Homoclinic orbits for $u^{iv} + au'' - u + f(u, b) = 0$

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Abstract

We consider the equation in the title where: $u \in \mathbb{R}$, $f(u,b) = \mathcal{O}(u^3)$ is real analytic, $f(\cdot, b)$ is odd, and $(a,b) \in \mathbb{R}^2$. The origin $u = 0$ is an equilibrium and has one-dimensional stable and unstable manifolds. Let a be fixed. We show that: if for $b = \bar{b}$ the equation has an orbit homoclinic to the origin ($u(t) \to 0$ as $t \to \pm\infty$) then the equation has orbits homoclinic to the origin for an infinite sequence of values of b and this sequence accumulates at $\bar{b}$.

1 Introduction

Let us consider the equation

$$u^{iv} + au'' - u + f(u,b) = 0 \tag{1}$$

where: $u \in \mathbb{R}$, $f(u,b) = \mathcal{O}(u^2)$ is real analytic, and a and b are real parameters. The origin $u = 0$ is an equilibrium of (1) and the characteristic equation related to it is $\lambda^4 + a\lambda^2 - 1 = 0$. This implies that $u = 0$ is an equilibrium of saddle-center type, i.e., it is related to pairs of real, $\lambda = \pm\nu \neq 0$, and imaginary, $\lambda \pm \omega i \neq 0$, eigenvalues. Therefore, the stable and unstable manifolds of the origin, W^s and W^u, respectively, are one dimensional. The problem we are interested in is: to find the set $\mathcal{A}$ of values of a and b such that there is an orbit homoclinic to $u = 0$. We point out that $(a,b) \in \mathcal{A}$ if W^s intersects W^u at some point distinct from the origin. This is a rather exceptional situation due to the one dimensionality of W^u and W^s. So, most values of (a,b) do not belong to $\mathcal{A}$. We will show that if $f(\cdot, b)$ is odd then $\mathcal{A}$ has no isolated points is some very strong sense.

Equation (1) appears in several branches of physics. The problem of finding a solution homoclinic to the origin is usually related to the existence of solitary waves, or to the existence of stationary solutions with finite energy (namely, solutions that are in $\mathrm{H}^2(\mathbb{R})$). For applications of (1) to the problem of existence of solitary waves in the presence of surface tension see [1]. In this case, f is usually approximated by u^2 and the parameter a is related to the wave speed. Equation (1) is also related to localized buckle patterns in elastic structures [6] and to models for spatial patterns appearing in phase transitions [3] (in this context (1) is called stationary extended Fisher-Kolmogorov equation).

2 Hamiltonian formulation

Equation (1) is the Euler-Lagrange equation of the functional $\int L dt$, namely, it is equivalent to

$$\frac{d^2}{dt^2}\partial_{u''}L - \frac{d}{dt}\partial_{u'}L + \partial_u L = 0, \quad \text{with} \quad L = \frac{(u'')^2}{2} - a\frac{(u')^2}{2} - \frac{u^2}{2} + F(u,b),$$

where $\partial_u F = f$. In order to write this equation in Hamiltonian form we introduce a new variable v ([2], paragraph 446) and consider the Lagrangian function $L = L(v', v, u)$ with the constraint $v = u'$. As it is usual in problems with constraints, we define a Lagrangian multiplier μ and write a new Lagrangian function as $\mathcal{L}(v', v, u', u) = L(v', v, u) + \mu(u' - v)$. Then, we find the Euler-Lagrange equations related to $\mathcal{L}$ assuming that u and v are independent, and, finally, use $v = u'$ to determine the Lagrange multiplier μ. In order to pass to the Hamilton formalism, we define the momenta $p_u = \partial_{u'}\mathcal{L} = \mu$ and $p_v = \partial_{v'}\mathcal{L} = \partial_{v'}L$ and the Hamiltonian function

$$H(p_u, p_v, u, v) = p_u u' + p_v v' - \mathcal{L}(v', v, u', u) = p_v v' + p_u v - L(v', v, u),$$

where we used that $p_u = \mu$ and that v' is implicitly given as a function of (p_v, u, v) by $p_v = \partial_{v'}L$. In the new variables, equation (1) is equivalent to $p_u' = -\partial_u H$, $p_v' = -\partial_v H$, $u' = \partial_{p_u}H$ and $v' = \partial_{p_v}H$, where H is given by

$$H = p_u v + \frac{p_v^2}{2} + a\frac{v^2}{2} + \frac{u^2}{2} - F(u,b). \tag{2}$$

3 "Persistence" of the homoclinic orbit

In order to explain one of the hypotheses in this section we start considering, as a model problem, two one-parameter families of Hamiltonian systems, which are related to the following Hamiltonian functions:

$$H_\pm = \frac{p_1^2}{2} - \frac{q_1^2}{2} + \frac{q_1^3}{3} \pm \left\{\frac{p_2^2}{2} + \frac{q_2^2}{2}\right\} + \epsilon h(p_1^2, p_2^2, q_1, q_2), \tag{3}$$

where $h = \mathcal{O}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$ is some analytic function. The Hamiltonian systems related to (2) and (3) are different. Nevertheless, they share the common feature of having a saddle-center equilibrium point. The symmetries of (3), for $\epsilon = 0$, imply that the analysis of system (3) is easier than that of system (2). Regardless the choice of sign in $H_\pm$ and the value of ϵ the origin, $(p_1, p_2, q_1, q_2) = (0,0,0,0) \stackrel{\text{def}}{=} \underline{0}$, is an equilibrium of system (3) of saddle-center type (it is related to the eigenvalues $\pm 1, \pm i$). For $\epsilon = 0$, system (3) has an orbit Γ homoclinic to $\underline{0}$ (it is contained in the plane $p_2 = q_2 = 0$). We are interested in the fate of Γ for $\epsilon \neq 0$. The Hamiltonian function $H_\pm$ ("energy") is a first integral of the system. Since $H_\pm(\underline{0}) = 0$ for any value of ϵ, we conclude that the stable and unstable manifolds of $\underline{0}$, W^s and W^u, respectively, are contained in the energy level $H_\pm = 0$. The dynamics in these energy levels differ drastically from case H_+ to case H_-. For instance, for $\epsilon = 0$, $H_\pm = 0$ implies

$$\mp\left\{\frac{p_1^2}{2} - \frac{q_1^2}{2} + \frac{q_1^3}{3}\right\} = \frac{p_2^2}{2} + \frac{q_2^2}{2} \geq 0.$$

So, for H_+ all orbits in the energy level $H_+ = 0$ are bounded, except for two branches of W^s and W^u, and for H_- all orbits in the energy level $H_- = 0$ are unbounded, except for Γ (see figure 1).

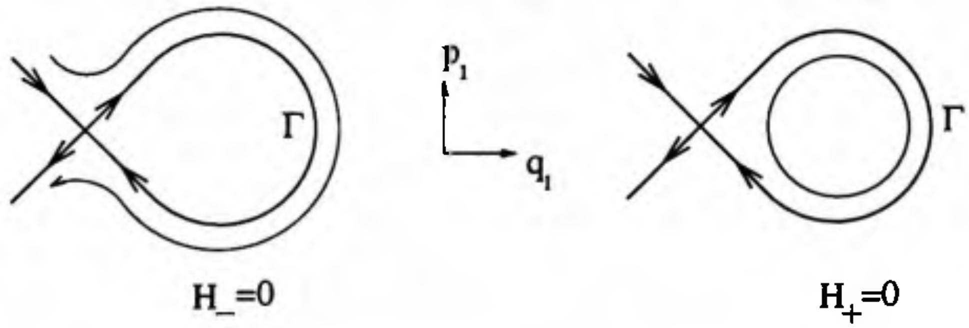

Figure 1

For $\epsilon \neq 0$, sufficiently small, the situation is not different. Thus, for $\epsilon \neq 0$ small, we very unlikely find an orbit of H_- homoclinic to $\underline{0}$ unless h has a very special form. For H_+ the situation is completely different. First of all, for any value of ϵ, H_+ generates a time-reversible flow. This implies that if W^u intersects the "zero velocity set", $\Lambda \stackrel{\text{def}}{=} \{p_1 = 0, p_2 = 0, q_1, q_2\}$, then there is an orbit homoclinic to $\underline{0}$. In particular, for $\epsilon = 0$, Γ intersects Λ at some point P with coordinate $q_1 \neq 0$. The remarkable fact found by Mielke et al. [5] is that, in general, W^u intersects Λ for an infinite sequence of values of ϵ, $(\epsilon_k)_{k\in\mathbb{Z}}$, that converges to zero. The idea behind this result is the following. For $\epsilon \neq 0$ small, W^u stays close to Γ for a very long time. In a first passage near P, W^u "misses" Λ but still get back close to $\underline{0}$. Then W^u stays near $\underline{0}$ for a very long time rotating due to the "center" part of $\underline{0}$ (or the imaginary eigenvalues of $\underline{0}$). This rotation has a "thickening" effect on W^u. This is eventually the cause for the infinitely many intersections of W^u and Λ (see [5] and [4] for more details).

In [5] the result on the "persistence" of Γ is presented for a model Hamiltonian that is exactly the H_+ above with a particular choice for h. There, the existence of the sequence $(\epsilon_k)_{k\in\mathbb{Z}}$ is shown under an additional hypothesis that certain integral related to Γ (a kind of Melnikov integral) is different from zero. Indeed, under this condition, they show that the sequence $(\epsilon_k)_{k\in\mathbb{Z}}$ contains infinitely many sub-sequences $(\epsilon_k^l)_{k\in\mathbb{Z}}$, $l = 2, 3, \ldots$, such that for every ϵ in $(\epsilon_k^l)_{k\in\mathbb{Z}}$ the system has an l-pulse homoclinic orbit to $\underline{0}$ (namely, an orbit that enters, and also leaves, some small neighborhood of $\underline{0}$ l-times). In [4], using the idea in Mielke et al [5], we prove a similar result for general analytic two degrees of freedom Hamiltonian systems reversible under general symplectic involutions. In [4], we also present a result on the "persistence" of Γ that is independent of the integral condition that appears in [5] (this is very convenient since in general it is quite difficult to check this condition). For the system related to H_+, our results in [4] imply the following statement: *let $\mathcal{A}$ be the set of values of ϵ where the system has a homoclinic orbit to $\underline{0}$, then:* 1- *any point of $\mathcal{A}$ is an accumulation point of $\mathcal{A}$ (in particular* 0 *is an accumulation point of $\mathcal{A}$),* 2- *two situations may occur, either $\mathcal{A}$ is an interval containing* 0 *or it is a countable set (in this case the system has l-pulse homoclinic orbits for all l).*

It happens, as it was remarked in [5], that if equation (1) has a time-reversible orbit homoclinic to the origin then the topology of its zero energy level is similar to the topology of the zero energy level of H_-. So, in general, the result presented above do not apply to (1). In order to change this picture we assume that $f(\cdot, b)$ is an odd function. Under this hypothesis, equation (1) has the symmetry $u \to -u$. In our model problem, this corresponds to change the term u^3 in H_- by u^4. In this case, for $\epsilon = 0$, H_- has an additional orbit homoclinic to $\underline{0}$ and in the level $H_- = 0$ all orbits of the system are bounded. For equation (1) a theorem in [4] implies the following result.

Theorem 1 *Consider equation* (1) *where* $f(u,b) = \mathcal{O}(u^3)$, *as* $u \to 0$, *is real analytic and* $f(\cdot, b)$ *is odd. Suppose that for* $(a,b) = (a_*, b_*)$ *equation* (1) *has an orbit homoclinic to the equilibrium* $u = 0$. *Let* $s \to (a(s), b(s))$ *be any analytic curve in the parameter space such that* $(a(0), b(0)) = (a_*, b_*)$. *Then for any* $\epsilon > 0$ *there are infinitely many values of* s, *with* $|s| < \epsilon$, *such that equation* (1) *has an orbit homoclinic to the equilibrium* $u = 0$.

In the sense of this theorem we can say that, for $f(\cdot, b)$ odd, orbits of (1) that are homoclinic to $u = 0$ are persistent under small perturbations.

We remark that in the problem of existence of solitary waves in the presence of surface tension $f(u,b)$ is usually taken as u^2. Here, the situation is the same as the one for H_- (W^u and W^s have unbounded branches). So, in this case, equation (1) may have homoclinic orbits to $u = 0$ only for isolated values of a.

4 Example

I learned the following algorithm for constructing equations of type (1) containing homoclinic orbits to $u = 0$ from Prof. J. F. Toland.

Suppose that $g(u) = \mathcal{O}(u^3)$ is an odd analytic function and that

$$u'' - c^2 u + g(u) = 0, \qquad c^2 = constant \neq 0, \tag{4}$$

has a solution Γ homoclinic to the equilibrium $u = 0$. Denoting by G the primitive of g, $G(0) = 0$, we have that Γ also satisfies the equation

$$(u')^2 - c^2 u^2 + 2G(u) = 0. \tag{5}$$

Differentiating equation (4) twice we get

$$u^{iv} - c^2 u'' + (u')^2 g''(u) + u'' g'(u) = 0.$$

Using equations (4) and (5) we eliminate u' and u'' in the last two terms of this equation to obtain

$$u^{iv} - c^2 u'' + [c^2 u^2 - 2G(u)]g''(u) + [c^2 u - g(u)]g'(u) = 0. \tag{6}$$

Multiplying equation (4) by c^{-2} and adding it to (6) we obtain

$$u^{iv} + (c^{-2} - c^2)u'' - u + \{c^{-2} g(u) + [c^2 u^2 - 2G(u)]g''(u) + [c^2 u - g(u)]g'(u)\} = 0. \tag{7}$$

The function inside brackets is odd and of third order in u. So, it can be taken as f. By construction, equation (7) has the solution Γ homoclinic to the equilibrium $u = 0$. In particular choosing $g(u) = u^3$ we get

$$u^{iv} + (c^{-2} - c^2)u'' - u + (c^{-2} + 9c^2)u^3 - 6u^5 = 0.$$

Equation (4) with $g(u) = u - \sin(u)$ has two heteroclinic orbits connecting the equilibria $u = 0$ and $u = 2\pi$. This and the reasoning above imply that equation (7) with the same g,

$$u^{iv} - \frac{3}{2}\sin(2u) = 0, \tag{8}$$

has also two heteroclinic orbits connecting the equilibria $u = 0$ and $u = 2\pi$. For this pair of heteroclinic orbits a result analogous to theorem 1 holds, provided we consider perturbations of equation (8) that are odd 2π-periodic functions of u.

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