RT-MAE 9712

ASYMPTOTIC RELATIVE EFFICIENCY OF WALD TESTS IN MEASUREMENT ERROR MODELS

by

Patricia Gimenez, Enrico A. Colosimo and Heleno Bolfarine

Palavras-Chave: Asymptotic tests; Corrected score function; naive Wald test; Tweedie class of models.

Classificação AMS: 62F05, 62J05.
(AMS Classification)

- Julho de 1997 -

ASYMPTOTIC RELATIVE EFFICIENCY OF WALD TESTS IN MEASUREMENT ERROR MODELS

PATRICIA GIMENEZ[a,b], ENRICO A. COLOSIMO[a,c]

HELENO BOLFARINE[a]

[a]Departamento de Estatística, IME/USP, Caixa Postal 20570, 01452-990, São Paulo-SP, Brazil

[b]Departamento de Matemática - FCEYN/UNMDP, Funes 3350, C.P. 7600, BS.AS., Argentina

[c]Departamento de Estatística, ICEx/UFMG, Caixa Postal 702, 30161-970, Belo Horizonte-MG, Brazil

SUMMARY

In this paper, asymptotic relative efficiency (ARE) of Wald tests for the Tweedie class of models with log-linear mean, is considered when the auxiliary variable is measured with error. Wald test statistics based on the naive maximum likelihood estimator and on a consistent estimator which is obtained by using Nakamura's (1990) corrected score function approach are defined. As shown analytically, the Wald statistics based on the naive and corrected score function estimators are asymptotically equivalents in terms of ARE. Comparisons with the Wald statistic based on the true covariate are also reported. A small scale numerical Monte Carlo study and an example illustrate the small sample size situation.

Keywords: Asymptotic tests; Corrected score function; naive Wald test; Tweedie class of models.

1. INTRODUCTION

The problem of testing the association between a response variable y and a covariate z when the variable x is observed instead of the true variable z, has been the subject of several papers in the statistical literature (Lagakos, 1988, Tosteson and Tsiatis, 1988). Suppose that the null hypothesis of no association can be tested by using the statistic Q_z when measurement error is not present. When z can not be observed directly due to measurement error, a naive statistics Q_x is obtained by replacing the unobserved variable z by x, in a test which would be used only if z were observed. The asymptotic relative efficiency can be used to evaluate the efficiency loss by using Q_x instead of Q_z, which we denote by $ARE(Q_x, Q_z)$, when considering local alternatives to the null hypothesis.

Tosteson and Tsiatis (1988) compared the local power of the optimal score test, the naive score test and the test obtained when is z is really observed, assuming several error structures. The maximum likelihood estimators are not necessary to evaluate the expression of the score test statistic and thus make it considerably easier to be obtained than the Wald test statistic.

It is well known that in generalized linear models the naive estimator is inconsistent when the covariate z is observed with error (Fuller, 1987, Stefanski, 1985). Several methods that can be used to produce estimators with reduced bias, has been reviewed by Carroll (1989). Stefanski (1985) proposed an estimator based on first order bias correction of the naive estimator which is valid for situations dealing with small measurement errors. Stefanski and Carroll (1990) compared the Wald test that follows from Stefanski (1985) correction approach under an additive error structure with the Wald test based on the naive estimator.

Lagakos (1988) studied the behavior of three types of naive statistics: Wald tests for linear models, likelihood ratio tests for logistic regression models and score tests for Cox proportional regression models. Several error structures are considered and the joint distribution of y and x is taken in an arbitrary way. An important result is shown by Lagakos (1988) and also by Tosteson and Tsiatis (1989) for score tests in generalized linear models. The asymptotic relative efficiency of the naive score test with respect to the score test based on the true covariate z equals the square of the correlation coefficient between x and z.

This paper is concerned with the properties of Wald test statistics for testing the association between z and y in a particular class of generalized linear models, usually called the Tweedie class of models (Jørgensen, 1987), with a log-linear mean. An additive error structure is considered and the joint distribution of x and z is normal. The corrected score estimator (Nakamura, 1990) and its asymptotic variance can be obtained for this class of models. We recall that most of the approaches considered for estimation in measurement error models produces only approximately unbiased estimates, with no formal theoretical justification. Nakamura's corrected score along with the conditional score of Stefanski and Carroll (1987) are probably the only approaches able to generate consistent and asymptotically normal estimators in such models without making assumptions concerning the true covariate z. The asymptotic local power of the corrected Wald statistic is compared with the naive Wald statistic.

Section 2 introduces the Tweedie class of models and the Wald tests of interest. In Section 3, under the assumptions considered in Section 2, the asymptotic bias of the naive estimator is obtained. It behaves like the bias of the naive estimator for the standard linear regression model with measurement error (Fuller, 1987). In Section 4, the asymptotic properties of the three tests are considered. It is shown that the asymptotic relative efficiency of the naive Wald statistic with respect to the true Wald statistic (z observed) equals the square of the correlation between x and z, as also obtained by Lagakos (1988) and Tosteson and Tsiatis (1988) for other types of tests. On the other hand, the asymptotic relative efficiency of the naive Wald statistic with respect to the Wald statistic based on the corrected score is equal to one. Similar asymptotic results are obtained by Stefanski and Carroll (1990) for the asymptotic relative efficiency of the Wald statistics based on the naive estimator and on Stefanki (1985) corrected estimator. In order to evaluate the behavior of the Wald statistics for small sample sizes, Section 5 presents results of a small scale Monte Carlo study. A numerical example in Section 6 illustrates the results obtained in the previous sections.

2. MODEL FORMULATION AND THE WALD TEST

Suppose that given z, y follows a generalized linear model so that

$$\mu(\boldsymbol{\theta}'\mathbf{Z}) = E[y|z] = e^{\boldsymbol{\theta}'\mathbf{Z}} \quad \text{and} \quad Var[y|z] = \phi V(\mu), \tag{2.1}$$

with $V(\mu) = \mu^p$, $p \in \Re - (0,1)$, where $\boldsymbol{\theta} = (\alpha, \beta)'$ is the vector of unknown parameters, $\mathbf{Z} = (1, z)'$, $\phi > 0$ is the dispersion parameter and $V(\mu)$ is the variance function. This class of models is known as the Tweedie models with log-linear mean, which can be summarized in Table 1 that follows.

Details on the properties of the elements of this class of distributions can be found in Jørgensen (1987).

In this paper, situations where z can not be recorded directly are considered. That is, the observed data are $(x_1, y_1), \ldots, (x_n, y_n)$, with

$$x_i = z_i + u_i, \tag{2.2}$$

where u_i, the measurement error associated with the true value z_i, has normal distribution with zero mean and variance σ_u^2, $i = 1, \ldots, n$. The variance σ_u^2 is assumed known or a consistent is available. The measurement error u_i is independent of z_i and y_i, and the true covariate z_i has a normal distribution with mean μ and variance σ_z^2, $i = 1, \ldots, n$. The following notation is considered in this context: $\mathbf{X} = (1, x)'$, $\mathbf{v} = (0, 1)'$, $\mathbf{y} = (y_1, \ldots, y_n)'$, $\mathbf{z} = (z_1, \ldots, z_n)'$ and $\mathbf{x} = (x_1, \ldots, x_n)'$.

Table 1 - Tweedie class of models

Distribution	p
Extreme stable	$p < 0$
Normal	$p = 0$
Poisson	$p = 1$
Compound Poisson	$1 < p < 2$
Gamma	$p = 2$
Positive stable	$2 < p < 3$
Inverse Gaussian	$p = 3$
Positive stable	$p > 3$

The main interest is on testing the hypothesis $H_0 : \beta = 0$, based on a sample of size n. Let $\boldsymbol{\theta} = (\alpha, \beta)'$ be the true parameter value and $\boldsymbol{\theta}_0 = (\alpha, 0)'$ be the parameter value under the null hypothesis. The generalized linear model theory (McCullagh and Nelder, 1989) establishes the following form for the score function

$$\mathbf{U}(\boldsymbol{\theta}; \mathbf{y}, \mathbf{z}) = \sum_{i=1}^{n} l_i(\boldsymbol{\theta}'\mathbf{Z}_i)\mathbf{Z}_i,$$

where

$$l_i(\boldsymbol{\theta}'\mathbf{Z}_i) = \frac{1}{\phi}\dot{\mu}(\boldsymbol{\theta}'\mathbf{Z}_i)V\{\mu(\boldsymbol{\theta}'\mathbf{Z}_i)\}^{-1}\{y_i - \mu(\boldsymbol{\theta}'\mathbf{Z}_i)\},$$

with $\dot{\mu}(.) = d\mu(.)$, the first derivative of the function μ. It follows, for the class of models (2.1), that

$$l_i(\boldsymbol{\theta}'\mathbf{Z}_i) = \frac{1}{\phi}e^{(1-p)\boldsymbol{\theta}'\mathbf{Z}_i}\{y_i - e^{\boldsymbol{\theta}'\mathbf{Z}_i}\}.$$

Given an estimator $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta})'$ such that under H_0, $n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{D} N(0, \Sigma)$, and $\Sigma_n(\hat{\boldsymbol{\theta}})$ a consistent estimator of Σ, the Wald statistic for testing $H_0 : \beta = 0$ is constructed in the usual way, that is,

$$Q = \frac{\sqrt{n}\hat{\beta}}{(\mathbf{v}'\Sigma_n(\hat{\boldsymbol{\theta}})\mathbf{v})^{1/2}}. \qquad (2.3)$$

The null hypothesis is rejected if Q is large when compared to the corresponding percentile of the standard normal distribution.

It will be denoted by $\hat{\boldsymbol{\theta}}_z = (\hat{\alpha}_z, \hat{\beta}_z)'$, the maximum likelihood estimator of $\boldsymbol{\theta}$ when z is measured without error. That is, $\hat{\boldsymbol{\theta}}_z$ is the solution to the equation

$$\mathbf{U}(\boldsymbol{\theta}; \mathbf{y}, \mathbf{z}) = 0.$$

The naive score function is obtained by replacing z by x, that is,

$$\mathbf{U}(\boldsymbol{\theta};\mathbf{y},\mathbf{x}) = \sum_{i=1}^{n} l_i(\boldsymbol{\theta}'\mathbf{X}_i)\mathbf{X}_i.$$

The naive maximum likelihood estimator is obtained as the solution to the equations $\mathbf{U}(\boldsymbol{\theta};\mathbf{y},\mathbf{x}) = \mathbf{0}$ and it will be denoted by $\hat{\boldsymbol{\theta}}_x = (\hat{\alpha}_x, \hat{\beta}_x)'$. Typically, $\hat{\boldsymbol{\theta}}_x$ is asymptotically biased. In Section 3, the bias of $\hat{\boldsymbol{\theta}}_x$ will be explicitly obtained for the class of models (2.1). A consistent estimator can be obtained by considering the corrected score function approach introduced by Nakamura (1990). The corrected score function, which we denote by $\mathbf{U}^*(\boldsymbol{\theta};\mathbf{y},\mathbf{x})$, satisfies

$$E[\mathbf{U}^*(\boldsymbol{\theta};\mathbf{y},\mathbf{x})|\mathbf{y},\mathbf{z}] = \mathbf{U}(\boldsymbol{\theta};\mathbf{y},\mathbf{z}),$$

for all $\mathbf{y},\mathbf{z},\boldsymbol{\theta}$. Thus, under the true model it follows that $E[\mathbf{U}^*(\boldsymbol{\theta};\mathbf{y},\mathbf{x})] = \mathbf{0}$, so that, under certain regularity conditions, there exists $\hat{\boldsymbol{\theta}}_c = (\hat{\alpha}_c, \hat{\beta}_c)'$, a solution to $\mathbf{U}^*(\boldsymbol{\theta};\mathbf{y},\mathbf{x}) = \mathbf{0}$ which is consistent and asymptotically normally distributed.

Under assumptions (2.1) and (2.2) it follows that the corrected score function is easily obtained and has the simple form

$$\mathbf{U}^*(\boldsymbol{\theta};\mathbf{y},\mathbf{x}) = \sum_{i=1}^{n} \mathbf{U}^*(\boldsymbol{\theta};y_i,x_i),$$

where $\mathbf{U}^*(\boldsymbol{\theta};y_i,x_i) = (U_1^*(\boldsymbol{\theta};y_i,x_i), U_2^*(\boldsymbol{\theta};y_i,x_i))'$, with

$$U_1^*(\boldsymbol{\theta};y_i,x_i) = \frac{1}{\phi}\{e^{(1-p)\boldsymbol{\theta}'\mathbf{X}_i-\psi}y_i - e^{(2-p)\boldsymbol{\theta}'\mathbf{X}_i-\gamma}\} \tag{2.4}$$

and

$$U_2^*(\boldsymbol{\theta};y_i,x_i) = \frac{1}{\phi}\left\{e^{(1-p)\boldsymbol{\theta}'\mathbf{X}_i-\psi}[x_i-(1-p)\sigma_u^2\beta]y_i \right. \tag{2.5}$$

$$\left. -e^{(2-p)\boldsymbol{\theta}'\mathbf{X}_i-\gamma}[x_i-(2-p)\sigma_u^2\beta]\right\},$$

where

$$\psi = \frac{(1-p)^2\beta^2\sigma_u^2}{2}, \quad \text{and} \quad \gamma = \frac{(2-p)^2\beta^2\sigma_u^2}{2}.$$

The key result for deriving the corrected score function is the identity $E[e^{\beta x}|z] = e^{\beta z + \frac{\beta^2\sigma_u^2}{2}}$, the moment generating function of x, which is normally distributed. Note that the corrected score function $\mathbf{U}^*$ depends also on σ_u^2, and when necessary, it will be denoted by $\mathbf{U}^*(\boldsymbol{\theta},\sigma_u^2;\mathbf{y},\mathbf{x})$ to emphasize this dependency. It will be denoted by Q_z, Q_x and Q_c the Wald statistics (see expression (2.3)) based on the estimators $\hat{\boldsymbol{\theta}}_z$, $\hat{\boldsymbol{\theta}}_x$, and $\hat{\boldsymbol{\theta}}_c$, respectively.

3. THE ASYMPTOTIC BIAS OF THE NAIVE ESTIMATOR

In this section, a closed expression is provided for the asymptotic bias of the naive estimator $\hat{\theta}_x$.

Lemma 3.1. *Under the assumptions (2.1) and (2.2), it follows that* $\hat{\theta}_x \xrightarrow{P} \theta_*$, *so that* $\theta_* = (\alpha_*, \beta_*)'$, *with*

$$\alpha_* = \alpha + (1-R)\mu\beta + \frac{1}{2}R\sigma_u^2\beta^2 \quad \text{and} \quad \beta_* = R\beta,$$

where $R = \sigma_z^2/(\sigma_z^2 + \sigma_u^2)$, *usually known as the reliability ratio.*

Proof. As considered in Stefanski (1985) (see also Mak, 1982), $\hat{\theta}_x \xrightarrow{P} \theta_*$, with $\theta_* = (\alpha_*, \beta_*)'$ satisfying

$$E[\mathrm{U}(\theta_*; \mathrm{y}, x)] = 0, \tag{3.1}$$

where the expectation is taken with respect to the true parameter value θ. From (3.1) it follows for the class of models (2.1)-(2.2) that

$$E[l(\theta_*'\mathrm{X})] = 0 \quad \text{and} \quad E[l(\theta_*'\mathrm{X})x] = 0, \tag{3.2}$$

where

$$l(\theta_*'\mathrm{X}) = \frac{1}{\phi}\mathrm{e}^{(1-p)\theta_*'\mathrm{X}}\{y - \mathrm{e}^{\theta_*'\mathrm{X}}\}.$$

Using properties of the moment generating function of normally distributed random variables, it follows from (3.2) that

$$a - b = 0 \tag{3.3}$$

and

$$a\{\mu + \sigma_z^2[(1-p)\beta_* + \beta] + \sigma_u^2(1-p)\beta_*\} - b\{\mu + \sigma_z^2(2-p)\beta_* + \sigma_u^2(2-p)\beta_*\} = 0, \tag{3.4}$$

where

$$a = \mathrm{e}^{(1-p)\alpha_* + \alpha + [(1-p)\beta_* + \beta]\mu + \psi + \eta}$$

and

$$b = \mathrm{e}^{(2-p)(\alpha_* + \beta_*\mu) + \gamma + \delta},$$

with

$$\psi = \frac{(1-p)^2\beta_*^2\sigma_u^2}{2}, \quad \gamma = \frac{(2-p)^2\beta_*^2\sigma_u^2}{2},$$

$$\eta = \frac{[(1-p)\beta_* + \beta]^2\sigma_z^2}{2}, \quad \text{and} \quad \delta = \frac{(2-p)^2\beta_*^2\sigma_z^2}{2}.$$

By solving (3.3) and (3.4) we arrive at

$$\beta_* = R\beta \quad \text{and} \quad \alpha_* = \alpha + (1-R)\mu\beta + \frac{\sigma_u^2}{2}R\beta^2,$$

where $R = \sigma_z^2/(\sigma_z^2 + \sigma_u^2)$, as was to be shown.

Notice that $R = \sigma_z^2/(\sigma_z^2 + \sigma_u^2) = corr^2[x, z]$. Moreover, the asymptotic bias of the naive estimator behave according to the usual "correction for attenuation" formula in the ordinary linear regression model (Fuller, 1987). If the true value is the parameter $\theta_0 = (\alpha, 0)'$ then $\beta_* = 0$ and $\alpha_* = \alpha$.

4. ASYMPTOTIC PROPERTIES OF WALD TESTS

In this section the Wald statistics based on $\hat{\theta}_z$, $\hat{\theta}_x$ and $\hat{\theta}_c$ are obtained. They have the usual form (2.3) and for each case the asymptotic variance Σ is established. Asymptotic relative efficiency is used to compare the Wald tests.

4.1. Wald test statistics

When the covariate z is truly observed, it follows that $\hat{\theta}_z$ is such that, under $H_0 : \theta = \theta_0$, $\hat{\theta}_z \xrightarrow{P} \theta_0$, and

$$\sqrt{n}(\hat{\theta}_z - \theta_0) \xrightarrow{D} N(0, \mathbf{K}^{-1}),$$

where $\mathbf{K}(\theta_0) = E[-\dot{l}(\theta_0'\mathbf{Z})\mathbf{Z}\mathbf{Z}']$, with $\dot{l}(.) = dl(.)$. Moreover,

$$\mathbf{K}_n(\hat{\theta}_z) = -\frac{1}{n}\sum_{i=1}^{n} \dot{l}_i(\hat{\theta}_z'\mathbf{Z}_i)\mathbf{Z}_i\mathbf{Z}_i'$$

is a consistent estimator of $\mathbf{K}(\theta)$. The true Wald test statistic can be written as

$$Q_z = \frac{\sqrt{n}\hat{\beta}_z}{\{\mathbf{v}'\mathbf{K}_n^{-1}(\hat{\theta}_z)\mathbf{v}\}^{1/2}}. \tag{4.1}$$

Let $\hat{\theta}_x$ be the naive maximum likelihood estimator of $\hat{\theta}_x$. Thus, from Lemma 3.1 it follows that, under $H_0 : \theta = \theta_0$, $\hat{\theta}_x \xrightarrow{P} \theta_* = \theta_0$ and using results in Stefanski (1985),

$$\sqrt{n}(\hat{\theta}_x - \theta_0) \xrightarrow{D} N(0, \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}),$$

where

$$\mathbf{A} = E[\dot{l}(\theta_0'\mathbf{X})\mathbf{X}\mathbf{X}']$$

and

$$\mathbf{B} = E[l^2(\theta_0'\mathbf{X})\mathbf{X}\mathbf{X}'].$$

It follows that $\mathbf{A}$ and $\mathbf{B}$ are consistently estimated by

$$\mathbf{A}_n(\hat{\theta}_x) = \frac{1}{n}\sum_{i=1}^{n} \dot{l}_i(\hat{\theta}_x'\mathbf{X}_i)\mathbf{X}_i\mathbf{X}_i'$$

and

$$\mathbf{B}_n(\hat{\theta}_x) = \frac{1}{n}\sum_{i=1}^{n} l_i^2(\hat{\theta}_x'\mathbf{X}_i)\mathbf{X}_i\mathbf{X}_i'.$$

Thus, the naive Wald statistic for testing $H_0 : \theta = \theta_0$ is given by

$$Q_x = \frac{\sqrt{n}\hat{\beta}_x}{\{\mathbf{v}'\mathbf{A}_n^{-1}(\hat{\theta}_x)\mathbf{B}_n(\hat{\theta}_x)\mathbf{A}_n^{-1}(\hat{\theta}_x)\mathbf{v}\}^{1/2}}.$$

Assuming that σ_u^2 is known, it follows that the corrected estimator introduced by Nakamura (1990) is, under H_0, such that

$$\sqrt{n}(\hat{\theta}_c - \theta_0) \xrightarrow{D} N(0, \Lambda^{-1}\Gamma\Lambda^{-1}),$$

where

$$\Lambda = E\left[\frac{\partial \mathbf{U}^*(\theta_0; y, x)}{\partial \theta}\right] \quad \text{and} \quad \Gamma = E\left[\mathbf{U}^*(\theta_0; y, x)\mathbf{U}^*(\theta_0; y, x)'\right].$$

It follows that Λ and Γ are consistently estimated by

$$\Lambda_n(\hat{\theta}_c) = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial \mathbf{U}^*(\hat{\theta}_c; y_i, x_i)}{\partial \theta} \quad \text{and} \quad \Gamma_n(\hat{\theta}_c) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{U}^*(\hat{\theta}_c; y_i, x_i)\mathbf{U}^*(\hat{\theta}_c; y_i, x_i)'.$$

Thus, the corrected Wald statistic is given by

$$Q_c = \frac{\sqrt{n}\hat{\beta}_c}{\{\mathbf{v}'\Lambda_n^{-1}(\hat{\theta}_c)\Gamma_n(\hat{\theta}_c)\Lambda_n^{-1}(\hat{\theta}_c)\mathbf{v}\}^{1/2}}. \tag{4.2}$$

In Appendix 1 is shown that in the case where σ_u^2 is unknown, replacing σ_u^2 for an available estimator $\hat{\sigma}_u^2$ results in a Wald statistic exactly as given by (4.2). That is, the estimation of σ_u^2 does not change the Wald statistic for testing $H_0 : \theta = \theta_0$. The main result of the section is presented next.

Theorem 4.1. *Under the assumptions (2.1) and (2.2), it follows that* $ARE(Q_x, Q_z) = ARE(Q_c, Q_z) = R$ *and* $ARE(Q_x, Q_c) = 1$, *where* R *is the coefficient of attenuation established in Lemma 3.1.*

Proof. The approach followed is the one considered in Sen and Singer (1993), for example. Under a sequence of local alternatives given by $\beta_n = \Delta n^{-1/2}$, the limiting value of the naive estimator $\hat{\beta}_x$ is $\beta_{*n} = R\beta_n = R\Delta n^{-1/2}$, as follows from Lemma 3.1. Thus, some algebraic manipulations yelds, as $n \to \infty$,

$$Q_z \xrightarrow{D} N\left(\frac{\Delta}{\{\mathbf{v}'\mathbf{K}^{-1}\mathbf{v}\}^{1/2}}, 1\right),$$

$$Q_x \xrightarrow{D} N\left(\frac{R\Delta}{\{\mathbf{v}'(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\mathbf{v}\}^{1/2}}, 1\right)$$

and

$$Q_c \xrightarrow{D} N\left(\frac{\Delta}{\{\mathbf{v}'(\Lambda^{-1}\Gamma\Lambda^{-1})\mathbf{v}\}^{1/2}}, 1\right).$$

Thus, the asymptotic relative efficiencies (ARE) (Cox and Hinkley, 1974) are given by

$$ARE(Q_x, Q_z) = \frac{R^2\{\mathbf{v}'\mathbf{K}^{-1}\mathbf{v}\}}{\mathbf{v}'(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\mathbf{v}},$$

$$ARE(Q_x, Q_c) = \frac{R^2\{\mathbf{v}'(\Lambda^{-1}\Gamma\Lambda^{-1})\mathbf{v}\}}{\mathbf{v}'(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\mathbf{v}}$$

and

$$ARE(Q_c, Q_z) = \frac{\mathbf{v}'\mathbf{K}^{-1}\mathbf{v}}{\mathbf{v}'(\Lambda^{-1}\Gamma\Lambda^{-1})\mathbf{v}}.$$

In Appendix 2 is shown that under the assumptions (2.1) and (2.2),

$$\mathbf{v}'\mathbf{K}^{-1}\mathbf{v} = \frac{\phi}{\sigma_z^2} e^{(p-2)\alpha}, \tag{4.3}$$

$$\mathbf{v}'(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\mathbf{v} = \frac{\phi}{\sigma_z^2 + \sigma_u^2} e^{(p-2)\alpha} \tag{4.4}$$

and

$$\mathbf{v}'(\Lambda^{-1}\Gamma\Lambda^{-1})\mathbf{v} = \frac{\phi(\sigma_z^2 + \sigma_u^2)}{\sigma_z^4} e^{(p-2)\alpha}. \tag{4.5}$$

Thus, it follows that

$$ARE(Q_x, Q_z) = ARE(Q_c, Q_z) = R \quad \text{and} \quad ARE(Q_x, Q_c) = 1.$$

5. SIMULATION STUDY

In this section we perform a Monte Carlo simulation study for comparing the empirical power of the test statistics Q_z, Q_x and Q_c. The simulation study is based on an exponential regression model for lifetime data.

A set of independent random variables $\mathbf{T}' = (T_1, ..., T_n)$ is generated for each repetition. $\mathbf{T}$ is a vector of realizations of a exponential distribution with parameter $exp(\alpha + \beta z)$ and the null hypothesis of interest is $\beta = 0$. The true covariate is generated as a standard normal and the error variable as a normal with mean 0 and variance σ_u^2. Several values of σ_u^2 are considered. The parameter α in this study is set equal to zero and 1000 replications are run for each simulation.

The simulations are performed for some values of the error variance ($\sigma_u^2 = 0.1, 0.3, 0.5$) and sample size ($n = 30, 50$). The simulations were run in a CRAY computer using the IMSL software available at CCE/USP.

An adjustment was performed using the empirical distributions of the statistics generated from the simulations such that the empirical size is corrected to be 5%. Figure 1 displays the empirical power of the three tests for the six combinations of $\sigma_u^2 = 0.1, 0.3, 0.5$ and $n = 30, 50$. As expected Q_x and Q_c lose power as the error variance increases. The power performance of Q_x and Q_c seems to be very similar with a slight superiority in favor of Q_x for $\sigma_u^2 = 0.5$.

6. AN EXAMPLE

Feigl and Zelen (1965) presented a data set of survival times of 17 leukemia patients. The response y is time to death measured in weeks from diagnosis and a covariate z is $\log_{10}$ initial white blood cell count. The association between y and z is the main aspect of interest.

It is well known in the medical literature that white blood cell count is usually measured with error. Nakamura (1992) also calls attention to this point. According to Cox and Snell (1981) there is no evidence against the exponential model for the analysis of this data set. The interest is on testing the null hypothesis $H_0 : \beta = 0$ for the exponential regression model, that in terms of the Tweedie class of models is represented by

$$\mu(\theta' Z) = e^{\alpha + \beta z}$$

and

$$p = 2, \phi = 1,$$

where z is the $\log_{10}$ white blood cell count. The true covariate values z are not recorded. The observed data x is measured with error according to (2.2). We can not use the corrected Wald test or the Stefanski's Wald test because σ_u^2 is unknown and replications are not available in order to obtain an estimate of it. However, the naive Wald test does not depend on σ_u^2 and according to the results in Section 4 is asymptotically equivalent to the ones just mentioned. Moreover, the Monte Carlo simulation studies in Section 5 showed some equivalence of the Wald tests for small sample sizes. Therefore, the calculated value of $Q_x = -2.77$, shows an strong prognostic effect of this covariate.

APPENDIX 1. UNKNOWN σ_u^2

In this appendix, we investigate the behavior of the Wald test statistic when the variance σ_u^2 is unknown and replaced by an estimator $\hat{\sigma}_u^2$ which satisfies

$$\sqrt{n}\begin{pmatrix} U^*(\theta, \sigma_u^2; y, x)/n \\ \hat{\sigma}_u^2 - \sigma_u^2 \end{pmatrix} \xrightarrow{D} N(0, V),$$

where

$$\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}' & \delta^2 \end{pmatrix}.$$

Lets denote by $\hat{\theta}_c(\hat{\sigma}_u^2)$ the corrected score estimator, obtained by solving the equation $\mathbf{U}^*(\theta, \hat{\sigma}_u^2; \mathbf{y}, \mathbf{x}) = 0$. It can be shown that

$$\sqrt{n}(\hat{\theta}_c(\hat{\sigma}_u^2) - \theta) \xrightarrow{D} N(0, \Sigma), \qquad (A.1)$$

where

$$\Sigma = \Lambda^{-1}\{\Gamma + 2\mathbf{D}V_{12} + \delta^2\mathbf{D}\mathbf{D}'\}\Lambda^{-1},$$

with

$$\mathbf{D} = E[\frac{\partial \mathbf{U}^*(\theta, \sigma_u^2; y, x)}{\partial \sigma_u^2}].$$

The proof of (A.1) is similar to that given in Gong and Samaniego (1981) and Liang and Zeger (1986) and will not be presented. Now, from (2.4) and (2.5) it follows that

$$\frac{\partial \mathbf{U}^*(\theta, \sigma_u^2; y, x)}{\partial \sigma_u^2} = (\frac{\partial U_1^*}{\partial \sigma_u^2}, \frac{\partial U_2^*}{\partial \sigma_u^2})',$$

with

$$\frac{\partial U_1^*}{\partial \sigma_u^2} = \frac{1}{2\phi}\left\{-e^{(1-p)\theta'\mathbf{X}-\psi}(1-p)^2\beta^2 y + e^{(2-p)\theta'\mathbf{X}-\gamma}(2-p)^2\beta^2\right\}$$

and

$$\frac{\partial U_2^*}{\partial \sigma_u^2} = -\frac{1}{2\phi}e^{(1-p)\theta'\mathbf{X}-\psi}y\{(1-p)^2\beta^2[x-(1-p)\sigma_u^2\beta] + 2(1-p)\beta\} +$$

$$\frac{1}{2\phi}e^{(1-p)\theta'\mathbf{X}-\gamma}\{(2-p)^2\beta^2[x-(2-p)\sigma_u^2\beta] + 2(2-p)\beta\},$$

so that, under H_0, $\mathbf{D} = 0$ and $\Sigma = \Lambda^{-1}\Gamma\Lambda^{-1}$, which implies that the corrected Wald statitic is given as in (4.2) with $\hat{\sigma}_u^2$ replaced by σ_u^2.

APPENDIX 2. ON THE DERIVATIONS OF EXPRESSIONS (4.3), (4.4) and (4.5)

Recall that

$$\mathbf{K}(\theta) = E[-\dot{l}(\theta'\mathbf{Z})\mathbf{Z}\mathbf{Z}'],$$

where

$$\dot{l}(\theta'\mathbf{Z}) = \frac{1}{\phi}(1-p)e^{(1-p)\theta'\mathbf{Z}}\{y - e^{\theta'\mathbf{Z}}\} - \frac{1}{\phi}e^{(2-p)\theta'\mathbf{Z}}. \qquad (A.2)$$

Thus, it follows that

$$\mathbf{K}(\theta_0) = \frac{1}{\phi}e^{(2-p)\alpha}E[\mathbf{Z}\mathbf{Z}'] \quad \text{and} \quad \mathbf{K}(\theta_0)^{-1} = \phi e^{(p-2)\alpha}\{E[\mathbf{Z}\mathbf{Z}']\}^{-1},$$

where
$$\{E[\mathbf{ZZ}']\}^{-1} = \frac{1}{\sigma_z^2}\begin{pmatrix} \sigma_z^2+\mu^2 & -\mu \\ -\mu & 1 \end{pmatrix}.$$

It then follows that $\mathbf{v}'\mathbf{K}^{-1}\mathbf{v} = \phi e^{(p-2)\alpha}/\sigma_z^2$,
$$\mathbf{A}(\theta) = -\frac{1}{\phi}e^{(2-p)\alpha}E[\mathbf{XX}'] \quad \text{and} \quad \mathbf{B}(\theta) = \frac{1}{\phi}e^{(2-p)\alpha}E[\mathbf{XX}']$$

and
$$\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} = \phi e^{(p-2)\alpha}\{E[\mathbf{XX}']\}^{-1},$$

with
$$\{E[\mathbf{XX}]\}^{-1} = \frac{1}{\sigma_z^2+\sigma_u^2}\begin{pmatrix} \sigma_z^2+\sigma_u^2+\mu^2 & -\mu \\ -\mu & 1 \end{pmatrix}.$$

Thus,
$$\mathbf{v}'(\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\mathbf{v} = \phi e^{(p-2)\alpha}/(\sigma_z^2+\sigma_u^2).$$

From (2.4) and (2.5) it follows that
$$\frac{\partial \mathbf{U}^*(\theta_0;y,x)}{\partial\theta} = \frac{1}{\phi}[(1-p)e^{(1-p)\alpha}y-(2-p)e^{(2-p)\alpha}]\begin{pmatrix} 1 & x \\ x & x^2-\sigma_u^2 \end{pmatrix}.$$

Thus, some algebraic manipulations lead to
$$\Lambda^{-1} = \{E[\frac{\partial \mathbf{U}^*(\theta_0;y,x)}{\partial\theta}]\}^{-1} = -\phi e^{(p-2)\alpha}\frac{1}{\sigma_z^2}\begin{pmatrix} \sigma_z^2+\mu^2 & -\mu \\ -\mu & 1 \end{pmatrix}.$$

Shortening the notation, we write
$$\mathbf{U}^*\mathbf{U}^{*\prime} = \begin{pmatrix} U_1^{*2} & U_1^*U_2^* \\ U_1^*U_2^* & U_2^{*2} \end{pmatrix},$$

where
$$U_1^{*2} = \frac{1}{\phi^2}[e^{2(1-p)\alpha}y^2 - 2e^{(3-2p)\alpha}y + e^{2(2-p)\alpha}],$$

$U_1^*U_2^* = U_1^{*2}x$ and $U_2^{*2} = U_1^{*2}x^2$. Now, since
$$E[U_1^{*2}] = \frac{1}{\phi}e^{(2-p)\alpha},$$

it follows that
$$\Gamma = E[\mathbf{U}^*\mathbf{U}^{*\prime}] = \frac{1}{\phi}e^{(2-p)\alpha}E[\mathbf{XX}'].$$

Finally, using the results above we have
$$\Lambda^{-1}\Gamma\Lambda^{-1} = \phi\frac{(\sigma_z^2+\sigma_u^2)}{\sigma_z^4}e^{(p-2)\alpha}\begin{pmatrix} \frac{\sigma_z^4}{\sigma_z^2+\sigma_u^2}+\mu^2 & -\mu \\ -\mu & 1 \end{pmatrix},$$

leading to

$$v'(\Lambda^{-1}\Gamma\Lambda^{-1})v = \phi\frac{(\sigma_z^2 + \sigma_u^2)}{\sigma_z^4}e^{(p-2)\alpha}.$$

REFERENCES

Carroll,R.J. (1989). Covariance analysis in generalized linear measurement error models. *Statistics in Medicine*, 8, 1075-1093.

Carroll,R.J., Ruppert,D. and Stefanski,L.A. (1995). *Measurement Error in Nonlinear Modesl.* Chapman and Hall, New York.

Cox,D.R. and Hinkley,D.V. (1974). *Theoretical Statistics.* Chapman and Hall, London.

Cox,D.R. and Snell,E.J. (1981). *Applied Statistics: Principles and Examples.* Chapman and Hall, London.

Feigl,P. and Zelen,M. (1965). Estimation of exponential survival probabilities with concomitant information. *Biometrics*, 21, 826-838.

Fuller,W.A. (1987). *Measurement Error Models.* J. Wiley, New York.

Gong,G. and Samaniego,F.J. (1981). Pseudo maximum likelihood estimation: theory and applications. *Ann. Statist.*, 9, 861-869.

Jørgensen,B. (1987). Exponential dispersion models (with discussion). *J. R. Statist. Soc. B*, 49, 127-162.

Lagakos,S.A. (1988). Effects of mismodelling and mismeasuring explanatory variables on tests of their association with a response variable. *Statistics in Medicine*, 7, 257-274.

Liang,K.Y. and Zeger,S.L. (1986). Longitudinal data analysis using generalized linear models. *Biometrika*, 73, 13-22.

Mak,T.K. (1982). Estimation in the presence of incidental parameters. *Canadian Journal of Statistics*, 10, 121-132.

McCullagh,P. & Nelder,J.A. (1989). *Generalized Linear Models (2nd. ed.).* Chapman and Hall, London.

Nakamura,T. (1990). Corrected score function for errors in variables models: Methodology and application to generalized linear models. *Biometrika*, 77, 127-137.

Nakamura,T. (1992). Proportional hazards model with covariates subject to measurement error. *Biometrics*, 48, 829-838.

Sen,P.K. and Singer,J.M. (1993). *Large Sample Methods in Statistics: An Introduction with Applications.* Chapman and Hall, New York.

Stefanski,L.A. (1985). The effects of measurement error on parameter estimation. *Biometrika*, **72**, 583-592.

Stefanski,L.A. and Carroll,R.J. (1987). Conditional scores and optimal scores for generalized linear measurement error models. *Biometrika*, **74**, 703-716.

Stefanski,L.A. and Carroll,R.J. (1990). Score tests in generalized linear measurement error models. *J. R. Statist. Soc. B*, **52**, 345-359.

Tosteson,T.D. and Tsiatis,A.A. (1988). The asymptotic relative efficiency of score tests in a generalized linear model with surrogate covariates. *Biometrika*, **75**, 507-514.

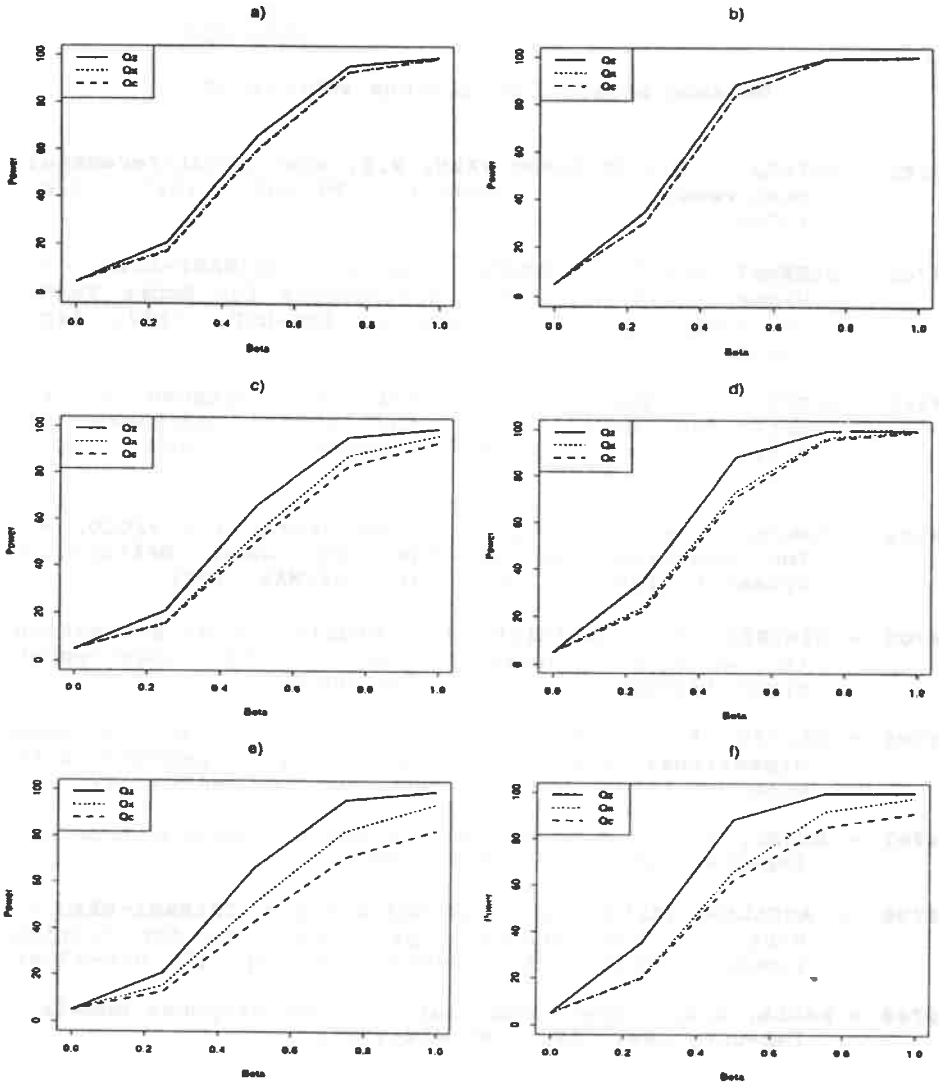

Figure 1: Power curves for the tests Q_z, Q_x e Q_c. a)$\sigma_u^2 = 0.1, n = 30$. b)$\sigma_u^2 = 0.1, n = 50$. c)$\sigma_u^2 = 0.3, n = 30$. d)$\sigma_u^2 = 0.3, n = 50$. e)$\sigma_u^2 = 0.5, n = 30$. f)$\sigma_u^2 = 0.5, n = 50$.

ÚLTIMOS RELATÓRIOS TÉCNICOS PUBLICADOS

9701 - **BOFARINE, H.; ARELLANO-VALE, R.B.** Weak nondifferential measurement error models. IME-USP, 1997. 12p. (RT-MAE-9701)

9702 - **FERRARI, S.L.P.; CORDEIRO, G. M.; CRIBARI-NETO, F.** Higher Order Asymptotic Refinements for Score Tests in Proper Dispersion Models. IME-USP, 1997. 14p. (RT-MAE-9702)

9703 - **DOREA, C.; GALVES, A.; KIRA, E.; ALENCAR, A. P.** Markovian modeling of the stress contours of Brazilian and European Portuguese. IME-USP, 1997. 11p. (RT-MAE-9703)

9704 - **FONTES, L.R.G.; ISOPI, M.; KOHAYAKAWA, Y.; PICCO, P.** The Spectral Gap of the REM under Metropolis Dynamics. IME-USP, 1997. 24p. (RT-MAE-9704)

9705 - **GIMENEZ, P.; BOLFARINE, H.; COLOSIMO, E.A.** Estimation in weibull regression model with measurement error.IME-USP, 1997. 17p. (RT-MAE-9705)

9706 - **GALVES, A.; GUIOL, H.** Relaxation time of the one-dimensional symmetric zero range process with constant rate. IME-USP, 1997. 10p. (RT-MAE-9706)

9707 - **GUIOL, H.** A note about a burton keane's theorem. IME-USP, 1997. 7p. (RT-MAE-9707)

9708 - **ARELLANO-VALLE,R.B.; FERRARI,S.L.P.; CRIBARI-NETO,F.** Bartlett and barblett-type corrections for testing linear restrictions. IME-USP, 1997. 5p. (RT-MAE-9708)

9709 - **PAULA, G.A.** One-Sided Tests in Dose-Response Models. IME-USP, 1997. 29p. (RT-MAE-9709)

9710 - **BRESSAUD, X.** Subshift on an infinite alphabet. IME-USP, 1997. 34p. (RT-MAE-9710)

9711 - **YOSHIDA, O.S.; LEITE, J.G.; BOLFARINE, H.** Stochastic Monotonicity properties of bayes estimation of the population size for capture-recapture data. IME-USP. 1997. 12p. (RT-MAE-9711)

The complete list of ''Relatórios do Departamento de Estatística'', IME-USP, will be sent upon request.

Departamento de Estatística
IME-USP
Caixa Postal 66.281
05315-970 - São Paulo, Brasil