ISSN 0012-9593

quatrième série - tome 56 fascicule 4 juillet-août 2023

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Jonathan CONEJEROS & Fábio Armando TAL

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 459 euros.
Abonnement avec supplément papier :
Europe : 646 €. Hors Europe : 730 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 56, 2023, p. 1155 à 1197

APPLICATIONS OF FORCING THEORY TO HOMEOMORPHISMS OF THE CLOSED ANNULUS

BY JONATHAN CONEJEROS AND FÁBIO ARMANDO TAL

ABSTRACT. – This paper studies homeomorphisms of the closed annulus that are isotopic to the identity from the viewpoint of rotation theory, using a newly developed forcing theory for surface homeomorphisms. Our first result is a solution to the so called strong form of Boyland's Conjecture on the closed annulus: Assume f is a homeomorphism of $\overline{\mathbb{A}} := (\mathbb{R}/\mathbb{Z}) \times [0,1]$ which is isotopic to the identity and preserves a Borel probability measure μ with full support. We prove that if the rotation set of f is a non-trivial segment, then the rotation number of the measure μ cannot be an endpoint of this segment. We also study the case of homeomorphisms such that $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times (0,1)$ is a region of instability of f. We show that, if the rotation numbers of the restriction of f to the boundary components lie in the interior of the rotation set of f, then f has uniformly bounded deviations from its rotation set. Finally, by combining this last result and recent work on realization of rotation vectors for annular continua, we obtain that if f is any area-preserving homeomorphism of $\overline{\mathbb{A}}$ isotopic to the identity, then for every real number ρ in the rotation set of f, there exists an associated Aubry-Mather set, that is, a compact f-invariant set such that every point in this set has a rotation number equal to ρ. This extends a result by P. Le Calvez previously known only for diffeomorphisms.

RÉSUMÉ. – Dans cet article, nous étudions les homéomorphismes de l'anneau compact qui sont isotopes à l'identité d'un point de vue de la théorie des rotations, en utilisant la notion de théorie de forçage récemment développée pour les homéomorphismes des surfaces. Notre premier résultat est une solution à la conjecture de Boyland sur l'anneau compact : Supposons que f est un homéomorphisme de $\overline{\mathbb{A}} := (\mathbb{R}/\mathbb{Z}) \times [0,1]$ qui est isotope à l'identité et qui préserve une mesure borélienne de probabilité μ à support total. Nous prouvons que si l'ensemble de rotation de f est un intervalle non trivial, le nombre de rotation de la mesure μ ne peut pas être une borne de cet intervalle. Nous étudions aussi les homéomorphismes f dont $\mathbb{A} := (\mathbb{R}/\mathbb{Z}) \times (0,1)$ est une région d'instabilité. Nous prouvons que si les nombres de rotation de la restriction de f aux composantes du bord appartiennent à l'intérieur de l'ensemble de rotation de f, alors la déviation de f de son ensemble de rotation est uniformément bornée. Enfin en combinant ce dernier résultat et des travaux récents de réalisation de vecteurs de rotation pour les anneaux continus, nous déduisons que si f est un homéomorphisme de $\overline{\mathbb{A}}$ qui est isotope à l'identité et qui préserve l'aire, alors pour tout nombre réel ρ dans l'ensemble de rotation de f il existe un ensemble d'Aubry-Mather, c'est-à-dire un ensemble compact et invariant tel que tout point dans cet ensemble a un nombre de rotation égal à ρ. Cela étend un résultat de P. Le Calvez connu auparavant uniquement pour les difféomorphismes.

0012-9593/04/ doi:10.24033/asens.2552

1. Introduction

This article studies homeomorphisms of the closed annulus that preserve the orientation and the boundary components, by the point of view of rotation theory. We denote by $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ the circle, by $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0,1]$ the closed annulus and by $\widehat{\overline{\mathbb{A}}} = \mathbb{R} \times [0,1]$ its universal covering. Let $\widehat{\pi} : \widehat{\overline{\mathbb{A}}} \to \overline{\mathbb{A}}$ be the corresponding covering map, and $p_1 : \widehat{\overline{\mathbb{A}}} \to \mathbb{R}$ the projection on the first coordinate. Let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a homeomorphism which preserves both orientation and boundary components and let $\widehat{f}$ be a lift of f to the universal covering. Inspired by the concept of Poincaré's rotation number for orientation-preserving homeomorphisms of the circle, one can define a similar object for $\widehat{f}$, called the *rotation set of* $\widehat{f}$, as follows: let μ be an f-invariant Borel probability measure on $\overline{\mathbb{A}}$. We can define the *rotation number of* μ *for* $\widehat{f}$ as

$$\mathrm{Rot}(\widehat{f}, \mu) := \int_{\overline{\mathbb{A}}} p_1(\widehat{f}(\widehat{z})) - p_1(\widehat{z})\, d\mu(z),$$

where $\widehat{z} \in \widehat{\pi}^{-1}(z)$. Note that this definition does not depend on the choice of $\widehat{z} \in \widehat{\pi}^{-1}(z)$. The *rotation set of* $\widehat{f}$, denoted by $\mathrm{Rot}(\widehat{f})$, is the set of all rotation numbers of f-invariant Borel probability measures. Since the set of f-invariant Borel probability measures is convex and compact in the weak-$*$topology, one shows that the rotation set of $\widehat{f}$ is a non-empty compact interval of $\mathbb{R}$.

We remark that the concept of rotation sets is not restricted to homeomorphisms of the annulus, and has been useful in the general study of homeomorphisms in the isotopy class of the identity of surfaces in general, and particularly for the two dimensional torus. One of the reasons for the growing interest in the subject is the variety of dynamical properties and phenomena that can be deduced from rotation sets; it is a useful tool in, for instance, estimating the topological entropy of a map in [35, 28] or determining the existence of periodic points with arbitrarily large prime periods and distinct rotational behavior in [11].

One of the driving problems in the understanding of the rotation theory for homeomorphisms of the closed annulus and of the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has been the Boyland's Conjecture, see for instance [3, 38]. In the original form, Boyland's Conjecture for the closed annulus claimed that, whenever $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ preserved the Lebesgue measure and had a lift $\widehat{f}$ such that the rotation number of the Lebesgue measure for $\widehat{f}$ was null, then either the rotation set of $\widehat{f}$ was a singleton, or 0 lied in the interior of the rotation set of $\widehat{f}$. A stronger version of this conjecture has also been proposed, saying that whenever f preserved the Lebesgue measure and the rotation set of $\widehat{f}$ was a nondegenerate interval, then the rotation number of the Lebesgue measure for $\widehat{f}$ always lies in the interior of the rotation set, and similar questions were posed for homeomorphisms of $\mathbb{T}^2$. In [1] the strong form of Boyland's Conjecture for $\mathbb{T}^2$ was shown to hold for $\mathcal{C}^{1+\epsilon}$-diffeomorphisms, a result later extended for the $\mathcal{C}^0$ case in [32]. The later paper also proved the original conjecture for the closed annulus, but the strong version remained untenable. Our first result of this paper is the solution to this problem.

THEOREM A. – *Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity and preserves a Borel probability measure μ with full support. Let $\widehat{f}$ be* *a lift of f to $\mathbb{R}\times[0,1]$. Suppose that $\mathrm{Rot}(\widehat{f})$ is a non-trivial segment. Then the rotation number of μ cannot be an endpoint of $\mathrm{Rot}(\widehat{f})$.*

Another research topic in rotation theory that has gathered substantial attention lately is the concept of bounded rotational deviations from rotation sets. It is a well known fact that, given an orientation-preserving homeomorphism $h:\mathbb{T}^1\to\mathbb{T}^1$ and a lift $\widehat{h}$ to the real line whose rotation number is α, one has that every orbit of $\widehat{h}$ remains at a bounded distance from the orbit of the associated rigid rotation. That is, there exists some constant $L>0$ such that, for all $\widehat{x}\in\mathbb{R}$ and all $n\in\mathbb{N}$, $|\widehat{h}^n(\widehat{x})-\widehat{x}-n\alpha|\le L$ (and in this case L can be taken as 1). A natural question is then to ask if some aspects of this property extend to similar situations for homeomorphisms of surfaces. For instance, one could pose the problem: consider a homeomorphism f of $\mathbb{T}^2$ in the isotopy class of the identity and say that f *has uniformly bounded deviations from its rotation set* if, given $\widehat{f}$ a lift of f to $\mathbb{R}^2$, the universal covering of $\mathbb{T}^2$, there is a constant $L>0$ such that, for all $\widehat{z}\in\mathbb{R}^2$ and all $n\in\mathbb{N}$, if d is the distance between a point and a set of $\mathbb{R}^2$, then $d(\widehat{f}^n(\widehat{z})-\widehat{z}, n\,\mathrm{Rot}(\widehat{f}))\le L$. One then asks if it always holds that f has uniformly bounded deviations. This question is false in general, particularly when the rotation set of $\widehat{f}$ is a singleton (see for instance [21, 25]), but it does hold in many situations, particularly when $\mathrm{Rot}(\widehat{f})$ has nonempty interior (see [7, 8, 1, 14, 26, 32]), and similar results also are valid for homeomorphisms of $\mathbb{T}^2$ isotopic to Dehn Twists (see [2]). Furthermore, bounded deviations have also shown to have relevant dynamical consequences, for instance it was used in the proof of Boyland's Conjecture on $\mathbb{T}^2$ in [1, 32]. In some particular cases it can also imply that the dynamics factors over ergodic rotations of $\mathbb{T}^2$ (see [17]) or $\mathbb{T}^1$ (see [18] and [20]). Furthermore, it was shown in [39] that, for a class of $\mathcal{C}^r$ diffeomorphisms of $\mathbb{T}^2$, bounded deviations imply $\mathcal{C}^{r-1}$-rigidity, that is, that there exists a sequence of positive iterates of the map converging in the $\mathcal{C}^{r-1}$-topology to the identity.

Our second theorem deals with bounded deviations from rotation sets for homeomorphisms of $\overline{\mathbb{A}}$ in the following relevant scenario. We will say that $\mathbb{A}=\mathbb{T}^1\times(0,1)$ is a *Birkhoff region of instability* for a homeomorphism f of $\overline{\mathbb{A}}$ if for any neighborhood U of $\mathbb{T}^1\times\{0\}$ and any neighborhood V of $\mathbb{T}^1\times\{1\}$ one can find points $x\in U$, $y\in V$ and positive integers n_1,n_2 such that $f^{n_1}(x)\in V$ and $f^{n_2}(y)\in U$.

THEOREM B. – *Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}}=\mathbb{T}^1\times[0,1]$ which is isotopic to the identity. Suppose that $\mathbb{A}=\mathbb{T}^1\times(0,1)$ is a Birkhoff region of instability for f. Let $\widehat{f}$ be a lift of f to $\mathbb{R}\times[0,1]$. Suppose that $\mathrm{Rot}(\widehat{f})=[\alpha,\beta]$ and that both boundary component rotation numbers are strictly larger than α. Then there exists a real constant $L>0$ such that for every $\widehat{z}\in\mathbb{R}\times[0,1]$ and every integer $n\ge 1$ we have*

$$p_1(\widehat{f}^n(\widehat{z}))-p_1(\widehat{z})-n\alpha\ge -L.$$

Likewise, if we assume that both boundary component rotation numbers are strictly smaller than β, then there exists a real constant $L>0$ such that for every $\widehat{z}\in\mathbb{R}\times[0,1]$ and every integer $n\ge 1$ we have

$$p_1(\widehat{f}^n(\widehat{z}))-p_1(\widehat{z})-n\beta\le L.$$

Interestingly, this is to our knowledge the first positive result on bounded deviations for homeomorphisms of $\overline{\mathbb{A}}$. The hypothesis that $\mathbb{A}$ is a Birkhoff region of instability is in many ways fundamental. One can easily create examples of homeomorphisms of $\overline{\mathbb{A}}$ that do not present bounded deviation when this condition is missing. It also cannot be replaced by assuming that f preserves the Lebesgue measure, as one can create examples without bounded deviation by having infinitely many distinct and invariant sub-annuli each with increasing large deviations. Likewise, the hypothesis that the rotation numbers of the boundary components of $\overline{\mathbb{A}}$ lie in the interior of the rotation set is also fundamental as shown by the following example, which we present in Section 5.

PROPOSITION 1.1. – *There exists a homeomorphism f of the closed annulus $\overline{\mathbb{A}}$ which is isotopic to the identity, such that $\mathbb{A}$ is a Birkhoff region of instability for f and such that f has a lift $\widehat{f}$ to $\mathbb{R} \times [0,1]$ satisfying:*

(i) $\mathrm{Rot}(\widehat{f}) = [0,1]$, *and*

(ii) *for every real number $L > 0$ there exist a point $\widehat{z}$ in $\mathbb{R} \times [0,1]$ and an integer n such that*

$$p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) < -L.$$

The third topic we deal with in this paper is the strong realization of rotation numbers. We say that a point $z \in \overline{\mathbb{A}}$ *has rotation number equal to* ρ if, for any $\widehat{z} \in \widehat{\pi}^{-1}(z)$, one has $\lim_{n\to\infty} p_1(\widehat{f}^n(\widehat{z}) - \widehat{z})/n = \rho$, and we note that if the limit exists, it is independent of which $\widehat{z}$ one chooses in $\widehat{\pi}^{-1}(z)$. We say that a number $\rho \in \mathrm{Rot}(\widehat{f})$ *is realized by an ergodic measure* if there exists some f-invariant ergodic measure ν such that $\mathrm{Rot}(\widehat{f}, \nu) = \rho$. Finally, one says that ρ *is realized by a compact invariant set* if there exists a compact invariant set Q such that all points in Q have rotation number equal to ρ. There is a natural hierarchy of realization. Any ρ that is realized by a compact set is also realized by an ergodic measure, any ρ that is realized by an ergodic measure is also the rotation number of some point, and the rotation number of points are clearly contained in $\mathrm{Rot}(\widehat{f})$. Note that, if f is an area-preserving twist map of the open annulus, then a result by Mather (see [36]) shows that every point ρ in the rotation set of $\widehat{f}$ is realized by a compact set, the so called Aubry-Mather set of ρ. A natural question is then to decide which points in the rotation set of $\widehat{f}$ were realized by compact subsets.

This turned out to be a difficult problem to tackle. An important result by Handel (see [15]) showed that the set of points that are realized by ergodic measures is a closed subset of $\mathrm{Rot}(\widehat{f})$ and he further showed that, except for a possible discrete subset, all were also realized by compact invariant sets. Franks (see [10, 12]) showed that, if f preserves a measure of full support, then every rational number in $\mathrm{Rot}(\widehat{f})$ is realized by a periodic orbit and Le Calvez ([29]) showed that, if f is an area-preserving diffeomorphism, then every point in the rotation set is realized by a compact invariant subset.

Our third theorem, that relies on Theorem B, shows that the answer to this problem is true for regions of instability of Mather. We will say that $\mathbb{A} = \mathbb{T}^1 \times (0,1)$ is a *Mather region of instability* for a homeomorphism f of $\overline{\mathbb{A}}$ if there exist points z_1, z_2 in $\mathbb{A}$ such that the α-limit set of z_1 is contained in $\mathbb{T}^1 \times \{0\}$ and while the ω-limit set of z_1 is contained in $\mathbb{T}^1 \times \{1\}$ and such that the α-limit set of z_2 is contained in $\mathbb{T}^1 \times \{1\}$ and while the ω-limit set of z_2 is contained in $\mathbb{T}^1 \times \{0\}$.

THEOREM C. – *Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity. Suppose that $\mathbb{A} = \mathbb{T}^1 \times (0,1)$ is a Mather region of instability for f. Let $\widehat{f}$ be a lift of f to $\mathbb{R} \times [0,1]$. For every ρ in $\mathrm{Rot}(\widehat{f})$ there exists a compact invariant set Q_ρ such that every point of Q_ρ has a well-defined rotation number and it is equal to ρ. Moreover, if $\rho = p/q$ is a rational number, written in an irreducible way, then Q_ρ can be taken to be the orbit of a periodic point of period q.*

The part in Theorem C where every rational point in the rotation set is realized by a periodic orbit is due to Franks ([10]). We note that, in reality, Theorem C is the direct consequence of a stronger result, Proposition 6.1, which we do not state here due to its more technical hypotheses.

Finally, by combining Theorem C and results from Koropecki (see [22]), Franks and Le Calvez (see [13]) and Koropecki, Le Calvez and Nassiri (see [23]), we are able to deduce the following extension of the above mentioned result by Le Calvez, by improving the smoothness requirements.

THEOREM D. – *Let f be an area-preserving homeomorphism of the closed annulus $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity. Let $\hat{f}$ be a lift of f to $\mathbb{R} \times [0,1]$. For every ρ in $\mathrm{Rot}(\hat{f})$ there exists a compact f-invariant set Q_ρ such that every point of Q_ρ has a well-defined rotation number and is equal to ρ.*

Theorems C and D and the results quoted above all go in the direction of showing that the realization by compact subsets of rotation vectors realized by ergodic measures should always hold, but the general question is still open. In particular, there is no known example where this fails to happen, including if, for instance, $\mathbb{A}$ is a Birkhoff region of instability but not a Mather region of instability.

This work relies heavily on both Le Calvez's Brouwer Equivariant Theory (see [29, 30]) and also from a forcing theory for surface homeomorphisms recently developed by Le Calvez and the second author. To be able to use these works, we need to introduce the concepts of maximal isotopies, Brouwer-Le Calvez transverse foliations and transverse paths to these foliations, which we do in Section 2. The main breakthrough that allows us to obtain our results is the use of these new techniques, coupled with a careful analysis of possible transverse paths of maps of the annulus and with classical ergodic theory. As stated before, in Section 2 we introduce the basic lemmas and results from the above mentioned forcing theory, as well as detail the concept of rotation set for annular homeomorphisms. Section 3 is devoted to showing Theorem A. Section 4 includes the proof of Theorem B and Section 5 provides an example displaying how tight are the hypotheses of Theorem B. Section 6 contains the proofs of Theorem C and Theorem D.

Acknowledgements

F. T. was partially supported by the Alexander von Humboldt foundation and by Fapesp and CNPq. J. C. was supported by CNPq-Brasil and by FONDECYT postdoctoral grant N 3170455 entitled "Algunos problemas para grupos de homeomorfismos de superficies". The authors would like to thank A. Koropecki for substantial discussions regarding this work in general and in particular about Theorem D. The authors would also like to thank the anonymous referees for their careful work and for the suggestions that helped improve this paper.

2. Preliminary results

In this section, we state different results and definitions that will be useful in the rest of the article. The main tool will be the "forcing theory" introduced recently by Le Calvez and the second author (see [32] for more details) and further developed in [33]. This theory will be expressed in terms of maximal isotopies, transverse foliations and transverse trajectories.

2.1. Transverse paths to surface foliations

Let M be an oriented surface. An *oriented singular foliation* $\mathcal{F}$ on M is a closed set $\mathrm{Sing}(\mathcal{F})$, called *the set of singularities of* $\mathcal{F}$, together with an oriented foliation $\mathcal{F}'$ on the complement of $\mathrm{Sing}(\mathcal{F})$, called *the domain of* $\mathcal{F}$ denoted by $\mathrm{dom}(\mathcal{F})$, i.e., $\mathcal{F}'$ is a partition of $\mathrm{dom}(\mathcal{F})$ into connected oriented 1-manifolds (circles or lines) called *leaves of* $\mathcal{F}$, such that for every z in $\mathrm{dom}(\mathcal{F})$ there exist an open neighborhood W of z, called *trivializing neighborhood* and an orientation-preserving homeomorphism called *trivialization chart at* z, $h : W \to (0,1)^2$ that sends the restricted foliation $\mathcal{F}|_W$ onto the vertical foliation oriented downward. If the singular set of $\mathcal{F}$ is empty, we say that the foliation $\mathcal{F}$ is *non singular*. For every $z \in \mathrm{dom}(\mathcal{F})$ we write ϕ_z for the leaf of $\mathcal{F}$ that contains z, ϕ_z^+ for the positive half-leaf and ϕ_z^- for the negative one. A leaf ϕ of $\mathcal{F}$ is said to be *wandering* if ϕ is not closed and if, for any z in ϕ, one can find a trivialization neighborhood W_z such that each leaf of $\mathcal{F}$ intersects W_z in at most a single connected component.

A *path* on M is a continuous map $\gamma : J \to M$ defined on an interval J of $\mathbb{R}$. In absence of ambiguity its image also will be called a path and denoted by γ. A path $\gamma : J \to \mathrm{dom}(\mathcal{F})$ is *positively transverse* [(1)] to $\mathcal{F}$ if for every $t_0 \in J$ there exists a trivialization chart h at $\gamma(t_0)$ such that the application $t \mapsto \pi_1(h(\gamma(t)))$, where $\pi_1 : (0,1)^2 \to (0,1)$ is the projection on the first coordinate, is increasing in a neighborhood of t_0. We note that if $\widehat{M}$ is a covering space of M and $\widehat{\pi} : \widehat{M} \to M$ the covering projection, then $\mathcal{F}$ can be naturally lifted to a singular foliation $\widehat{\mathcal{F}}$ of $\widehat{M}$ such that $\mathrm{dom}(\widehat{\mathcal{F}}) = \widehat{\pi}^{-1}(\mathrm{dom}(\mathcal{F}))$. We will denote $\widetilde{\mathrm{dom}}(\mathcal{F})$ the universal covering space of $\mathrm{dom}(\mathcal{F})$ and $\widetilde{\mathcal{F}}$ the foliation lifted from $\mathcal{F}|_{\mathrm{dom}(\mathcal{F})}$. We note that $\widetilde{\mathcal{F}}$ is a non singular foliation of $\widetilde{\mathrm{dom}}(\mathcal{F})$. Moreover if $\gamma : J \to \mathrm{dom}(\mathcal{F})$ is *positively transverse* to $\mathcal{F}$, every lift $\widehat{\gamma} : J \to \mathrm{dom}(\widehat{\mathcal{F}})$ of γ is *positively transverse* to $\widehat{\mathcal{F}}$. In particular, every lift $\widetilde{\gamma} : J \to \widetilde{\mathrm{dom}}(\mathcal{F})$ of γ to the universal covering space $\widetilde{\mathrm{dom}}(\mathcal{F})$ of $\mathrm{dom}(\mathcal{F})$ is *positively transverse* to the lifted non singular foliation $\widetilde{\mathcal{F}}$.

2.1.1. *Transverse paths intersecting $\mathcal{F}$-transversally.* – A *line* on M is an injective and proper path $\lambda : J \to M$, that is, the interval J is open and the pre-image of every compact subset of M is compact. It inherits a natural orientation induced by the usual orientation of $\mathbb{R}$. Let λ be a line of the plane $\mathbb{R}^2$. The complement of λ has two connected components, $R(\lambda)$ which is on the right of λ and $L(\lambda)$ which is on its left. We will say that a line λ *separates* X from Y, if X and Y belong to different connected components of the complement of λ. Let us consider three pairwise disjoint lines λ_0, λ_1 and λ_2 in $\mathbb{R}^2$. We say that λ_2 is *above* λ_1 relative to λ_0 (and

(1) In the whole text "transverse" will mean "positively transverse"

λ_1 is *below* λ_2 relative to λ_0) if none of the lines separates the two others; and if γ_1 and γ_2 are two disjoint paths that join $z_1 = \lambda_0(t_1)$, $z_2 = \lambda_0(t_2)$ to $z'_1 \in \lambda_1$, $z'_2 \in \lambda_2$ respectively, and that do not meet the three lines but at the ends, then $t_2 > t_1$. This notion does not depend on the orientation of λ_1 and λ_2 but depends on the orientation of λ_0 (see Figure 1).

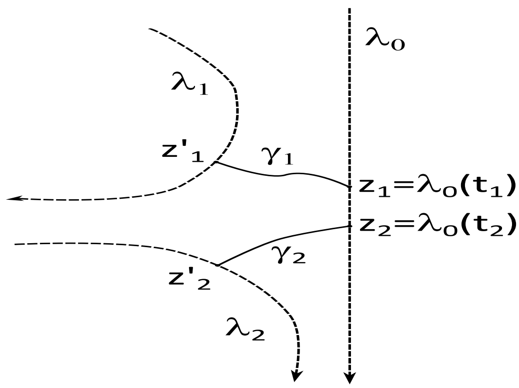

FIGURE 1. λ_2 is above λ_1 relative to λ_0.

Let $\mathcal{F}$ be an oriented singular foliation on an oriented surface M. Let $\gamma_1 : J_1 \to \mathrm{dom}(\mathcal{F})$ and $\gamma_2 : J_2 \to \mathrm{dom}(\mathcal{F})$ be two transverse paths. Suppose that there exist $t_1 \in J_1$ and $t_2 \in J_2$ such that $\gamma_1(t_1) = \gamma_2(t_2)$. We say that γ_1 *intersects* γ_2 *$\mathcal{F}$-transversally and positively at* $\gamma_1(t_1) = \gamma_2(t_2)$, if there exist a_1, b_1 in J_1 satisfying $a_1 < t_1 < b_1$, and a_2, b_2 in J_2 satisfying $a_2 < t_2 < b_2$ such that if $\widetilde{\gamma}_1 : J_1 \to \widetilde{\mathrm{dom}}(\mathcal{F})$ and $\widetilde{\gamma}_2 : J_2 \to \widetilde{\mathrm{dom}}(\mathcal{F})$ are lifts of γ_1 and γ_2 respectively, satisfying $\widetilde{\gamma}_1(t_1) = \widetilde{\gamma}_2(t_2)$ then

— $\phi_{\widetilde{\gamma}_2(a_2)}$ is below $\phi_{\widetilde{\gamma}_1(a_1)}$ relative to $\phi_{\widetilde{\gamma}_1(t_1)}$; and

— $\phi_{\widetilde{\gamma}_2(b_2)}$ is above $\phi_{\widetilde{\gamma}_1(b_1)}$ relative to $\phi_{\widetilde{\gamma}_2(t_2)}$.

See Figure 2. In this situation we also say that γ_2 intersects γ_1 $\mathcal{F}$-transversally and negatively at $\gamma_1(t_1) = \gamma_2(t_2)$, and that γ_1 and γ_2 have a $\mathcal{F}$-transverse intersection at $\gamma_1(t_1) = \gamma_2(t_2)$.

If $\gamma_1 = \gamma_2$ in the situation above, we will say that γ_1 has a $\mathcal{F}$-transverse self-intersection. This means that if $\widetilde{\gamma}_1$ is a lift of γ_1 to the universal covering of $\mathrm{dom}(\mathcal{F})$, then there exists a covering automorphism T such that $\widetilde{\gamma}_1$ and $T(\widetilde{\gamma}_1)$ have a $\widetilde{\mathcal{F}}$-transverse intersection at $\widetilde{\gamma}_1(t_1) = T(\widetilde{\gamma}_1(t_2))$.

2.2. Maximal isotopies and transverse foliations

2.2.1. *Isotopies, maximal isotopies.* – Let M be an oriented surface. Let f be a homeomorphism of M. An *identity isotopy* of f is a path that joins the identity to f in the space of homeomorphisms, furnished with the C^0-topology. We will say that f *is isotopic to the identity* if the set of identity isotopies of f is not empty. Let $I = (f_t)_{t\in[0,1]}$ be an identity isotopy

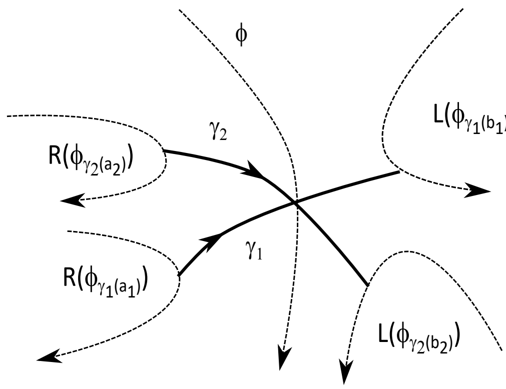

FIGURE 2. The paths γ_1 and γ_2 intersect $\mathcal{F}$-transversally and positively at $\gamma_1(t_1) = \gamma_2(t_2)$.

of f. Given $z \in M$ we can define the *trajectory of* z as the path $I(z) : t \mapsto f_t(z)$. For every integer $n \geq 1$ we define $I^n(z) = \prod_{0 \leq k < n} I(f^k(z))$ by concatenation. Furthermore, we define

$$I^{\mathbb{N}}(z) = \prod_{k\in\mathbb{N}} I(f^k(z)), \quad I^{-\mathbb{N}}(z) = \prod_{k\in\mathbb{N}} I(f^{-k}(z)), \quad I^{\mathbb{Z}}(z) = \prod_{k\in\mathbb{Z}} I(f^k(z)).$$

The last path will be called the *whole trajectory of* z. One can define the fixed point of I as $\mathrm{Fix}(I) = \cap_{t\in[0,1]} \mathrm{Fix}(f_t)$, which is the set of points with trivial whole trajectory. The complement of $\mathrm{Fix}(I)$ will be called the *domain of* I, and it will be denoted by $\mathrm{dom}(I)$.

In general, let us say that an identity isotopy of f is a maximal isotopy, if there is no fixed point of f whose trajectory is contractible relative to the fixed point set of I. A very recent result of F. Béguin, S. Crovisier and F. Le Roux (see [4]) asserts that such an isotopy always exists if f is isotopic to the identity. A slightly weaker result was previously proved by O. Jaulent (see [19]).

THEOREM 2.1 ([19], [4]). – *Let M be an oriented surface. Let f be a homeomorphism of M which is isotopic to the identity and let I' be an identity isotopy of f. Then there exists an identity isotopy I of f such that:*

(i) $\mathrm{Fix}(I') \subset \mathrm{Fix}(I)$;

(ii) *I is homotopic to I' relative of* $\mathrm{Fix}(I')$;

(iii) *there is no point $z \in \mathrm{Fix}(f) \setminus \mathrm{Fix}(I)$ whose trajectory $I(z)$ is homotopic to zero in $M \setminus \mathrm{Fix}(I)$.*

We will say that an identity isotopy I satisfying the conclusion of Theorem 2.1 is a *maximal isotopy*. We note that the last condition of the above theorem can be stated in the following equivalent form:

(iii′) if $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ is the identity isotopy that lifts $I|_{M\setminus \mathrm{Fix}(I)}$ to the universal covering space of $M \setminus \mathrm{Fix}(I)$, then $\widetilde{f}_1$ is fixed point free.

2.2.2. *Transverse foliations.* – Let us recall the equivariant foliation version of the Plane Translation Theorem due to P. Le Calvez (see [30]).

THEOREM 2.2 ([30]). – *Let M be an oriented surface. Let f be a homeomorphism of M which is isotopic to the identity and let I be a maximal identity isotopy of f. Then there exists an oriented singular foliation $\mathcal{F}$ with* $\mathrm{dom}(\mathcal{F}) = \mathrm{dom}(I)$*, such that for every $z \in \mathrm{dom}(I)$ the trajectory $I(z)$ is homotopic on* $\mathrm{dom}(I)$*, relative to the endpoints, to a positively transverse path to $\mathcal{F}$.*

We will say that a foliation $\mathcal{F}$ satisfying the conclusion of Theorem 2.2 is *transverse* to I. Observe that if $\widehat{M}$ is a covering space of M and $\widehat{\pi} : \widehat{M} \to M$ the covering projection, a foliation $\mathcal{F}$ transverse to a maximal identity isotopy I lifts to a foliation $\widehat{\mathcal{F}}$ transverse to the lifted isotopy $\widehat{I}$. In particular, the foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathrm{dom}(\mathcal{F})}$, which is non singular, is transverse to the isotopy $\widetilde{I}$. This last property is equivalent to say that every leaf $\widetilde{\phi}$ of $\widetilde{\mathcal{F}}$ is a Brouwer line of $\widetilde{f}$, that is $\widetilde{f}(\widetilde{\phi}) \subset L(\widetilde{\phi})$ and $\widetilde{f}^{-1}(\widetilde{\phi}) \subset R(\widetilde{\phi})$, where $L(\widetilde{\phi})$ and $R(\widetilde{\phi})$ are the left and right connected components of the complement of $\widetilde{\phi}$, defined so that they are compatible with the orientation of the line.

Given $z \in M$ we will write $I^1_{\mathcal{F}}(z)$ for the subset of paths that are positively transverse to $\mathcal{F}$, that join z to $f(z)$ and that are homotopic in $\mathrm{dom}(\mathcal{F})$ to $I(z)$, relative to the endpoints. We will also use the notation $I^1_{\mathcal{F}}(z)$ for every path in this set and we will call it the transverse trajectory of z. More generally, for every integer $n \geq 1$ we can define $I^n_{\mathcal{F}}(z) = \prod_{0 \leq k < n} I^1_{\mathcal{F}}(f^k(z))$ by concatenation, that is either a transverse path passing through the points $z, f(z), \cdots, f^n(z)$, or a set of such paths. Furthermore, we define

$$I^{\mathbb{N}}_{\mathcal{F}}(z) = \prod_{k \in \mathbb{N}} I^1_{\mathcal{F}}(f^k(z)), \quad I^{-\mathbb{N}}_{\mathcal{F}}(z) = \prod_{k \in \mathbb{N}} I^1_{\mathcal{F}}(f^{-k}(z)), \quad I^{\mathbb{Z}}_{\mathcal{F}}(z) = \prod_{k \in \mathbb{Z}} I^1_{\mathcal{F}}(f^k(z)).$$

The last path will be called the *whole transverse trajectory of z*.

Given two transverse paths $\gamma_i : I_i \to \mathrm{dom}(I)$, $i \in \{1,2\}$, we say that the paths are *$\mathcal{F}$-equivalent* if there exist paths $\widetilde{\gamma_i} : I_i \to \widetilde{\mathrm{dom}(I)}$, $i \in \{1,2\}$ with $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ lifting γ_1 and γ_2 respectively, and such that $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2}$ intersect the same set of leaves from $\widetilde{\mathcal{F}}$. In particular, if $I_i = [a_i, b_i]$, $i \in \{1,2\}$ are closed intervals, this is the same as requiring that $\phi_{\widetilde{\gamma_1}(a_1)} = \phi_{\widetilde{\gamma_2}(a_2)}$ and that $\phi_{\widetilde{\gamma_1}(b_1)} = \phi_{\widetilde{\gamma_2}(b_2)}$. If no confusion is present, we will just refer to equivalent paths instead of $\mathcal{F}$-equivalent. Let us state the following result that will be useful later.

LEMMA 2.3 ([32]). – *Fix $z \in \mathrm{dom}(I)$, an integer $n \geq 1$, and parameterize $I^n_{\mathcal{F}}(z)$ by* $[0,1]$. *For every $0 < a < b < 1$, there exists a neighborhood V of z such that for every z' in V, the path $I^n_{\mathcal{F}}(z)|_{[a,b]}$ is equivalent to a subpath of $I^n_{\mathcal{F}}(z')$. Moreover, there exists a neighborhood W of z such that for every z' and z'' in W, the path $I^n_{\mathcal{F}}(z')$ is equivalent to a subpath of $I^{n+2}_{\mathcal{F}}(f^{-1}(z''))$.*

2.3. Forcing theory

2.3.1. *Admissible paths.* – We will say that a transverse path $\gamma : [a,b] \to \mathrm{dom}(I)$ is *admissible of order n* ($n \geq 1$ is an integer) if it is equivalent to a path $I^n_{\mathcal{F}}(z)$, z in $\mathrm{dom}(I)$. Note that this implies that, if $\widetilde{\gamma} : [a,b] \to \widetilde{\mathrm{dom}(I)}$ is a lift of γ, there exists a point $\widetilde{z}$ in $\widetilde{\mathrm{dom}(I)}$ such that $\widetilde{z} \in \phi_{\widetilde{\gamma}(a)}$ and $\widetilde{f}^n(\widetilde{z}) \in \phi_{\widetilde{\gamma}(b)}$, or equivalently, that

$$\widetilde{f}^n(\phi_{\widetilde{\gamma}(a)}) \cap \phi_{\widetilde{\gamma}(b)} \neq \emptyset.$$

The fundamental proposition (Proposition 20 from [32]) is a result about maximal isotopies and transverse foliations that permits us to construct new admissible paths from a pair of admissible paths.

PROPOSITION 2.4 ([32]). – *Suppose that* $\gamma_1 : [a_1, b_1] \to M$ *and* $\gamma_2 : [a_2, b_2] \to M$ *are two transverse paths that intersect* $\mathcal{F}$*-transversally at* $\gamma_1(t_1) = \gamma_2(t_2)$*. If* γ_1 *is admissible of order* n_1 *and* γ_2 *is admissible of order* n_2*, then the paths* $\gamma_1|_{[a_1,t_1]}\gamma_2|_{[t_2,b_2]}$ *and* $\gamma_2|_{[a_2,t_2]}\gamma_1|_{[t_1,b_1]}$ *are admissible of order* $n_1 + n_2$*.*

One deduces immediately the following result.

LEMMA 2.5 ([32]). – *Let* $\gamma_i : [a_i, b_i] \to M$*,* $1 \leq i \leq r$*, be a family of* $r \geq 2$ *transverse paths. We suppose that for every* $i \in \{1, \ldots, r\}$ *there exist* $s_i \in [a_i, b_i]$ *and* $t_i \in [a_i, b_i]$ *such that:*

(i) $\gamma_i|_{[s_i,b_i]}$ *and* $\gamma_{i+1}|_{[a_{i+1},t_{i+1}]}$ *intersect* $\mathcal{F}$*-transversally at* $\gamma_i(t_i) = \gamma_{i+1}(s_{i+1})$ *if* $i < r$*;*

(ii) *one has* $s_1 = a_1 < t_1 < b_1$*,* $a_r < s_r < t_r = b_r$ *and* $a_i < s_i < t_i < b_i$ *if* $1 < i < r$*;*

(iii) γ_i *is admissible of order* n_i*.*

Then $\prod_{1\leq i\leq r} \gamma_i|_{[s_i,t_i]}$ *is admissible of order* $\sum_{1\leq i\leq r} n_i$*.*

The following result is a consequence of Proposition 23 from [32].

COROLLARY 2.6 ([32]). – *Let* $\gamma : [a, b] \to M$ *be a transverse path admissible of order* n*. Then there exists* $\gamma' : [a, b] \to M$ *a transverse path, also admissible of order* n*, such that* γ' *has no* $\mathcal{F}$*-transverse self-intersection and* $\phi_{\gamma(a)} = \phi_{\gamma'(a)}$*,* $\phi_{\gamma(b)} = \phi_{\gamma'(b)}$*.*

2.4. Rotational horseshoe

The presence of topological horseshoes has been a paradigmatic feature of dynamical systems, and its prevalence as a phenomenon is well-known. In the article [33], which is a natural continuation of the article [32], the authors develop a new criteria for the existence of topological horseshoes for surface homeomorphisms in the isotopy class of the identity based on the notions of maximal isotopies, transverse foliations and transverse trajectories. The fundamental result of [33] is that the existence of an admissible path with a $\mathcal{F}$-transverse self-intersection implies the existence of a horseshoe.

Moreover, the horseshoe obtained in the main theorem from [33] is rotational, in the sense that it is the projection in $\mathrm{dom}(I)$ of a rotational horseshoe (also called rotary horseshoes, see [16, 5]) for some annular covering of $\mathrm{dom}(I)$, implying the existence of nontrivial rotational behavior. To be precise, and adapting those results to our need, let us consider f a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0, 1]$ which is isotopic to the identity. Let $\widehat{\overline{\mathbb{A}}} := \mathbb{R} \times [0, 1]$ be the universal covering of $\overline{\mathbb{A}}$ and let $\widehat{f}$ be a lift of f to $\widehat{\overline{\mathbb{A}}}$. Let I' be an identity isotopy of f, such that its lift to $\widehat{\overline{\mathbb{A}}}$ is an identity isotopy of $\widehat{f}$. Let I be a maximal identity isotopy of f larger than I' and let $\mathcal{F}$ be a singular foliation transverse to I. Let $\widehat{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widehat{\overline{\mathbb{A}}}$. The following result is a direct consequence of Theorem M of [33].

PROPOSITION 2.7. – *Suppose that there exist an admissible transverse path* $\widehat{\gamma} : [a,b] \to \widehat{\widehat{\mathbb{A}}}$ *of order* $q \geq 1$ *and an integer* $p \in \mathbb{Z}$ *such that* $\widehat{\gamma}$ *and* $\widehat{\gamma} + (p,0)$ *intersect* $\widehat{\mathcal{F}}$*-transversally at* $\phi_{\widehat{\gamma}(t)} = \phi_{(\widehat{\gamma}+(p,0))(s)}$, *with* $a < s < t < b$. *Then for any* $0 < \theta \leq p/q$, *there exists a nonempty* f*-invariant compact subset* Q_θ *of* $\overline{\mathbb{A}}$ *such that, for each* $z \in Q_\theta$, *one has that* $\mathrm{Rot}(\widehat{f}, z) = \theta$. *Moreover, if* $\theta = r/s$ *is a rational number, written in an irreducible way, then* Q_θ *can be taken to be an orbit of a periodic point of period* s.

2.5. Transverse paths with $\mathcal{F}$-transverse self-intersection

We will also need the following result, which can be directly derived from Proposition 24 of [33]:

PROPOSITION 2.8. – *Let* $\mathcal{F}$ *be a singular foliation of* $\mathbb{R}^2$, *and let* $\gamma : [a,b] \to \mathbb{R}^2$ *be transverse to* $\mathcal{F}$. *Assume that there exist* $a < t_0 < t_1 < b$ *such that* $\phi_{\gamma(t_0)} = \phi_{\gamma(t_1)}$ *and that* $\gamma|_{[t_0,t_1]}$ *is* $\mathcal{F}$*-equivalent to a simple closed curve. If* $U = \bigcup_{t_0 \leq s \leq t_1} \phi_{\gamma(s)}$ *and if* $\gamma(a)$ *and* $\gamma(b)$ *belong to the same connected component of the complement of* U, *then* γ *has a* $\mathcal{F}$*-transverse self-intersection.*

2.6. Rotation set of annular homeomorphisms

2.6.1. *Rotation set.* – We will denote by $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ the circle and by $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0,1]$ the closed annulus. We endow the annulus $\overline{\mathbb{A}}$ with its usual topology and orientation. Let $\widehat{\pi} : \widehat{\widehat{\mathbb{A}}} = \mathbb{R} \times [0,1] \to \overline{\mathbb{A}}$ be the universal covering map of $\overline{\mathbb{A}}$ defined by $\widehat{\pi}(x,y) = (x+\mathbb{Z}, y)$. Let f be a homeomorphism of $\overline{\mathbb{A}}$ that is isotopic to the identity (that is f preserves the orientation and the boundary components) and let $\widehat{f}$ be a lift of f to $\widehat{\widehat{\mathbb{A}}}$, i.e ($f \circ \widehat{\pi} = \widehat{\pi} \circ \widehat{f}$). We can define the *displacement function* $\rho_1 : \overline{\mathbb{A}} \to \mathbb{R}$ as

$$\rho_1(x+\mathbb{Z}, y) = p_1(\widehat{f}(x,y)) - x,$$

where $p_1 : \widehat{\widehat{\mathbb{A}}} \to \mathbb{R}$ is the projection on the first coordinate and $(x,y) \in \widehat{\pi}^{-1}(x+\mathbb{Z}, y)$. Let $X \subset \overline{\mathbb{A}}$ be a compact f-invariant set. We will denote by $\mathcal{M}_f(\overline{\mathbb{A}}, X)$ the set of all f-invariant Borel probability measures supported in X. If μ is in $\mathcal{M}_f(\overline{\mathbb{A}}, X)$, we define its *rotation number* as

$$\mathrm{Rot}(\widehat{f}, \mu) := \int_{\overline{\mathbb{A}}} \rho_1 \, d\mu.$$

Then we define the *rotation set of* $\widehat{f}$ *in* X as

$$\mathrm{Rot}(\widehat{f}, X) := \{\mathrm{Rot}(\widehat{f}, \mu) : \mu \in \mathcal{M}_f(\overline{\mathbb{A}}, X)\}.$$

REMARK 1. – For every $p \in \mathbb{Z}$ and every $q \in \mathbb{Z}$, the map $\widehat{f^q} + (p,0)$ is a lift of f^q and we have $\mathrm{Rot}(\widehat{f^q} + (p,0), X) = q\,\mathrm{Rot}(\widehat{f}, X) + p$.

REMARK 2. – Note also that, as $\mathcal{M}_f(\overline{\mathbb{A}}, X)$ is a compact convex set in the weak-* topology and $\mathrm{Rot}(\widehat{f}, a\mu_1 + (1-a)\mu_2) = a\,\mathrm{Rot}(\widehat{f}, \mu_1) + (1-a)\,\mathrm{Rot}(\widehat{f}, \mu_2)$, for all $\mu_1, \mu_2 \in \mathcal{M}_f(\overline{\mathbb{A}}, X)$ and $a \in [0,1]$. This implies that $\mathrm{Rot}(\widehat{f}, X)$ is always a compact interval.

For every measure μ in $\mathcal{M}_f(\overline{\mathbb{A}}, X)$, by the Ergodic Decomposition Theorem, there is a unique Borel probability measure τ on $\mathcal{M}_f(\overline{\mathbb{A}}, X)$ supported in $\mathcal{M}_f^e(\overline{\mathbb{A}}, X)$ (the set of all ergodic measure in $\mathcal{M}_f(\overline{\mathbb{A}}, X)$) such that for every continuous function $\varphi : \overline{\mathbb{A}} \to \mathbb{R}$ we have

$$\int_{\overline{\mathbb{A}}} \varphi \, d\mu = \int_{\mathcal{M}_f^e(\overline{\mathbb{A}}, X)} \left(\int_{\overline{\mathbb{A}}} \varphi \, d\nu \right) d\tau(\nu).$$

Hence, if $\mathrm{Rot}(\widehat{f}, \mu)$ is an endpoint of $\mathrm{Rot}(\widehat{f}, X)$, then τ-almost all ergodic measures appearing in the ergodic decomposition of μ must have a well-defined rotation number, and this number must be equal to $\mathrm{Rot}(\widehat{f}, \mu)$. Moreover, if ν in $\mathcal{M}_f(\overline{\mathbb{A}}, X)$ is ergodic, then Birkhoff Ergodic Theorem implies that ν-almost every point has a well-defined rotation number which is equal to the rotation number of ν. That is, for ν-almost all points z, given any lift $\hat{z}$ of z, the limit of the sequence $\left((p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}))/n\right)_{n \in \mathbb{N}}$ exists, and will be denoted $\mathrm{Rot}(\widehat{f}, z)$. Furthermore,

$$\mathrm{Rot}(\widehat{f}, z) = \lim_{n \to +\infty} \frac{p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z})}{n} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho_1(f^k(z)) = \mathrm{Rot}(\widehat{f}, \nu).$$

If $X = \overline{\mathbb{A}}$, we denote the rotation set of $\widehat{f}$ by $\mathrm{Rot}(\widehat{f})$ instead of $\mathrm{Rot}(\widehat{f}, \overline{\mathbb{A}})$. We have the following theorem, which can be deduced from [10].

THEOREM 2.9 (Franks Theorem). – *Let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a homeomorphism of $\overline{\mathbb{A}}$ which is isotopic to the identity. Suppose that f preserves a Borel probability measure of full support. Let $\widehat{f}$ be a lift of f to $\mathbb{R} \times [0, 1]$. Then for every rational number r/s, written in an irreducible way, in the interior of the rotation set of $\widehat{f}$ there exists a point $\widehat{z} \in \mathbb{R} \times [0, 1]$ such that $\widehat{f}^s(\widehat{z}) = \widehat{z} + (r, 0)$.*

2.6.2. *Dynamics near a boundary component with positive rotation number*. – We recall the local dynamics of a homeomorphism of the close annulus near a boundary component of $\overline{\mathbb{A}}$ with positive rotation number. Let us consider a homeomorphism f of the closed annulus which is isopotic to the identity. Let $\widehat{f}$ be a lift of f to $\mathbb{R} \times [0, 1]$. We will suppose that the rotation number of the lower boundary component of $\overline{\mathbb{A}}$,

$$\rho_0 := \lim_{n \to +\infty} \frac{p_1(\widehat{f}^n(x, 0)) - x}{n},$$

where $x \in \mathbb{R}$, is positive. Therefore $\widehat{f}|_{\mathbb{R} \times \{0\}}$ has no fixed point, and so we can consider

$$m := \inf_{\widehat{z} \in \mathbb{R} \times \{0\}} (p_1(\widehat{f}(\widehat{z})) - p_1(\widehat{z})) > 0.$$

We deduce the next result.

LEMMA 2.10. – *There exists a real number $\delta > 0$ such that for every $\widehat{z} \in \mathbb{R} \times [0, \delta]$ we have*

$$\frac{m}{2} < p_1(\widehat{f}(\widehat{z})) - p_1(\widehat{z}).$$

We deduce the following result.

COROLLARY 2.11. – *For every real number $M > 0$, there exist a real number $\delta > 0$ and an integer $n \geq 1$ such that for every $\hat{z} \in \mathbb{R} \times [0, \delta]$ we have*

$$M < p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}).$$

2.7. Dynamics of an oriented foliation in a neighborhood of an isolated singularity

In this subsection, we consider an oriented singular foliation $\mathcal{F}$ on an oriented surface M which has an isolated singularity z_0. A *sink* (resp. a *source*) of $\mathcal{F}$ is an isolated singularity point z_0 of $\mathcal{F}$ such that there is a homeomorphism h from a neighborhood U of z_0 to the open unit disk $\mathbb{D}$ of $\mathbb{R}^2$ which sends z_0 to $0 \in \mathbb{D}$ and sends the restricted foliation $\mathcal{F}|_{U\setminus\{z_0\}}$ to the radial foliation on $\mathbb{D} \setminus \{0\}$ with the leaves towards (resp. backwards) to 0. We recall that for every $z \in \mathrm{dom}(\mathcal{F})$ we will write ϕ_z for the leaf of $\mathcal{F}$ that contains z, ϕ_z^+ for the positive half-leaf and ϕ_z^- for the negative one.

We can define the α-limit and ω-limit sets of each leaf ϕ of $\mathcal{F}$ as follows:

$$\alpha(\phi) := \bigcap_{z\in\phi} \overline{\phi_z^-}, \quad \text{and} \quad \omega(\phi) := \bigcap_{z\in\phi} \overline{\phi_z^+}.$$

We will use the following result due to Le Roux that describes the dynamics of an oriented singular foliation $\mathcal{F}$ near an isolated singularity (see [34]). For our purpose we state a simplified version of his result.

PROPOSITION 2.12 ([34]). – *Let $\mathcal{F}$ be an oriented singular foliation on an oriented surface M. Let z_0 be an isolated singularity of $\mathcal{F}$. Then at least one of the following cases holds:*

(1) *every neighborhood of z_0 contains a closed leaf of $\mathcal{F}$;*

(2) *there exist a leaf of $\mathcal{F}$ whose α-limit set is reduced to z_0 and a leaf of $\mathcal{F}$ whose ω-limit set is reduced to z_0; or*

(3) *z_0 is either a sink or a source of $\mathcal{F}$.*

Let z_0 be a point in an oriented surface M and let f be a homeomorphism of M which fixes z_0. A *local identity isotopy* of f is a path that joins the identity to f in the space of homeomorphisms of M fixing z_0, furnished with the C^0-topology.

We will say that the foliation $\mathcal{F}$ is *locally transverse* to I at z_0 (see [31]), if for every neighborhood V_{z_0} of z_0 there exists a neighborhood W_{z_0} of z_0 contained in V_{z_0} such that for every $z \in W_{z_0}$, $z \neq z_0$, its transverse trajectory $I^1_{\mathcal{F}}(z)$ is contained in $V_{z_0} \setminus \{z_0\}$. We will use also the following result due to Le Calvez.

PROPOSITION 2.13 ([31]). – *Let f be a homeomorphism of an oriented surface M which fixes z_0. Let I be a local identity isotopy of f. Suppose that $\mathcal{F}$ is an oriented singular foliation on M which is transverse to I. If $M \setminus \{z_0\}$ is not a topological sphere, then $\mathcal{F}$ is also locally transverse to I at z_0.*

2.8. Periodic disks for area-preserving homeomorphisms of $\overline{\mathbb{A}}$

In this subsection, let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be an area-preserving homeomorphism, and let $\hat{f}$ be a lift of f to the universal covering space. A set $O \subset \overline{\mathbb{A}}$ is a topological disk if it is homeomorphic to an open disk of $\mathbb{R}^2$. We need the following result which can be deduced from Theorem 4 of [24].

LEMMA 2.14. – *Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving homeomorphism and let $z \in \mathbb{R}^2$ be such that, for every neighborhood U of z, there exist three disjoint and invariant topological disks O_1, O_2, O_3 such that $O_i \cap U \neq \emptyset$, $i \in \{1, 2, 3\}$. If each O_i does not contain wandering points, then $g(z) = z$.*

As a consequence we obtain

LEMMA 2.15. – *Let $O \subset \overline{\mathbb{A}}$ be a topological disk and $\widehat{O}$ a connected component of $\widehat{\pi}^{-1}(O)$. If $\widehat{O}$ is unbounded, and if there exists an integer p, q with $q > 0$ such that $\widehat{f}^q(\widehat{O}) = \widehat{O} + (p, 0)$, then there exists $\widehat{x}$ such that $\hat{f}^q(\widehat{x}) = \widehat{x} + (p, 0)$ and such that $\widehat{\pi}(\widehat{x}) \in \partial O$.*

Proof. – Note that, since O is a topological disk, the sets $\widehat{O}_i = \widehat{O} + (i, 0)$, with $i \in \mathbb{Z}$ are all disjoint, and since $\widehat{O}$ is unbounded one may find a sequence of points $(\widehat{z}_i)_{i \in \mathbb{Z}}$ of points in $\widehat{O}_i$ that accumulates on a point $\widehat{x}$. Note that each $\widehat{O}_i$ is invariant by $\widehat{g} = \widehat{f}^q - (p, 0)$. Furthermore, since each $\widehat{O}_i$ projects to a topological disk and is invariant by $\widehat{g}$, the dynamics of $\widehat{g}$ restricted to each $\widehat{O}_i$ is conjugated to the dynamics of f^q restricted to O. Since f^q is area-preserving, it has no wandering points and therefore $\widehat{g}$ has no wandering points in each $\widehat{O}_i$. The result follows from the previous lemma. □

2.9. Essential sets on the annulus, prime ends and realization of rotation vectors in continua

We say that an open subset O of $\mathbb{A} = \mathbb{T}^1 \times (0, 1)$ is *essential* if it contains a simple closed curve which is not homotopic to a point. If O is open, connected and essential, then the *filling of O*, that is the union of O and all the compact connected components of its complement, is a topological open annulus homeomorphic to $\mathbb{A}$. We say that $K \subset \mathbb{A}$ is an *essential continuum* if it is a continuum (i.e., connected and compact) which separates the two ends of $\mathbb{A}$. Likewise, if $K \subset \overline{\mathbb{A}} = \mathbb{T}^1 \times [0, 1]$, then we say that K is an *essential continuum* if it is contained in $\mathbb{A}$ and is an essential continuum for $\mathbb{A}$. If $K \subset \mathbb{A}$, we denote by $U_+ = U_+(K)$ and $U_- = U_-(K)$ the components of $\overline{\mathbb{A}} \setminus K$ containing $\mathbb{T}^1 \times \{1\}$ and $\mathbb{T}^1 \times \{0\}$ respectively.

2.9.1. *Prime ends rotation numbers.* – We start recalling a very brief description of prime ends rotation numbers (for a more complete description see [23]). Let f be a homeomorphism of $\overline{\mathbb{A}}$, and let K be an essential f-invariant continuum. Collapsing the lower and upper boundary components of $\overline{\mathbb{A}}$ to points S and N, respectively, we obtain a topological 2-dimensional sphere and the dynamics induced by f fixes these two points. We consider U_+ and U_- as defined above. The sets $U_+^* = U_+ \cup \{N\}$ and $U_-^* = U_- \cup \{S\}$ are invariant open topological disks. It is known that one may define a prime end compactification $\widetilde{U_+^*}$ (respectively $\widetilde{U_-^*}$) of U_+^* (resp. U_-^*) which, as a set, is the disjoint union of U_+^* (resp. U_-^*) with a topological circle, called the circle of prime ends. This compactification can be endowed with a suitable topology making it homeomorphic to the closed unit disk $\overline{\mathbb{D}}$ of the plane. Furthermore, and more relevantly, this compactification is such that the restriction of f to U_+ (resp U_-) extends in a unique way to a homeomorphism f_+ of $\widetilde{U}_+^*$ (resp. a homeomorphism f_- of $\widetilde{U}_-^*$).

The set $\widetilde{U}_+^* \setminus \{N\}$ is homeomorphic to $\mathbb{T}^1 \times [0, \infty)$, whose universal covering space is $\overline{H}_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$, and denote $H_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Let $\hat{\pi}_+ : \overline{H}_+ \to \widetilde{U}_+^* \setminus \{N\}$ be the projection map, and note that $\hat{\pi}_+(H_+)$ is the image of the inclusion map $i : U_+ \to \widetilde{U}_+^* \setminus \{N\}$. One may then obtain a map $\pi_+ : \widehat{\pi}^{-1}(U_+) \to H_+$ and a homeomorphism $\widehat{f}_+ : \overline{H}_+ \to \overline{H}_+$ such that $\pi_+ \widehat{f}|_{\widehat{\pi}^{-1}(U_+)} = \widehat{f}_+ \pi_+$ and $\widehat{f}_+(x+1, y) = \widehat{f}_+(x, y) + (1, 0)$. Similarly, we obtain a map $\pi_- : \widehat{\pi}^{-1}(U_-) \to H_- := \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and a homeomorphism $\widehat{f}_- : \overline{H}_- \to \overline{H}_-$ such that $\pi_- \widehat{f}|_{\widehat{\pi}^{-1}(U_-)} = \widehat{f}_- \pi_-$ and $\widehat{f}_-(x+1, y) = \widehat{f}_-(x, y) + (1, 0)$. The *upper* (respectively *lower*) *prime end rotation number* of K associated to $\widehat{f}$ is defined as

$$\rho^{\pm}(\widehat{f}, K) := \lim_{n \to +\infty} \frac{p_1(\widehat{f}_{\pm}^{n}(x, 0)) - x}{n},$$

which is independent of x. If $\rho^+(\widehat{f}, K) = \rho^-(\widehat{f}, K)$ we call this number *the prime end rotation number of* K. We note that for every integers p, q, the map $\widehat{f}^q + (p, 0)$ is a lift of f^q and we have

$$\rho^{\pm}(\widehat{f}^q + (p, 0), K) = q\rho^{\pm}(\widehat{f}, K) + p.$$

If f is a homeomorphism of $\mathbb{A}$ and if $O \subset \mathbb{A}$ is a pre-compact essential open annulus which is f-invariant, one can likewise define the prime ends compactification of O in the following way. Let S, N be the two ends of $\mathbb{A}$. Since O is an essential open annulus, its complement has exactly two connected components. Let K_N be the subset of the boundary of O that is contained in the connected component that is a neighborhood of N, and let $K_S = \partial O \setminus K_N$. One notes that both K_N and K_S are f-invariant essential continua and that $O \subset U_-(K_N) \cap U_+(K_S)$. The prime ends compactification O^* of O is the disjoint union of O with two topological circles C_S and C_N with an appropiate topology such that there exist neighborhoods V_S, V_N of C_S and C_N respectively in O^*, a neighborhood W_S of the prime ends circle in $\widetilde{U}_+^*(K_S)$ and W_N neighborhood of the prime ends circle in $\widetilde{U}_-^*(K_N)$, such that V_S is homeomorphic to W_S and such that V_N is homeomorphic to W_N. Done in this way, O^* is homeomorphic to $\overline{\mathbb{A}}$, and f has a unique continuous extension f^* that is a homeomorphism of O^*. One can then verify that, if $\widehat{f^*}$ is a lift of f^* to the universal covering, then the rotation number of the restriction of f^* to C_N is the same as $\rho^-(\widehat{f}, K_N)$ and that the rotation number of the restriction of f^* to C_S is the same as $\rho^+(\widehat{f}, K_S)$.

2.9.2. *Realization of rotation vectors in continua.* – We will need the following result, which can be derived from [22] and [23].

PROPOSITION 2.16. – *Let $V \subset \mathbb{A}$ be an essential open annulus, K' be a connected component of ∂V, and K be the union of K' and all the pre-compact connected components of its complement. Let $f : \mathbb{A} \to \mathbb{A}$ be an area-preserving homeomorphism such that $f(V) = V$ and $\widehat{f}$ a lift of f to its universal covering. Then there exists ρ such that every point in K has the same rotation number ρ. Furthermore, ρ is the prime end rotation number of K.*

Proof. – The same proof as Theorem 2.8 of [22] shows that the rotation number of any point in K' is the same, and that it is precisely the prime end rotation number of K'. It remains to show the same is true for any point in K. Suppose, for a contradiction, that there are points with two different rotation numbers, ρ_- and ρ_+ in K. By Proposition 5.4 of [13] one has that for every rational p/q in (ρ_-, ρ_+) there exists a point $z_{p/q}$ in K such that, if $\widehat{z_{p/q}} \in \widehat{\pi}^{-1}(z_{p/q})$, then $\widehat{f}^q(\widehat{z_{p/q}}) = \widehat{z_{p/q}} + (p, 0)$. By Theorem A of [22], since the rotation interval of the restriction of f to K is a non-degenerate closed interval, one deduces that there are two ergodic measures, μ_1 and μ_2, supported in K, such that both $\mathrm{Rot}(\widehat{f}, \mu_1)$ and $\mathrm{Rot}(\widehat{f}, \mu_2)$ are irrational numbers and such that $\mathrm{Rot}(\widehat{f}, \mu_1) \neq \mathrm{Rot}(\widehat{f}, \mu_2)$. This implies that there exist recurrent points z_1 and z_2 in K such that the rotation number of z_1 is $\mathrm{Rot}(\widehat{f}, \mu_1)$ and the rotation number of z_2 is $\mathrm{Rot}(\widehat{f}, \mu_2)$. But if O is a pre-compact connected component of the complement of K', then O is an open topological disk in $\mathbb{A}$, and since f preserves area there exists some integer q_0 such that $f^{q_0}(O) = O$. This implies that every recurrent point of O that has a rotation number must have a rational rotation number. Therefore neither z_1 nor z_2 can lie in pre-compact connected components of the complement of K', and so both points belong to K'. But this is a contradiction, since every point in K' has the same rotation number. □

LEMMA 2.17. – *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism, $\widehat{f}$ be a lift of f and assume that there exist a real number ρ and an f-invariant compact set K such that, for every f-invariant ergodic measure ν supported on K, the rotation number of ν is ρ. Then for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that for all $\widehat{z} \in \widehat{\pi}^{-1}(K)$ and all $n \geq N_0$, $|p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\rho| < n\varepsilon/2$. Furthermore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if y is a point whose whole orbit lies in the δ-neighborhood of K and has rotation number, then $|\mathrm{Rot}(\widehat{f}, y) - \rho| \leq \varepsilon$.*

Proof. – Suppose, by contradiction, that there exist a sequence of points $\widehat{z_k} \in \widehat{\pi}^{-1}(K)$ and an increasing sequence of integers n_k such that $|p_1(\widehat{f}^{n_k}(\widehat{z}_k)) - p_1(\widehat{z}_k) - n_k\rho| \geq n_k\varepsilon/2$. Let $\nu_k = \frac{1}{n_k}\sum_{i=0}^{n_k-1} \delta_{f^i(\widehat{\pi}(\widehat{z}_k))}$, and let $\overline{\nu}$ be an accumulation point of this sequence in the weak-$*$ topology. It follows that $\overline{\nu}$ is an f-invariant measure supported on K and that $|\mathrm{Rot}(\widehat{f}, \overline{\nu}) - \rho| \geq \varepsilon/2$. By the Ergodic Decomposition Theorem, we obtain that there exists an f-invariant ergodic measure ν supported on K whose rotation number is not ρ, a contradiction. For the second assertion, given $\varepsilon > 0$, if $N_0 = N_0(\varepsilon)$, note that the continuous function $g : \mathbb{A} \to \mathbb{R}$, $g(z) = \frac{1}{N_0}\left[p_1(\widehat{f}^{N_0}(\widehat{z})) - p_1(\widehat{z})\right] - \rho$, where $\widehat{z}$ is any point in $\widehat{\pi}^{-1}(z)$, takes values in $(-\varepsilon/2, \varepsilon/2)$ for $z \in K$. Therefore, there exists some $\delta > 0$ such that, for every y in a δ-neighborhood of K, the function g takes values in $[-\varepsilon, \varepsilon]$. This implies that, if the whole orbit of y lies in the δ-neighborhood of K, then

$$\left|\frac{1}{iN_0}\left[p_1(\widehat{f}^{iN_0}(\widehat{y})) - p_1(\widehat{y})\right] - \rho\right| = \left|\frac{1}{i}\sum_{j=0}^{i-1} g(f^{jN_0}(y))\right| \leq \varepsilon$$

where $\widehat{y} \in \pi^{-1}(y)$. Therefore, if y has a well-defined rotation number then $|\mathrm{Rot}(\widehat{f}, y) - \rho| \leq \varepsilon$. □

PROPOSITION 2.18. – *Let $f : \mathbb{A} \to \mathbb{A}$ be an area-preserving homeomorphism, let K' be an f-invariant essential continuum and let K be the union of K' and all the pre-compact connected components of its complement. If the interior of K is inessential, then every point in K has the same rotation number.*

Proof. – Since K' is essential, so is K. By Proposition 2.16, one knows that there exist ρ_1 and ρ_2 such that every point in $\partial U_{-}(K)$ has rotation number ρ_1, and that any point in $\partial U_{+}(K)$ has rotation number ρ_2. If the interior of K is inessential, we have that $\partial U_{-}(K)$ and $\partial U_{+}(K)$ are not disjoint, and so ρ_1 and ρ_2 are the same. Note that $\partial K = \partial U_{-}(K) \cup \partial U_{+}(K)$. Let O be a connected component of the interior of K, and note that again O is an f-periodic open topological disk in $\mathbb{A}$, and there exist integers p_0, q_0 with q_0 positive, such that if $\widehat{O}$ is a connected component of $\widehat{\pi}^{-1}(O)$, then $\widehat{f}^{q_0}(\widehat{O}) = \widehat{O} + (p_0, 0)$. We claim that $\rho_1 = p_0/q_0$. Indeed, if $\widehat{O}$ is bounded, then every recurrent point in the closure of O has the same rotation number and it is p_0/q_0, and since $\partial O \subset \partial K$ we get the claim, and if $\widehat{O}$ is unbounded, then by Lemma 2.15 there exists a point $\bar{z} \in \partial O$ such that $f^{q_0}(\bar{z}) = \bar{z}$ and such that the rotation number of $\bar{z}$ is p_0/q_0. Since every point in ∂O is in ∂K, and every point in ∂K has the same rotation number, we deduce that $\rho_1 = p_0/q_0$. Since O was arbitrary, one deduces that any recurrent point in K has rotation number ρ_1, and therefore any ergodic measure supported in K has rotation number ρ_1. But if a compact invariant set is such that every ergodic measure supported on it has the same rotation number, this implies that every point in the set has a well-defined rotation number and that this number is ρ_1. □

2.10. Regions of instability

Let A be an open topological annulus, denote by S, N the two topological ends of A and let $f : A \to A$ be a homeomorphism preserving both the orientation and the ends of A. There are two classical definitions of regions of instability: we say that A is a *Birkhoff region of instability* if for any U, V, neighborhoods of S and N respectively, there exist $n_1, n_2 > 0$ such that $f^{n_1}(U) \cap V \neq \emptyset$ and such that $f^{n_2}(V) \cap U \neq \emptyset$. We note that, if the dynamics of f is such that every point is non-wandering, then equivalently A is a Birkhoff region of instability if for any U, V, neighborhoods of S and N respectively, the full orbit of U intersects V. This implies that, if the dynamics of f is non-wandering and A is not a Birkhoff region of instability, then there exist O_S, O_N, neighborhoods of S and N respectively, which are f-invariant and disjoint.

We say that A is a *Mather region of instability* if there exist points z_1, z_2 in A such that both $f^{-n}(z_1)$ and $f^n(z_2)$ converge to S when n goes to infinity, and such that both $f^n(z_1)$ and $f^{-n}(z_2)$ converge to N when n goes to infinity. One clearly has that a Mather region of instability is also a Birkhoff region of instability. Let us introduce a new definition, which is stronger than the first one but weaker than the second. We say that A is a *SN mixed region of instability* if, for every neighborhood U of S, one can find points z_1, z_2 in U such that both $f^n(z_1)$ and $f^{-n}(z_2)$ converge to N when n goes to infinity. One defines similarly a NS mixed region of instability if for every neighborhood V of N, one can find points z_1, z_2 in V such that both $f^n(z_1)$ and $f^{-n}(z_2)$ converge to S when n goes to infinity.

A general question whose answer is unknown is to give conditions such that any Birkhoff region of instability must also be a Mather region of instability. This was shown to hold for twists maps, and also generically for area-preserving diffeomorphisms ([37] and [13] respectively). One should note that there are examples of homeomorphisms for which the annulus is a Birkhoff region of instability but not a Mather region of instability. For instance one can find homeomorphisms of the closed annulus whose rotation segment is a single

Liouville point that are weak-mixing ([9]) but also rigid ([6]) and as such every point is recurrent.

Our next proposition shows that, in the area-preserving context and assuming that the rotation set of the homeomorphism is not a singleton, the existence presence of a Birkhoff region of instability, ensures that a stronger form of instability must also hold.

PROPOSITION 2.19. – *Let $f:\overline{\mathbb{A}}\to\overline{\mathbb{A}}$ be a homeomorphism which is isotopic to the identity that preserves a measure of full support such that $\mathbb{A}$ is a Birkhoff region of instability for the restriction of f. Assume further that the rotation set of the restriction of f to $\overline{\mathbb{A}}$ is not a single point. Then there exists some f-invariant open topological annulus $A_2\subset\mathbb{A}$, such that $\overline{A_2}$ meets the upper boundary and contains the lower boundary of $\overline{\mathbb{A}}$. Moreover A_2 is a SN mixed region of instability and, if A_2^* is the prime-ends compactification of A_2, then the rotation set of the restriction of f to A_2^* is the same as that of f.*

(See the left side of Figure 3.)

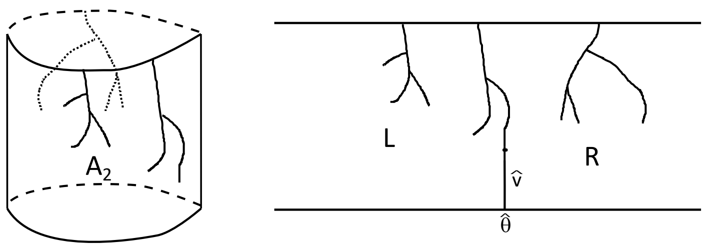

FIGURE 3. Left side: the annulus A_2 in the statement of Proposition 2.19. Right side: The segment $\widehat{v}$ on the proof of Lemma 2.20

Before we begin with the proof of this proposition, let us repeat a construction that dates back to Birkhoff itself, and has more recently been used in the study of homeomorphisms of the annulus and of the 2-dimensional torus in [3, 38, 26]. We refer the reader to Section 4.3 of [26] for more details of the constructions. For the remainder of this section we assume that we are under the hypotheses of Proposition 2.19.

For every $0<\varepsilon<1$, let $B_\varepsilon(f)$ be the connected component of the set

$$\bigcap_{i\in\mathbb{N}} f^{-i}\left(\mathbb{T}^1\times[\varepsilon,1]\right)$$

that contains $\mathbb{T}^1\times\{1\}$, and we omit the dependence on f whenever the context permit. It is a classical result that, since $\mathbb{A}$ is a Birkhoff region of instability, B_ε intersects the circle $\mathbb{T}^1\times\{\varepsilon\}$ and is forward invariant. We also define the set $\omega_\varepsilon(f)$ as the connected component of

$$\bigcap_{i\in\mathbb{Z}} f^{-i}\left(\mathbb{T}^1\times[\varepsilon,1]\right)$$

that contains $\mathbb{T}^1\times\{1\}$, and again we omit the dependence on f whenever the context permit. Let us point remark that $\omega_\varepsilon(f)=\omega_\varepsilon(f^{-1})$. One verifies trivially that ω_ε is closed, invariant, and that the ω-limit set of any point in B_ε is contained is ω_ε. Furthermore, since we are assuming that $\mathbb{A}$ is a Birkhoff region of instability, one has that for each $\varepsilon > 0$, the interior of B_ε does not contain an essential annulus.

One also verifies that if $\varepsilon_1 > \varepsilon_2$, then $\omega_{\varepsilon_1} \subset \omega_{\varepsilon_2}$. On the other hand, if in this situation $\omega_{\varepsilon_2} \subset \mathbb{T}^1 \times [\varepsilon_1, 1]$, then $\omega_{\varepsilon_2} = \omega_{\varepsilon_1}$.

LEMMA 2.20. – *There exists ε_0 such that, if $\varepsilon < \varepsilon_0$, then $\omega_\varepsilon = \omega_{\varepsilon_0}$.*

Proof. – First note that, for every positive integer p, $\omega_\varepsilon(f) \subset \omega_\varepsilon(f^p)$. Let g be a power of f and $\widehat{g}$ be a lift of g such that the $\mathrm{Rot}(\widehat{g})$ contains the interval $[-1, 1]$, and such that the rotation number of the restriction of g to both boundaries do not belong to $\{-1, 0, 1\}$. There exists some ε_0 such that, if $\widehat{z} \in \mathbb{R}\times\{0\}$, then the ball with radius ε_0 and center $\widehat{z}$, $B_{\epsilon_0}(\widehat{z})$ is free for $\widehat{g}$, meaning that $\widehat{g}(B_{\epsilon_0}(\widehat{z})) \cap B_{\epsilon_0}(\widehat{z}) = \emptyset$. Suppose, for a contradiction that there exists some $\varepsilon_1 < \varepsilon_0$ such that $\omega_{\varepsilon_1} = \omega_{\varepsilon_1}(g)$ is not equal to $\omega_{\varepsilon_0} = \omega_{\varepsilon_0}(g)$. If this is true, and if

$$\delta = \min_{s\in[0,1]} \{\exists\theta \in \mathbb{T}^1, (\theta, s) \in \omega_{\varepsilon_1}\},$$

then $\varepsilon_1 \leq \delta \leq \varepsilon_0$, because otherwise one would have that ω_{ε_1} is g-invariant and contained in $\bigcap_{i\in\mathbb{Z}} g^{-i}\left(\mathbb{T}^1 \times [\varepsilon_0, 1]\right)$, and therefore must be equal to ω_{ε_0}. There exists some $\theta \in \mathbb{T}^1 \times \{0\}$ such that $(\theta, \delta) \in \omega_{\varepsilon_1}$. Let $v = \theta \times [0, \delta)$ be a line segment. Note that v is disjoint from ω_{ε_1}.

Let $\widehat{v}$ be the connected component of $\widehat{\pi}^{-1}(v)$ that contains the point $(\widehat{\theta}, 0)$ with $0 \leq \widehat{\theta} < 1$, let $\widehat{\omega_{\varepsilon_1}} = \widehat{\pi}^{-1}(\omega_{\varepsilon_1})$ and let $F = \widehat{v} \cup \widehat{\omega_{\varepsilon_1}}$. First note that the complement of $\widehat{\omega_{\varepsilon_1}}$ has a connected component A that contains the strip $\mathbb{R} \times [0, \delta)$. We claim that A is the unique connected component of the complement of $\widehat{\omega_{\varepsilon_1}}$. Indeed, the complement of A is invariant and contained in $\mathbb{R} \times [\varepsilon_1, 1]$. Furthermore, since any connected component of $\widehat{\pi}(A)^C$ has a point of ω_{ε_1} and the later is connected, we have that $\widehat{\pi}(A)^C$ is connected, and therefore it is contained in ω_{ε_1}.

The complement of F can have at most two connected components, L, which contains $(-\infty, \widehat{\theta}) \times \{0\}$, and R, which contains $(\widehat{\theta}, \infty) \times \{0\}$. Let us show that these are different connected components. (See Figure 3 Right). If not, there would be an arc γ joining $(-1, 0)$ and $(1, 0)$ entirely contained in F^C. If $\beta = \gamma \cup ([-1, 1] \times \{0\})$, then β is the image of a simple closed curve, which is disjoint from $\widehat{\omega_{\varepsilon_1}}$ but such that the point $(\widehat{\theta}, \delta)$ is in a different connected component from the complement of β than $\mathbb{R} \times \{1\}$. This contradicts the fact that ω_{ε_1} is connected and also contains $\mathbb{T}^1 \times \{1\}$.

Note also that, if $\widehat{z}$ is a point in the complement of $\widehat{\omega_{\varepsilon_1}}$, one can find an arc α in $(\widehat{\omega_{\varepsilon_1}})^C$ joining $\hat{z}$ to a point $(a, 0)$. Since $\widehat{\omega_{\varepsilon_1}}$ is invariant by integer horizontal translations, one gets that if $p > |a| + 1$ is sufficiently large, $\alpha + (p, 0)$ and $\alpha - (p, 0)$ are both disjoint from $\widehat{v}$ and $\widehat{\omega_{\varepsilon_1}}$. One checks that $\alpha + (p, 0) \subset R$, since it contains $(a + p, 0)$ and that $\alpha - (p, 0) \subset L$, since it contains $(a - p, 0)$. We get that, for each $\widehat{z} \notin \widehat{\omega_{\varepsilon_1}}$, there exists a sufficiently large p such that $\widehat{z} + (p, 0)$ is in R and $\widehat{z} - (p, 0)$ is in L.

There are two cases to consider. Either the rotation number of $\mathbb{T}^1 \times \{0\}$ for $\widehat{g}$ is positive or it is negative. The rest of the proof is similar in both situations, so we will assume that it is positive. This implies that $\widehat{g}((\widehat{\theta}, 0))$ belongs to R and $\widehat{g}^{-1}((\widehat{\theta}, 0))$ belongs to L. Note that $\widehat{v}$ is contained in a ball of radius ε_0 centered at $(\theta, 0)$ and therefore $\widehat{v}$ is free. Since $\widehat{v}$ is also disjoint from $\widehat{\omega_{\varepsilon_1}}$, we get that the image of $\widehat{v}$ by $\widehat{g}$ is contained in R, and that its preimage is contained in L. This implies that $\widehat{g}(R)$ does not intersect $\widehat{v}$. Since $\widehat{g}(R)$ is also disjoint from $\widehat{\omega_{\varepsilon_1}}$ and is connected, one has that it must be contained in a connected component of the complement of F, and as the image by $\widehat{g}$ of $(\widehat{\theta}, \infty) \times \{0\}$ intersects itself, we deduce that $\widehat{g}(R) \subset R$.

Finally, since g preserves a measure of full support, by the hypothesis on the rotation set of $\widehat{g}$, one can find some $\bar{z}$ in $\overline{\mathbb{A}}$ such that, if $\widehat{\bar{z}}$ is a lift of $\bar{z}$, then by Franks' Theorem (Theorem 2.9) $\widehat{g}(\widehat{\bar{z}}) = \widehat{\bar{z}} - (1,0)$. Let $A' = \mathbb{T}^1 \times [0,2]$ where we view $\overline{\mathbb{A}}$ as a subset of A', let $T : A' \to A'$, $T(x,y) = (x, 2-y)$ and consider the extension $g' : A' \to A'$ of g obtained by requiring that g' commutes with T. Note that g' is also area-preserving and we can apply Proposition 2.18 to obtain that every point of ω_{ε_1} has the same rotation number for any lift $\widehat{g'}$ of g'. This implies that every point in ω_{ε_1} has the same rotation number for $\widehat{g}$, and since $\mathbb{T}^1 \times \{1\}$ belongs to ω_{ε_1}, this number is not -1. This implies that $\bar{z}$ does not belong to ω_{ε_1}, and therefore there exists some integer p such that $\widehat{\bar{z}} + (p,0)$ belongs to R, and such that $\widehat{g}^{2p}(\widehat{\bar{z}} + (p,0)) = \widehat{\bar{z}} - (p,0)$ belongs to L, which is a contradiction since R is positively invariant. This shows that $\omega_\varepsilon(g) = \omega_{\varepsilon_0}(g)$ for all $\varepsilon < \varepsilon_0$.

Now, if $\varepsilon < \varepsilon_0$, since $\omega_\varepsilon(f) \subset \omega_\varepsilon(g)$, we get that $\omega_\varepsilon(f) \subset \omega_{\varepsilon_0}(g) \subset \mathbb{T}^1 \times [\varepsilon_0, 1]$, and since $\omega_\varepsilon(f)$ is f-invariant, connected and contains $\mathbb{T}^1 \times \{1\}$, we deduce that it is contained in $\omega_{\varepsilon_0}(f)$. Since it holds that $\omega_{\varepsilon_0}(f) \subset \omega_\varepsilon(f)$, we have the result. □

End of the proof of Proposition 2.19. – Let A_2 be the complement of $\omega_{\varepsilon_0} \cup (\mathbb{T}^1 \times \{0\})$. Note that A_2 is open, contains $\mathbb{T}^1 \times (0, \varepsilon_0)$ and therefore separates $\mathbb{T}^1 \times \{0\}$ and $\mathbb{T}^1 \times \{1\}$, and its complement has exactly two connected components. Therefore A_2 is an essential open topological annulus. Let S be the end of A_2 corresponding to $\mathbb{T}^1 \times \{0\}$, and let N be the other end.

If U is a neighborhood in A_2 of S, then there exists $\varepsilon > 0$ such that $\mathbb{T}^1 \times (0, 2\varepsilon)$ is contained in U. As noted before, $B_\varepsilon(f)$ has a point in z_1 in $\mathbb{T}^1 \times \{\varepsilon\} \subset U$ and the ω-limit set of any point in $B_\varepsilon(f)$ is contained in $\omega_\varepsilon = \omega_{\varepsilon_0}$, one has that the future orbit of z_1 in U converges to N. Likewise, one knows that $\omega_\varepsilon(f^{-1}) = \omega_\varepsilon(f) = \omega_{\varepsilon_0}(f)$, and since $B_\varepsilon(f^{-1})$ has a point z_2 in U whose ω-limit set for f^{-1}, and therefore whose α-limit set for f, is contained in $\omega_{\varepsilon_0}(f)$. This shows that A_2 is a SN mixed region of instability.

Finally, since the rotation set of f is not a single point, it has non-empty interior and since f preserves a measure of full support, one knows that for every p/q in the rotation set of f there exists a periodic point $z_{p/q}$ such that the rotation number of $z_{p/q}$ is p/q. Since every point in ω_{ε_0} has the same rotation number which is equal to the rotation number of the restriction of f to $\mathbb{T}^1 \times \{1\}$, and since every point in $\mathbb{T}^1 \times \{0\}$ has the same rotation number, we get that for all but possibly two values of p/q in the rotation set of f, $z_{p/q}$ must belong to A_2. Since the rotation set of the restriction of f to A_2^* must be closed, we deduce it must be equal to the full rotation set of f. □

3. Proof of Theorem A

In this section, we prove Theorem A. Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity and preserves a Borel probability measure μ with full support. Let $\widehat{f}$ be a lift of f to $\mathbb{R} \times [0,1]$. Replacing f by a power f^q and the lift $\widehat{f}$ by a lift $\widehat{f}^q + (p,0)$, one can suppose that $\mathrm{Rot}(\widehat{f}) = [\alpha, \beta]$ with $\alpha < 0 < 1 < \beta$. We can also assume that the rotation numbers of both boundary components of $\overline{\mathbb{A}}$ are not equal to 1.

Suppose by contradiction that $\mathrm{Rot}(\widehat{f},\mu)=\alpha$. We will fix a maximal identity isotopy I of f that lifts to a maximal identity isotopy of $\widehat{f}$, and we also fix an oriented singular foliation $\mathcal{F}$ transverse to I and its lift $\widehat{\mathcal{F}}$.

3.1. Construction of a set with good recurrent properties

PROPOSITION 3.1. – *Let $(Y,\mathcal{B},\nu)$ be a probability space, and let $T: Y\to Y$ be an ergodic automorphism. If $\varphi: Y\to\mathbb{R}$ is an integrable map such that $\int_Y \varphi\, d\nu = 0$, then for every $B\in\mathcal{B}$ and every real number $\epsilon>0$, one has*

$$\nu\left(\left\{y\in B,\ \exists n\geq 0,\ T^n(y)\in B \text{ and } \left|\sum_{k=0}^{n-1}\varphi(T^k(y))\right|<\epsilon\right\}\right)=\nu(B).$$

As a corollary we have the following result (see Corollary 4.6 of [26]).

COROLLARY 3.2. – *Let ν be an ergodic invariant measure for f and $\varphi:\overline{\mathbb{A}}\to\mathbb{R}$ be an integrable map such that $\int_{\overline{\mathbb{A}}}\varphi\, d\nu=0$. Then for ν-almost every point $z\in\overline{\mathbb{A}}$, there exists an increasing sequence $q_l\to+\infty$ such that $f^{q_l}(z)\to z$ and $\sum_{j=0}^{q_l-1}\varphi(f^j(z))\to 0$.*

We also have the following result which can be derived directly from Proposition 4.3 of [26].

LEMMA 3.3. – *For every Borel subset $B\subset\overline{\mathbb{A}}$ such that $\mu(B)>0$, there exists an f-invariant ergodic measure ν such that* $\mathrm{Rot}(\widehat{f},\nu)=\alpha$ *and such that $\nu(B)>0$.*

Finally, we deduce the following proposition.

PROPOSITION 3.4. – *There exists a set X_α in $\overline{\mathbb{A}}$ with full μ measure such that, for every $z\in X_\alpha$ we have*

(i) *z is a bi-recurrent point of f;*

(ii) *the rotation number of z is well-defined and* $\mathrm{Rot}(\widehat{f},z)=\alpha$*; and*

(iii) *one can find a sequence $(p_l,q_l)_{l\in\mathbb{N}}$ in $(-\mathbb{N})\times\mathbb{N}$ such that, if $\widehat{z}$ belongs to $\widehat{\pi}^{-1}(z)$:*

$$\lim_{l\to+\infty} q_l=+\infty,\quad \lim_{l\to+\infty}(p_l-\alpha q_l)=0,\quad \lim_{l\to+\infty}\widehat{f}^{q_l}(\widehat{z})-\widehat{z}-(p_l,0)=0$$

and a sequence $(p'_l,q'_l)_{l\in\mathbb{N}}$ in $\mathbb{N}\times\mathbb{N}$ satisfying:

$$\lim_{l\to+\infty} q'_l=+\infty,\quad \lim_{l\to+\infty}(p'_l+\alpha q'_l)=0,\quad \lim_{l\to+\infty}\widehat{f}^{-q'_l}(\widehat{z})-\widehat{z}+(p'_l,0)=0.$$

Proof. – Let B be the complement of the set of points that satisfy properties (i) to (iii), and assume for a contradiction that $\mu(B)>0$. By Lemma 3.3, we can find an f-invariant ergodic measure ν such that $\mathrm{Rot}(\widehat{f},\nu)=\alpha$ and such that $\nu(B)>0$. Since ν is ergodic, ν-almost every point in B is bi-recurrent and has rotation number equal to α. Applying Corollary 3.2 using $\varphi(z)=p_1(\widehat{f}(\widehat{z})-\widehat{z})-\alpha$, one has that there exists a sequence of integers $q_l\to+\infty$ such that $f^{q_l}(z)\to z$ and $\sum_{j=0}^{q_l-1}\varphi(f^j(z))=p_1(\widehat{f}^{q_l}(\widehat{z})-\widehat{z})-q_i\alpha\to 0$. Since $f^{q_l}(z)\to z$ one deduces that there exists a sequence of integers p_l such that $\widehat{f}^{q_l}(\widehat{z})-\widehat{z}-(p_l,0)\to(0,0)$ and so $p_l-q_l\alpha\to 0$. So ν-almost every point in B satisfies the first part of property (iii). A similar argument, using Corollary 3.2 with f^{-1} in place of f and $\varphi'(z)=p_1(\widehat{f}^{-1}(\widehat{z})-\widehat{z})+\alpha$, gives us that ν-almost every point in B satisfies the second part of property (iii). Therefore ν-almost every point of B satisfies properties (i)–(iii), a contradiction. □

We have the following result, whose proof is immediate.

LEMMA 3.5. – *Let $\alpha < 0$ and r be a real number and let $(p_l, q_l)_{l\in\mathbb{N}}$ be a sequence in $(-\mathbb{N}) \times \mathbb{N}$ such that:*

$$\lim_{l\to+\infty} q_l = +\infty, \quad \lim_{l\to+\infty} (p_l - \alpha q_l) = 0.$$

If $k > -\alpha r$, then there exists some integer $l(k) \in \mathbb{N}$ such that for every integer $l \geq l(k)$, we have

$$\frac{p_l - k}{q_l + r} < \alpha.$$

3.2. Construction of a good admissible transverse path

Note that $\mathrm{Rot}(\widehat{f}) = [\alpha, \beta]$ and $\alpha < 0 < 1 < \beta$. We recall that we are assuming that f preserves a Borel probability measure of full support, so by Franks' Theorem (Theorem 2.9) one can find a fixed point z of f in the interior of the annulus such that, if $\widehat{z}$ is a lift of z, then $\widehat{f}(\widehat{z}) = \widehat{z} + (1, 0)$. Let $I^{\mathbb{Z}}_{\mathcal{F}}(z)$ be the whole transverse trajectory of $z = \widehat{\pi}(\widehat{z})$. We start by recalling some facts about the transverse path $I^{\mathbb{Z}}_{\mathcal{F}}(z)$. By definition, we have $I^{\mathbb{Z}}_{\mathcal{F}}(z)(0) = I^{\mathbb{Z}}_{\mathcal{F}}(z)(1)$. Let $\gamma' : \mathbb{R} \to \mathrm{dom}(I)$ be such that $\gamma' : [0,1] \to \mathrm{dom}(I)$ is in $I^{\mathbb{Z}}_{\mathcal{F}}(z)|_{[0,1]}$ and such that $\gamma'(t+1) = \gamma'(t)$, for all t. Then γ' is the natural lift of a transverse loop Γ'. We know that every leaf that meets Γ' is wandering. Consequently, if t and t' are sufficiently close, one has $\phi_{\Gamma'(t)} \neq \phi_{\Gamma'(t')}$. Moreover, because Γ' is positively transverse to $\mathcal{F}$, one cannot find an increasing sequence $(a_n)_{n\in\mathbb{N}}$ and a decreasing sequence $(b_n)_{n\in\mathbb{N}}$, such that $\phi_{\Gamma'(a_n)} = \phi_{\Gamma'(b_n)}$. So, there exist real numbers a, b with $0 \leq a < b \leq 1$ such that $t \mapsto \phi_{I^{\mathbb{Z}}_{\mathcal{F}}(z)(t)}$ is injective on $[a, b)$ and satisfies $\phi_{I^{\mathbb{Z}}_{\mathcal{F}}(z)(a)} = \phi_{I^{\mathbb{Z}}_{\mathcal{F}}(z)(b)}$. Replacing $I^{\mathbb{Z}}_{\mathcal{F}}(z)$ by an equivalent transverse path, one can suppose that $I^{\mathbb{Z}}_{\mathcal{F}}(z)(a) = I^{\mathbb{Z}}_{\mathcal{F}}(z)(b)$. Let Γ be the loop naturally defined by the closed path $I^{\mathbb{Z}}_{\mathcal{F}}(z)|_{[a,b]}$. The set $U_\Gamma = \bigcup_{t\in[a,b]} \phi_{I^{\mathbb{Z}}_{\mathcal{F}}(z)(t)}$ is an open annulus and Γ is a simple loop. Note that we can collapse the two boundary components of $\overline{\mathbb{A}}$ to two points S, N to obtain a sphere, and that there exist natural extensions of f and I to this sphere, as well as a new transversal foliation that has singularities in S and N, which for simplicity we denote by f^*, I^* and $\mathcal{F}^*$. As z is a periodic point we have the following result. This lemma is contained in the proof of Proposition 2 from [32].

LEMMA 3.6 ([32]). – *Suppose that there exists $t < a$ such that $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)(t) \notin U_\Gamma$. Then there exists $t' \in \mathbb{R}$ with $b < t'$ such that $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)(t)$ and $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)(t')$ are in the same connected component of the complement of U_Γ. Moreover $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)|_{[t,t']}$ has a $\mathcal{F}^*$-transverse self-intersection.*

Proof. – See the proof of Proposition 2 from [32]. □

Therefore, as z is a periodic point, there are two possibilities for the whole transverse trajectory of z, $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)$, namely:

I) $I^{*1}_{\mathcal{F}^*}(z)$ has a $\mathcal{F}$-transverse self-intersection; or

II) $I^{*\mathbb{Z}}_{\mathcal{F}^*}(z)$ is equivalent to the natural lift of a simple loop Γ of $\mathbb{A}$.

We will analyze each case separately.

3.2.1. *Case I.* – Since for this induced map $\Gamma'^{*} = I^{*1}_{\mathcal{F}^*}(z)$ is a loop homologous to zero with a $\mathcal{F}^*$-transverse self-intersection, one can apply Proposition 7 of [32] and deduce that $I^{*2}_{\mathcal{F}^*}(z)$, which is admissible of order 2, has a $\mathcal{F}^*$-transverse self-intersection. It follows that $I^2_{\mathcal{F}}(z)$ has a $\mathcal{F}$-transverse self-intersection and that, if $\widehat{z}$ is a lift of z, one can find an integer p_0 such that the transverse path $\widehat{\gamma} := I^2_{\widehat{\mathcal{F}}}(\widehat{z})$ has a $\widehat{\mathcal{F}}$-transverse intersection with $\widehat{\gamma} + (p_0, 0)$. Lemma 2.3 provides us with a sufficiently small neighborhood W of $\widehat{z}$ such that, for every $\widehat{y} \in W$, one has that

(a) $I^4_{\widehat{\mathcal{F}}}(\widehat{f}^{-3}(\widehat{y}))$ contains a subpath equivalent to $I^2_{\widehat{\mathcal{F}}}(\widehat{f}^{-2}(\widehat{z})) = \widehat{\gamma} - (2, 0)$, and

(b) $I^4_{\widehat{\mathcal{F}}}(\widehat{f}^{-1}(\widehat{y}))$ contains a subpath equivalent to $\widehat{\gamma}$.

Let $k > 0$ be an integer such that $k - p_0 > -2\alpha$. Let $W_k \subset W$ be a neighborhood of $\widehat{z}$ such that $\widehat{f}^k(W_k) \subset W + (k, 0)$, and let $\widehat{z_*}$ be a point in $W_k \cap \widehat{\pi}^{-1}(X_\alpha)$, where X_α is the set provided by Proposition 3.4. By Proposition 3.4 and Lemma 3.5, one finds integers p, q, with $q > k + 8$ sufficiently large, such that

— $\widehat{f}^q(\widehat{z_*})$ belongs to $W + (p, 0)$;

— $\frac{p+p_0-k}{q+1} < \alpha$.

Now, if $\widehat{y_*} = \widehat{f}^{k-1}(\widehat{z_*})$, then $\widehat{f}(\widehat{y_*}) \in W + (k, 0)$, which implies that $I^4_{\widehat{\mathcal{F}}}(\widehat{y_*})$ has $\gamma_1 = \widehat{\gamma} + (k, 0)$ as a subpath. Furthermore, as $\widehat{f}^q(\widehat{z_*})$ belongs to $W + (p, 0)$, the transverse path $I^4_{\widehat{\mathcal{F}}}(\widehat{f}^{q-3}(\widehat{z_*}))$ has $\gamma_2 = \widehat{\gamma} + (p-2, 0)$ as a subpath. Note that the path $\gamma^* = I^{q+1}_{\widehat{\mathcal{F}}}(\widehat{z_*})$ is admissible of order $q + 1$, and γ_2 has a $\widehat{\mathcal{F}}$-transverse intersection with $\gamma_2 + (p_0, 0) = \gamma_1 + (p + p_0 - k - 2, 0)$. Therefore the path $\gamma^*|_{[q-3,q+1]} = I^4_{\widehat{\mathcal{F}}}(\widehat{f}^{q-3}(\widehat{z_*}))$ has a $\widehat{\mathcal{F}}$-transverse intersection with $(\gamma^* + (p + p_0 - k - 2, 0))|_{[k-1,k+3]} = I^4_{\widehat{\mathcal{F}}}(\widehat{y_*}) + (p + p_0 - k - 2, 0)$. By Proposition 2.7, $\frac{p+p_0-k-2}{q+1}$ is in the rotation set of $\widehat{f}$. As $\frac{p+p_0-k-2}{q+1} < \alpha$ we have a contradiction. Therefore Case I cannot happen.

3.2.2. *Case II.* – Assume now that $I^{\mathbb{Z}}_{\mathcal{F}}(z)$ is equivalent to the natural lift of a simple loop Γ in $\mathbb{A}$. Let us consider $\gamma : \mathbb{R} \to \mathbb{A}$ the natural lift of Γ such that $\gamma(t+1) = \gamma(t)$ for every $t \in \mathbb{R}$ and $\widehat{\gamma}$ the lift of γ to $\widehat{\mathbb{A}}$. One gets that the set of leaves intersecting Γ is an open topological sub-annulus U_Γ of $\mathbb{A}$, and since $\widehat{f}(\widehat{z}) = \widehat{z} + (1, 0)$, one gets that $\widehat{U_\Gamma} = \widehat{\pi}^{-1}(U_\Gamma)$ has a single connected component, and that U_Γ is an essential annulus.

Proposition 3.7. – *There exist an admissible transverse path γ^* and real numbers $a < a' < b' < b$ such that, if $\widehat{\gamma^*}$ is a lift of γ^* to $\widehat{\mathbb{A}}$, then*

— *$\widehat{\gamma^*}|_{[a',b']}$ is equivalent to $\widehat{\gamma}|_{[s,s+1]}$ for some $s \in \mathbb{R}$ and $\gamma^*(a') = \gamma^*(b')$;*

— *$\widehat{\gamma^*}|_{(a,a')}$ is included in $\widehat{U_\Gamma}$ but it does not meet $\phi_{\widehat{\gamma}(s)} - (1, 0)$ and $\widehat{\gamma^*}|_{(b',b)}$ is included in $\widehat{U_\Gamma}$ but it does not meet $\phi_{\widehat{\gamma}(s+1)} + (1, 0)$; and*

— *$\widehat{\gamma^*}(a)$ and $\widehat{\gamma^*}(b)$ belong to the same connected component of the complement to $\widehat{U_\Gamma}$.*

(See Figure 4.)

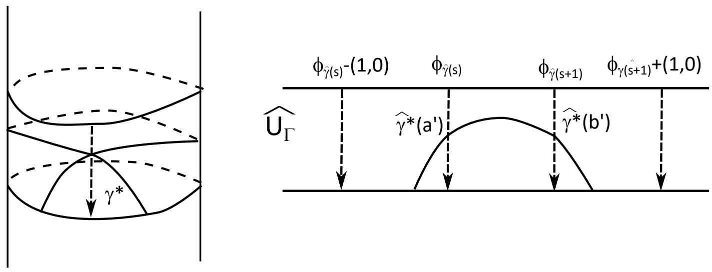

FIGURE 4. Left side: The transverse path γ^*. Right side: A lift of γ^*

Proof of Proposition 3.7. – By density of the set X_α, provided by Proposition 3.4, and Lemma 2.3, we can suppose that $\widehat{\gamma}|_{[0,2]}$ is equivalent to a subpath of $I^{\mathbb{Z}}_{\widehat{\mathcal{F}}}(\widehat{z_0})$, the whole transverse trajectory of a point $\widehat{z_0}$ that lifts a point z_0 in X_α. We will denote $\widehat{\gamma_0} = I^{\mathbb{Z}}_{\widehat{\mathcal{F}}}(\widehat{z_0})$. Since the point z_0 has a negative rotation number, the transverse path $\widehat{\gamma_0}$ cannot be contained in $\widehat{U_\Gamma}$. Hence one can find real numbers

$$t_1 < t'_1 < t'_2 < t_2$$

and integers j^- and j^+ uniquely determined such that

— $\widehat{\gamma_0}|_{[t'_1,t'_2]}$ is equivalent to $\widehat{\gamma}|_{[j^-,j^+]}$;

— $\widehat{\gamma_0}|_{(t_1,t'_1)}$ and $\widehat{\gamma_0}|_{(t'_2,t_2)}$ are included in $\widehat{U_\Gamma}$ but do not meet $\phi_{\widehat{\gamma}(0)} + (j^- - 1, 0)$ and $\phi_{\widehat{\gamma}(0)} + (j^+ + 1, 0)$ respectively;

— $\widehat{\gamma_0}(t_1)$ and $\widehat{\gamma_0}(t_2)$ do not belong to $\widehat{U_\Gamma}$.

We claim that, if $\widehat{\gamma_0}(t_1)$ and $\widehat{\gamma_0}(t_2)$ belong to the same connected component of the complement of $\widehat{U_\Gamma}$ then, we are done. Indeed, if we have that $j^+ - j^- \geq 2$, we know that $\widehat{\gamma_0}|_{[t_1,t'_2]} + (j^+ - 1 - j^-, 0)$ and $\widehat{\gamma_0}|_{[t'_1,t_2]}$ intersect $\widehat{\mathcal{F}}$-transversally at

$$\widehat{\gamma_0}(t) + (j^+ - 1 - j^-, 0) = \widehat{\gamma_0}(t'_2).$$

By Proposition 2.4 the transverse path

$$\widehat{\gamma} := (\widehat{\gamma_0} + (j^+ - 1 - j^-, 0))|_{[t_1,t'_1]}\widehat{\gamma_0}|_{[t'_2,t_2]}$$

is a subpath of an admissible transverse path $\widehat{\gamma^*}$ which satisfies the proposition. In what follows we assume that

— $\widehat{\gamma_0}(t_1)$ and $\widehat{\gamma_0}(t_2)$ belong to different connected components of the complement of $\widehat{U_\Gamma}$.

Since the point z_0 is bi-recurrent and the complement of the annulus $\widehat{U_\Gamma}$ saturated, that is, it is the union of singular point and leaves of $\widehat{\mathcal{F}}$, one can find real numbers

$$t_2 \leq t_3 < t_4$$

such that

— $\widehat{\gamma_0}(t_4)$ belongs to the same connected component of the complement of $\widehat{U_\Gamma}$ with $\widehat{\gamma_0}(t_1)$;

— $\widehat{\gamma_0}|_{[t_2,t_4)}$ does not meet this connected component of the complement of $\widehat{U_\Gamma}$;

— $\widehat{\gamma_0}|_{(t_3,t_4)}$ is included in $\widehat{U_\Gamma}$; and

— $\widehat{\gamma_0}(t_3)$ belongs to the same connected component of the complement of $\widehat{U_\Gamma}$ with $\widehat{\gamma_0}(t_2)$.

Observe now that, by Lemma 3.6, there exists a non-zero integer j, such that $\widehat{\gamma}|_{[t_1,t_2]}$ and $\widehat{\gamma}|_{[t_3,t_4]} + (j, 0)$ intersect $\widehat{\mathcal{F}}$-transversally at $\widehat{\gamma}(s) = \widehat{\gamma}(t)$ with $t_1 < s < t_2 < t_3 < t < t_4$. Hence by Proposition 2.4, one has that the path

$$\widehat{\gamma_0''} := \widehat{\gamma}|_{[t_1,s]}(\widehat{\gamma}|_{[t,t_4]} + (j, 0)),$$

is a subpath of an admissible transverse path. We can construct an admissible transverse path $\widehat{\gamma^*}$ as above. □

Let us consider a lift $\widehat{\gamma^*} : [a, b] \to \widehat{\widehat{\mathbb{A}}}$ of γ^*, provided by Proposition 3.7, to $\widehat{\widehat{\mathbb{A}}}$ and let $\widehat{\gamma}$ be a lift of the natural lift of Γ such that $\widehat{\gamma}(0) = \widehat{\gamma}^*(a')$ and $\widehat{\gamma}(1) = \widehat{\gamma}^*(b')$. We suppose that $\widehat{\gamma^*}$ is admissible of order $N \geq 1$. We will denote by $\widehat{\phi_0}$ the leaf $\phi_{\widehat{\gamma}(0)} = \phi_{\widehat{\gamma^*}(a')}$ of $\widehat{\mathcal{F}}$. Let $\widehat{U_\Gamma}$ be a lift of U_Γ which is homeomorphic to $\mathbb{R}^2$. We have the following corollary to Proposition 3.7.

Corollary 3.8. – *For every integer $k \geq 1$, there exist an admissible transverse path of order kN $\widehat{\gamma_k} : [a, b] \to \widehat{\widehat{\mathbb{A}}}$, real numbers $a < a'_k < b'_k < b$ such that*

— *$\widehat{\gamma_k}|_{[a'_k,b'_k]}$ is equivalent to $\widehat{\gamma}|_{[0,k]}$;*

— *$\widehat{\gamma_k}|_{(a,a'_k)}$ is included in $\widehat{U_\Gamma}$ but it does not meet $\widehat{\phi_0} - (1, 0)$ and $\widehat{\gamma_k}|_{(b'_k,b)}$ is included in $\widehat{U_\Gamma}$ but it does not meet $\widehat{\phi_0} + (k + 1, 0)$;*

— *$\widehat{\gamma_k}(a)$ and $\widehat{\gamma_k}(b)$ belong to the same connected component of the complement to $\widehat{U_\Gamma}$.*

Proof. – Note that the paths $\widehat{\gamma^*}|_{[a',b]}$ and $(\widehat{\gamma^*} + (1, 0))|_{[a,b']}$ intersect $\widehat{\mathcal{F}}$-transversally at $\widehat{\gamma^*}(b') = \widehat{\gamma^*}(a') + (1, 0)$. It follows that for every integer $i \in \{1, \ldots, k - 1\}$ the paths $(\widehat{\gamma^*} + (i, 0))|_{[a',b]}$ and $(\widehat{\gamma^*} + (i + 1, 0))|_{[a,b']}$ intersect $\widehat{\mathcal{F}}$-transversally at $\widehat{\gamma^*}(b') + (i, 0) = \widehat{\gamma^*}(a') + (i + 1, 0)$. Applying Lemma 2.5, one knows that the path

$$\widehat{\gamma_k} := \widehat{\gamma^*}|_{[a,b']} \left(\prod_{i=1}^{k-2} (\widehat{\gamma^*} + (i, 0))|_{[a',b']} \right) (\widehat{\gamma^*}|_{[a',b]} + (k - 1, 0))$$

is admissible of order kN. This completes the proof of the corollary. □

For every integer k large enough consider real numbers a''_k and b''_k with $a'_k < a''_k < b''_k < b'_k$ such that $\phi_{\widehat{\gamma_k}(a''_k)} = \widehat{\phi_0} + (1, 0)$ and $\phi_{\widehat{\gamma_k}(b''_k)} = \widehat{\phi_0} + (k - 1, 0)$. Let $\widehat{z_k}$ be a point in $\mathrm{dom}(\widehat{I})$ such that $\widehat{\gamma_k}$ is a subpath of the path $I^{kN}_{\widehat{\mathcal{F}}}(\widehat{z_k})$, and consider the smallest integer i_k and the largest integer N_k in $\{1, \ldots, kN\}$ such that

— $\widehat{\gamma_k}|_{[a,a''_k]}$ is a subpath of $I^{i_k}_{\widehat{\mathcal{F}}}(\widehat{z_k})$;

— $\widehat{\gamma_k}|_{[b''_k,b]}$ is a subpath of $I^{kN-i_k-N_k}_{\widehat{\mathcal{F}}}(\widehat{f}^{i_k+N_k}(\widehat{z_k}))$.

Let put $\widehat{y_k} = \widehat{f}^{i_k}(\widehat{z_k})$ and $\widehat{f}^{N_k}(\widehat{y_k}) = \widehat{f}^{i_k+N_k}(\widehat{z_k})$. We have the following result.

Corollary 3.9. – *For every integer k large enough the paths $I^{i_k}_{\widehat{\mathcal{F}}}(\widehat{f}^{-i_k}(\widehat{y_k}))$ and $I^{kN-N_k}_{\widehat{\mathcal{F}}}(\widehat{f}^{N_k}(\widehat{y_k})) - (k, 0)$ intersect $\widehat{\mathcal{F}}$-transversally.*

By density of the set X_α given by Proposition 3.4 and Lemma 2.3, considering a point close to $\widehat{y_k}$, we can suppose that $\widehat{y_k}$ belongs to X_α. We have the following consequence, which results of Proposition 3.4.

LEMMA 3.10. – *For every positive real number L, there exist two integers $p \in -\mathbb{N}$ and $q \in \mathbb{N}$ satisfying $q - i_k > L$ such that:*

— $\widehat{\gamma_k}|_{[a,a''_k]}$ *is a subpath of* $I^{i_k+2}_{\widehat{\mathcal{F}}}(\widehat{f}^{q-i_k-1}(\widehat{y_k})) - (p,0)$.

Proof. – By Proposition 3.4, one can find an integer l sufficiently large such that $q_l > L + i_k$ and such that $\widehat{f}^{q_l}(\widehat{y_k}) - (p_l,0)$ is close to $\widehat{y_k}$. Lemma 2.3 permits us to conclude that the path $I^{i_k}_{\widehat{\mathcal{F}}}(\widehat{f}^{-i_k}(\widehat{y_k}))$ is a subpath of $I^{i_k+2}_{\widehat{\mathcal{F}}}(\widehat{f}^{q_l-i_k-1}(\widehat{y_k})) - (p_l,0)$. Hence choosing $q = q_l$ and $p = p_l$ we have that $\widehat{\gamma_k}|_{[a,a''_k]}$ is a subpath of $I^{i_k+2}_{\widehat{\mathcal{F}}}(\widehat{f}^{q-i_k-1}(\widehat{y_k})) - (p,0)$. This completes the proof of the lemma. □

Let $\widehat{\gamma^{**}}$ be the transverse path $I^{q+1-N_k}_{\widehat{\mathcal{F}}}(\widehat{f}^{N_k}(\widehat{y_k}))$. We deduce the next result.

COROLLARY 3.11. – *The paths $\widehat{\gamma^{**}}$ and $\widehat{\gamma^{**}} + (p-k,0)$ intersect $\widehat{\mathcal{F}}$-transversally at some leaf $\phi_{\widehat{\gamma^{**}}(t)} = \phi_{\left(\widehat{\gamma^{**}}+(p-k,0)\right)(s)}$ where $s < t$.*

Proof. – One knows that the path $\widehat{\gamma_k}|_{[a,a''_k]} + (p,0)$ is a subpath of $I^{i_k+2}_{\widehat{\mathcal{F}}}(\widehat{f}^{q-1-i_k}(\widehat{y_k})) = \widehat{\gamma^{**}}|_{[q-1-i_k-N_k,q+1-N_k]}$ and $\widehat{\gamma_k}|_{[b''_k,b]} + (p-k,0)$ is a subpath of $\widehat{\gamma^{**}}|_{[0,kN-N_k]} + (p-k,0)$, which implies that $\widehat{\gamma^{**}}|_{[q-1-i_k-N_k,q+1-N_k]}$ and $\widehat{\gamma^{**}}|_{[0,kN-N_k]}+(p-k,0)$ have a $\widehat{\mathcal{F}}$-transverse intersection. The result follows since we took q sufficiently large so that $q-1-i_k-N_k$ is larger than $kN - N_k$. □

As a consequence of Proposition 2.7 we deduce the following result.

COROLLARY 3.12. – *We have that $\frac{p-k}{q+1-N_k}$ belongs to* $\mathrm{Rot}(\widehat{f})$.

As $N_k \geq 1$, this implies also that $\frac{p-k}{q+1-N_k} < \frac{p-k}{q}$, but since $p - \alpha q < 1$, we have that $\frac{p-k}{q} - \alpha < 0$, a contradiction since we assumed that $\mathrm{Rot}(\widehat{f}) = [\alpha,\beta]$. Therefore Case II also cannot happen, thus concluding the proof of Theorem A.

4. Proof of Theorem B

In this section, we will prove Theorem B. We start by proving a result of uniformly boundedness for the diameter of the projection onto the first coordinate of the leaves of $\widehat{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is the lift of a foliation that is transverse to a maximal identity isotopy whose endpoint is a homeomorphism that satisfies hypotheses of Theorem B. This result plays a key role in the proof of Theorem B.

4.1. Uniform boundedness of leaves

Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity, that is, f preserves the orientation and each boundary component of $\overline{\mathbb{A}}$. Suppose that $\mathbb{A} := \mathbb{T}^1 \times (0,1)$ is a Birkhoff region of instability of f. Let I' be an identity isotopy of f and let $\widehat{f}$ be the lift of f to $\mathbb{R} \times [0,1]$ associated to I'. Suppose that $\mathrm{Rot}(\widehat{f}) = [\alpha, \beta]$ with $\alpha < 0 < \beta$ and that both boundary component rotation numbers are positive.

We consider the open annulus $\mathbb{A} := \mathbb{T}^1 \times (0,1)$. We will denote by N (resp. by S) the upper (resp. lower) end of $\mathbb{A}$. We recall that the homeomorphism f restricted to the open annulus $\mathbb{A}$ can be extended to a homeomorphism, which we still denote f, of the end compactification of $\mathbb{A}$, which is a topological sphere, and this homeomorphism fixes both ends of $\mathbb{A}$. Let I be a maximal identity isotopy larger than I' (isotopy associated to the lift $\widehat{f}$) and let $\widehat{I}$ be a lift of I. Let $\mathcal{F}$ be a singular foliation transverse to I, and let $\widehat{\mathcal{F}}$ be a lift of $\mathcal{F}|_{\mathbb{A}}$. We note that N and S are in $\mathrm{Fix}(I)$ and that these are isolated singularities of $\mathcal{F}$. The next result follows from Proposition 2.12 which describes the dynamics of a foliation near an isolated singularity and the fact that the boundary component rotation numbers are positive.

Lemma 4.1. – *The isolated singularity S (resp. N) is a sink (resp. a source) of $\mathcal{F}$.*

Proof. – We will prove that S is a sink of $\mathcal{F}$ (one proves analogously that N is a source). By Proposition 2.12 it is sufficient to prove that both Cases (1) and (2) of Proposition 2.12 do not hold and that S is not a source. We will prove it by contradiction.

Suppose first that Case (1) of Proposition 2.12 holds, that is, there exists an open topological disk D containing S and contained in a small neighborhood of S whose boundary is a closed leaf of $\mathcal{F}$. By transversality of the foliation either $f(D) \subset D$ or $f^{-1}(D) \subset D$. This contradicts the fact that $\mathbb{A}$ is a Birkhoff region of instability of f.

Suppose now that Case (2) of Proposition 2.12 holds, that is, there exist leaves ϕ_S^+ and ϕ_S^- of $\mathcal{F}$ whose ω-limit and α-limit set are reduced to S respectively. We will prove that the existence of ϕ_S^- implies that the rotation number of the boundary component $\mathbb{R} \times \{0\}$ is negative or zero. This contradicts our hypothesis.

Claim. – Suppose that there exists a leaf ϕ_S^- of $\mathcal{F}$ whose α-limit set is reduced to S. Then the rotation number of the boundary component $\mathbb{R} \times \{0\}$ is negative or zero.

Proof. – Let us parameterize the leaf $\phi_S^- : \mathbb{R} \to \mathbb{A}$. Conjugating f by a homeomorphism given by Schoenflies' Theorem, we may suppose that $\phi_S^-|_{(-\infty,0]}$ is contained in $\{0\} \times (0,1)$. Let U be a Euclidean circle centered at S whose boundary meets $\phi_S^-|_{(-\infty,0]}$. By Proposition 2.13, $\mathcal{F}$ is locally transverse to I at S. Let V be a neighborhood of S contained in U given by the local transversality of $\mathcal{F}$ to I at S: the trajectory of each $z \in V$, $z \neq S$ along I is homotopic, with fixed endpoints, to an arc $I^1_{\mathcal{F}}(z)$ which is transverse to $\mathcal{F}$ and included in U. In particular, the arc $I^1_{\mathcal{F}}(z)$ must cross ϕ_S^- from right to left. More precisely, let $\widehat{f}$ be the lift of $f|_{\mathbb{A}}$ associated to I, and let $\widehat{\mathcal{F}}$ be the lift of $\mathcal{F}|_{\mathbb{A}}$. Let $\widehat{U}$ and $\widehat{V}$ be lifts of the sets $U \setminus \{S\}$ and $V \setminus \{S\}$ respectively. Let $\hat{\phi}_S^-$ be the lift of ϕ_S^- contained in the line $\{0\} \times (0,1)$. For every $n \in \mathbb{N}$, choose $z_n \in V \setminus \{S\}$ such that $\{z_, \dots, f^{n-1}(z_n)\} \subset V$. Let $\widehat{z}_n$ be the lift of z_n such that $-1 < p_1(\hat{z}_n) \leq 0$ and $I^1_{\widehat{\mathcal{F}}}(\widehat{z}_n)$ the lift of the arc $I^1_{\mathcal{F}}(z_n)$ from $\widehat{z}_n$. Since the path $I^n_{\widehat{\mathcal{F}}}(\hat{z}_n)$ is transverse to $\widehat{\mathcal{F}}$ and does not meet the boundary of $\widehat{U}$, we obtain that $p_1(\widehat{f}^n(\widehat{z}_n)) < 0$ and thus

$$\rho_n(\widehat{f}, z_n) := \frac{1}{n}(p_1(\widehat{f}^n(\widehat{z}_n)) - p_1(\widehat{z}_n)) \leq \frac{1}{n}.$$

This implies, by Corollary 2.11, that the rotation number of the boundary component $\mathbb{R} \times \{0\}$ is negative or zero. □

We note finally that if S is a source, then by the above claim we deduce that the rotation number of the boundary component $\mathbb{R} \times \{0\}$ is negative or zero. This contradicts again our hypothesis. This completes the proof of the lemma. □

Let Γ_S and Γ_N be two $\mathcal{F}$-transverse loops close enough to S and N respectively, and let γ_S and γ_N be their respective natural lifts such that the annuli

$$U_S := \bigcup_{t \in \mathbb{R}} \phi_{\gamma_S(t)} \quad \text{and} \quad U_N := \bigcup_{t \in \mathbb{R}} \phi_{\gamma_N(t)}.$$

coincide with the attracting and repelling basin of S and N for $\mathcal{F}$ respectively. Let $\widehat{\mathcal{F}}$ be a lift of $\mathcal{F}|_{\mathbb{A}}$ to $\mathbb{R} \times (0,1)$. Now we can state the main result of this subsection, which will also be useful in the proof of Theorem B.

Proposition 4.2. – *Up to a suitable change of coordinate, the diameters (on the first coordinate) of the leaves of $\widehat{\mathcal{F}}$ are uniformly bounded.*

Proof. – Up to a suitable change of coordinate we can suppose that the foliation $\mathcal{F}$ restricted to a neighborhood of S (resp. N) coincides with the foliation of vertical lines downward (resp. upward) on $\mathbb{T}^1 \times (0,1)$. By Lemma 2.10 and Corollary 2.11 (applied at both ends of $\mathbb{A}$), there exist a neighborhood $V_S \subset U_S$ (resp. $V_N \subset U_N$) of S (resp. of N) and integers $n_S \geq 1$ and $n_N \geq 1$ such that for every $z \in V_S$, $z \neq S$ (resp. $z \in V_N$, $z \neq N$) the closed path $\gamma_S|_{[0,1]}$ (resp. $\gamma_N|_{[0,1]}$) is a subpath of $I^{n_S}_{\mathcal{F}}(z)$ (resp. $I^{n_N}_{\mathcal{F}}(z)$). On the other hand, since $\mathbb{A}$ is a Birkhoff region of instability of f one can find two points z_0 and z_1 and integers $n_0 \geq 1$ and $n_1 \geq 1$ satisfying:

$$z_0, f^{n_1}(z_1) \in \cap_{i=0}^{n_S} f^{-i}(V_S) \quad \left(\text{resp. } z_1, f^{n_0}(z_0) \in \cap_{i=0}^{n_N} f^{-i}(V_N)\right).$$

We will write $\gamma_0 := I^{n_0+n_S}_{\mathcal{F}}(z_0)$ and $\gamma_1 := I^{n_1+n_N}_{\mathcal{F}}(z_1)$ for convenience. Therefore there exist leaves $\phi_S \subset U_S$ and $\phi_N \subset U_N$ of $\mathcal{F}$ and real numbers $s_0 < t_0$ and $s_1 < t_1$ such that:

— $\phi_{\gamma_0(s_0)} = \phi_S$ and $\phi_{\gamma_0(t_0)} = \phi_N$;

— $\phi_{\gamma_1(s_1)} = \phi_N$ and $\phi_{\gamma_1(t_1)} = \phi_S$.

Replacing γ_0 by an equivalent transverse path, one can suppose that $\gamma_0(s_0) = \gamma_1(t_1)$ and $\gamma_0(t_0) = \gamma_1(s_1)$. Let Γ be the loop naturally defined by the closed path $\gamma_0|_{[s_0,t_0]}\gamma_1|_{[s_1,t_1]}$ which is transverse to $\mathcal{F}$. Since the loop Γ is homologous to zero in the sphere, one can define a *dual function* δ defined up to an additive constant on $\mathbb{S}^2 \setminus \Gamma$ as follows: for every z and z' in $\mathbb{S}^2 \setminus \Gamma$, the difference $\delta(z) - \delta(z')$ is the algebraic intersection number $\Gamma \wedge \gamma'$, where γ' is any path from z to z'. As Γ is transverse to $\mathcal{F}$ the function δ decreases along each leaf with a jump at every intersection point. One proves that δ is bounded and that the space of leaves that meet Γ, furnished with the quotient topology is a (possibly non Hausdorff) one dimensional manifold (see [32] for more details). In particular, there exists an integer $K \geq 1$ such that

Γ intersects each leaf of $\mathcal{F}$ at most K times. Moreover any lift of the set $\phi_S \cup \gamma_0|_{[s_0,t_0]}\gamma_1|_{[s_1,t_1]} \cup \phi_N$ separates the plane $\mathbb{R}^2$. More precisely, every set whose diameter on the first coordinate is large enough must intersect a lift of $\phi_S \cup \gamma_0|_{[s_0,t_0]}\gamma_1|_{[s_1,t_1]} \cup \phi_N$. Hence, we have that each leaf of $\widehat{\mathcal{F}}$ intersects at most K translates of a lift of the closed path $\gamma_0|_{[s_0,t_0]}\gamma_1|_{[s_1,t_1]}$. Hence one deduces that for every leaf $\hat{\phi}$ of $\widehat{\mathcal{F}}$ the diameter of the projection on the first coordinate of $\hat{\phi}$ is smaller than the constant $\mathrm{diam}(\gamma_0|_{[s_0,t_0]}\gamma_1|_{[s_1,t_1]}) + K + 2$. This completes the proof of the proposition. □

4.2. Construction of two admissible transverse paths

Let f be a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0,1]$ which is isotopic to the identity. Let $\hat{f}$ be a lift of f to $\mathbb{R} \times [0,1]$. We assume that $\mathrm{Rot}(\hat{f}) = [\alpha, \beta]$ and that both boundary components have rotation number strictly larger than α, the case where both boundary components have rotation number strictly smaller than β is similar. Hence considering a rational number p/q between the left endpoint of $\mathrm{Rot}(\hat{f})$ and the minimum of the two boundary component rotation numbers, we can replace f by a power f^q and the lift $\hat{f}$ by a lift $\hat{f}^q + (p, 0)$, and so suppose that $\mathrm{Rot}(\hat{f}) = [\alpha, \beta]$ with $\alpha < 0 < \beta$ and that both boundary component rotation numbers are strictly positive.

We consider the open annulus $\mathbb{A} := \mathbb{T}^1 \times (0,1)$. We will denote by N (resp. S) the upper (resp. lower) end of $\mathbb{A}$. We recall that the homeomorphism f restricted to the open annulus $\mathbb{A}$ can be extended to a homeomorphism, denoted still f, of the end compactification of $\mathbb{A}$, which is a topological sphere, and this homeomorphism fixes both ends of $\mathbb{A}$. Let I be a maximal identity isotopy larger than I' (isotopy associated to the lift $\hat{f}$) and let $\hat{I}$ be a lift of I. Let $\mathcal{F}$ be a singular foliation transverse to I, and let $\widehat{\mathcal{F}}$ be a lift of $\mathcal{F}|_{\mathbb{A}}$.

We know from the previous subsection that S and N are a sink and a source of $\mathcal{F}$ respectively and that up to a conjugation the leaves of $\widehat{\mathcal{F}}$ are uniformly bounded on the first coordinate. Moreover for a positive real number δ the foliation $\widehat{\mathcal{F}}$ restricted to $\mathbb{R}\times(0,\delta)$ (resp. $\mathbb{R}\times(1-\delta,1)$) is the foliation in vertical lines on $\mathbb{R}\times(0,1)$ oriented downwards (resp. upwards). Let $\hat{\gamma}_N : \mathbb{R} \to \mathbb{R} \times (0,1)$ and $\hat{\gamma}_S : \mathbb{R} \to \mathbb{R} \times (0,1)$ be the transverse paths defined by

$$\hat{\gamma}_N(t) := (t, 1-\delta/2) \quad \text{and} \quad \hat{\gamma}_S(t) := (t, \delta/2).$$

Let us consider

$$\widehat{U}_N := \cup_{t\in\mathbb{R}}\phi_{\hat{\gamma}_N(t)} \quad \text{and} \quad \widehat{U}_S := \cup_{t\in\mathbb{R}}\phi_{\hat{\gamma}_S(t)}.$$

We will begin by proving the following result.

LEMMA 4.3. – *There exist two admissible transverse paths* $\hat{\gamma}_0^* : [a_0, b_0] \to \widehat{\mathbb{A}}$ *and* $\hat{\gamma}_1^* : [a_1, b_1] \to \widehat{\mathbb{A}}$, *real numbers* $a_0 < t_0 < b_0$, $a_1 < t_1 < b_1$ *and a real number* $K^* > 0$ *such that:*

(i) $\hat{\gamma}_0^*|_{[a_0,t_0]}$ *and* $\hat{\gamma}_1^*|_{[t_1,b_1]}$ *intersect* $\hat{\mathcal{F}}$*-transversally; and*

(ii) *for every transverse path* $\hat{\gamma} : [a,b] \to \widehat{\mathbb{A}}$ *such that* $p_1(\hat{\gamma}(b) - \hat{\gamma}(a)) < -K^*$, *there exist two integers* p_0 *and* p_0' *such that* $\hat{\gamma}$ *intersects* $\hat{\mathcal{F}}$*-transversally both* $\left(\hat{\gamma}_0^* + (p_0, 0)\right)|_{[t_0,b_0]}$ *and* $\left(\hat{\gamma}_1^* + (p_0', 0)\right)|_{[a_1,t_1]}$.

Proof. – Let us prove (i). As in the proof of Proposition 4.2, one can find two points $\hat{z}_0$, $\hat{z}_1$ in $\widehat{\mathbb{A}} = \mathbb{R} \times (0,1)$ and two positive integers n_0 and n_1 such that, if $\hat{\gamma}_0^* := I_{\hat{\mathcal{F}}}^{n_0}(\hat{z}_0)$ and $\hat{\gamma}_1^* := I_{\hat{\mathcal{F}}}^{n_1}(\hat{z}_1)$, then there exist real numbers

$$a_0 < s_0^- < s_0^+ < t_0 \leq r_0^- < r_0^+ < b_0$$

satisfying (see Figure 5):

- $\hat{\gamma}_0^*|_{[s_0^-, s_0^+]}$ is equivalent to the path $\hat{\gamma}_S|_{[-1,0]}$;
- $\hat{\gamma}_0^*|_{(s_0^+, t_0)}$ is contained in $\widehat{U}_S$ but it does not meet $\phi_{\hat{\gamma}_S(1)}$;
- $\hat{\gamma}_0^*(t_0)$ does not belong to $\widehat{U}_S$;
- $\hat{\gamma}_0^*|_{[r_0^-, r_0^+]}$ belongs to the complement of $\overline{\widehat{U}_S \cup \widehat{U}_N}$;
- If $\varepsilon > 0$ is sufficiently small, then $\hat{\gamma}_0^*(r_0^- - \varepsilon) \in \overline{\widehat{U}_S}$ and $\hat{\gamma}_0^*(r_0^+ + \varepsilon) \in \overline{\widehat{U}_N}$,

and real numbers

$$a_1 < r_1^- < r_1^+ \leq t_1 < s_1^- < s_1^+ < b_1$$

satisfying:

- $\hat{\gamma}_1^*|_{[s_1^-, s_1^+]}$ is equivalent to the path $\hat{\gamma}_S|_{[0,1]}$;
- $\hat{\gamma}_1^*|_{(t_1, s_1^-)}$ is contained in $\hat{U}_S$ but it does not meet $\phi_{\hat{\gamma}_S(-1)}$;
- $\hat{\gamma}_1^*(t_1)$ does not belong to $\widehat{U}_S$;
- $\hat{\gamma}_1^*|_{[r_1^-, r_1^+]}$ belongs to the complement of $\overline{\widehat{U}_S \cup \widehat{U}_N}$;
- If $\varepsilon > 0$ is sufficiently small, then $\hat{\gamma}_1^*(r_1^- - \varepsilon) \in \overline{\widehat{U}_N}$ and $\hat{\gamma}_1^*(r_1^+ + \varepsilon) \in \overline{\widehat{U}_S}$.

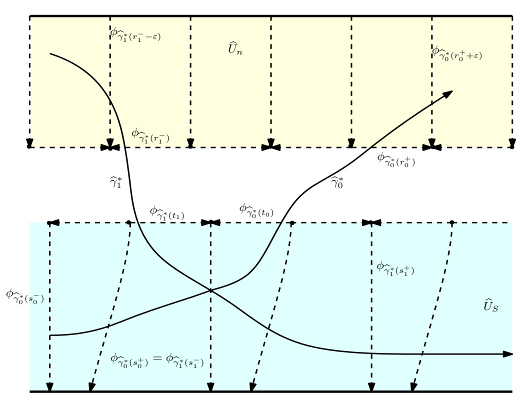

FIGURE 5. The paths $\hat{\gamma}_0^*$ and $\hat{\gamma}_1^*$. Dashed lines are leafs of the $\widehat{\mathcal{F}}$.

We can also assume, by using Corollary 2.6, that neither the path $\hat{\gamma}_0^*|_{[a_0,b_0]}$ nor the path $\hat{\gamma}_1^*|_{[a_1,b_1]}$ have $\widehat{\mathcal{F}}$-transverse self-intersection. We note that the paths $\hat{\gamma}_0^*|_{[s_0^-,t_0]}$ and $\hat{\gamma}_1^*|_{[t_1,s_1^+]}$ have a $\hat{\mathcal{F}}$-transverse intersection at $\phi_{\hat{\gamma}_S(0)}$, and so item (i) is proved.

Let us prove (ii). First note that, since $\widehat{\mathbb{A}}$ has null genus, every leaf of $\hat{\mathcal{F}}$ is either closed or it does not accumulate on itself. Consequently, if β is any path transversal to $\hat{\mathcal{F}}$ defined on a compact interval, then β intersects any leaf of the foliation a finite number of times. Let then $\varepsilon_0 > 0$ be sufficiently small such that, if $a_0 \leq t \leq b_0 - \varepsilon_0$, then $\hat{\gamma}_0^*|_{[t,t+\varepsilon_0]}$ does not intersect a leaf twice.

Now, note that $\partial\widehat{U}_S$ is a foliated set with empty interior, so that no transverse path can be contained in it. Therefore, there exists an open and dense subset $I_- \subset [0, \varepsilon_0]$ such that, if $t \in I_-$, then $\hat{\gamma}_0^*(r_0^- - t)$ belongs to $\widehat{U}_S$. Using a similar argument for $\widehat{U}_N$, one can find $0 < \varepsilon < \varepsilon_0$ such that $\hat{\gamma}_0^*(r_0^- - \varepsilon)$ belongs to $\widehat{U}_S$ and such that $\hat{\gamma}_0^*(r_0^+ + \varepsilon)$ belongs to $\widehat{U}_N$. By Proposition 4.2 there exists a real number $K_0 > 0$ such that for each leaf $\hat{\phi}$ of $\widehat{\mathcal{F}}$ the diameter of $p_1(\hat{\phi})$ is bounded by K_0 and by compactness there exists a real number $K_0' > 0$ such that $\hat{\gamma}_0^*|_{[r_0^- -\varepsilon, r_0^+ +\varepsilon]}$ is contained in $(-K_0', K_0') \times (0,1)$. Let $K = K_0 + K_0'$, and consider a transverse path $\hat{\gamma}$ with diameter on the first coordinate larger than $K^* := 2K+1$. Then there exists an integer p_0 such that $\hat{\gamma}$ meets both $(-\infty, p_0 - K) \times (0,1)$ and $(p_0 + K, +\infty) \times (0,1)$.

We claim that $\hat{\gamma}_0^*|_{[r_0^- -\varepsilon, r_0^+ +\varepsilon]}$ meets each leaf of $\widehat{\mathcal{F}}$ at most once. Indeed, if by contradiction $\hat{\gamma}_0^*|_{[r_0^- -\varepsilon, r_0^+ +\varepsilon]}$ meets a leaf twice, then one must have that $\hat{\gamma}_0^*|_{(r_0^-, r_0^+)}$ must also meet a leaf twice, and there must exist a closed interval $J \subset (r_0^-, r_0^+)$ such that the restriction of $\hat{\gamma}_0^*$ to this interval is equivalent to a simple closed curve. But both $\widehat{U}_S$ and $\widehat{U}_N$ belong to the same connected component of the complement of $\bigcup_{t\in J} \phi_{\hat{\gamma}_0^*(t)}$ so that applying Proposition 2.8 to $\hat{\gamma}_0^*|_{[r_0^- -\varepsilon, r_0^+ +\varepsilon]}$ implies this path has a $\widehat{\mathcal{F}}$-transverse self-intersection, a contradiction with our assumption.

Let B be the union of leaves met by $\hat{\gamma}_0^*|_{(r_0^- -\varepsilon, r_0^+ +\varepsilon)}$. Then B is a foliated subset of $\widehat{\mathbb{A}}$ and, by the argument above, it is homeomorphic to the plane and the space of leaves of $\widehat{\mathcal{F}}$ in B is homeomorphic to an open interval. Furthermore, since B contains a leaf of $\widehat{U}_S$ and a leaf of $\widehat{U}_N$, B separates $\mathbb{R}^2$ and its complement has exactly two connected components, one denoted $L(B)$ that contains $(-\infty, -K) \times (0,1)$ and the other, denoted $R(B)$, contains $(K, +\infty) \times (0,1)$. Finally, note that $\phi_{\gamma_0^*(r_0^- -\varepsilon)}$ belongs to $\partial L(B)$ and B is locally to the left of this leaf. Likewise, $\phi_{\gamma_0^*(r_0^+ +\varepsilon)}$ belongs to $\partial R(B)$ and B is locally to the right of this leaf (see Figure 6).

Since $\hat{\gamma}(a)$ belongs to $R(B) - (p_0, 0)$ and $\hat{\gamma}(b)$ belongs to $L(B) - (p_0, 0)$, one find some $a < a' < b' < b$ such that $\hat{\gamma}(a')$ belongs to $\partial\,(R(B) - (p_0, 0))$, $\hat{\gamma}(b')$ belongs to $\partial\,(L(B) - (p_0, 0))$, and $\hat{\gamma}|_{(a',b')}$ is contained in $B - (p_0, 0)$. Note that $\phi_{\hat{\gamma}(a')} + (p_0, 0)$ is not $\phi_{\hat{\gamma}_0^*(r_0^+ +\varepsilon)}$, since the latter has B on its right, and likewise $\phi_{\hat{\gamma}(b')} + (p_0, 0)$ is not $\phi_{\hat{\gamma}_0^*(r_0^- -\varepsilon)}$, since the latter has B on its left. One concludes that $\widehat{\gamma}|_{[a',b']}$ has a $\widehat{\mathcal{F}}$-transverse intersection with $\left(\hat{\gamma}_0^*|_{[r_0^- -\varepsilon, r_0^+ +\varepsilon]} + (p_0, 0)\right)$ at any leaf $\phi_{\hat{\gamma}(t)}$ for $a' < t < b'$. Since the same construction holds for every $0 < \varepsilon' < \varepsilon$, we get that $\hat{\gamma}$ and $\hat{\gamma}_0^*|_{[r_0^-, r_0^+]} + (p_0, 0)$ intersect

$\widehat{\mathcal{F}}$-transversally, as claimed. The $\widehat{\mathcal{F}}$-transverse intersection with $\hat{\gamma}_1^*|_{[r_1^-,r_1^+]} + (p_0', 0)$ can be obtained in a similar way. $\square$

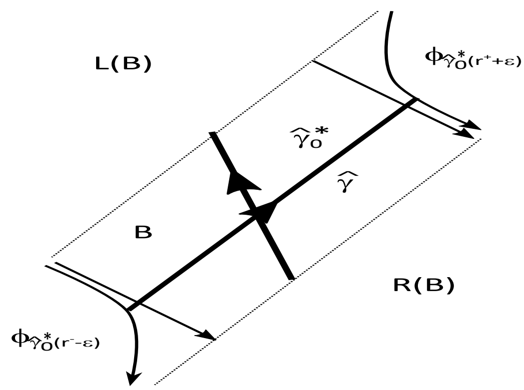

FIGURE 6. The set B

End of the proof of Theorem B. – One knows by Proposition 4.2 that the leaves of $\widehat{\mathcal{F}}$ are uniformly bounded on the first coordinate. Hence in order to prove Theorem B it is sufficient to prove that there exists a real constant $L > 0$ such that for every admissible transverse path $\hat{\gamma} : [a, b] \to \widehat{\mathbb{A}}$ of order $n \geq 1$, one has

$$p_1(\hat{\gamma}(b)) - p_1(\hat{\gamma}(a)) - n\alpha \geq -L.$$

Let $\hat{\gamma} : [a, b] \to \mathbb{R} \times (0, 1)$ be a transverse path such that

$$p_1(\hat{\gamma}(b)) - p_1(\hat{\gamma}(a)) < -2K^*.$$

One can find c, d in (a, b) with $c < d$ such that

$$p_1(\hat{\gamma}(b)) - p_1(\hat{\gamma}(d)) = -K^* \quad \text{and} \quad p_1(\hat{\gamma}(c)) - p_1(\hat{\gamma}(a)) = -K^*.$$

By item (ii) from Lemma 4.3 there exist p_0 and p_0' in $\mathbb{Z}$, $a < l_0 < c$, $d < l_1 < b$, $r_0^- < w_0 < r_0^+$, $r_1^- < w_1 < r_1^+$ such that

— $\hat{\gamma}|_{[a,c]}$ and $\hat{\gamma}_0^* + (p_0, 0)$ intersect $\widehat{\mathcal{F}}$-transversally at $\hat{\gamma}(l_0) = \hat{\gamma}_0^*(w_0) + (p_0, 0)$;

— $\hat{\gamma}|_{[d,b]}$ and $\hat{\gamma}_1^* + (p_0', 0)$ intersect $\widehat{\mathcal{F}}$-transversally at $\hat{\gamma}(l_1) = \hat{\gamma}_1^*(w_1) + (p_0', 0)$.

If $\hat{\gamma}$ is admissible of order $n \geq 1$, then the path

$$\hat{\gamma}' := \left(\hat{\gamma}_0^*|_{[a_0,w_0]} + (p_0, 0)\right) \hat{\gamma}|_{[l_0,l_1]} \left(\hat{\gamma}_1^*|_{[w_1,b_1]} + (p_0', 0)\right)$$

is admissible of order $n + n_0 + n_1$ by Corollary 2.5 and one has

$$p_1(\hat{\gamma}(b)) - p_1(\hat{\gamma}'(1)) < -K^* \quad \text{and} \quad p_1(\hat{\gamma}'(0)) - p_1(\hat{\gamma}(a)) < -K^*.$$

Recall that $\hat{\gamma}_0^*$ and $\hat{\gamma}_1^*$ intersect $\widehat{\mathcal{F}}$-transversally (item (i) from Lemma 4.3). One deduces that $\hat{\gamma}'$ intersects $\widehat{\mathcal{F}}$-transversally $\hat{\gamma}' + (p', 0)$, where $p' = p_0 - p_0'$. Proposition 2.7 tells us that $p'/(n + n_0 + n_1)$ belongs to $\mathrm{Rot}(\hat{f})$, which implies that

$$\alpha \leq \frac{p'}{n + n_0 + n_1}.$$

We write K^{**} by the diameter on the first coordinate of $\hat{\gamma}_0^*$. Observe now that

$$p_1(\hat{\gamma}'(1)) + p' - p_1(\hat{\gamma}'(0)) > -K^{**}.$$

So, one deduces

$$p_1(\hat{\gamma}(b)) - p_1(\hat{\gamma}(a)) - n\alpha \geq -2K^* - K^{**} + \alpha n_0 + \alpha n_1.$$

This completes the proof of Theorem B. □

5. Example

Recall that, by Theorem B, if f is a homeomorphism of the closed annulus $\overline{\mathbb{A}} := \mathbb{T}^1 \times [0, 1]$ that is isotopic to the identity and having $\mathbb{A} := \mathbb{T}^1 \times (0, 1)$ as Birkhoff region of instability, and $\hat{f}$ is a lift of f to $\mathbb{R} \times [0, 1]$ with a nontrivial rotation set $\mathrm{Rot}(\hat{f}) = [\alpha, \beta]$, such that the rotation numbers of both boundary components of $\overline{\mathbb{A}}$ are in the interior of $[\alpha, \beta]$, then there exists a real constant $L > 0$ such that for every $\hat{z} \in \mathbb{R} \times [0, 1]$ and every integer $n \geq 1$ we have

$$p_1(\hat{f}^n(\hat{z})) - p_1(\hat{z}) - \alpha n \geq -L \quad \text{and} \quad p_1(\hat{f}^n(\hat{z})) - p_1(\hat{z}) - \beta n \leq L.$$

The following example shows that the hypothesis "the rotation numbers of both boundary components of $\overline{\mathbb{A}}$ are contained in the interior of the rotation set" is essential in the conclusion of Theorem B.

Proposition 5.1. – *There exists a homeomorphism f of the closed annulus $\overline{\mathbb{A}}$ which is isotopic to the identity, such that $\overline{\mathbb{A}}$ is a Birkhoff region of instability of f and has a lift $\hat{f}$ to $\mathbb{R} \times [0, 1]$ satisfying:*

(i) $\mathrm{Rot}(\hat{f}) = [0, 1]$*, and*

(ii) *for every real number $L > 0$ there exist a point $\hat{z}$ in $\mathbb{R} \times [0, 1]$ and an integer n such that*

$$p_1(\hat{f}^n(\hat{z})) - p_1(\hat{z}) < -L.$$

Proof. – For every real number r, we write T_r the homeomorphism of $\mathbb{R} \times [0, 1]$ defined by $T_r : (x, y) \mapsto (x + r, y)$. Let $h : (0, 1) \to \mathbb{R}$ be such that $h(y) = 1/y$ if $y < 1/2$ and $h(y) = 2$ if $y \geq 1/2$.

For every $y \in (0, 1)$, let

$$I_y' := \left(h(y) + \frac{1}{16}, h(y) + \frac{3}{16}\right) \times \{y\} \subset \left(h(y), h(y) + \frac{1}{4}\right) \times \{y\} := I_y$$

be two subsets of $\mathbb{R} \times [0, 1]$ and consider

$$\hat{U}_0' := \bigcup_{y \in (0,1)} I_y' \subset \bigcup_{y \in (0,1)} I_y := \hat{U}_0 \quad \text{and} \quad \hat{V}_0' := T_{1/2}(\hat{U}_0') \subset T_{1/2}(\hat{U}_0) := \hat{V}_0.$$

Let, finally,

$$\hat{U}' := \bigcup_{n\in\mathbb{Z}} T_1^n(\hat{U}_0') \subset \bigcup_{n\in\mathbb{Z}} T_1^n(\hat{U}_0) := \hat{U} \quad \text{and} \quad \hat{V}' := \bigcup_{n\in\mathbb{Z}} T_1^n(\hat{V}_0') \subset \bigcup_{n\in\mathbb{Z}} T_1^n(\hat{V}_0) := \hat{V}.$$

Let $g : [0,1] \to [0,1]$ be a homeomorphism of $[0,1]$ satisfying:

— $g(0) = 0$, $g(1) = 1$;

— for every $y \in (0,1)$ we have $g(y) < y$;

— $\lim_{y\to 0}\left(\frac{1}{y} - \frac{1}{g(y)}\right) = 0$.

We note that the dynamics of g is well-known, that is, for every $y \in (0,1)$ we have

$$\lim_{n\to+\infty} g^n(y) = 0 \quad \text{and} \quad \lim_{n\to+\infty} g^{-n}(y) = 1. \tag{1}$$

To construct $\hat{f}$ we start with a homeomorphism $\hat{f}_0$ satisfying:

$$\hat{f}_0(x,y) = \begin{cases} (x - h(y) + h(g(y)), g(y)), & \text{if} \quad (x,y) \in \hat{U}_0' \subset \hat{U}_0, \\ \left(x - h(y) + h(g^{-1}(y)), g^{-1}(y)\right), & \text{if} \quad (x,y) \in \hat{V}_0' \subset \hat{V}_0. \end{cases}$$

Note that $\hat{f}_0$ leaves both $\hat{U}_0'$ and $\hat{V}_0'$ invariant. Let us begin by extending $\hat{f}_0$ to $\hat{U}_0$ and to $\hat{V}_0$ such that $\hat{f}_0$ leaves both these sets invariant, such that

$$\begin{cases} p_2(\hat{f}_0(x,y)) < y, & \text{if} \quad (x,y) \in \hat{U}_0, \\ y < p_2(\hat{f}_0(x,y)), & \text{if} \quad (x,y) \in \hat{V}_0, \end{cases}$$

where $p_2 : \mathbb{R}\times[0,1] \to [0,1]$ is the projection on the second coordinate, such that $\hat{f}_0$ extends continuously as the identity to the boundary of $\hat{U}_0 \cup \hat{V}_0 \cap \mathbb{A}$ and finally that $p_1(\hat{f}_0(x,y)) = x$ if $g(y) \geq 1/2$. Now extend $\hat{f}_0$ to $\hat{U}$ and $\hat{V}$ by the formula $\hat{f}_0 T_1 = T_1 \hat{f}_0$. Finally we extend $\hat{f}_0$ to $\mathbb{R}\times[0,1]$ such that $\hat{f}_0$ coincides with the identity on the complement of the union of $\hat{U}$ and of $\hat{V}$. Note that $\hat{f}_0$ extends continuously to $\mathbb{R}\times\{0,1\}$ since $\lim_{y\to 0}(h(y) - h(g(y))) = \lim_{y\to 1}(h(y) - h(g(y))) = 0$.

We note that by construction $\hat{f}_0$ commutes with T_1, and as such it induces a homeomorphism f_0 of $\overline{\mathbb{A}}$. Furthermore, for every $i \in \mathbb{Z}$ the sets $T_i(\hat{U}_0')$ and $T_i(\hat{V}_0')$ are $\hat{f}_0$-invariant and project to topological disks on $\overline{\mathbb{A}}$. Since $\hat{f}_0$ has no recurrent point in these sets, then the same is true for f_0 in their projections. Therefore the only recurrent points of f_0 are those points lifted to fixed points of $\hat{f}_0$ which implies that $\mathrm{Rot}(\hat{f}_0) = \{0\}$. Note also that if $(x,y) \in \hat{V}'$ and $n \geq 1$, then

$$\hat{f}_0^n(x,y) = \left(x - \frac{1}{y} + \frac{1}{g^{-n}(y)}, g^{-n}(y)\right).$$

Thus,

$$p_1(\hat{f}_0^n(x,y)) - x = \frac{1}{g^{-n}(y)} - \frac{1}{y},$$

so one knows that the function $p_1(\hat{f}_0^n(x,y) - (x,y))$ is not bounded from below. Moreover from (1), $\mathbb{A} = \mathbb{T}^1 \times (0,1)$ is a Birkhoff region of instability of f_0, the projection of $\hat{f}_0$ to $\overline{\mathbb{A}}$. More precisely, the f_0-orbit of each point $z \in U' = \hat{\pi}(\hat{U}_0')$ goes from the upper boundary component of $\overline{\mathbb{A}}$ to the lower one, that is

$$\lim_{n\to+\infty} p_2(f_0^{-n}(z)) = 1 \quad \text{and} \quad \lim_{n\to+\infty} p_2(f_0^n(z)) = 0, \tag{2}$$

and the f_0-orbit of each point $z \in V' = \hat{\pi}(\hat{V}_0')$ goes from the lower boundary component of $\overline{\mathbb{A}}$ to the upper one. This is almost what we need, but as the rotation set of f_0 is a singleton, we need to make another small perturbation.

Let $y_1 = \frac{g^{-1}(1/2)+g^{-2}(1/2)}{2}$. Consider $\hat{f}_1(x,y) := T_{\varphi(y)}(x,y)$, where $\varphi : [0,1] \to [0,1]$ is a continuous function satisfying:

— $\varphi(y) = 0$ if $y \in [0, g^{-1}(1/2)] \cup [g^{-2}(1/2), 1]$; and

— $\varphi(y_1) = 1$.

We note that $\hat{f}_1$ commutes with T_1, the compact strip

$$S_0 := \mathbb{R} \times [g^{-1}(1/2), g^{-2}(1/2)]$$

is an $\hat{f}_1$-invariant set, and $\hat{f}_1$ acts as the identity on the complement of S_0. We now consider $\hat{f} := \hat{f}_1 \circ \hat{f}_0$ which is a homeomorphism of $\mathbb{R}\times[0,1]$ and which commutes with T_1. It remains to prove that $\hat{f}$ satisfies the properties described in the proposition.

To see that $\mathbb{A}$ is a Birkhoff region of instability of f, note that if we choose some point $\hat{z}_0 = (x_0, g^{-1}(1/2)) \in \hat{V}'$ then $\hat{f}^n(\hat{z}_0)$ belongs to $\mathbb{R} \times [g^{-2}(1/2), 1]$ for all $n > 0$, and to $\mathbb{R}\times[0, g^{-1}(1/2)]$ for all $n \leq 0$. In particular, $\hat{f}^n(\hat{z}_0) = \hat{f}_0^n(\hat{z}_0)$. Therefore $z_0 = \hat{\pi}(\hat{z}_0)$ goes from the lower boundary to the upper boundary component of $\overline{\mathbb{A}}$. We can likewise choose a point $\hat{z}_1 = (x_1, g^{-1}(1/2)) \in \hat{U}'$ and then $\hat{f}^n(\hat{z}_1)$ belongs to $\mathbb{R} \times [g^{-2}(1/2), 1]$ for all $n < 0$, and to $\mathbb{R} \times [0, g^{-1}(1/2)]$ for all $n \geq 0$. Therefore $z_1 = \hat{\pi}(\hat{z}_1)$ goes from the upper boundary component to the lower boundary component of $\overline{\mathbb{A}}$.

We note also that any point in $\mathbb{R} \times [0, g^{-1}(1/2)]$ belongs to either $\hat{V}$, to $\hat{U}$ or is a fixed point of $\hat{f}_0$. One verifies trivially that the set $\mathbb{R} \times [0, g^{-1}(1/2)] \cap \hat{V}$ is forward invariant for $\hat{f}_0$ and therefore also for $\hat{f}$, and every point is a wandering point. One also verifies that $\mathbb{R} \times [0, g^{-1}(1/2)] \cap \hat{U}$ is backward invariant for $\hat{f}$, and thus all its points are also wandering. Therefore any point in $\mathbb{R}\times[0, g^{-1}(1/2)]$ that is not wandering must be fixed by $\hat{f}$. A similar argument shows that any point of $\mathbb{R} \times [g^{-2}(1/2), 1]$ that is not wandering for $\hat{f}$ is also fixed. Thus any periodic point of f that does not lift to a fixed point of $\hat{f}$ must have its whole orbit contained in the annulus $\mathbb{T}^1 \times [g^{-1}(1/2), g^{-2}(1/2)]$. Now, for any point (x,y) in S_0, we have that $0 \leq p_1(\hat{f}(x,y) - (x,y)) \leq 1$, so one gets that $\mathrm{Rot}(\hat{f}) \subset [0,1]$. Finally, take the point $(1/2, y_1)$, which lies in the boundary of $\hat{U}$. Therefore $\hat{f}_0(1/2, y_1) = (1/2, y_1)$ and therefore $\hat{f}(1/2, y_1) = (1/2 + 1, y_1)$. Therefore $1 \in \mathrm{Rot}(\hat{f})$ and since 0 is also in the rotation interval, we deduce that $\mathrm{Rot}(\hat{f}) = [0,1]$. □

6. Realization results

In this section, we will prove Theorems C and D.

6.1. Proof of Theorem C

Theorem C will be a consequence of the following stronger proposition, since being a Mather region of instability is stronger than being a SN mixed region of instability.

PROPOSITION 6.1. – *Let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a homeomorphism which is isotopic to the identity, and let $\widehat{f}$ be a lift of f to the universal covering. Suppose that $\mathbb{T}^1 \times (0,1)$ is a SN mixed region of instability. For every ρ in $\mathrm{Rot}(\widehat{f})$ there exists a compact invariant set Q_ρ such that every point of Q_ρ has a well-defined rotation number and it is equal to ρ. Moreover, if $\rho = p/q$ is a rational number, written in an irreducible way, then Q_ρ can be taken to be the orbit of a periodic point of period q.*

The proof of the proposition is immediate when $\mathrm{Rot}(\widehat{f})$ is a single point, as in this case every point of $\overline{\mathbb{A}}$ has a well-defined rotation number and this number is unique. Therefore we can assume that $\mathrm{Rot}(\widehat{f}) = [\alpha, \beta]$, with $\alpha < \beta$. We divide the proof of Proposition 6.1 in two cases, when ρ is in the boundary of the rotation set, and when it is in the interior.

6.1.1. *When ρ is a boundary point.* – We show that α is realized by a compact invariant set, the case for β is similar. If any of the boundary components rotation numbers is α it suffices to take Q_α as the boundary component, therefore we assume that α is strictly smaller than the rotation numbers of the boundary components and thus that f satisfies the hypotheses of Theorem B, since being a SN mixed region of instability is stronger than being a Birkhoff region of instability. Let us consider

$$\mathcal{M}_\alpha := \{\mu \in \mathcal{M}_f(\overline{\mathbb{A}}) : \mathrm{Rot}(\mu) = \alpha\} \quad \text{and} \quad X_\alpha := \overline{\bigcup_{\mu \in \mathcal{M}_\alpha} \mathrm{Supp}(\mu)}.$$

By the following proposition we can take Q_α as the set X_α, completing the proof of the proposition in this case.

PROPOSITION 6.2. – *Let α be in the boundary of the rotation set of $\widehat{f}$. Every measure supported on X_α belongs to $\mathcal{M}_\alpha$. Moreover, if $\widehat{z}$ lifts a point z of X_α, then for every integer $n \geq 1$, we have*

$$\left| p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\alpha \right| \leq L,$$

where L is the constant given by Theorem B.

Proof. – We note that it is sufficient to prove the second statement. We recall that, if $\mu \in \mathcal{M}_\alpha$, then every ergodic measure ν that appears on the ergodic decomposition of μ also belongs to $\mathcal{M}_\alpha$. We claim that, for every ergodic measure ν in $\mathcal{M}_\alpha$, there exists a set A of full measure such that if $\widehat{z}$ lifts a point z of X_α, then for every integer $n \geq 1$, we have

$$\left| p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\alpha \right| \leq L.$$

The claim is sufficient to show the result as, if the inequality holds in A, then it must also hold on the closure of A, and therefore in the support ν. Furthermore, given any $\mu \in \mathcal{M}_\alpha$, since any point in the support of μ is accumulated by points in the support of an ergodic measure in the decomposition of μ, one also deduces that the inequality above must hold in the whole support of μ. Therefore the claim implies that the inequality holds in $\bigcup_{\mu \in \mathcal{M}_\alpha} \mathrm{Supp}(\mu)$ and so by closure it also holds in X_α.

So, let ν in $\mathcal{M}_\alpha$ be an ergodic measure. By Atkinson's Lemma, Proposition 3.1, there exists a set A of ν-full measure, such that if $\widehat{z}$ lifts a point z of A, then there exists a sub-sequence of integers $(q_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to+\infty} \widehat{f}^{q_l}(\widehat{z}) - \widehat{z} - (q_l\alpha, 0) = 0.$$

Nevertheless, by Theorem B we know that for every $\widehat{z} \in \mathbb{R}\times[0,1]$, and every integer $n \geq 1$ we have

$$p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\alpha \geq -L. \tag{3}$$

It remains to check that, if $\widehat{z}$ is a lift of $z \in A$, then

$$p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\alpha \leq L. \tag{4}$$

Indeed, if l is large enough such that q_l is greater than n, one can write

$$\begin{aligned} &p_1(\widehat{f}^n(\widehat{z})) - p_1(\widehat{z}) - n\alpha \\ &\quad = p_1(\widehat{f}^{q_l}(\widehat{z})) - p_1(\widehat{z}) - q_l\alpha - \Big(p_1(\widehat{f}^{q_l-n}(\widehat{f}^n(\widehat{z}))) - p_1(\widehat{f}^n(\widehat{z})) - (q_l-n)\alpha\Big) \\ &\quad \leq p_1(\widehat{f}^{q_l}(\widehat{z})) - p_1(\widehat{z}) - q_l\alpha + L. \end{aligned}$$

Letting l tend to $+\infty$, we obtain inequality (4). The proposition follows from inequalities (3) and (4). □

6.1.2. *When ρ is an interior point*. – We already know the existence of an f-invariant compact set with rotation number equal to ρ if $\rho \in \{\alpha, \beta\}$ or if ρ is the rotation number of one of the boundary components of $\overline{\mathbb{A}}$. So we assume $\rho \in (\alpha, \beta)$ and ρ is not the rotation number of the upper boundary. We can also assume that the rotation number of the upper boundary is ρ_1 larger than ρ, the case where it is smaller is again similar. We pick some rational number p/q such that

$$\alpha < p/q < \rho < (p+1)/q < (p+2)/q < \rho_1 \leq \beta.$$

Let $g = f^q$, and $\widehat{g} = \widehat{f}^q - (p,0)$. Let I' be an identity isotopy of g, such that its lift to $\widehat{\mathbb{A}}$, $\widehat{I}'$, is an identity isotopy of $\widehat{g}$. By Theorem 2.1 one can find a maximal identity isotopy larger than I'. By Theorem 2.2 one can find an oriented singular foliation $\mathcal{F}$ on $\mathbb{A}$ which is transverse to I, its lift to $\widehat{\mathbb{A}}$, denoted by $\widehat{\mathcal{F}}$ is transverse to $\widehat{I}$ (the lift of I). Note that the rotation number of the upper boundary for $\widehat{g}$ is larger than 2. Our goal is to show that there exist some integer $n > 0$ and some transverse trajectory $\widehat{\gamma}$ such that $\widehat{\gamma}$ is admissible of order n and such that $\widehat{\gamma}$ has a $\widehat{\mathcal{F}}$-transverse intersection with $\widehat{\gamma} + (j, 0)$ where $j > n$. If we do this, Proposition 2.7 implies that for all $0 < \theta \leq j/n$, there exists a g-invariant compact set with rotation number θ for $\widehat{g}$, and we deduce that f has a compact set with rotation number ρ.

To construct the transverse path γ, let us first note that as in the proof of Theorem B for a positive real number $\delta > 0$ the foliation $\widehat{\mathcal{F}}$ restricted to $\mathbb{R}\times(1-\delta, 1)$ can be assumed to be foliation by vertical lines oriented upwards. Let $\widehat{\gamma}_N : \mathbb{R} \to \mathbb{R}\times(0,1)$ be the transverse path defined by $\widehat{\gamma}_N(t) = (t, 1-\delta/2)$. Let $\epsilon > 0$ be such that $2+\epsilon < q\rho_1 - p$. Then, given a fixed positive integer m, it holds that for every $\widehat{z}$ sufficiently close to the upper boundary the transverse path $I^{\mathbb{Z}}_{\widehat{\mathcal{F}}}(\widehat{z})|_{[0,m]}$ contains a subpath equivalent to a translate of $\widehat{\gamma}_N|_{[0,(2+\epsilon/2)m]}$ (this last property comes from the fact that the rotation number of the upper boundary is larger than $2+\epsilon$). Let U_Γ be the annulus of the leaves crossed by $\pi(\widehat{\gamma}_N)$. We claim that there exists a neighborhood V of the lower boundary component of $\overline{\mathbb{A}}$ that is disjoint from U_Γ.

Indeed, as $\mathrm{Rot}(\widehat{g})$ has negative real numbers and $\mathbb{A}$ is a SN mixed region of instability for g, Franks' Theorem (Theorem 2.9) can be applied. One finds a periodic point y in $\mathbb{A}$ with negative rotation number. Lifting y to a point $\widehat{y}$, the transverse trajectory of $\widehat{y}$ contains a sub-path β' that is positively transverse to $\widehat{\mathcal{F}}$, and that joins $\widehat{y}$ to $\widehat{y}-(1,0)$. By removing closed loops from β', one finds another path β, whose image lies in the image of β', that is simple, positively transverse to $\widehat{\mathcal{F}}$, and joins $\widehat{y}$ to $\widehat{y}-(L,0)$. One gets a transverse line $\beta^* = \prod_{i=-\infty}^{\infty} \beta - (i,0)$. The left of β^* projects to the neighborhood V. Since every leaf of $\widehat{\mathcal{F}}$ is oriented in such a way that its future is completely contained in the left of β^*, one gets that any leaf lifted from U_Γ is disjoint from β^*, proving that U_Γ is disjoint from V.

Since $\mathbb{A}$ is a SN mixed region of instability for f, it is also a SN mixed region of instability for g , we have that there exist points $z_{S,N}$ and $z_{N,S}$, both lying in V, such that the ω-limit set of $z_{S,N}$ for g is contained in $\mathbb{T}^1 \times \{1\}$ and such that the α-limit set for g of $z_{N,S}$ is also contained in $\mathbb{T}^1 \times \{1\}$. This implies that there exists an integer n_0 sufficiently large such that, for all integer $n \geq n_0$, $g^n(z_{S,N})$ and $g^{-n}(z_{N,S})$ are very close to the upper boundary. Let $\widehat{U_\Gamma}$ be a lift of U_Γ. Taking some lift $\widehat{z_{N,S}}$ of $z_{N,S}$, one gets that, if m is an integer sufficiently large, the transverse path $I^{\mathbb{Z}}_{\widehat{\mathcal{F}}}(\widehat{z_{N,S}})|_{[-m-n_0,0]}$ ends outside of $\widehat{U_\Gamma}$ but contains a subpath that is equivalent to a translate of $\widehat{\gamma}_N|_{[0,(2+\epsilon/2)m]}$. Similarly, taking some lift $\widehat{z_{S,N}}$ of $z_{S,N}$, one gets that, if m is an integer sufficiently large, the transverse path $I^{\mathbb{Z}}_{\widehat{\mathcal{F}}}(\widehat{z_{S,N}})|_{[0,m+n_0]}$ starts outside of $\widehat{U_\Gamma}$ but contains a subpath that is equivalent to a translate of $\widehat{\gamma}_N|_{[0,(2+\epsilon/2)m]}$. Each of those paths is admissible of order n_0+m. By Proposition 2.4, if m is large enough, one can construct (as in the proof of Proposition 3.7) a transverse path $\widehat{\gamma}$ that starts and ends outside of $\widehat{U_\Gamma}$, and that contains as a subpath a translate of $\widehat{\gamma}_N|_{[0,(2+\epsilon/2)m]}$. The path $\widehat{\gamma}$ is admissible of order $2(n_0+m)$ and if $0 < j < (2+\epsilon/2)m$ is an integer, then $\widehat{\gamma}$ has a $\mathcal{F}$-transverse intersection with $\widehat{\gamma}+(j,0)$ at some point in $\widehat{U_\Gamma}$. Note that n_0 is fixed and so, if m is sufficiently large, there exists an integer $2(n_0+m) < j_0 < (2+\epsilon/2)m$. This ends the proof of Proposition 6.1.

6.2. Proof of Theorem D

Before we start the proof of Theorem D, let us state a simple corollary of Proposition 6.1.

Corollary 6.3. – *Let $f : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a homeomorphism which is isotopic to the identity, and let $\widehat{f}$ be a lift of f to the universal covering. Suppose that f preserves a measure of full support, and that $\mathbb{A} = \mathbb{T}^1 \times (0,1)$ is a Birkhoff region of instability. For every ρ in* $\mathrm{Rot}(\widehat{f})$ *there exists a compact invariant set Q_ρ such that every point of Q_ρ has a well-defined rotation number and it is equal to ρ. Moreover, if $\rho = p/q$ is a rational number, written in an irreducible way, then Q_ρ can be taken to be the orbit of a periodic point of period q.*

Proof. – Since $\mathbb{T}^1 \times (0,1)$ is a Birkhoff region of instability we can find, by Proposition 2.19, that there exists an essential open annulus $A \subset \overline{\mathbb{A}}$ which is a SN mixed region of instability, and such that, if A^* is the prime ends compactification of A, f^* is the extension of f to A^*, then there exists $\widehat{f^*}$ a lift of f^* such that $\mathrm{Rot}(\widehat{f^*}) = \mathrm{Rot}(\widehat{f})$. By Proposition 6.1 we deduce that for any ρ in $\mathrm{Rot}(\widehat{f})$ there exists a closed subset Q^*_ρ which is f^*-invariant and such that the $\widehat{f^*}$ rotation number of any point in Q^*_ρ is ρ. If Q^*_ρ is contained in A, then it suffices to take $Q_\rho = Q^*_\rho$. Otherwise there exists a point in the boundary of A^* with rotation number ρ, which implies that the rotation number of the restriction of f^* to one of these boundaries is ρ. But this implies that there exists a connected component K of the boundary of A which is an essential continuum, such that the prime end rotation number of K is exactly ρ, and as the dynamics preserves a measure of full support, every point in K has the same rotation number by Proposition 2.16, in which case it suffices to take $K = Q_\rho$. □

In the following let f be an area-preserving homeomorphism of $\overline{\mathbb{A}}$ which is isotopic to the identity. Let $\widehat{f}$ be a lift of f to $\widehat{\overline{\mathbb{A}}}$. If the rotation set of $\widehat{f}$ is a singleton, then the result is obvious as every point in the annulus will have the same rotation number. If not, then by a result from Franks (see [10, 12]), for every rational number p/q in the interior of $\mathrm{Rot}(\widehat{f})$ there exists a point $z_{p/q}$ in $\overline{\mathbb{A}}$ whose rotation number is p/q. Therefore it suffices to show the result for ρ in $\mathrm{Rot}(\widehat{f})$ which is an irrational number. We can also assume that the rotation number of each boundary of $\overline{\mathbb{A}}$ is not ρ, otherwise we are also done.

We will consider the extension f' of f to the open annulus $\mathbb{A}' = \mathbb{T}^1 \times \mathbb{R}$ by assuming that $f'(x, y) = f(x, 1) + (0, y-1)$ if $y > 1$ and $f'(x, y) = f(x, 0) + (0, y)$ if $y < 0$, and by choosing a compatible lift $\widehat{f}'$ that is also an extension of $\widehat{f}$ to $\mathbb{R}^2$. We still denote f for this extension. Let $(p_n/q_n)_{n\in\mathbb{N}}$ be a sequence of rational points in the interior of the rotation set of $\widehat{f}$ converging to ρ, and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of periodic points of $\overline{\mathbb{A}}$ such that for every integer n, z_n has rotation number p_n/q_n. We assume that $(z_n)_{n\in\mathbb{N}}$ is converging, otherwise we take a subsequence, and let $\overline{z}$ be the limit of this sequence. We repeat a construction from [27]. For each real number $\varepsilon > 0$, consider the set

$$U'_\varepsilon = \bigcup_{i\in\mathbb{Z}} f^i(B_\varepsilon(\overline{z})),$$

where as usual $B_\varepsilon(\overline{z})$ is the ε open ball centered at $\overline{z}$. One can show, since $\overline{z}$ is non-wandering, that U'_ε is an open set with finitely many connected components which are all periodic. Furthermore, the connected component of U'_ε that contains $\overline{z}$ cannot be contained in a topological disk, because every recurrent point in a periodic topological disk must have the same rational rotation number. As this component contains z_n for any sufficiently large n, we would obtain a contradiction. This implies that the connected component O_ε of U'_ε that contains $\overline{z}$ (and consequently $B_\varepsilon(\overline{z})$) is essential. Since f permutes the connected components of U'_ε, one has that either O_ε is invariant or it is disjoint from its image by f. But the later cannot happen, as it would imply that every point in O_ε is wandering since O_ε is essential. Therefore O_ε is invariant, and as it contains $B_\varepsilon(\overline{z})$ one deduces that $O_\varepsilon = U'_\varepsilon$. Finally note that, since $\overline{z}$ belongs to $\overline{\mathbb{A}}$ and since $B_\varepsilon(\overline{z}) \subset \mathbb{T}^1 \times [-\varepsilon, 1+\varepsilon]$, by the construction of the extension of f one has that $U'_\varepsilon \subset \mathbb{T}^1 \times [-\varepsilon, 1+\varepsilon]$ and thus it separates the two ends of $\mathbb{A}'$.

Let U_ε be the filling of U'_ε, that is, the union of U'_ε with all connected components of its complement that are bounded in $\mathbb{A}'$. Note that U_ε is a topological open annulus, also contained in $\mathbb{T}^1 \times [-\varepsilon, 1+\varepsilon]$. Note also that $U_{\varepsilon_1} \subset U_{\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$. We will consider the set $K'_0 = \bigcap_{\varepsilon\in(0,1)} \overline{U_\varepsilon}$. Now let K_0 be the filling of K'_0, and note that $K_0 \subset \overline{\mathbb{A}}$, that its complement contains only two connected components, U_+ which contains $\mathbb{T}^1 \times (1, \infty)$ and U_-, which contains $\mathbb{T}^1 \times (-\infty, 0)$, and both these components are topological sub-annuli of $\mathbb{A}'$.

LEMMA 6.4. – *Given a neighborhood W of K_0, there exists some $n_0 = n_0(W)$ such that for all $n > n_0$ the orbit of z_n remains in W for all time.*

Proof. – We may assume that W is open and contained in $\mathbb{T}^1 \times (-1, 2)$, because $W \cap (\mathbb{T}^1 \times (-1, 2))$ is also a neighborhood of K_0. Note that, if $\varepsilon < 1$, then $\mathbb{T}^1 \times (-1, 2)^C \subset \overline{U_\varepsilon}^C$. As $W^C \cap (\mathbb{T}^1 \times [-1, 2])$ is compact and as $(\overline{U_{\frac{1}{n}}}^C)_{n\in\mathbb{N}}$ is an increasing sequence of open sets whose union contains W^C, one deduces that there must exist some $n_0 > 1$ such that $\overline{U_{\frac{1}{n_0}}}^C$ contains both $W^C \cap (\mathbb{T}^1 \times [-1, 2])$ and $\mathbb{T}^1 \times (-1, 2)^C$. Therefore $W \supset \overline{U_{\frac{1}{n_0}}}$. Since the later set is invariant and contains z_n for sufficiently large n, the result follows. □

If K_0 has empty interior, then Theorem 2.8 of [22] shows that the rotation number of any point in K_0 is the same, and this number must be ρ by the previous lemma and Lemma 2.17, and so we can take $Q_\rho = K_0$. So we can assume that the interior of K_0 is not empty. If the interior of K_0 is inessential, then every point in K_0 has the same rotation number by Proposition 2.18, and this number must be ρ by the previous lemma and Lemma 2.17, and so we can take $Q_\rho = K_0$. So we can assume that the interior of K_0 is essential. Therefore ∂U_- is disjoint from ∂U_+, and their union is ∂K_0. We also remark that, as K_0 is contained in $\overline{\mathbb{A}}$, the restriction of the dynamics to K_0 is non-wandering. Let $\mathbb{A}^*$ be the interior of K_0, which is an open sub-annulus. In the following, we examine several different possibilities, and we conclude the result holds in each of them.

Case 1. – Assume $\overline{z}$ belongs to ∂K_0. We claim that in this case, $\mathbb{A}^*$ must be a Birkhoff region of instability. Let us first examine the case where $\overline{z}$ belongs to ∂U_-. If by contradiction the interior of K_0 is not a Birkhoff region of instability, as the restriction of f to the interior of K_0 is non-wandering, one can find V_-, V_+ open invariant and disjoint neighborhoods of ∂U_- and ∂U_+ respectively. But there must exist some $\delta > 0$ such that $B_\delta(\overline{z}) \subset V_-$ and therefore U'_δ, and U_δ, are disjoint from V_+. But this contradicts the fact that $\partial U_+ \subset \overline{U_\delta}$. A similar argument shows that $\mathbb{A}^*$ is a Birkhoff region of instability if $\overline{z} \in \partial U_+$.

Case 1.1. – Assume that there exists an infinite subsequence $(z_{n_k})_{k\in\mathbb{N}}$ such that, for all k, z_{n_k} does not belong to K_0. We can assume with no loss in generality that z_{n_k} belongs to U_+, and this implies that $\overline{z}$ belongs to ∂U_+. Note that every point in ∂U_+ has the same rotation number by Proposition 2.16. Further note that if W is a neighborhood of ∂U_+, then there exists a neighborhood W' of K_0 such that $W \cap U_+ = W' \cap U_+$ and therefore for sufficiently large k one gets by Lemma 6.4 that the whole orbit of z_{n_k} is contained in W. This again implies that the rotation number of every point in ∂U_+ must be ρ and we can take $Q_\rho = \partial U_+$.

Case 1.2. – Assume that for all but finitely many $n \in \mathbb{N}$, z_n lie in K_0. Since $\partial K_0 = \partial U_- \cup \partial U_+$, one deduces that the rotation number of points in ∂K_0 can have at most two values, and so we can assume that all z_n lie in $\mathbb{A}^*$. Let $\widetilde{f}$ be the extension of f to the prime ends compactification of $\mathbb{A}^*$, which is homeomorphic to $\overline{\mathbb{A}}$ and $\widehat{\widetilde{f}}$ a lift of $\widetilde{f}$ which is compatible with $\widehat{f}$. Note that the rotation set of $\widehat{\widetilde{f}}$ contains p_n/q_n for sufficiently large n and, by being closed, must also contain ρ. By Corollary 6.3 one finds a set Q'_ρ in the prime ends compactification of $\mathbb{A}^*$. If Q'_ρ lies in $\mathbb{A}^*$, we are done, just taking $Q_\rho = Q'_\rho$. If not, this implies that the prime end rotation number of f' at one of the ends of $\mathbb{A}^*$ must be ρ. In this case, one deduces, again by Proposition 2.16, that there exists a connected component Q of the boundary of $\mathbb{A}^*$ in $\mathbb{A}$ such that every point in Q has rotation number ρ and we are done.

Case 2. – Assume $\overline{z}$ belongs to $\mathbb{A}^*$. Again, if $\mathbb{A}^*$ is a Birkhoff region of instability, we can repeat the same argument as in the previous paragraph. Assume then that $\mathbb{A}^*$ is not a Birkhoff region of instability. In this case, one can find disjoint invariant neighborhoods V_- and V_+ of the ends of $\mathbb{A}^*$. By "filling", if necessary, V^- and V^+ with the connected components of its complement that are contained in $\mathbb{A}^*$, we can assume that both sets are essential open topological annuli.

We claim first that $\overline{z}$ belongs to the boundary of V^-. Indeed, if by contradiction one has that $\overline{z}$ is in the interior of V^-, then there exists some $\varepsilon > 0$ such that U_ε is contained in V^-, which contradicts $V^+ \subset K_0$. Likewise, if $\overline{z}$ lies is in the interior of the complement of V^-, then again there exists $\varepsilon > 0$ such that U_ε is disjoint from V^-, which contradicts $V^- \subset K_0$. A similar argument shows that $\overline{z} \in \partial V^+$. Let us then consider $K' = \partial V^- \cup \partial V^+$ and let K be the union of K' with the connected components of its complements that are contained in $\mathbb{A}^*$, and note that $\mathbb{A}^* \setminus K = V^- \cup V^+$ and that the interior of K is inessential, otherwise $\partial V^- \cap \partial V^+$ would be empty. Finally, note that both V^- and V^+ are Birkhoff regions of instability, the argument here being the same as in the case where $\overline{z}$ belonged to ∂K_0.

Since the interior of K is inessential, we know that every point in it has the same rotation set. We can therefore assume, by possibly erasing a term of the sequence, that z_n does not lie in K. Therefore either infinitely many of the points z_n lie in V^-, or infinitely many of them lie in V^+. But both V^- and V^+ are Birkhoff regions of instability, and the same reasoning as in the case where $\overline{z}$ belonged to ∂K_0 and the z_n lied in $\mathbb{A}^*$ can be applied to deduce the result.

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(Manuscrit reçu le 1er octobre 2019 ;
accepté le 8 septembre 2021.)

Jonathan Conejeros
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Avenida Vicuña Mackenna 4860
Macul, Santiago, Chile
E-mail: jaconeje@uc.cl

Fábio Armando Tal
Instituto de Matématica e Estatística
Universidade de São Paulo
Rua de Matão 1010, Cidade Universitária
05508-090 São Paulo, SP, Brazil
E-mail: fabiotal@ime.usp.br