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# Vortex on Surfaces and Brownian Motion in Higher Dimensions: Special Metrics

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**Abstract**

A single hydrodynamic vortex on a surface will in general move unless its Riemannian metric is a special "Steady Vortex Metric" (SVM). Metrics of constant curvature are SVM only in surfaces of genus zero and one. In this paper:

1. I show that K. Okikiolu's work on the regularization of the spectral zeta function leads to the conclusion that each conformal class of every compact surface with a genus of two or more possesses at least one steady vortex metric (SVM).
2. I apply a probabilistic interpretation of the regularized zeta function for surfaces, as developed by P. G. Doyle and J. Steiner, to extend the concept of SVM to higher dimensions.

The new special metric, which aligns with the Steady Vortex Metric (SVM) in two dimensions, has been termed the "Uniform Drainage Metric" for the following reason: For a compact Riemannian manifold $M$, the "narrow escape time" (NET) is defined as the expected time for a Brownian motion starting at a point $p$ in $M \backslash B_\epsilon(q)$ to remain within this region before escaping through the small ball $B_\epsilon(q)$, which is centered at $q$ with radius $\epsilon$ and acts as the escape window. The manifold is said to possess a uniform drainage metric if, and only if, the spatial average of NET, calculated across a uniformly distributed set of initial points $p$, remains invariant regardless of the position of the escape window $B_\epsilon(q)$, as $\epsilon$ approaches 0.

**Keywords** Point vortex · Riemann surfaces · Diffusion process · Brownian motion · Special metrics · Spectral zeta function

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## 1 Introduction

The motion of point vortices on the plane is a classical subject in fluid mechanics that goes back to Helmholtz, Kelvin, and Kirchhoff. The first to consider the motion of point vortices on a curved surface, the sphere embedded in $\mathbb{R}^3$, was Zermello in 1902. The paper Borisov et al. (2010) has a historical review on the early research on hydrodynamic vortices on surfaces. An intrinsic definition of the motion of vortices on a surface, which is independent of the embedding of the surface in $\mathbb{R}^3$ and on coordinates, started with Boatto and Koiller (2008) (see also Boatto and Koiller (2015); Dritschel and Boatto (2015); Ragazzo and de Barros Viglioni (2017)) and was recently completed by Gustafsson (2019, 2022).

A single vortex in the Euclidean plane, or in the round sphere, or in a flat torus does not move, and this motivated the definition of "Steady Vortex Metric" (Ragazzo and de Barros Viglioni 2017): a Riemannian metric for which a single vortex does not move regardless of its position. J. Koiller conjectured that a single vortex in a compact surface of constant curvature and of a genus greater than one does move. In Ragazzo (2017); Grotta-Ragazzo (2022), Koiller's conjecture was numerically verified for a particular surface of constant curvature of genus two: the Bolza surface. This result motivated the first main question to be answered in this work: Does a steady vortex metric exist on any orientable compact surface of a genus greater than one?

K. Okikiolu proved that a certain functional on the space of Riemannian metrics, which is an analog for closed surfaces of the ADM mass from general relativity, has a minimizing metric on each conformal class. It turns out that the special metrics of Okikiolu are steady vortex metrics, which gives a positive answer to the question in the paragraph above. This raises the question about the "meaning" (or properties) of this special metric. The steady vortex metric minimizes a certain functional (Okikiolu 2009) and has the property in its name, but does it have any other interesting geometrical property besides those? This question was the second motivation for this work.

The special metric found by Okikiolu is a critical point of a functional related to the regularized Green's function of the Laplacian: the "Robin function". Doyle and Steiner (2017) gave a probabilistic interpretation to the Robin function that is related to the concept of "Narrow-Escape-Time"(NET) (Holcman and Schuss 2014). The NET is defined as the expected time for a Brownian motion starting at $p$ in $M\backslash B_\epsilon(q)$ to remain within this region before escaping through the small ball $B_\epsilon(q)$, which is centered at $q$ with radius $\epsilon$ and acts as the escape window.

The NET is an important abstraction in science, as argued by Holcman and Schuss in the Introduction of Holcman and Schuss (2014): "The narrow escape problem in diffusion theory, which goes back to Helmholtz (Helmholtz (1860)) and Lord Rayleigh (Rayleigh (1945)) in the context of the theory of sound, is to calculate the mean first passage time of Brownian motion to a small absorbing window.... The renewed interest in the problem is due to the emergence of the narrow escape time (NET) as a key to the determination of biological cell function from its geometrical structure. The NET is ubiquitous in molecular and cellular biology and is manifested in stochastic models of chemical reactions..."

The average NET, with respect to a uniform distribution of initial positions (volume measure), that a particle takes to escape from $S\backslash B_\epsilon(q)$ through the small window $B_\epsilon(q)$ is proportional to $-\log\epsilon + R(q) + \mathcal{O}(\epsilon)$, where $R$ is the Robin function. So, for small $\epsilon$ the Robin function indicates the drainage capacity of different points $q$ in $S$. The Robin function is constant if, and only if, the metric is a steady vortex metric (SVM). Therefore, in a surface with a $SVM$, the drainage capacity of different points is the same and this leads to the alternative name "*uniform drainage metric*", a property that makes sense in dimensions larger than two. Note: the notion of hydrodynamic *point* vortex cannot be generalized to dimensions greater than two.

The main contribution in this paper is the definition of uniform drainage metric in dimensions greater than two and its geometric characterization in dimensions 3 and 4.

Following the same steps given in this paper, a characterization of a uniform drainage metric in higher dimensions can be accomplished by means of certain coefficients that appear in the so-called Minakshisundaram-Pleijel asymptotic expansion of the heat kernel. I prefer not to state any results in this direction because, in higher dimensions, it is necessary to compute more of these coefficients, which can be expressed in terms of powers of the Laplacian and the distance function $\ell$, and they become very complicated (Polterovich 2000).

The existence of uniform drainage surfaces of arbitrary finite genus in any conformal class is guaranteed by the theorem of Okikiolu. In higher dimensions any compact Riemannian manifold that is a homogeneous space is a uniform drainage manifold.[1] Does there exist a closed (compact and boundaryless) manifold that does not admit a uniform drainage metric?

This paper is organized as follows.

In Sect. 2, I give a precise definition of the steady vortex metric and present two fundamental theorems that stem from Okikiolu's work. I then use these theorems to compare the steady vortex metric with other natural Riemannian metrics: of constant curvature, canonical or Bergman, and Arakelov. The proofs of the two theorems are presented in Appendix A in a slightly different way than those given by Okikiolu. These theorems plus some simple arguments imply: "No orientable surface of genus 2 and of constant curvature is a Steady Vortex Surface."

In Sect. 3, I present a regularization of the Green's function in dimensions greater than two using the Minakshisundaram-Pleijel asymptotic expansion of the heat kernel. This provides a definition of the Robin function in higher dimensions. In Appendix B, I show that the Robin function can be written in terms of the analytic extension of the Minakshisundaram-Pleijel zeta function, and therefore uniform drainage manifolds have a special spectral property derived from this relation. The relation between the Robin function and the Minakshisundaram-Pleijel zeta function appeared in Steiner (2005), for surfaces, and in Bilal and Ferrari (2013), in a more general context and in dimension greater than two.

[1] There is a special class of Riemannian metrics on closed manifolds that are critical metrics of the trace of the heat kernel under conformal variations of the metric (El Soufi and Ilias 2002). A metric in this special class is always a uniform drainage metric (a consequence of Theorem 4.1 (ii) in El Soufi and Ilias (2002)). The metric of any Riemannian homogeneous space is critical for the trace of the heat kernel.

In Sect. 4, I give a characterization of uniform drainage metric in dimensions 2, 3 and 4. In dimension 4, a uniform drainage metric has constant Robin function (a global property) and constant scalar curvature (a local property).

In Sect. 5, I present a family of non-flat tori found by Okikiolu (2008), and which will be called Okikiolu's tori, that are uniform drainage surfaces. These tori are the only non-constant curvature uniform drainage surface that are explicitly known. For any $a > \sqrt{\pi/2}$, there is an Okikiolu's torus that is conformally equivalent to the flat torus $\mathbb{R}^2/(a\mathbb{Z} \times a^{-1}\mathbb{Z})$. Therefore, uniform drainage metrics may not be unique in a conformal class. The curvature of the Okikiolu's tori was computed in ibid., where it was realized that in the limit as $a \to \infty$ the curvature at almost every point of the torus tends to $1/\sqrt{4\pi}$. In Sect. 5, I embed a cylinder in $\mathbb{R}^3$ whose quotient under a discrete group of translations along the cylinder axis is an Okikiolu's torus. In this way, one can visualize the deformation of a flat torus into a pinched torus that is isometric to a round sphere with two opposite points being identified. The deformation is done along an interesting family of uniform drainage surfaces.

I finally remark about a possible upshot of the relation between the Robin function and the drainage capacity of different points. The importance of the NET in cellular biology is partially due to diffusion processes that occur in membranes toward special exit gates (escape windows). The minimum of the Robin function is an equilibrium position of a single vortex (Grotta-Ragazzo 2022) and also a point where the drainage capacity of the surface, as defined above, is maximum. Equilibrium positions of systems of point vortices, an issue that has been extensively studied, also have a probabilistic interpretation. If the position of a vortex is related to an entrance or exit gate, depending on the vortex sign, then some equilibrium configurations will certainly be more efficient in connecting different gates by means of diffusion than others. If this idea is correct, then the importance of equilibrium configurations on surfaces of spheres, including those which are not round, will be greatly enhanced.

## 2 Steady Vortex Metrics on Orientable Closed Surfaces

The definition of a hydrodynamic vortex requires some preliminaries (see Ragazzo and de Barros Viglioni (2017)). The fundamental equations of hydrodynamics on a surface $S$, Euler's equations, necessitate that $S$ be endowed with a Riemannian metric $g$. Here, $g$ represents a smooth family of inner products on the tangent spaces of $S$. In local coordinates, the Riemannian metric is given by $g = \sum_{jk} g_{jk} dx_j \otimes dx_k$. The associated volume form is $\mu = \sqrt{|g|} dx_1 \wedge dx_2$, where $|g|$ denotes the absolute value of the determinant of the matrix $g_{jk}$

In a neighborhood of each point of $S$, there exist coordinates (sometimes called isothermal coordinates) in which $g = \lambda^2(x)(dx_1^2 + dx_2^2)$ and $\mu = \lambda^2(x) dx_1 \wedge dx_2$. The existence of isothermal coordinates is a manifestation of the fact that any surface is locally conformal to the Euclidean plane. In this paper, I will also use $\lambda^2$ to denote the conformal factor between arbitrary given metrics $g_0$ and $g_1$. This will be explicitly stated when used.

The one-forms $\theta_1 = \lambda dx_1$ and $\theta_2 = \lambda dx_2$ constitute an orthonormal moving coframe. The Hodge-star operator acts linearly on forms and is defined by

$$*1 = \theta_1 \wedge \theta_2 = \mu, \qquad *\theta_1 = \theta_2, \qquad *\theta_2 = -\theta_1, \qquad *\mu = 1.$$

The Laplace operator acting on functions is given by $\Delta = *d*d = \frac{1}{\lambda^2}(\partial_{x_1}^2 + \partial_{x_2}^2)$ and the Gaussian curvature by $K = -\frac{1}{\lambda^2}\Delta \log \lambda$.

Let $V = \int_S \mu$ be the total area of $S$. The Green's function of $(S, g)$ is the unique solution in distribution sense to the equation

$$-\Delta_q G(q, p) = \delta_p(q) - V^{-1}, \tag{2.1}$$

that has the following properties (see Aubin (2013), theorem 4.13):

- for all functions $\phi \in C^2$

$$\phi(p) = \frac{1}{V}\int_S \phi\mu - \int_S G(q, p)\Delta\phi(q)\mu(q), \tag{2.2}$$

- $G(q, p)$ is $C^\infty$ on $S \times S$ minus the diagonal,
- $G$ is symmetric $G(q, p) = G(p, q)$,
- $G$ is bounded from below and $\int_S G(q, p)\mu(q) = 0$.

A point vortex of intensity $\Gamma \in \mathbb{R}$ at the point $p$ is the fluid velocity field defined on $S - \{p\}$ given by $q \to *\nabla\Gamma G(q, p)$, where $\nabla$ is the gradient operator and $*$ is the operator that rotates a vector by $\pi/2$.

The Robin function (the regularization of $G$) is a $C^\infty$ function on $S$ (Ragazzo and de Barros Viglioni 2017 Theorem 5.1) defined as

$$R(p) = \lim_{\ell(q,p)\to 0}\left[G(q, p) + \frac{1}{2\pi}\log \ell(q, p)\right], \tag{2.3}$$

where $\ell(q, p)$ is the Riemannian distance between $p$ and $q$.

The motion of a single vortex depends not only on its initial position but also on the initial value of a harmonic velocity field (a background flow) (Gustafsson 2022). In the following statement (Ragazzo and de Barros Viglioni 2017; Grotta-Ragazzo 2022), the initial background flow is assumed to be equal to zero:
A vortex initially placed at any point on a surface $S$ with Riemannian metric $g$ remains at rest if, and only if, the Robin function $R$ associated with $g$ is constant. A Riemannian metric with this property is called a "Steady Vortex Metric."

The first main result in this paper is the following.

**Theorem 2.1** *(*Steady Vortex Metric*) Let $S$ be a compact Riemann surface. There exists at least one steady vortex metric $g$ compatible with the conformal structure of $S$. There are examples where $g$ is not unique.*

The theorem effectively says that there always exist a metric for which the Robin function is constant.

This theorem is a direct consequence of a theorem proven by Okikiolu (2008, 2009), and its proof is given in Appendix A.

The second theorem in this section requires some definitions. A one-form $\theta$ on $S$ is harmonic if $d\theta = 0$ and $d * \theta = 0$. Since $*$ rotates one-forms by $\pi/2$, harmonic forms are conformal invariants. The vector space of harmonic forms on $S$ is finite and has dimension $2\mathcal{G}$ (De Rham 2012), where $\mathcal{G}$ is the genus of $S$. Let $\{\theta_1, \ldots \theta_{2\mathcal{G}}\}$ be an arbitrary orthonormal basis of harmonic one-forms in the sense that

$$(\theta_j, \theta_k) = \int_S \theta_j \wedge *\theta_k = \delta_{jk}. \tag{2.4}$$

Note: this definition of orthonormality depends only on the conformal structure.

**Theorem 2.2** *Let $(S, g)$ be a compact oriented Riemannian surface and $\sigma$ be the two-form*

$$\sigma = \sum_{k=1}^{2\mathcal{G}} \theta_k \wedge *\theta_k.$$

*Then, the Robin function $R$ is the only solution, up to an additive constant, of the equation*

$$\left( \Delta R + \frac{K}{2\pi} - \frac{2}{V} \right) \mu = -\sigma. \tag{2.5}$$

If the genus $\mathcal{G}$ of $S$ is zero, then $\sigma = 0$. So a metric on the sphere is a Steady Vortex Metric if, and only if, it is of constant curvature.

If the genus of $S$ is greater than zero, then $\sigma$ is the area form of the Bergman metric. The most common definition of the Bergman metric (Jost 2009 eqs. 1.4.22 and 1.4.23) uses a basis of holomorphic differentials $\{\omega_1, \ldots, \omega_{\mathcal{G}}\}$ that satisfy the orthonormality conditions $\frac{i}{2}\int_S \omega_j \wedge \overline{\omega}_k = \delta_{jk}$ (here the overbar denotes complex conjugation). A form $\omega_j$ is holomorphic if, and only if, $\omega_j = \theta_j + *\theta_j$ for some harmonic differential $\theta_j$ (Farkas and Kra 1992, Theorem I.3.11). If we define $\theta_{j+\mathcal{G}} = *\theta_j$, $j = 1, \ldots, \mathcal{G}$, then the orthogonality condition for holomorphic differentials implies the orthogonality condition for harmonic differentials (2.4) and

$$\sigma = \sum_{k=1}^{2\mathcal{G}} \theta_k \wedge *\theta_k = i \sum_{j=1}^{\mathcal{G}} \omega_j \wedge \overline{\omega}_j, \quad \text{with} \quad \int_S \sigma = 2\mathcal{G}. \tag{2.6}$$

The Bergman metric normalized as $\sigma/(2\mathcal{G})$ can also be defined using the Jacobian variety associated with $S$ (see Wentworth (1991); Jost (2009), or equation 1.25 in Fay (1992)). In several references (Wentworth 1991; Okikiolu 2009; Jorgenson and Kramer 2009), the normalized Bergman metric $\sigma/(2\mathcal{G})$ is called by the alternative name “canonical metric.”

Theorem 2.2 appeared in the work of Okikiolu (2009) (proposition 2.3) as a "well known" result related to the Arakelov Green's function. (In Appendix A, I give a more self-contained proof of Theorem 2.2 than that in Okikiolu (2009).) The Arakelov Green's function is used in the definition of the "Arakelov metric" that is characterized by the equation (see Jost (2009) eq. 1.4.24):

$$\frac{K_A}{2\pi}\mu_A = (2-2\mathcal{G})\frac{\sigma}{2\mathcal{G}}, \qquad \mathcal{G} \geq 1, \tag{2.7}$$

where $\mu_A$ and $K_A$ denote the area form and the curvature of the Arakelov metric.

Equation (2.5) implies that the several "natural" metrics considered in this paper satisfy the following relations:

$$\begin{aligned}
&\left(\tfrac{K_{svm}}{2\pi} - \tfrac{2}{V}\right)\mu_{svm} = -\sigma && \text{(SVM)}\\
&\left(\Delta R_{cc} - \tfrac{2\mathcal{G}}{V}\right)\mu_{cc} = -\sigma && \text{(constant curvature=CC)}\\
&\left(\Delta R_B + \tfrac{K_B}{2\pi}\right)\sigma = (2-2\mathcal{G})\tfrac{\sigma}{2\mathcal{G}} && \text{(Bergman)}\\
&\left(\Delta R_A - \tfrac{2}{V}\right)\mu_A = -\tfrac{\sigma}{\mathcal{G}} && \text{(Arakelov)}
\end{aligned} \tag{2.8}$$

From equations (2.7) and (2.8), we obtain

$$\begin{aligned}
CC = SVM \Leftrightarrow Bergman = CC \Leftrightarrow Bergman = SVM,\\
Arakelov = SVM \Leftrightarrow Bergman = Arakelov \Leftrightarrow Arakelov = CC.
\end{aligned} \tag{2.9}$$

For $\mathcal{G} \geq 1$, therefore, a constant curvature metric is a Steady Vortex Metric if and only if the Bergman metric has constant curvature. For $\mathcal{G} = 1$, this is the case, since the flat metric is the Bergman metric and also the Arakelov metric.

In any closed surface $S$ of genus $\mathcal{G} \geq 2$, the curvature of the Bergman metric is non-positive (Jost 2013) (theorem 5.5.1). If the curvature of the Bergman metric $K_B$ is non-constant in every $S$, which as far as I know has not been proved, then constant curvature metrics will never be SVM for $\mathcal{G} \geq 2$. The last theorem in Lewittes (1969) states that $K_B(p) = 0$ if and only if $S$ is hyperelliptic and $p$ is one of the $2\mathcal{G}+2$ classical Weierstrass points on $S$. Therefore, $K_B$ is not constant in hyperelliptic surfaces. Since every surface of genus 2 is hyperelliptic, the following theorem holds.

**Theorem 2.3** *No orientable surface of genus 2 and of constant curvature is a Steady Vortex Surface.*

The Gauss–Bonnet theorem implies that the average curvature of the Bergman metric is

$$K_{Ba} := \frac{1}{V_B}\int_S K_B\sigma = 2\pi\frac{2-2\mathcal{G}}{2\mathcal{G}}, \quad \text{where} \quad V_B = \int_S \sigma = 2\mathcal{G}. \tag{2.10}$$

We define the deviatoric part $K_{B\delta}$ of $K_B$ as:

$$K_{B\delta} := K_B - K_{Ba} \quad \text{with} \quad \int_S K_{B\delta}\,\sigma = 0. \tag{2.11}$$

Equation (2.8) then implies that the Robin function of the Bergman metric satisfies the simple relation

$$-\Delta R_B = \frac{1}{2\pi} K_{B\delta}. \tag{2.12}$$

This equation implies that in any conformal coordinates, $\{z, \overline{z}\}$, $R_B$ has a simple expression in terms of the potentials $F_j(z)$ of the holomorphic differentials $\omega_j = dF_j = F_j'(z)dz = \partial_z F_j(z)dz$ that appear in the definition of $\sigma$ in equation (2.6). Indeed: $i\sum_{j=1}^{\mathcal{G}} \omega_j \wedge \overline{\omega}_j = i\sum_{j=1}^{\mathcal{G}} F_j'(z)\overline{F_j'(z)}dz \wedge d\overline{z} = \lambda_B^2 \frac{i}{2} dz \wedge d\overline{z}$, with $\lambda_B^2 = 2\sum_{j=1}^{\mathcal{G}} F_j'(z)\overline{F_j'(z)}$, $\Delta = \frac{4}{\lambda_B^2}\partial_z\partial_{\overline{z}}$, $\lambda_B^2 = 2\partial_z\partial_{\overline{z}}\sum_{j=1}^{\mathcal{G}} F_j(z)\overline{F_j(z)}$, and $K_B = -\frac{4}{\lambda_B^2}\partial_z\partial_{\overline{z}}\log\lambda_B$; imply

$$R_B(z, \overline{z}) = \frac{1}{4\pi}\log\left[\sum_{j=1}^{\mathcal{G}} F_j'(z)\overline{F_j'(z)}\right] + \frac{1-\mathcal{G}}{2\mathcal{G}}\sum_{j=1}^{\mathcal{G}} F_j(z)\overline{F_j(z)} + constant. \tag{2.13}$$

The Riemann sphere admits a six-dimensional group of conformal transformations (the Moebius group) and a three-dimensional group of isometries. The pull-back metric $g_1$ of the round metric $g_0$ by a Moebius transformation that is not an isometry satisfies $g_1 = \lambda^2 g_0$ with $\lambda^2 \neq 1$ almost everywhere. The Robin function associated with $g_1$ is constant because, although different from $g_0$, $g_1$ is isometric to $g_0$. This type of "nonuniqueness" of a steady vortex metric within a conformal class will happen whenever the group of diffeomorphisms that preserves the conformal structure is larger than the group of isometries. Since all spheres with constant curvature are isometric to the round sphere, we conclude that $g_0$ is the only steady vortex metric modulo isometries. The question about the uniqueness of steady vortex metrics on tori will be postponed to Sect. 5.

## 3 Generalization to Higher Dimensions

The definition of hydrodynamic point vortex is restricted to two dimensions. There is an analogy between vortex and electric charges in two dimensions (Ragazzo and de Barros Viglioni 2017). Since the theory of electrostatics can be generalized to higher dimensions, electrostatics could be the physical guide to the definition of an "eletrostactic force-free metric" in dimension $n$. The idea although interesting leads to some difficulties, which will be discussed in the next paragraph, and it will not be pursued any further.

The Green's function $G(q, p)$, solution to equation (2.1), can be understood as the electric potential due to a positive point charge at the point $p$ plus a uniform distribution of negative charges. The Robin function $R(p)$ defined in equation (2.3) is the overall potential energy $G(q, p)$ minus the "singular potential of the point charge", $-(2\pi)^{-1} \log \ell(q, p)$, the difference being evaluated at $p$. The force upon the point charge is $dR(p)$. The most natural definition of Robin function in dimension $n \geq 3$ would be

$$\lim_{\ell(q,p)\to 0} \left[ G(q, p) - a_n \ell^{n-2}(q, p) \right], \tag{3.14}$$

where $a_n$ is some constant that depends on $n$. Unfortunately, the Robin function defined in this way is not a smooth function unless additional hypotheses are imposed on the Riemannian metric (see Habermann and Jost (1999) for a discussion about this definition in the context of the conformal Laplacian). Another way to define the Robin function would be first to compute the force upon a small Riemannian ball of radius $\epsilon$ at $p$ and then to take the limit as $\epsilon \to 0$ to obtain $dR(p)$. This procedure may lead to quite complicated computations as $n$ increases.

From a mathematical point of view, regularity is the key property of the Robin function, which in two dimensions is used in the definitions of vortex motion and force upon an electric charge. In order to define the Robin function in dimension greater than two, we will regularize the $\delta$-distribution, to do the computations in the regularized setting, and then to take the limit back to recover the $\delta$-distribution. In order to do all these limits independently of coordinates, we use the heat equation. This procedure naturally associates the Robin function with diffusion and Brownian motion. This association will be further addressed in Sect. 4.

Let $(M, g)$ be a compact Riemannian manifold. The heat kernel $K : M \times M \times \mathbb{R}_+ \to M$ is the fundamental solution to the heat equation

$$\left( \frac{\partial}{\partial t} - \Delta_q \right) K(q, p, t) = 0, \quad \text{with} \quad K(q, p, 0) = \delta_p(q). \tag{3.15}$$

The initial condition is understood as a distribution, namely, for any $\phi \in C^\infty(M)$

$$\int_M K(q, p, t)\phi(q)\mu(q) \to \phi(p) \quad \text{as} \quad t \to 0_+.$$

The heat kernel is a $C^\infty$ symmetric, $K(q, p, t) = K(p, q, t)$, function. Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ be the nontrivial eigenvalues to the problem $\Delta\phi + \lambda\phi = 0$ and $\phi_1, \phi_2, \ldots$ be a corresponding $L_2$–orthonormal basis of eigenfunctions for functions that integrate to zero over $M$. Then, the spectral decomposition of the heat kernel is

$$K(q, p, t) = \frac{1}{V} + \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(q)\phi_k(p),$$

with pointwise convergence (see, for instance, Rosenberg (1997) for basic properties of the heat kernel). The Green's function $G(q, p)$ is related to the heat kernel in the following way:

$$G(q, p) = \int_0^\infty \left( K(q, p, t) - \frac{1}{V} \right) dt$$

This is the formula that allows for the definition of the Robin function in dimension $n > 2$ by means of the regularization of the heat kernel.

As before, let $\ell(q, p)$ denote the Riemannian distance between $q$ and $p$. There exists $\epsilon > 0$ and a set of functions $u_0(q, p), u_1(q, p), \ldots$ such that for any given integer $N \geq 0$ the following estimate holds ( the so-called Minakshisundaram-Pleijel asymptotic expansion (Minakshisundaram and Pleijel 1949); see Equations (7)–(9) and the accompanying text)

$$\left| K(q, p, t) - \frac{e^{\frac{-\ell^2(q,p)}{4t}}}{(4\pi t)^{n/2}} \sum_{k=0}^{N} u_k(q, p) t^k \right| \leq C_N t^{N+1-n/2}, \tag{3.16}$$

for all $(q, p)$ with $\ell(q, p) < \epsilon$ and all $t \in (0, 1)$, where $C_N$ is a constant that depends only on $N$ ( see, for instance, Rosenberg (1997) exercise 5 in Section 3.3). The functions $u_k$ are $C^\infty$ and symmetric $u_k(q, p) = u_k(p, q)$ (Moretti 1999). If $q = p$, then the above expression implies

$$K(p, p, t) = \frac{1}{(4\pi t)^{n/2}} \left[ a_0(p) + t a_1(p) + \ldots + t^N a_N(p) \right] + E_N(p, t) \tag{3.17}$$

where $|E_N(p, t)| < C_N t^{N+1-n/2}$ for all $p \in M$ and $t \in (0, 1)$. The functions $a_k(p) = u_k(p, p)$ are local heat invariants of $M$ that can be expressed in terms of powers of the Laplacian and the distance function $\ell$ (Polterovich 2000, Theorem 1.2.1). For instance, $a_0(p) = 1$ and $a_1(p) = s(p)/6$, where $s(p)$ is the scalar curvature (Rosenberg 1997, proof of Lemma 3.26 and Proposition 3.29, respectively).

Suppose that $n \geq 2$ is even and $N$ in equation (3.17) is chosen as $\frac{n}{2} - 1$. Then, for $0 < \epsilon < 1$, equation (3.17) implies

$$\int_\epsilon^1 K(p, p, t) dt = \frac{1}{(4\pi)^{n/2}} \left[ \sum_{k=0}^{\frac{n}{2}-2} a_k(p) \frac{\epsilon^{k+1-\frac{n}{2}}}{\frac{n}{2} - k - 1} \quad - a_{\frac{n}{2}-1}(p) \log \epsilon \right] + C(p, \epsilon)$$

where $\lim_{\epsilon \to 0_+} C(\cdot, \epsilon)$ is a $C^\infty$ function on $M$. Similarly, if $n \geq 3$ is odd and $N$ in equation (3.17) is chosen as $\frac{n}{2} - \frac{3}{2}$, then

$$\int_\epsilon^1 K(p, p, t) dt = \frac{1}{(4\pi)^{n/2}} \left[ \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} a_k(p) \frac{\epsilon^{k+1-\frac{n}{2}}}{\frac{n}{2} - k - 1} \right] + \tilde{C}(p, \epsilon)$$

where $\lim_{\epsilon\to 0_+} \tilde{C}(\cdot,\epsilon)$ is a $C^\infty$ function on $M$. These computations motivate the following definition of the Robin function $R(p)$. If $n \geq 2$ is even, then

$$R(p) = \lim_{\epsilon\to 0_+}\Bigg\{ \int_\epsilon^\infty \left(K(p,p,t) - \frac{1}{V}\right) dt - \frac{1}{(4\pi)^{n/2}} \left[\sum_{k=0}^{\frac{n}{2}-2} a_k(p)\frac{\epsilon^{k+1-\frac{n}{2}}}{\frac{n}{2}-k-1} - a_{\frac{n}{2}-1}(p)\big(\log(4\epsilon)-\gamma\big)\right]\Bigg\}, \tag{3.18}$$

where $\gamma = -\int_0^\infty e^{-x}\log x\,dx = 0.577215\ldots$ is the Euler's constant. The constant term $a_{\frac{n}{2}-1}(p)(\log 4-\gamma)/(4\pi)^{n/2}$ was added to the right-hand side of equation (3.18) to preserve the definition of the Robin function given in equation (2.3).

If $n \geq 3$ is odd, then

$$R(p) = \lim_{\epsilon\to 0_+}\Bigg\{ \int_\epsilon^\infty \left(K(p,p,t) - \frac{1}{V}\right) dt - \frac{1}{(4\pi)^{n/2}} \left[\sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} a_k(p)\frac{\epsilon^{k+1-\frac{n}{2}}}{\frac{n}{2}-k-1}\right]\Bigg\}. \tag{3.19}$$

**Theorem 3.1** *The Robin function can be written in the following alternative way: for $n \geq 3$ odd*

$$R(p) = \lim_{q\to p}\left\{ G(q,p) - \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p)\left(\frac{4}{\ell^2(q,p)}\right)^{\frac{n}{2}-k-1}\Gamma(\frac{n}{2}-k-1)\right\}$$

*and for $n \geq 2$ even*

$$R(p) = \lim_{q\to p}\Bigg\{ G(q,p) - \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-2} u_k(q,p)\left(\frac{4}{\ell^2(q,p)}\right)^{\frac{n}{2}-k-1}\Gamma(\frac{n}{2}-k-1) + \frac{1}{(4\pi)^{n/2}} u_{\frac{n}{2}-1}(q,p)\Big[\log \ell^2(q,p)\Big]\Bigg\}.$$

***Proof*** At first we prove the statement for $n$ odd. From equation (3.19)

$$R(p) = \lim_{\epsilon\to 0_+}\lim_{q\to p}\Bigg\{ \int_0^\infty \left(K(q,p,t) - \frac{1}{V}\right) dt - \int_0^\epsilon \left(K(q,p,t) - \frac{1}{V}\right) dt - \frac{1}{(4\pi)^{n/2}} \left[\sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p)\frac{\epsilon^{k+1-\frac{n}{2}}}{\frac{n}{2}-k-1}\right]\Bigg\} \tag{3.20}$$

The term $\int_0^\epsilon K(q,p,t)dt$ is estimated in the following way. From equation (3.16)

$$K(q,p,t) = \frac{\mathrm{e}^{\frac{-\ell^2(q,p)}{4t}}}{(4\pi t)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p)t^k + \Phi_1(q,p,t)$$

such that $|\Phi_1(p,p,t)| \leq C_1 t^{-1/2}$, $C_1 > 0$, for $t \in (0,1)$ (the constants $C_1, C_2 \ldots$ do not depend on $t, q, p$, or $\epsilon$). Therefore,

$$\int_0^\epsilon K(q,p,t)dt = \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p) \int_0^\epsilon \mathrm{e}^{\frac{-\ell^2(q,p)}{4t}} t^{-\frac{n}{2}+k} dt + \Phi_2(q,p,\epsilon)$$

such that $|\Phi_2(p,p,\epsilon)| < C_2 \epsilon^{1/2}$, $C_2 > 0$. With the change of variables $t = \frac{\ell^2}{4s}$, we obtain

$$\int_0^\epsilon \mathrm{e}^{\frac{-\ell^2}{4t}} t^{-\frac{n}{2}+k} dt = \left(\frac{\ell^2}{4}\right)^{k+1-\frac{n}{2}} \left[\int_0^\infty \mathrm{e}^{-s} s^{\frac{n}{2}-k-2} ds - \int_0^{\frac{\ell^2}{4\epsilon}} \mathrm{e}^{-s} s^{\frac{n}{2}-k-2} ds\right]$$

where $\int_0^\infty \mathrm{e}^{-s} s^{\frac{n}{2}-k-2} ds = \Gamma(\frac{n}{2}-k-1)$ is the Gamma function. Using that $\mathrm{e}^{-s} = 1 - sF(s)$, where $F(s) = \int_0^1 \mathrm{e}^{-\eta s} d\eta$, an explicit computation gives

$$\left(\frac{\ell^2}{4}\right)^{k+1-\frac{n}{2}} \int_0^{\frac{\ell^2}{4\epsilon}} \mathrm{e}^{-s} s^{\frac{n}{2}-k-2} ds = \frac{\epsilon^{-\frac{n}{2}+k+1}}{\frac{n}{2}-k-1} - \frac{\ell^2}{4} \Phi_3\left(\frac{\ell^2}{4\epsilon}\right)$$

where $|\Phi_3\left(\frac{\ell^2}{4\epsilon}\right)| < 1$. Therefore,

$$\int_0^\epsilon \mathrm{e}^{\frac{-\ell^2}{4t}} t^{-\frac{n}{2}+k} dt = \left(\frac{\ell^2}{4}\right)^{k+1-\frac{n}{2}} \Gamma(\frac{n}{2}-k-1) - \frac{\epsilon^{-\frac{n}{2}+k+1}}{\frac{n}{2}-k-1} + \frac{\ell^2}{4} \Phi_3\left(\frac{\ell^2}{4\epsilon}\right)$$

Finally, using that $\int_0^\infty \left(K(q,p,t) - \frac{1}{V}\right) dt = G(q,p)$ and substituting all the previous estimates into equation (3.20), we obtain

$$R(p) = \lim_{\epsilon \to 0_+} \lim_{q \to p} \left\{ G(q,p) - \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p) \left(\frac{\ell^2}{4}\right)^{k+1-\frac{n}{2}} \Gamma(\frac{n}{2}-k-1) \right.$$
$$\left. + \frac{\epsilon}{V} - \Phi_2(q,p,\epsilon) - \frac{\ell^2(q,p)}{4} \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\frac{n}{2}-\frac{3}{2}} u_k(q,p) \Phi_3\left(\frac{\ell^2(q,p)}{4\epsilon}\right) \right\}.$$

The limit as $q \to p$ of the second line of this equation is $\frac{\epsilon}{V}$. Since for a fixed $\epsilon$, the limit as $q \to p$ of the expression inside brackets exists then the limit as $q \to p$ of the sum in the first line also exists and does not depend on $\epsilon$. So, the proof for $n$ odd is finished.

The proof for $n$ even is similar. The only difference is that it is necessary to estimate the additional integral

$$\int_0^{\epsilon} \mathrm{e}^{\frac{-\ell^2}{4t}} t^{-1} dt = \int_{\frac{\ell^2}{4\epsilon}}^{\infty} \mathrm{e}^{-s} s^{-1} ds = -\log\left(\frac{\ell^2}{4\epsilon}\right) + \int_{\frac{\ell^2}{4\epsilon}}^{\infty} \mathrm{e}^{-s} \log s \, ds.$$

The last integral is equal to minus the Euler's constant as $q \to p$. □

We remark that for $n > 2$, the term of highest order in $\ell^{-2}$ is

$$\frac{1}{(4\pi)^{n/2}} \left(\frac{4^{\frac{n}{2}-1}}{\ell^{n-2}(q, p)}\right) \Gamma(\frac{n}{2} - 1),$$

where we used that $u_0(p, p) = a_0(p) = 1$, is minus the "Newtonian potential" that appears in equation (3.14).

For $n = 2$, Theorem 3.1 states that the Robin function defined by equation (3.18) coincides with that given in equation (2.3).

Theorem 3.1 can also be obtained from the Hadamard parametrix, see Garabedian (1986) section 5.3.

The Robin function as given in Theorem 3.1 can be written in terms of the analytic extension of the Minakshisundaram-Pleijel zeta function, Steiner (2005) (dimension two) and Bilal and Ferrari (2013) (dimension greater than one). The relation between the Robin function and the zeta function is presented in Appendix B.

## 4 The "Narrow Escape Time (NET)".

In the context of a compact boundaryless manifold $M$, the narrow escape problem can be described in the following way. Consider a Brownian motion on $M$, whose infinitesimal generator is the Laplace–Beltrami operator $\Delta$. Let $B_\epsilon(q) \subset M$ be a geodesic ball of small radius $\epsilon > 0$. This ball will be the absorbing set or the small window through which a particle can escape. The amount of time that a particle initially at $p$ is expected to spend in $M \backslash B_\epsilon(q)$ (the mean sojourn time) will be denoted as $v_\epsilon(p, q)$. This function is the "narrow escape time" (NET) since it measures the mean time it takes for a particle initially at $p$ to escape through the narrow window $B_\epsilon(q)$. The NET is the solution to the problem (see Holcman and Schuss (2014), equation 3.1):

$$\begin{aligned} &\nu \Delta_p v_\epsilon(p, q) = -1, \quad p \in M \backslash B_\epsilon(q), \\ &\text{with} \quad v_\epsilon(p, q) = 0 \quad \text{for} \quad p \in \partial B_\epsilon(q), \end{aligned} \tag{4.21}$$

where $\nu$ is a diffusion coefficient with dimensional units length$^2$/time. The NET averaged against a uniform distribution of initial points in $M\backslash B_\epsilon(q)$,

$$\overline{v}_\epsilon(q) = \frac{1}{V - |B_\epsilon(q)|} \int_{M\backslash B_\epsilon(q)} v_\epsilon(p, q)\mu(p), \tag{4.22}$$

gives the expected time a particle randomly placed in the manifold remains in it until it scapes through $B_\epsilon(q)$.

In dimension 2, the following theorem was proved in Doyle and Steiner (2017) (Lemma 4.1 and Theorem 4.2 part 2).

**Theorem 4.1** *In dimensions 2, 3, and 4, the "Narrow Escape Time" (NET) is given by*

$$v_\epsilon(p, q) = -\frac{V}{\nu} G(p, q) + \overline{v}_\epsilon(q) + E(p, q, \epsilon), \tag{4.23}$$

*where* $\lim_{\epsilon\to 0} E(p, q, \epsilon) = 0$. *The average NET, equation (4.22), is given by*

$$\begin{aligned}
\overline{v}_\epsilon(q) &= \tfrac{V}{\nu}\left\{-\tfrac{1}{2\pi}\log\epsilon + R(q) + E_2(q,\epsilon)\right\} && n=2\\
\overline{v}_\epsilon(q) &= \tfrac{V}{\nu}\left\{\tfrac{1}{4\pi}\epsilon^{-1} + R(q) + E_3(q,\epsilon)\right\} && n=3\\
\overline{v}_\epsilon(q) &= \tfrac{V}{\nu}\left\{\tfrac{1}{4\pi^2}\epsilon^{-2} - \tfrac{1}{48\pi^2}S(q)\log\epsilon + \tfrac{1}{192\pi^2}S(q) + R(q) + E_4(q,\epsilon)\right\} && n=4,
\end{aligned}$$

*where* $S(q)$ *is the scalar curvature at* $q$ *and* $\lim_{\epsilon\to 0} E_n(p, q, \epsilon) = 0$.

*Remarks:*

- The normalization $\int_M G(p, q)\mu(p) = 0$ (equation (2.2)) ensures the compatibility of both sides of equation (4.23).
- The NET increases as $\epsilon$ decreases in the same way as the Newtonian potential in $\mathbb{R}^n$ increases as the distance to the singularity decreases (see e.g., Holcman and Schuss (2014), Sect. 3, for the same result for surfaces). This is true in all dimensions, not only $n = 2, 3, 4$.
- In dimensions 2 and 3, the divergent terms of $\overline{v}_\epsilon(q)$ with respect to $\epsilon$ do not depend on $q$. For $n = 4$, this is no longer true, since $\overline{v}_\epsilon(q)$ contains a logarithmic divergent term that is proportional to the mean curvature $S(q)$. If the mean curvature is constant on $M$, then the dependence of $\overline{v}_\epsilon(q)$ on $q$ as $\epsilon \to 0$ is determined by the Robin function, as it is in dimensions 2 and 3.

***Proof*** We will prove only the case n=4. The proof of the cases n=2 and n=3 is simpler and goes along the same lines.

We write $v_\epsilon(p, q) = -\frac{V}{\nu} G(p, q) + \frac{V}{\nu} h_\epsilon(p, q)$, and from equations (2.1) and (4.21), we obtain

$$\Delta_p h_\epsilon(p, q) = 0, \quad p \in M\backslash B_\epsilon(q),$$

$$\text{with} \quad h_\epsilon(p,q) = G(p,q) \quad \text{for} \quad p \in \partial B_\epsilon(q). \tag{4.24}$$

Let $x \in \mathbb{R}^n$ be an orthonormal coordinate system on the tangent space of $M$ at $q$. Let $x(p) := \exp_q^{-1} p$ be geodesic normal coordinates in $M$ defined in a neighborhood of $q$. The metric tensor in this coordinates is given by $g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + \mathcal{O}(|x|^3)$. Theorem 3.1 implies that for $p$ sufficiently close to $q$

$$G(p,q) = \frac{u_0(p,q)}{(4\pi)^2}\left(\frac{4}{|x|^2}\right) - \frac{u_1(p,q)}{(4\pi)^2}\log|x|^2 + R(q) + \mathcal{R}_1(p,q).$$

From $u_0(p,q) = 1/\sqrt{\det(\exp_q p)}$ (Rosenberg 1997 equation (3.11)), we obtain

$$\begin{aligned} u_0(q,p) &= \frac{1}{(\det g)^{1/4}} = 1 - \frac{1}{12} R^i_{kil}(q) x^k x^l + \mathcal{O}(|x|^3) \\ &= 1 - \frac{1}{48} S(q)|x|^2 - \frac{1}{12} Z_{kl}(q) x^k x^l + \mathcal{O}(|x|^3) \end{aligned}$$

where $R^i_{jkl}$ is the Riemman curvature tensor and $Z_{kl}$ is the traceless Ricci tensor. From Rosenberg (1997) Proposition 3.29,

$$u_1(p,q) = u_1(q,q) + \mathcal{O}(|x|) = \frac{1}{6} S(q) + \mathcal{O}(|x|).$$

The expressions in the previous paragraph imply that for $p$ sufficiently close to $q$

$$G(p,q) = \underbrace{\frac{1}{4\pi^2|x|^2} - \frac{S(q)}{192\pi^2} - \frac{S(q)}{48\pi^2}\log|x| + R(q)}_{\text{term a}} - \underbrace{\frac{Z_{kl}(q)}{48\pi^2}\frac{x^k x^l}{|x|^2}}_{\text{term b}} + \underbrace{\mathcal{R}_2(p,q)}_{\text{term c}}, \tag{4.25}$$

where $\lim_{|x|\to 0} \mathcal{R}_2(p,q) = 0$.

The solution $h_\epsilon$ to the problem in equation (4.24) can be split into three terms a, b, and c, according to the decomposition of the boundary conditions as given in equation (4.25). The term a is constant for $|x| = \epsilon$. This term appears in the expression for $\overline{v}_\epsilon(q)$ in the statement of the theorem.

The maximum principle dictates that the maximum of the function $p \mapsto |h_c(p,q,\epsilon)|$, where $h_c(p,q,\epsilon)$ is the solution to

$$\begin{aligned} \Delta_p h_{c,\epsilon}(p,q) = 0, \quad p \in M \setminus B_\epsilon(q), \quad \text{with} \\ h_{c,\epsilon}(p,q) = \mathcal{R}_2(p,q) \quad \text{for} \quad p \in \partial B_\epsilon(q), \end{aligned}$$

is attained on $\partial B_\epsilon(q)$. Given that $h_{c,\epsilon}(p,q) = \mathcal{R}_2(p,q)$ for $p \in \partial B_\epsilon(q)$, the limit $\epsilon \to 0$ implies $p \to q$, and since $\lim_{p\to q} \mathcal{R}_2(p,q) = 0$, it follows that $\lim_{\epsilon\to 0} |h_c(p,q,\epsilon)| \to 0$ for $p \in M\backslash B_\epsilon(q)$. Consequently, the component of $h_\epsilon$

corresponding to term (c) in equation (4.25) contributes to the function $E(q, p, \epsilon)$ as stated in the theorem.

The part of $h_\epsilon$ associated with the term b in equation (4.25) will be denoted as $H_\epsilon(p, q)$. It satisfies the problem

$$\Delta_p H_\epsilon(p, q) = 0 \quad \text{with} \quad H_\epsilon(p, q) = -\frac{Z_{kl}(q)}{192\pi^2}\frac{x^k x^l}{\epsilon^2} \quad \text{for} \quad |x| = \epsilon. \tag{4.26}$$

In order to finish the proof, we must show that $|H_\epsilon(p, q)| \to 0$ as $\epsilon \to 0$. The proof has several steps.

**Proposition 4.1**

$$\int_{|x|=\epsilon} Z_{kl}(q)\frac{x^k x^l}{\epsilon^2} d\sigma(x) = \mathcal{O}(\epsilon^5),$$

*where $d\sigma(x)$ is the "area form" on the geodesic sphere $\partial B_\epsilon(q)$.*

***Proof*** The area form on $\partial B_\epsilon(q)$ satisfies $d\sigma(x) = [1 - \frac{1}{6}R^i_{kil}x^k x^l + \mathcal{O}(|x|^3)]d\sigma_E(x)$, where $d\sigma_E(x)$ is the Euclidean area form on the sphere $|x| = \epsilon$. The function $Z_{kl}(q)x^k x^l$ is harmonic with respect to the Euclidean Laplacian, since the trace of $Z$ is zero, and therefore its integral over $|x| = \epsilon$ with respect to $d\sigma_E(x)$ is zero. The proposition follows from the expression for $d\sigma(x)$ and $|d\sigma_E| = \mathcal{O}(\epsilon^3)$ on $|x| = \epsilon$. □

The identity $\Delta_z[G(z, q) - G(z, p)] = \delta_p(z)$, for $z$ and $p$ in $M\backslash B_{2\epsilon}(q)$, and Green's second identity imply that for $\epsilon$ sufficiently small

$$\begin{aligned} H_\epsilon(p, q) = &\int_{\partial B_{2\epsilon}} H_\epsilon(x, q)\nabla_x[G(x, q) - G(x, p)] \cdot \frac{x}{|x|} d\sigma(x) \\ &- \int_{\partial B_{2\epsilon}} [G(x, q) - G(x, p)]\nabla_x H_\epsilon(x, q) \cdot \frac{x}{|x|} d\sigma(x). \end{aligned} \tag{4.27}$$

We will first estimate the integral in the second line of equation (4.27). Equation (4.25) implies that $G(x, q) - G(x, p)$ with $|x| = 2\epsilon$ can be written as a term $A_1 = \mathcal{O}(\epsilon^2)$ that does not depend on $x$ and a term $A_2$ that is bounded by a constant $C_1(q)$ that is independent of $x$ and $\epsilon$. The integral $A_1 \int_{\partial B_{2\epsilon}} \nabla_x H_\epsilon(x, q) \cdot \frac{x}{|x|} d\sigma(x) = 0$ because $\int_{M\backslash B_{2\epsilon}(q)} \Delta_p H_\epsilon(p, q)\mu(p) = 0$. In order to estimate the integral that contains $A_2$ we will use one of the Schauder interior estimates (Gilbarg and Trudinger 2001) (Corollary 6.3)[2]

$$\epsilon \max_{|x|=2\epsilon} \left|\nabla_x H_\epsilon(z, q) \cdot \frac{x}{|x|}\right| \le \max_{p\in M\backslash B_\epsilon(q)} |H_\epsilon(p, q)| \le \max_{|x|=\epsilon} |H_\epsilon(x, q)| = C_2(q),$$

where the second inequality follows from the maximum principle and the constant $C_2(q)$ does not depend on $\epsilon$. So $\left|\int_{|x|=2\epsilon} A_2\nabla_x H_\epsilon(x, q) \cdot \frac{x}{|x|} d\sigma(x)\right| = \mathcal{O}(\epsilon^2)$ and the

[2] Here is the reason for having integrated over the domain $M\backslash B_{2\epsilon}(q)$ and not $M\backslash B_\epsilon(q)$.

integral in the second line of equation (4.27) is at most of the order of $\epsilon^2$. It remains to estimate the integral in the first line of equation (4.27).

For a fixed $p \in M \backslash B_{2\epsilon}(q)$ the function $\nabla_x[G(x,p)] \cdot \frac{x}{|x|}$ restricted to $|x| = 2\epsilon$ is uniformly bounded with respect to $\epsilon$. Therefore, using that $|H_\epsilon(x,q)| < C_2(q)$, we obtain $\int_{|x|=2\epsilon} H_\epsilon(x,q)\nabla_x[G(x,p)] \cdot \frac{x}{|x|} d\sigma(x) = \mathcal{O}(\epsilon^3)$ and it remains to estimate $\int_{\partial B_{2\epsilon}} H_\epsilon(x,q)\nabla_x[G(x,q)] \cdot \frac{x}{|x|} d\sigma(x)$.

The term $\frac{1}{4\pi^2|x|^2}$ in equation (4.25) is the leading order term of a parametrix for the Laplace equation (see Garabedian (1986), Equation (5.79), or Aubin (2013), Theorem 4.13, Equation (17)). This implies that $G(x,q) - \frac{1}{4\pi^2|x|^2}$, where $G(x,q)$ is given in equation (4.25), can be differentiated for $x \neq 0$ and the derivative of $\mathcal{R}_2(x,q)$ is dominated by those of the other terms, so that $|\nabla_x[G(x,q) - \frac{1}{4\pi^2|x|^2}] \cdot \frac{x}{|x|}| = \mathcal{O}(1/|x|)$. This and $|H_\epsilon(x,q)| < C_2(q)$ imply

$$\int_{|x|=2\epsilon} H_\epsilon(x,q)\nabla_x G(x,q) \cdot \frac{x}{|x|} d\sigma = \int_{|x|=2\epsilon} H_\epsilon(x,q)\nabla_x \left[\frac{1}{4\pi^2|x|^2}\right] \cdot \frac{x}{|x|} d\sigma + \mathcal{O}(\epsilon^2).$$

It remains to estimate the integral in the right-hand side of this equation.

Green's second identity with $\Delta_x H_\epsilon(x,q) = 0$ and $\nabla_x \left[\frac{1}{4\pi^2|x|^2}\right] \cdot \frac{x}{|x|} = -\frac{1}{2\pi^2|x|^3}$, which is valid because $x$ are normal coordinates, imply

$$\begin{aligned} &\int_{B_{2\epsilon}(q)\backslash B_\epsilon(q)} H_\epsilon(x,q)\Delta\left[\frac{1}{4\pi^2|x|^2}\right] dx^4 = \\ &-\frac{1}{16\pi^2\epsilon^3}\int_{|x|=2\epsilon} H_\epsilon(x,q) d\sigma + \frac{1}{2\pi^2\epsilon^3}\int_{|x|=\epsilon} H_\epsilon(x,q) d\sigma \\ &-\frac{1}{16\pi^2|\epsilon|^2}\int_{|x|=2\epsilon} \nabla H_\epsilon(x,q) \cdot \frac{x}{|x|} d\sigma + \frac{1}{4\pi^2|\epsilon|^2}\int_{|x|=\epsilon} \nabla H_\epsilon(x,q) \cdot \frac{x}{|x|} d\sigma. \end{aligned}$$

The integrals in the last line are zero because $\int_{M\backslash B_{s\epsilon}(q)} \Delta_p H_\epsilon(p,q)\mu(p) = 0$, for $s = 1, 2$. Due to equation (4.26) and Proposition 4.1, $\frac{1}{2\pi^2\epsilon^3}\int_{|x|=\epsilon} H_\epsilon(x,q) d\sigma = \mathcal{O}(\epsilon^2)$. A computation using the expression for the Laplacian in geodesic normal coordinates (Rosenberg 1997) (Theorem 2.63) gives $\Delta\left[\frac{1}{4\pi^2|x|^2}\right] = \mathcal{O}(|x|^{-2})$. This and $|H_\epsilon(x,q)| < C_2(q)$ imply $\int_{B_{2\epsilon}(q)\backslash B_\epsilon(q)} H_\epsilon(x,q)\Delta\left[\frac{1}{4\pi^2|x|^2}\right] dx^4 = \mathcal{O}(\epsilon^2)$. In conclusion, all these estimates imply

$$\int_{|x|=2\epsilon} H_\epsilon(x,q)\nabla_x\left[\frac{1}{4\pi^2|x|^2}\right] \cdot \frac{x}{|x|} d\sigma = -\frac{1}{16\pi^2\epsilon^3}\int_{|x|=2\epsilon} H_\epsilon(x,q) d\sigma = \mathcal{O}(\epsilon^2),$$

which finishes the proof. □

## 5 Examples of Non-constant Curvature Uniform Drainage Surfaces: Okikiolu's Tori

The flat metric $g_0$ on any two-dimensional torus is a steady vortex metric (SVM). Equation (2.5) implies that there exists a second SVM $g_1$ conformal to $g_0$, $g_1 = \lambda^2 g_0$, if and only if

$$\left(\frac{K_1}{2\pi} - \frac{2}{V_1}\right)\mu_1 = -\frac{2}{V_0}\mu_0 \tag{5.28}$$

Normalizing the volumes $\mu_0$ and $\mu_1$ such that $V_0 = V_1 = 1$, using $-\Delta_0 \log\lambda = \lambda^2 K_1$ and $\mu_1 = \lambda^2\mu_0$, and defining $f = \log\lambda^2$, we get the following equation for $f$

$$\Delta_0 f = 8\pi - 8\pi \mathrm{e}^f \tag{5.29}$$

Each nontrivial solution to this equation corresponds a SVM $g_1$ conformal to $g_0$.

In the following, we present a family of examples due to Okikiolu (2008) of non-flat two-dimensional tori that have constant Robin function, and so are uniform drainage surfaces. Each non-flat torus in the family is conformal to a flat torus, which is also a uniform drainage surface. The Robin function of the non-flat tori are smaller than those of the conformally equivalent flat tori, and so the narrow escape time of the non-flat tori are smaller than those of the conformally equivalent flat tori. There are two differences between our presentation and that of Okikiolu. We simplify the proof that the Robin functions of the non-flat tori are smaller than those of the flat tori, and we represent the non-flat tori in $\mathbb{R}^3$ as the quotient of an isometrically embedded cylinder.

Consider the torus $\mathbb{R}^2/(a\mathbb{Z}\times a^{-1}\mathbb{Z})$, $a \geq 1$, with the conformal structure induced by the flat metric $g_0$. If $a \leq 2/\sqrt{\pi}$, then $g_0$ is the unique uniform drainage metric (Nolasco and Tarantello 1998), and if $a > \sqrt{\pi/2}$, then $g_0$ is not unique (Lin and Lucia 2006). When $a > \sqrt{\pi/2}$, a second natural vortex metric can be constructed in the following way (Okikiolu 2008). Let $(x, y)$ be Cartesian coordinates on $\mathbb{R}^2$. We will look for a nontrivial solution to equation (5.29) that depends only on the variable $x$, $\partial_y f = 0$, with $f(x+a) = f(x)$. Then, $f$ must satisfy $\ddot{f} := \frac{d^2 f}{dx^2} = 8\pi(1-\mathrm{e}^f)$. This ordinary differential equation has a single equilibrium and a first integral

$$H(f, p) = p^2/2 + 8\pi(\mathrm{e}^f - f - 1), \quad p = \dot{f}. \tag{5.30}$$

This shows that all solutions $f$ are periodic with a period $T(E)$, where $E$ is the value of the first integral associated to the solution. The linearized period at $(f, \dot{f}) = (0, 0)$ is $T(0) = \sqrt{\pi/2}$.

The period function $E \to T(E)$ of equation $\ddot{f} = 8\pi(1-\mathrm{e}^f)$ was studied in Chicone (1987) (p. 315), where it is shown that $\frac{d}{dE}T(E) > 0$. We will additionally show that $\lim_{E\to\infty} T(E) = \infty$. Consider the solution associated to the initial condition $f(0) = 0$, $\dot{f}(0) = -\sqrt{2E}$ and integrate the equation $\ddot{f} = 8\pi(1-\mathrm{e}^f)$ on the interval $[0, \beta]$, where $\beta > 0$ is the smallest value such that $f(\beta) = 0$. Since $\dot{f}(\beta) = \sqrt{2E}$, the result is

$$\sqrt{2E}/(4\pi) = \beta - \int_0^\beta \mathrm{e}^f \, dt < \beta < T(E), \tag{5.31}$$

and therefore $\lim_{E\to\infty} T(E) = \infty$. As a result, equation (5.29) has nontrivial solutions for all $a > \sqrt{\pi/2}$ such that $f(x+a) = f(x)$ (indeed as many different solutions as we wish provided $a$ is sufficiently large).

For a given $a > \sqrt{\pi/2}$, let $g_1 = \mathrm{e}^{f(x)}(dx^2 + dy^2)$ be the metric associated to a periodic solution to $\ddot{f} = 8\pi(1 - \mathrm{e}^f)$ with minimal period $a$. We will use lemma A.1 to show that the Robin function $R_1$ associated with $g_1$ has a smaller value than the Robin function $R_0$ of the flat metric. The area form associated with $g_1$ is given by $\mu_1 = \mathrm{e}^f \mu_0 = \left[1 - \frac{\ddot{f}}{8\pi}\right] dx \wedge dy$ and the equation that determines the function $\phi$ in lemma A.1 becomes

$$\Delta_0 \phi \;\; dx \wedge dy = \mu_1 - \mu_0 = -\frac{\ddot{f}}{8\pi} dx \wedge dy, \qquad \int_S \phi \mu_0 = 0$$

that implies

$$\phi(x) = -\frac{f(x)}{8\pi} + \frac{1}{8\pi a} \int_0^a f(x) dx.$$

The constant $c = -\frac{1}{V}\int_S \phi(p)\mu_1(p)$ in lemma A.1 can be easily computed and is equal to $c = \frac{1}{(8\pi)^2 a}\int_0^a \dot{f}^2 dx$. These computations and equation (A.38) imply

$$R_1 - R_0 = \frac{1}{4\pi a} \int_0^a f dx + \frac{1}{(8\pi)^2 a} \int_0^a \dot{f}^2 dx \tag{5.32}$$

If we use the first integral $H$ in equation (5.30) to eliminate $f$ in the right-hand side of this equation and then use $\frac{1}{a}\int_0^a \mathrm{e}^f dx = 1$, which we obtain integrating $\ddot{f} = 8\pi(1 - \mathrm{e}^f)$ over the interval $[0, a]$, then

$$R_1 - R_0 = -\frac{H}{32\pi^2} + \frac{1}{32\pi^2 a} \int_0^a \dot{f}^2 dx. \tag{5.33}$$

The equation $\ddot{f} = 8\pi(1 - \mathrm{e}^f)$ can be written in Hamiltonian form with Hamiltonian function $H$. Using the definition of the action $I(E) = \frac{1}{2\pi}\oint p df$ from Hamiltonian mechanics (Arnol'd 2013), we can write

$$\frac{1}{32\pi^2 a} \int_0^a \dot{f}^2 dx = \frac{1}{32\pi^2 a} \int_0^a p \dot{f} dx = \frac{1}{16\pi a} \frac{1}{2\pi} \oint p df = \frac{I(E)}{16\pi a}$$

In this expression, $a$ is the period of $f$, and therefore $a = T(E)$ where $E$ is the value of $H$ associated with $f$. The Hamiltonian function can be written as a function of the

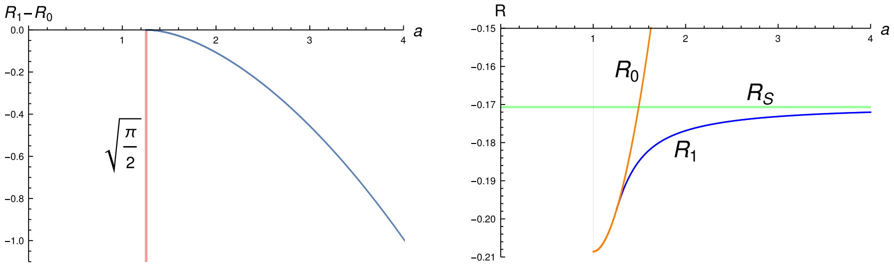


**Fig. 1** LEFT: Difference $R_1 - R_0$ as a function of $a$, where $R_1$ ($R_0$) is the Robin function of the non-flat torus $\{\mathbb{R}^2/(a\mathbb{Z} \times a^{-1}\mathbb{Z}), g_1\}$ (flat torus $\{\mathbb{R}^2/(a\mathbb{Z} \times a^{-1}\mathbb{Z}), g_0\}$). RIGHT: Graphs of $R_1$ and $R_0$ as a function of $a$. The horizontal line represents the value of the Robin function $R_S$ for a round sphere of area 1. According to Okikiolu (2008) (Appendix): $R_0(a) = -\frac{\log(2\pi)}{2\pi} - \frac{\log(|\eta(ia^2)|^4 a^2)}{4\pi}$ and $R_S = -\frac{1+\log\pi}{4\pi}$, where $\eta$ is the Dedekind eta function

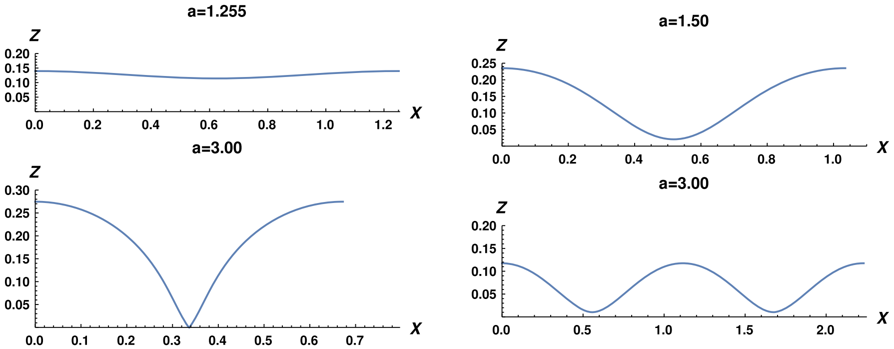


**Fig. 2** Generating functions of four periodic cylinders (each cylinder is constructed rotating the graph of $X \to Z(X)$ about the $X$-axis). The quotient of a cylinder by the group of periodic translations gives a torus that is isometric to a non-flat torus with a steady vortex metric. The value of the period $a$ of each torus is shown in the corresponding figure. There are two different tori with $a = 3$: one for which the minimal period of $f$ is 3 and another for which the minimal period of $f$ is 1.5, and so $f$ oscillates twice inside a fundamental cell

action $E = H(I)$ with $H'(I) = 2\pi/T(I)$. All these results imply that equation (5.33) can be written as

$$R_1 - R_0 = \frac{1}{32\pi^2}\left(IH'(I) - H(I)\right) \tag{5.34}$$

Since $H'(I) = 2\pi/T(I) > 0$ and $T'(I) > 0$ (Chicone 1987) (p. 315), we conclude that $H''(I) = -2\pi T'(I)/T^2(I) < 0$. This fact and $H(0) = 0$ imply that $R_1 - R_0 < 0$. In Fig. 1, we present a numerical estimate of the difference $R_1 - R_0$.

The torus $\left\{\mathbb{R}^2/(a\mathbb{Z} \times a^{-1}\mathbb{Z}), g_1\right\}$ can be represented as the quotient of a cylinder that is infinite along the $x$-axis and periodic with period $a$. We will show that this cylinder can be isometrically embedded in the Euclidean three-space. Let $X, Y, Z$ be Cartesian coordinates in $\mathbb{R}^3$. We will look for an embedding of the form $X = X(x)$, $Y = F(x)\sin(2\pi a y)$ and $Z = F(x)\cos(2\pi a y)$, where $x \in \mathbb{R}$, $y \in \mathbb{R}/a^{-1}\mathbb{Z}$. The pull-back of the Euclidean metric by the embedding is $(\dot{X}^2 + \dot{F}^2)dx^2 + 4\pi^2 a^2 F^2 dy^2$.

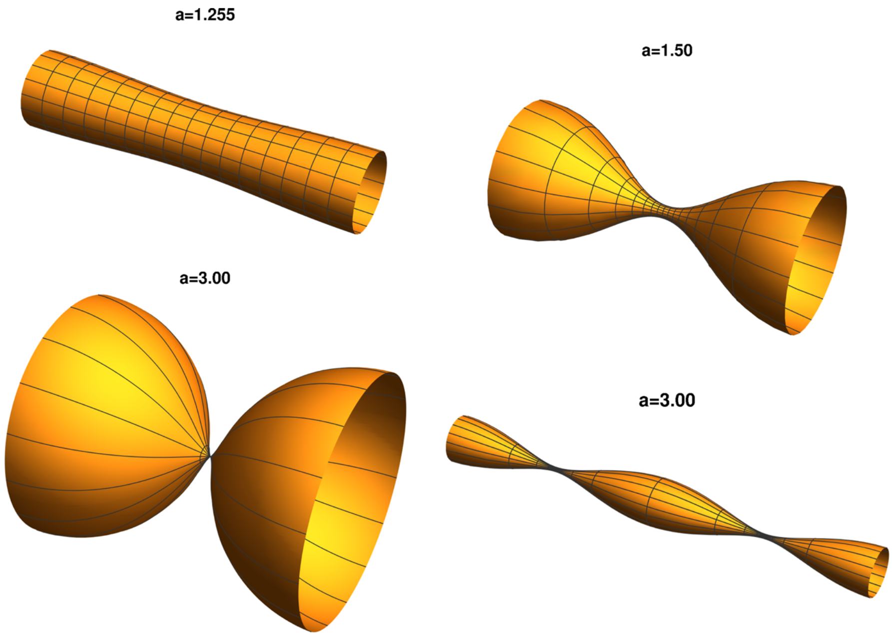


**Fig. 3** Three-dimensional representation of the tori whose generators are shown in Fig. 2. See the caption of Fig. 2 for explanations

We impose that the pull-back coincides with $g_1 = e^{f(x)}(dx^2 + dy^2)$ and obtain that $4\pi^2 a^2 F^2 = e^f$ and $\dot{X}^2 + \dot{F}^2 = e^f$. This implies that $F(x) = e^{f(x)/2}/(2\pi a)$ and

$$\dot{X}^2 = e^f \left(1 - \frac{\dot{f}^2}{16\pi^2 a^2}\right) \tag{5.35}$$

Since $X : \mathbb{R} \to \mathbb{R}$ must be a diffeomorphism, the right-hand side of equation (5.35) must be strictly positive for all $x \in [0, a]$. We will show this in the following paragraph.

The first integral (5.30) and $(e^f - f - 1) \geq 0$ imply that $\dot{f}^2(x) \leq 2E$ for $x \in [0, a]$, where $E$ is the value of $H$ for the solution with period $T(E) = a$. This and inequality (5.31) imply

$$1 - \frac{\dot{f}^2}{16\pi^2 a^2} \geq 1 - \frac{2E}{16\pi^2 T(E)^2} > 0.$$

In Fig. 2, we show the curves $x \to \{X(x), Z(x)\}$, $x \in [0, a]$ and $y = 0$, that when rotated about the $X-$axis generate the embedded cylinders. These curves were obtained by the numerical integration of equations $\ddot{f} = 8\pi(1 - e^f)$ and (5.35) for: $a = 1.255$, $a = 1.50$, and $a = 3.0$. Only one fundamental cell of the periodic cylinder is shown. There are two different tori with $a = 3$: one for which the minimal period of $f$ is 3 and another for which the minimal period of $f$ is 1.5, and so $f$ oscillates twice inside a fundamental cell. In Fig. 3 we show a 3-dimensional representation of a single cell of each one of the cylinders whose generators are in Fig. 2. It is clear from Fig. 3 that for $a \gg 1$, the cylinder becomes a collection of aligned spheres each one touching its neighbors at a single point. This is in agreement with the interpretation given in Okikiolu (2008): (the non-flat torus) "is approximately spherical except for a short wormhole joining the poles." Note: as shown in the right panel of Fig. 1, in the limit as $a \to \infty$, the tori converge to a punctured sphere and $R_1(a) \to R_S$ where $R_S$ is the Robin function of the round sphere.

**Acknowledgements** This paper is dedicated to Jair Koiller who introduced me to the subject of vortices on surfaces and presented to me the work of Okikiolu and Steiner. Jair has been a constant source of inspiration.

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

## A Proofs of theorems 2.1 and 2.2.

**Lemma A.1** *Let $g_0$ and $g_1$ be two different Riemannian metrics on $S$ in the same conformal class, $g_1 = \lambda^2 g_0$. Let $G_j$, $R_j$, $\mu_j$, $K_j$, $\Delta_j$, $j = 0, 1$, be the: Green's function, Robin function, volume form, Gaussian curvature, and Laplace operator, of $g_j$. Let the conformal factor $\lambda$ be normalized such that the volumes $\int_S \mu_0 = \int_S \mu_1 = V$ are the same. Let $\phi$ be the unique solution of*

$$d * d\phi = \frac{\mu_1 - \mu_0}{V} \quad \textit{with} \quad \int_S \phi\mu_0 = 0,$$

*that is given by*

$$\phi(p) = -\frac{1}{V}\int_S G_0(q, p)\lambda^2(q)\mu_0(q) = -\frac{1}{V}\int_S G_0(q, p)\mu_1(q) \tag{A.36}$$

*Then $G_0$, $G_1$, $R_0$ and $R_1$ satisfy the following relations:*

$$G_1(q, p) - G_0(q, p) = \phi(q) + \phi(p) + c \tag{A.37}$$

$$R_1(p) = R_0(p) + \frac{1}{2\pi}\log\lambda(p) + 2\phi(p) + c \tag{A.38}$$

*where*

$$c = -\frac{1}{V}\int_S \phi(p)\mu_1(p) = \frac{1}{V^2}\int_S\int_S G_0(q, p)\mu_1(q)\mu_1(p)$$

*is a constant. Equation (A.37) is in Morpurgo (1996) (equation (8)) and Equation (A.38) is in Steiner (2005) (Theorem 4).*

***Proof*** Let $p$ and $q$ be sufficiently close to be in a domain $U$ of a local uniformizer $z$. Suppose that $U$ is such that any two points in $U$ are connected by a single geodesic in $U$. In this coordinates the length elements of the metrics $g_0$ and $g_1$ are $\lambda_0|dz|$ and

$\lambda_1|dz|$, respectively. Notice that $\lambda_1 = \lambda\lambda_0$. If $\mu = dx \wedge dy$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the area form and the usual Laplacian in the coordinates $z = (x, y)$, respectively, then

$$\Delta_j = \frac{1}{\lambda_j^2}\Delta, \qquad -\Delta \log \lambda_j = \lambda_j^2 K_j, \qquad \mu_j = \lambda_j^2 \mu, \qquad j = 0, 1. \quad \text{(A.39)}$$

The Dirac-delta distributions associated with the volume forms $\mu_0$ and $\mu_1$ satisfy

$$\delta_{j,w} = \frac{1}{\lambda_j^2}\delta_w, \quad \text{where} \quad \psi(w) = \int \psi(x, y)\delta_w(x, y) dx \wedge dy.$$

To simplify the notation we write $z(q) = z$ and $z(p) = w$. In the coordinates $(z, w)$, equation (2.1) becomes

$$-\Delta_z G_j(z, w) = \delta_w(z) - \frac{\lambda_j^2(z)}{V}. \quad \text{(A.40)}$$

The Green's function can be written as

$$G_j(z, w) = -\frac{1}{2\pi} \log|z - w| + f_j(z, w) \quad \text{(A.41)}$$

where $f_j(z, w) = f_j(w, z)$. Since $\Delta_z \log|z - w| = 2\pi\delta_w$, we obtain

$$\Delta_z f_j(z, w) = \frac{\lambda_j^2(z)}{V}. \quad \text{(A.42)}$$

Let $\ell_j(z, w)$ be the length with respect to the metric $g_j$ of the unique geodesic connecting $z$ to $w$. It can be shown that (see for instance (Ragazzo and de Barros Viglioni 2017) proof of Theorem 5.1):

$$\ell_j(z, w) = |w - z|\sqrt{\lambda_j(z)\lambda_j(w)}[1 + \mathcal{O}(|z - w|)]$$

Therefore,

$$G_j(z, w) + \frac{1}{2\pi} \log \ell_j(z, w) = f_j(z, w) + \frac{1}{4\pi} \log[\lambda_j(z)\lambda_j(w)] + \mathcal{O}(|z - w|).$$

Taking the limit as $|z - w| \to 0$, we obtain

$$R_j(z) = f_j(z, z) + \frac{1}{2\pi} \log \lambda_j(z). \quad \text{(A.43)}$$

If we subtract equation (A.40) for $j = 0$ from that for $j = 1$, we obtain

$$\Delta_z G_1(z, w) - \Delta_z G_0(z, w) = \frac{\lambda_1^2(z) - \lambda_0^2(z)}{V}. \quad \text{(A.44)}$$

This equation can be written intrinsically in terms of two-forms as

$$d_q * d_q G_1(q, p) - d_q * d_q G_2(q, p) = \frac{\mu_1 - \mu_0}{V} = d_q * d_q \phi(q)$$

where $\phi$ is the function in the statement of the theorem. Thus, $G_1(q, p) - G_0(q, p) = \phi(q) + \psi(p)$ that, due to the symmetry $G_j(p, q) = G_j(q, p)$, implies equation (A.37). Equation $d_q * d_q \phi(q) = \frac{\mu_1 - \mu_0}{V}$ can be written as $\Delta_0 \phi = (\lambda^2 - 1)/V$. The representation formula (2.2) for $\phi$ plus the relations $\int_S \phi \mu_0 = 0$ and $\int_S G_0(q, p)\mu_0(q) = 0$ imply that $\phi$ can be written as in equation (A.36). Integrating both sides of equation (A.37) with respect to $\mu_1(q)$ over $S$ we obtain the expression for $c$ in the lemma. In the $z$-coordinates, equation (A.37) implies $f_1(z, w) - f_0(z, w) = \phi(z) + \phi(w) + c$. This equation and equation (A.43) imply equation (A.38). □

**Lemma A.2** *Let $g_0$ and $g_1$ be two different Riemannian metrics on $S$ in the same conformal class, as in Lemma A.1. Let $z = x + iy$ be a local uniformizer and to simplify the notation write $z(q) = w$ and $z(p) = z$. Then,*

$$\left(\Delta_1 R_1 + \frac{K_1}{2\pi} - \frac{2}{V}\right)\mu_1 = \left(\Delta_0 R_0 + \frac{K_0}{2\pi} - \frac{2}{V}\right)\mu_0 = -\tilde{\sigma}, \tag{A.45}$$

*where*

$$-\tilde{\sigma} = 8h(z)dx \wedge dy = 4ih(z)dz \wedge d\overline{z},$$

*with*

$$h(z) = \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z}G_0(z, w)\Big|_{w=z} = \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z}G_1(z, w)\Big|_{w=z}.$$

***Proof*** In this proof, we follow the notation of the proof of Lemma A.1. In the $z$-coordinates, equation (A.38) becomes

$$R_1(z) = R_0(z) + \frac{1}{2\pi}\log\lambda_1(z) - \frac{1}{2\pi}\log\lambda_0(z) + 2\phi(z) + c.$$

Taking the Laplacian $\Delta_z$ of both sides of this equation, using $\Delta_z \phi_z = (\lambda_1^2 - \lambda_0^2)/V$, and the relations (A.39) for conformal metrics, we obtain the first equality in equation (A.45). We recall that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, $\Delta_z = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$, and $dx \wedge dy = \frac{i}{2}dz \wedge d\overline{z}$. From equation (A.43), we obtain for $j = 0, 1$

$$\begin{aligned}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}R_j(z) &= \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}f_j(z, w)\Big|_{w=z} + \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial w}f_j(z, w)\Big|_{w=z} \\ &\quad + \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial w}f_j(z, w)\Big|_{w=z} + \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z}f_j(z, w)\Big|_{w=z} + \frac{1}{2\pi}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\log\lambda_j(z).\end{aligned}$$

From equation (A.42) and $f_j(z,w) = f_j(w,z)$, we get

$$\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} f_j(z,w)\Big|_{w=z} = \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial w} f_j(z,w)\Big|_{w=z} = \frac{1}{4}\Delta_z f_j(z,w)\Big|_{w=z} = \frac{1}{4}\frac{\lambda_j^2(z)}{V}$$

From equation (A.41) and from the symmetry $\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial w} f_j(z,w) = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial w} f_j(w,z)$, we get

$$\frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z} G_j(z,w)\Big|_{w=z} = \frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z} f_j(z,w)\Big|_{w=z} = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial w} f_j(z,w)\Big|_{w=z}. \quad \text{(A.46)}$$

Finally, from the above equations and from equation (A.39), we obtain

$$\Delta_z R_j(z) + \frac{\lambda_j^2(z)}{2\pi}K_j(z) - \frac{2\lambda_j^2(z)}{V} = 8\frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z} G_j(z,w)\Big|_{w=z}$$

If we multiply both sides of this equation by $dx \wedge dy$, we obtain

$$\left(\Delta_j R_j + \frac{K_j}{2\pi} - \frac{2}{V}\right)\mu_j = 8\frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z} G_j(z,w)\Big|_{w=z} dx \wedge dy$$

for $j = 0, 1$. Since we have already shown that the left-hand side of this equation gives the same 2-form for $j = 0$ and $j = 1$, then the right-hand side has the same property. □

The expression $\frac{\partial}{\partial \overline{w}}\frac{\partial}{\partial z} G_j(z,w)$ is formally analogous to the traditional Bergman kernel for bounded domains in the complex plane. Indeed, equation (A.41), which represents the decomposition of the Green's function into its singular and regular parts, applies as well to the Green's function for bounded domains in the plane. The distinction between the two situations lies in the regular part $f$, which is harmonic in bounded domains, whereas in this paper, the non-harmonicity of $f$ stems from the additional term of constant "background vorticity."

Following (Royden 1967), let $\partial$ be an operator defined on complex valued functions by $\partial = \frac{1}{2}(d + i * d)$ and $\overline{\partial} = \frac{1}{2}(d - i * d)$. In terms of a local uniformizer $z$, we have $\partial f = \frac{\partial f}{\partial z}dz$ and $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}}d\overline{z}$.

**Lemma A.3** *If $G(q,p)$ is the Green's function associated with a given metric and $\{\theta_1, \ldots, \theta_{2\mathcal{G}}\}$ is an orthonormal basis of harmonic forms, then*

$$-2(\partial_p\overline{\partial}_q G + \overline{\partial}_p\partial_q G) = -(d_p d_q G + *_p *_q d_p d_q G) = \sum_{k=1}^{2\mathcal{G}} \theta_k(q)\theta_k(p) \quad \text{(A.47)}$$

*is the Bergman reproducing kernel for harmonic forms in $S$. Moreover, if $q$ and $p$ are in the domain of a local uniformizer with $z(q) = w$ and $z(p) = z$, then*

$$2(\partial_p\overline{\partial}_q G + \overline{\partial}_p\partial_q G) = 4\mathrm{Re}\{\partial_p\overline{\partial}_q G\} = 4\mathrm{Re}\left\{\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w,z)d\overline{w}dz\right\} \quad \text{(A.48)}$$

***Proof*** The equality $2(\partial_p\overline{\partial}_q G+\overline{\partial}_p\partial_q G) = d_p d_q G + *_p *_q d_p d_q G$ and equation (A.48) are direct consequences of the definition of the operators $\partial$ and $\overline{\partial}$. Due to equation (A.46), the function $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w, z)$ is $C^\infty$ for all values of $z$ and $w$ including $z = w$. So the double one-form $d_p d_q G + *_p *_q d_p d_q G$ is $C^\infty$ on $S \times S$.

The Bergman reproducing kernel for harmonic forms $H(q, p) = \sum_{k=1}^{2\mathcal{G}} \theta_k(q)\theta_k(p)$ is characterized by the following properties:

For an arbitrary function $\psi$ on $S$:

$$\int_S d\psi(p) \wedge H(q, p) = 0$$
$$\int_S *_p d\psi(p) \wedge H(q, p) = 0,$$

where the integrations are with respect to the variable $p$; and for any harmonic one-form $\nu$ on $S$

$$\nu(q) = \int_S \nu(p) \wedge *_p H(q, p) = \sum_{k=1}^{2\mathcal{G}} \theta_k(q) \int_{S(p)} \nu(p) \wedge *_p \theta_k(p).$$

In order to prove the equality $d_p d_q G + *_p *_q d_p d_q G = -H(q, p)$, we use the regularity of $d_p d_q G + *_p *_q d_p d_q G$ on $S \times S$. So, for any function $\psi$ on $S$

$$\int_S d_p\psi(p) \wedge (d_p d_q G + *_p *_q d_p d_q G) = -\int_S \psi(p) \wedge d_p(d_p d_q G + *_p *_q d_p d_q G)$$
$$= -\lim_{\epsilon\to 0} \int_{S-B_\epsilon(q)} \psi(p) \wedge d_p(d_p d_q G + *_p *_q d_p d_q G)$$

where $B_\epsilon(q)$ is a small ball (with respect to any local uniformizer) of radius $\epsilon$ with center at $q$. For $p$ outside $B_\epsilon(q)$,

$$d_p(d_p d_q G + *_p *_q d_p d_q G) = *_q d_q (d_p *_p d_p G) = *_q d_q \left(\frac{\mu(p)}{V}\right) = 0,$$

so $\int_S d_p\psi(p) \wedge (d_p d_q G + *_p *_q d_p d_q G) = 0$. In the same way it is possible to prove that $\int_S *_p d\psi(p) \wedge (d_p d_q G + *_p *_q d_p d_q G) = 0$.

It remains to show that $\nu(q) = -\int_S \nu(p) \wedge *_p(d_p d_q G + *_p *_q d_p d_q G)$ for any harmonic one-form $\nu$ on $S$. This is a consequence of

$$\int_S \nu(p) \wedge *_p(d_p d_q G + *_p *_q d_p d_q G)$$
$$= \lim_{\epsilon\to 0} \int_{S-B_\epsilon(q)} \nu(p) \wedge *_p(d_p d_q G + *_p *_q d_p d_q G)$$
$$= \lim_{\epsilon\to 0} \int_{-\partial B_\epsilon(q)} *_p\nu(p) d_q G + \int_{-\partial B_\epsilon(q)} \nu(p) *_q d_q G$$

An explicit computation using a local uniformizer gives that this last integral is equal to $-\nu(q)$. □

Theorem 2.2 is a consequence of Lemmas A.2 and A.3 and the following reasoning. Let $z(p) = z = x + iy$ and $z(q) = w = \xi + i\eta$ be the components of the local uniformizer used in Lemma A.3 and $\theta_k(p) = \theta_{k1}(z)dx + \theta_{k2}(z)dy$ and $\theta_k(q) = \theta_{k1}(w)d\xi + \theta_{k2}(w)d\eta$ be the components of $\theta_k$. Lemma A.3 implies that

$$\begin{aligned}\sum_{k=1}^{2\mathcal{G}} \theta_k(q)\theta_k(p) &= \left(\sum_{k=1}^{2\mathcal{G}} \theta_{k1}(w)\theta_{k1}(z)\right) dxd\xi + \left(\sum_{k=1}^{2\mathcal{G}} \theta_{k2}(w)\theta_{k2}(z)\right) dyd\eta \\ &+ \left(\sum_{k=1}^{2\mathcal{G}} \theta_{k2}(w)\theta_{k1}(z)\right) dxd\eta + \left(\sum_{k=1}^{2\mathcal{G}} \theta_{k1}(w)\theta_{k2}(z)\right) dyd\xi \\ &= -4\mathrm{Re}\left\{\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w,z)d\overline{w}dz\right\}\end{aligned}$$

For $q = p$ and $dz = dw$, the right-hand side of this equation becomes

$$-4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w,z)\Big|_{w=z}(dx^2 + dy^2)$$

that implies

$$\sum_{k=1}^{2\mathcal{G}} \theta_{k1}^2(z) = \sum_{k=1}^{2\mathcal{G}} \theta_{k2}^2(z) = -4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w,z)\Big|_{w=z}, \quad \text{and} \quad \sum_{k=1}^{2\mathcal{G}} \theta_{k1}(z)\theta_{k2}(z) = 0$$

So, the form $\sigma$ in theorem 2.2 can be written as

$$\begin{aligned}\sigma(z) &= \sum_{k=1}^{2\mathcal{G}} \theta_k(z) \wedge *\theta_k(z) = \sum_{k=1}^{2\mathcal{G}} [\theta_{k1}(z)dx + \theta_{k2}(z)dy] \wedge [\theta_{k1}(z)dy - \theta_{k2}(z)dx] \\ &= \sum_{k=1}^{2\mathcal{G}} [\theta_{k1}^2(z) + \theta_{k2}^2(z)]dx \wedge dy = -8\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{w}} G(w,z)\Big|_{w=z} dx \wedge dy = \tilde{\sigma},\end{aligned}$$

where $\tilde{\sigma}$ is the form in equation (A.45). This proves that equation (2.5) holds and finishes the proof of theorem 2.2. □

Now, we prove theorem 2.1. The Robin function on a Riemannian manifold $(S, g)$ is constant whenever $(S, g)$ admits a transitive Lie group action of isometries. So, the Robin function is constant for the round sphere and for all flat tori. Let $S$ be a sphere (torus) endowed with a Riemannian metric $g_0$. The uniformization theorem implies the existence of a diffeomorphism from $(S, g_0)$ to the round sphere (a flat torus) $(\mathbb{S}^2, g_1)$ such that the pull-back of $g_1$ is conformal to $g_0$. So, the existence of a steady vortex metric on the sphere (torus) is proved.

The proof is more complicated when $S$ is compact and has a genus larger than one. Equation (A.38) implies:

$$\Delta_0 R_1(p) = \Delta_0 R_0(p) + \frac{1}{2\pi}\Delta_0 \log\lambda(p) + 2\frac{\lambda^2 - 1}{V} \tag{A.49}$$

Imposing that $R_1$ is constant, normalizing the volume $V$ of $S$ to be equal to one, and defining

$$u = 4\pi R_0 + \log\lambda^2$$

we get the following equation for $u$

$$\Delta_0 u = 8\pi - 8\pi h \mathrm{e}^u \tag{A.50}$$

where $h = \mathrm{e}^{-4\pi R_0}$. To each solution of this equation corresponds a Riemannian metric $g_1$ conformal to $g_0$ such that $\Delta R_1 = 0$ and therefore $R_1$ is constant. Equation (A.50) was very much studied for several reasons. It appears in the problem of finding a Riemannian metric on the sphere with a prescribed curvature $h$ that is conformal to the standard metric with curvature $4\pi$ (the conformal factor is $\mathrm{e}^u$). It also appears in the so-called Chern-Simons-Higgs theory (see Ding et al. (1997) for references). The following theorem was taken from Ding et al. (1997) (it is a combination of their theorem 1.2 plus their remark 1.3).

**Theorem A.1** *(*Ding, Jost, Li,and Wang*) Let* $(S, g_0)$ *be a compact Riemann surface and let* $K_0$ *be its Gauss curvature. Let* $h$ *be a positive smooth function on* $S$*. Suppose that the function* $8\pi R_0 + 2\log h$ *achieves its maximum at* $p$*. If* $\Delta_0 \log h(p) > -(8\pi - 2K_0(p))$ *then equation* (A.50) *has a smooth solution.*

It is remarkable that in the case we are interested in $h = \mathrm{e}^{-4\pi R_0}$ and $8\pi R_0 + 2\log h = 0$. So, any point in $S$ is a point of maximum and therefore to finish the proof it is sufficient to show the existence of a point $p$ in $S$ where the inequality $0 > -\Delta_0 \log h(p) - (8\pi - 2K_0(p))$ holds. The Gauss–Bonet theorem implies $\int_S K_0\mu_0 = 2\pi(2 - 2\mathcal{G})$, where $\mathcal{G}$ is the genus of $S$. Since $\int \mu_0 = 1$, the integral of the right-hand side of the inequality above is $-8\pi\mathcal{G} < 0$. This finishes the proof of existence of a natural vortex metric if $\mathcal{G} > 1$. □

## B The Robin Function and the Minakshisundaram-Pleijel Zeta Function.

The Minakshisundaram-Pleijel zeta function, which will be referred as the zeta function, is defined as

$$\zeta(q, p, s) = \sum_{k=1}^{\infty} \frac{\phi_k(q)\phi_k(p)}{\lambda_k^s} = \frac{1}{\Gamma(s)}\int_0^{\infty}\left(K(q, p, t) - \frac{1}{V}\right)t^{s-1}dt, \tag{B.51}$$

where $s \in \mathbb{C}$ and $\mathrm{Re}\, s > n/2$ (the convergence is a consequence of inequality (3.16)).

According to the theorem in Sect. 5 of Minakshisundaram and Pleijel (1949), the function

$$\zeta(p, s) := \zeta(p, p, s)$$

can be extended as a meromorphic function to the whole complex plane. If dimension $n \geq 3$ is odd, then the only possible poles of $\zeta(p, s)$ are located at $s = n/2, n/2 - 1, \ldots, 3/2, 1/2, -1/2, \ldots$. If the dimension $n$ is even, then $\zeta(p, s)$ has at most a finite number of poles that are possibly located at $s = n/2, n/2-1, \ldots, 2, 1$ and the residue at the poles can be computed (Minakshisundaram and Pleijel 1949). In particular, if $n$ is even and $s$ is close to $s = 1$, then

$$\zeta(p, s) = \frac{1}{(4\pi)^{n/2}} \frac{a_{n/2-1}(p)}{s-1} + \text{convergent power series in } (s-1), \quad \text{(B.52)}$$

where $a_{n/2-1}(p)$ is the function that appears in equation (3.17).

If $s$ is made equal to one in equation (B.51), then we obtain a formal expression

$$G(q, p) = \int_0^\infty \left( K(q, p, t) - \frac{1}{V} \right) dt = \sum_{k=1}^{\infty} \frac{\phi_k(q)\phi_k(p)}{\lambda_k} \;{}'\!=\!{}'\; \zeta(q, p, 1) \quad \text{(B.53)}$$

that indicates a possible relation between the regularization of $G(q, p)$ and $\zeta(q, p, s)$ as $q \to p$ and $s \to 1$. Indeed, for $n = 2$, the following result holds (see, e.g., Steiner (2005), Proposition 2 and Appendix):

$$\begin{aligned} R(p) &= \lim_{\ell(q,p)\to 0} \left[ G(q, p) + \frac{1}{2\pi} \log \ell(q, p) \right] \\ &= \lim_{s\to 1} \left[ \zeta(p, s) - \frac{1}{(4\pi)} \frac{1}{s-1} \right] + \frac{\log 4 - 2\gamma}{4\pi} \end{aligned} \quad \text{(B.54)}$$

where $\gamma$ is the Euler's constant. In the following theorem, we show that this result can be generalized to higher dimensions. The same result, for an elliptic operator that appears in the context of quantum field theory in curved spacetime, was obtained by Bilal and Ferrari in Bilal and Ferrari (2013) (Sect. 3). If the parameters $m$ and $\psi$ that appear in their elliptic operator are set equal to zero, then the formulas in equations (3.45) and (3.46) of Bilal and Ferrari (2013) are exactly ours in theorem (B.1).

**Theorem B.1** *The Robin function can be written in terms of the analytic extension of the Minakshisundaram-Pleijel zeta function as*

$$\begin{aligned} R(p) &= \lim_{s\to 1} \left[ \zeta(p, s) - \tfrac{1}{(4\pi)} \tfrac{1}{s-1} \right] + \tfrac{\log 4 - 2\gamma}{(4\pi)^{n/2}} && \text{if } n \text{ is even,} \\ R(p) &= \zeta(p, 1) && \text{if } n \text{ is odd.} \end{aligned} \quad \text{(B.55)}$$

***Proof*** We will prove the theorem only for $n$ even, since the proof for $n$ odd is similar. For $s > n/2$, both sides of equation (B.51) converge. The idea is to add terms to both sides of that equation such that the integral in the right-hand side of equation (B.51) converges when $s = 1$. In analogy to what we did to define the Robin function, we rewrite equation (B.51) for $s > n/2$ as

$$\begin{aligned}\zeta(p,s) - \sum_{k=0}^{n/2-1} \frac{a_k(p)}{(4\pi)^{n/2}\Gamma(s)} \int_0^1 t^{k-n/2} t^{s-1} dt &= \frac{1}{\Gamma(s)} \int_1^\infty \left( K(p,p,t) - \frac{1}{V} \right) t^{s-1} dt \\ &+ \lim_{\epsilon\to 0_+} \frac{1}{\Gamma(s)} \int_\epsilon^1 \left( K(p,p,t) - \frac{1}{V} - \sum_{k=0}^{n/2-1} \frac{a_k(p)}{(4\pi)^{n/2}} t^{k-n/2} \right) t^{s-1} dt.\end{aligned} \tag{B.56}$$

For $s > n/2$, the left-hand side of this equation can be written as

$$\zeta(p,s) - \frac{a_{n/2-1}(p)}{(4\pi)^{n/2}\Gamma(s)} \frac{1}{s-1} - \sum_{k=0}^{n/2-2} \frac{a_k(p)}{(4\pi)^{n/2}\Gamma(s)} \frac{1}{s-n/2+k}. \tag{B.57}$$

Due to equations (3.16) and (3.17), the integrand in the last line of equation (B.56) is bounded by a constant times $t^{s-1}$, and therefore the right-hand side of equation (B.56) is an analytic function of $s$ for $\operatorname{Re} s > 0$. This implies that the analytic continuation of $\zeta(p,s)$ to $\operatorname{Re} s > 0$ is given by the regular function at the right-hand side of equation (B.56) plus the poles at $s = 1, 2, \ldots, n/2$ explicitly given in the left-hand side of the same equation. With this understanding, we can compute the regularized value of $\zeta(p,s)$ at $s = 1$ as

$$\begin{aligned}&\lim_{s\to 1}\left[\zeta(p,s) - \frac{a_{n/2-1}(p)}{(4\pi)^{n/2}\Gamma(s)} \frac{1}{s-1}\right] \\ &= \lim_{s\to 1}\left[\zeta(p,s) - \frac{a_{n/2-1}(p)}{(4\pi)^{n/2}} \frac{1}{s-1}\right] + \frac{a_{n/2-1}(p)}{(4\pi)^{n/2}}\Gamma'(1) \\ &= \sum_{k=0}^{n/2-2} \frac{a_k(p)}{(4\pi)^{n/2}} \frac{1}{1-n/2+k} + \int_1^\infty \left( K(p,p,t) - \frac{1}{V} \right) dt \\ &\quad + \lim_{\epsilon\to 0_+} \int_\epsilon^1 \left( K(p,p,t) - \frac{1}{V} - \sum_{k=0}^{n/2-1} \frac{a_k(p)}{(4\pi)^{n/2}} t^{k-n/2} \right) dt,\end{aligned} \tag{B.58}$$

where we used that the integrand in the last line of equation (B.56) is bounded by a constant times $t^{s-1}$ to exchange the order of the limits. Performing the integrals of the terms that are polynomials in $t$ in the right-hand side of equation (B.58), using the definition of the Robin function given in equation (3.18), and that $\Gamma'(1) = -\gamma$, we obtain the result in the statement of the theorem. □

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