global inversion (or global injectivity) of self-maps of $\mathbb{R}^n$ that are true for C^1 local diffeomorphisms are also true for the non-singular differentiable maps, we may mention that this is the case (as indicated by Campbell himself) for those results of [4]. Meanwhile, the referred extension is not obvious for the main result of both [3] and [9]. In the following, we shall state some results in which the question above has a positive answer.

In this section we are suppose to use the Theorems 3.5 and 4.3 wherever they are necessary.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. We denote by Spec(F) the set of (complex) eigenvalues of the derivative DF_p, as p varies in $\mathbb{R}^n$. Using the same arguments as those of [9], we obtain the following results:

Theorem 5.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map without critical points such that* Spec(F) *is disjoint of a sequence $\{t_m\}$ of real numbers which converges to 0 as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbb{R}^n$ with $\|x\| > R$, $\|F(x)\| \leq \|x\|^\alpha$, then F is injective.*

Theorem 5.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz differentiable map without critical points. Suppose that for some $\epsilon > 0$,*

$$\liminf_{|x|\to\infty} (|x| \cdot |\det(DF(x))|) > \epsilon.$$

Then F is a global homeomorphism (and F^{-1} is differentiable and has no critical points).

Theorem 5.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz differentiable map without critical points. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (with non-empty interior), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and*

$$\text{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F is injective.

Theorem 5.4. *Let F be a differentiable map without critical points such that there is a compact subset of $\mathbb{R}^n$ outside of which the jacobian matrix $JF(x)$ commutes with its transpose. If* Spec(F) $\subset \{z \in \mathbb{C} : \Re(z) < 0\}$, *then F is injective.*

Theorem 5.5. *Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map without critical points. Suppose that, if $n \geq 2$, there exist $K > 0$ and a compact* *subset of* $\mathbb{R}^n$ *outside of which*

$$||DF(x)|| \leq K|x|^{\frac{1}{n-1}}.$$

If $\mathrm{Spec}(F) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$, *then* F *is injective.*

Observe that, by the Mean Value Theorem, a differentiable map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map if, and only if, there is a constant $K > 0$ such that, for all $x \in \mathbb{R}^n$, $||DF(x)|| \leq K$. Therefore, Theorem 5.5 applies to Lipschitz maps.

6. Černavskii's Theorem

In this section, the considered homologies and cohomologies will be that of Cěch having $\mathbb{Z}$ as the coefficient group.

Theorem 6.1. *(Hurewicz–Wallman) Let* Y *be a compact space and let* p *be a positive integer. Then the* topological dimension *of* Y, $\dim(Y)$, *is less or equal than* p *if, and only if, for all closed subset* C *of* Y, $H^{p+1}(Y, C) = 0$.

We shall need the following

Corollary 6.2. *Let* Y *be a locally compact Haussdorf space. Them,* $\dim(Y) \leq p$ *if, and only if, for all closed subset* C *of* Y, $H^{p+1}(Y, C) = 0$.

Proof: Consider the Alexandroff Compactification $Y^\infty = Y \cup \{\infty\}$ of Y. Let $C \subset Y$ be closed; then $C = C \cup \{\infty\}$ is a compact subset of Y^∞. In this way, by Hurewicz–Wallman Theorem, $H^{p+1}(Y, C) = H^{p+1}(Y^\infty, C^\infty) = 0$

Theorem 6.3. *Let* X *be an open subset of connected generalized manifold of dimension* $n > 0$. *Let* A *be a proper subset of* X *which is closed in* X. *Suppose that, for every* $x \in A$ *and for every small contractible neighborhood* V *of* x *in* X, $V \setminus A$ *is a nonempty connected set. Then* $\dim(A) \leq n - 2$.

Proof: Let $C \subset A$ be closed in A. Observe that as $X \setminus A \neq \emptyset$,

$$H^n(A^\infty/C^\infty) \cong H^n(A^\infty, C^\infty) \cong H_0(X \setminus C, X \setminus A) = 0 \tag{6.1}$$

Also, as $n > 0$,

$$H^n(X^\infty/C^\infty, A^\infty/C^\infty) \cong H^n(X^\infty/C^\infty) \cong \mathbb{Z}. \tag{6.2}$$

In fact, $H^n(X^\infty/C^\infty) \cong H^n(X^\infty, C^\infty) \cong H_0(X^\infty \setminus C^\infty) \cong H_0(X \setminus C) = \mathbb{Z}$. Similarly, $H^n(X^\infty/C^\infty, A^\infty/C^\infty) \cong H_0(X \setminus A) = \mathbb{Z}$.

By the assumptions of this theorem, $H_1(X \setminus A) \to H_1(X \setminus C)$ is onto. Hence, as the following diagram is conmutative,

$$\begin{array}{ccc} H^n(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^n(X^\infty/C^\infty) \\ \updownarrow & & \updownarrow \\ H_1(X \setminus A) & \to & H_1(X \setminus C) \end{array}$$

where the vertical arrows are isomorphisms, we obtain, that

$$(6.3) \qquad H^{n-1}(X^\infty/C^\infty, A^\infty/C^\infty) \to H^{n-1}(X^\infty/C^\infty) \qquad \text{is onto}$$

Now, using (6.1) - (6.3) above and considering the exact sequence associated to $C^\infty \subset A^\infty \subset X^\infty$:

$$\begin{array}{ccccccc} H^{n-1}(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^{n-1}(X^\infty/C^\infty) & \to & H^{n-1}(A^\infty/C^\infty) & \to \\ H^n(X^\infty/C^\infty, A^\infty/C^\infty) & \to & H^n(X^\infty/C^\infty) & \to & H^n(A^\infty/C^\infty) & \end{array}$$

we obtain $H^{n-1}(A^\infty/C^\infty) = H^{n-1}(A^\infty, C^\infty) = 0$. This together with the Hurewicz–Wallman Theorem imply this theorem. □

Theorem 6.4. *(Černavskii). Let $U \subset I\!R^n$ be an open set and let $f : U \to I\!R^n$ be an open, discrete, continuous map. Then* $\dim(B_f) \leq n-2$.

Proof: It follows directly from Theorem 6.3 and Theorem 3.1. ■

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