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Identities on Units of Algebraic Algebras.

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IDENTITIES ON UNITS OF ALGEBRAIC ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be an algebraic algebra over an infinite field K and $\mathcal{U}(\mathcal{A})$ be its group of units. We prove a stronger version of Hartley's Conjecture for $\mathcal{A}$, namely, if a Laurent polynomial identity (LPI, for short) $f = 0$ is satisfied in $\mathcal{U}(\mathcal{A})$, then $\mathcal{A}$ satisfies a polynomial identity (PI). We also show that, if $\mathcal{A}$ is non-commutative, then $\mathcal{A}$ is PI, provided $f = 0$ is satisfied by the non-central units of $\mathcal{A}$. In particular, $\mathcal{A}$ is locally finite and, thus, the Kurosh Problem has a positive answer for K-algebras whose unit group is LPI. Moreover, $f = 0$ holds in $\mathcal{U}(\mathcal{A})$ if and only if the same identity is satisfied in $\mathcal{A}$. The last fact remains true for generalized Laurent polynomial identities, provided that $\mathcal{A}$ is locally finite.

1. INTRODUCTION

Let K be an infinite field, $\mathcal{A}$ be a unitary associative K-algebra and $\mathcal{U}(\mathcal{A})$ be its group of units. For algebras with "many units" several recent papers considered the following conjecture: *if $\mathcal{U}(\mathcal{A})$ satisfies a group identity (GI), or a semigroup identity (SI), then $\mathcal{A}$ is PI* [1], [3], [4], [6], [7], [8], [9], [10], [11], [14], [16]. In the case of group algebras of torsion groups this conjecture is attributed to Brian Hartley. In this note we consider a stronger version of Hartley's Conjecture for algebraic algebras, substituting GI or SI by an arbitrary Laurent polynomial identity (LPI). Moreover, we want to transfer the law from $\mathcal{U}(\mathcal{A})$ to $\mathcal{A}$. Namely, we consider the following question: *if $\mathcal{A}$ is an algebra with "many" units such that $\mathcal{U}(\mathcal{A})$ is LPI (in particular, GI or SI), is it true that $\mathcal{A}$ satisfies the same identity?* We also consider such transferring from the non-central units of $\mathcal{A}$ to $\mathcal{A}$, when $\mathcal{A}$ is non-commutative.

By a Laurent polynomial $f = f(\zeta_1, \zeta_2, \ldots, \zeta_k)$ we mean a non-zero element of KF_k, the group algebra over the field K of the free group F_k freely generated by $\zeta_1, \zeta_2, \ldots, \zeta_k$. A generalized Laurent polynomial is a non-zero element of the free product (coproduct) $\mathcal{A} *_K KF_k$ of $\mathcal{A}$ and KF_k over K.

Let $f = f(\zeta_1, \zeta_2, \ldots, \zeta_k)$ be a generalized Laurent polynomial which does not involve the inverses of $\zeta_1, \ldots, \zeta_s$ $(0 \leq s \leq k)$ and the inverses of $\zeta_{s+1}, \ldots, \zeta_k$ do appear in f. We say that $\mathcal{A}$ satisfies the identity $f = 0$ if this equality always holds after substituting the $\zeta_1, \ldots, \zeta_s$ by arbitrary elements of $\mathcal{A}$ and the $\zeta_{s+1}, \ldots, \zeta_k$ by arbitrary units of $\mathcal{A}$. Then we come to the concept of a group identity $u = 1$ in its usual sense if we take $f = u - 1$, where $u = u(\zeta_1, \ldots, \zeta_k)$ is an element of F_k. If all the inverses of ζ_i's are involved in u

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(or f), then $s=0$ and only units of $\mathcal{A}$ are allowed to be substituted in u (or f).

Depending on the way that an identity is written, we can extract more or less information on the algebra structure, when transferring the identity from $\mathcal{U}(\mathcal{A})$ to $\mathcal{A}$. Let us illustrate this with the following example.

Let $\mathcal{A}$ be an algebra such that the identities (GI, LPI or GLPI) are transferable from $\mathcal{U}(\mathcal{A})$ to $\mathcal{A}$. If $\mathcal{U}(\mathcal{A})$ satisfies the identity $x^{-1}y^{-1}xy = 1$ then the same will be true for $\mathcal{A}$, however, since only invertible elements are allowed, this does not give anything new about $\mathcal{A}$. On the other hand, if we say that $\mathcal{U}(\mathcal{A})$ satisfies $xy = yx$ then $\mathcal{A}$ is also commutative, the conclusion becomes different.

This note is structured in the following way. In Section 2 we consider locally finite algebras $\mathcal{A}$ over an infinite field and prove that if $\mathcal{U}(\mathcal{A})$ satisfies a GLPI $f=0$ then the same identity holds in $\mathcal{A}$. Moreover, if $\mathcal{A}$ is non-commutative then $f=0$ holds in $\mathcal{A}$ if and only if $f=0$ is satisfied by the non-central units of $\mathcal{A}$ (Theorem 1). In particular, a generalized polynomial, group or semigroup law can be transferred from the non-central units to $\mathcal{A}$. We also give easy applications to group algebras, twisted group algebras and nil-generated unitary algebras (Corollary 4).

In Section 3 we prove that if $\mathcal{A}$ is an algebraic algebra over an infinite field K such that $\mathcal{U}(\mathcal{A})$ satisfies an LPI $f=0$, then a PI holds in $\mathcal{A}$. Moreover, if $\mathcal{A}$ is non-commutative, then $\mathcal{A}$ is PI, provided $f=0$ is satisfied on the non-central units of $\mathcal{A}$ (Theorem 5). This establishes a strong version of Hartley's Conjecture. It follows that $\mathcal{A}$ is locally finite and thus $\mathcal{A}$ satisfies $f=0$. In particular, a PI can be transferred from $\mathcal{U}(\mathcal{A})$ to $\mathcal{A}$ (Corollary 9), and the Kurosh Problem has a positive answer for K-algebras whose non-central units satisfy an LPI.

2. Locally finite algebras

Theorem 1. *Let $\mathcal{A}$ be a locally finite algebra over an infinite field K. Then*
(i) $\mathcal{A}$ satisfies a GLPI if and only if the same identity holds in $\mathcal{U}(\mathcal{A})$.
(ii) If $\mathcal{A}$ is non-commutative, then a GLPI holds in $\mathcal{A}$ if and only if the same identity is satisfied by the non-central units of $\mathcal{A}$. In particular, $\mathcal{A}$ satisfies a generalized polynomial identity $f=0$ if and only if $f=0$ holds on the non-central units of $\mathcal{A}$.

Proof. (i) The 'only if' part is obvious, so suppose that $\mathcal{U}(\mathcal{A})$ satisfies a GLPI $f=0$ where $f = f(\zeta_1, \zeta_2, \ldots, \zeta_k)$ does not involve the inverses of $\zeta_1, \zeta_2, \ldots, \zeta_s$ $(0 \le s \le k)$. If $s=0$ this means that the inverses of all variables appear in f. In this case the claim is true by vacuity, since we are not allowed to make substitutions in f by non-invertible elements of $\mathcal{A}$. One may assume, therefore, that s is at least 1.

It is also clear that we may suppose that $\mathcal{A}$ is of finite dimension n. Fixing a K-basis of $\mathcal{A}$ we consider an arbitrary element $x \in \mathcal{A}$ as a point $(x_1, \ldots, x_n)$ in the affine space

$\mathcal{A} = K^n$. For a subset $Y \subset \mathcal{A}$ denote by $I(Y)$ the ideal of all polynomials on n variables $x_1, \ldots, x_n$ which annihilate each $y \in Y$. Then it is well known that $I(Y) = I(\overline{Y})$, where $\overline{Y}$ is the closure of Y in $\mathcal{A}$ with respect to the Zariski topology on $\mathcal{A}$.

It is easily seen that $\mathcal{U}(\mathcal{A})$ is open in $\mathcal{A}$. Indeed, let $\Gamma : \mathcal{A} \to M_n(K)$ be the regular representation of $\mathcal{A}$. If $det(\Gamma(x)) \neq 0$ then $\Gamma(x)$ is invertible in $M_n(K)$, thus it can not be a zero divisor in $\Gamma(\mathcal{A})$. Since an element in a finite dimensional algebra is either a zero divisor or invertible, it follows that $\Gamma(x)$ is invertible in $\Gamma(\mathcal{A})$. Now, $det(\Gamma(x))$ can be considered as a polynomial h in coordinates $x_1, \ldots, x_n$ of x. Hence

$$\mathcal{U}(\mathcal{A}) = \{(x_1, \ldots, x_n) \in \mathcal{A} \; : \; h(x_1, \ldots, x_n) \neq 0\},$$

and $\mathcal{U}(\mathcal{A})$ is an open subset in $\mathcal{A}$.

Fix elements $a_1, \ldots, a_{s-1}, b_1, \ldots, b_{s'} \in \mathcal{U}(\mathcal{A})$ ($s' = k - s$). If we substitute in

(1) $$f(a_1, \ldots, a_{s-1}, x, b_1, \ldots, b_{s'}) = 0$$

the elements $a_1, \ldots, a_{s-1}, x, b_1, \ldots, b_{s'}$ by their coordinates with respect to the fixed basis, we obtain n equalities $f_1 = 0, \ldots, f_n = 0$, where each f_i is a polynomial on variables $x_1, ..., x_n$. Observe that for a given element $y = (y_1, \ldots, y_n) \in \mathcal{A}$ the equality (1) is satisfied when substituting x by y if and only if $f_i(y_1, \ldots, y_n) = 0$ for each $i = 1, \ldots, n$. Since all f_i annihilate $\mathcal{U}(\mathcal{A})$, then

$$f_i(x_1, \ldots, x_n) \in I(\mathcal{U}(\mathcal{A})) = I(\overline{\mathcal{U}(\mathcal{A})}) \quad (i = 1, \ldots, n).$$

Because $\mathcal{A}$ is an irreducible topological space and $\mathcal{U}(\mathcal{A})$ is open in $\mathcal{A}$, it follows that $\overline{\mathcal{U}(\mathcal{A})} = \mathcal{A}$ and, thus, each f_i annihilates $\mathcal{A}$. Coming back to our original identity this means that

$$f(a_1, \ldots, a_{s-1}, c_s, b_1, \ldots, b_{s'}) = 0$$

holds for all $a_1, \ldots, a_{s-1}, b_1, \ldots, b_{s'} \in \mathcal{U}(\mathcal{A})$, and each $c_s \in \mathcal{A}$. Suppose that we already proved that the identity

$$f(a_1, \ldots, a_t, c_{t+1}, \ldots, c_s, b_1, \ldots, b_{s'}) = 0$$

holds for all $a_1, \ldots, a_t \in \mathcal{U}(\mathcal{A}), c_{t+1}, \ldots, c_s \in \mathcal{A}$ and $b_1, \ldots, b_{s'} \in \mathcal{U}(\mathcal{A})$. Fixing units $a_1, \ldots, a_{t-1}, b_1, \ldots, b_{s'} \in \mathcal{U}(\mathcal{A})$ and elements $c_{t+1}, \ldots, c_s \in \mathcal{A}$ and taking on the t^{th} place a variable element $x = (x_1, \ldots, x_n)$, we see, again, that

(2) $$f(a_1, \ldots, a_{t-1}, x, c_{t+1}, \ldots, c_s, b_1, \ldots, b_{s'}) = 0$$

gives rise to n equalities $g_1 = 0, \ldots, g_n = 0$, where each $g_i(x_1, \ldots, x_n)$ is a polynomial, which annihilates $\mathcal{U}(\mathcal{A})$. As above, we obtain that it has to annihilate $\mathcal{A}$. Hence, (2) holds for all $x \in \mathcal{A}$. Thus, by induction, we conclude that the identity $f = 0$ is satisfied on $\mathcal{A}$.

(ii) Suppose now that $\mathcal{A}$ is non-commutative and the identity $f = 0$ is satisfied by the non-central units of $\mathcal{A}$. By item (i) it is enough to show that $f = 0$ is satisfied on $\mathcal{U}(\mathcal{A})$.

Observe first that $\mathcal{A}$ is generated, as a vector space over K, by $\mathcal{U}(\mathcal{A})$. Indeed, the K-linear span $\mathcal{B}$ of $\mathcal{U}(\mathcal{A})$ is a subspace of $\mathcal{A}$ and, therefore, is closed in $\mathcal{A}$ with respect to the Zariski topology. Thus, $\mathcal{A} = \overline{\mathcal{U}(\mathcal{A})} \subset \mathcal{B}$ which gives $\mathcal{A} = \mathcal{B}$, as desired.

Since $\mathcal{A}$ is non-commutative, the set $\mathcal{V}$ of the non-central units of $\mathcal{A}$ is non-empty. It can be easily seen that the centralizer $C_{\mathcal{U}(\mathcal{A})}(y)$ of an element $y \in \mathcal{U}(\mathcal{A})$ in $\mathcal{U}(\mathcal{A})$ is closed in $\mathcal{U}(\mathcal{A})$, since the condition $xy = yx$ can be written as a system of polynomial equations in coordinates of $x \in \mathcal{U}(\mathcal{A})$. Hence the centre of $\mathcal{U}(\mathcal{A})$ is closed as an intersection of closed sets and thus $\mathcal{V}$ is a non-empty open subset of $\mathcal{U}(\mathcal{A})$. Because $\mathcal{A} = \overline{\mathcal{U}(\mathcal{A})}$ is an irreducible space, it follows that $\mathcal{U}(\mathcal{A})$ is also irreducible and, consequently, $\mathcal{V}$ is dense in $\mathcal{U}(\mathcal{A})$. This yields that $I(\mathcal{V}) = I(\mathcal{U}(\mathcal{A}))$.

Fix elements $a_1, \dots, a_{k-1} \in \mathcal{V}$ and write $f(a_1, \dots, a_{k-1}, x) = 0$ as a system of polynomial equations $h_1 = 0, \dots, h_n = 0$ in variables $x_1, \dots, x_n$. Since each h_i $(i = 1, \dots, n)$ annihilates $\mathcal{V}$, it follows that $h_i \in I(\mathcal{U}(\mathcal{A}))$ and, consequently,

$$f(a_1, \dots, a_{k-1}, x) = 0$$

for all $x \in \mathcal{U}(\mathcal{A})$. As above, going by induction, we conclude that $f = 0$ holds on $\mathcal{U}(\mathcal{A})$. □

Remark 2. Our proof shows that for a finite dimensional K-algebra $\mathcal{A}$ a GLPI can be transferred to $\mathcal{A}$ from an arbitrary non-empty open subset (in, particular, from a non-empty open subset of units).

The above result suggests the following problem:

Problem. *Let $\mathcal{A}$ be an algebra over an infinite field K such that $\mathcal{A}$ is generated by its units as a vector space over K. Is it true that if $\mathcal{U}(\mathcal{A})$ satisfies a semigroup (group, polynomial, Laurent polynomial etc.) identity then $\mathcal{A}$ satisfies the same identity?*

Remark 3. Since a finite dimensional K-algebra is generated as a vector space over K by its units this fact also holds for algebraic K-algebras.

Corollary 4. *Let K be an infinite field and suppose that one of the following conditions is satisfied:*

(i) $\mathcal{A}$ is a nil-generated unitary K-algebra.

(ii) $\mathcal{A}$ is the group algebra KG of a torsion group G.

(iii) K is algebraically closed and $\mathcal{A}$ is a twisted group algebra K^tG of a torsion group G.

Then $\mathcal{U}(\mathcal{A})$ satisfies a GI if and only if the same identity is satisfied in $\mathcal{A}$. In particular, $\mathcal{U}(\mathcal{A})$ satisfies a SI if and only if a binomial identity holds in $\mathcal{A}$.

Proof. In view of the above theorem it suffices to show in all cases that $\mathcal{A}$ is locally finite.

(i) By [1], $\mathcal{A}$ satisfies a polynomial identity. Moreover, it follows from [1, Lemma 2.2] that $\mathcal{A}$ is algebraic. Hence by [12, Theor. 6.4.3] $\mathcal{A}$ is locally finite.

(ii) By [8] $\mathcal{A}$ satisfies a polynomial identity. Therefore, [15, pp. 196-197] implies that G is locally finite and hence $\mathcal{A}$ is locally finite too.

(iii) By Theorem 5.1 and Theorem 3.8 of [3] $[G : \Delta(G)] < \infty, |\Delta'(G)| < \infty$, where $\Delta(G)$ is the FC-centre of G and $\Delta'(G)$ its commutator subgroup, Since G is torsion, it follows that G is locally finite and, consequently, $\mathcal{A}$ is locally finite. □

3. Algebraic algebras

Theorem 5. *Let $\mathcal{A}$ be an algebraic algebra over an infinite field K. If $\mathcal{U}(\mathcal{A})$ is LPI then $\mathcal{A}$ is PI. Moreover, if $\mathcal{A}$ is non-commutative then $\mathcal{A}$ is PI, provided that the non-central units of $\mathcal{A}$ satisfy an LPI.*

We fix first some notation. Let KF be the group K-algebra of the free group F on two free generators x_1 and x_2, and $K\langle y_1, y_2\rangle$ the free algebra on y_1 and y_2 over K. Denote by $K\langle y_1, y_2\rangle[[\lambda_1, \lambda_2]]$ the ring of formal power series in the commutative indeterminates λ_1 and λ_2 with coefficients in $K\langle y_1, y_2\rangle$.

We also need to introduce the ring $\mathcal{A}[\lambda_1, \lambda_2]_S$, the localization of the polynomial ring $\mathcal{A}[\lambda_1, \lambda_2]$ in the two commutative indeterminates λ_1 and λ_2, by the multiplicative set S of all polynomials in λ_1 and λ_2 with coefficients in K, having non-zero constant term.

Let $a \in \mathcal{A}$ be such that $p(a) = 0$, where $p(x) = x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n \in K[x], n > 1$, and let λ be a non-zero element of K. Set $a_\lambda = 1 + \lambda a$. Then a_λ is a root of the polynomial

$$h_\lambda(x) = p((x-1)/\lambda).$$

Moreover, a_λ is invertible, except for finitely many values of λ in K. In fact, if a is nilpotent, then for each $\lambda \in K$

$$a_\lambda^{-1} = 1 - \lambda a + \lambda^2 a^2 + \ldots + (-1)^{n-1}\lambda^{n-1}a^{n-1}. \tag{3}$$

If a is not nilpotent, then

$$a_\lambda^{-1} = -[a_\lambda^{n-1} + \beta_1(\lambda)a_\lambda^{n-2} + \ldots + \beta_{n-1}(\lambda)]/[\lambda^n\beta_n(\lambda)], \tag{4}$$

where

$$h_\lambda(x) = \lambda^{-n}[x^n + \beta_1(\lambda)x^{n-1} + \ldots + \beta_{n-1}(\lambda)x + \beta_n(\lambda)],$$

$$\lambda^{-n}\beta_n(\lambda) = h_\lambda(0) = p(-1/\lambda),$$

and each $\beta_i(\lambda)$ is a polynomial in λ over K.

We start with an easy preliminary result.

Lemma 6. *Let $f(\lambda_1, \lambda_2)$ be an element of $\mathcal{A}[\lambda_1, \lambda_2]_S$. Assume that there are two infinite subsets T_1 and T_2 in K, such that for all $\gamma_i \in T_i, 1 \leq i \leq 2, f(\gamma_1, \gamma_2) = 0$. Then f is identically zero.*

Proof. It follows by applying the standard Vandermonde argument. □

We shall essentially use the following fact due to Makar-Limanov [13].

Lemma 7. *The homomorphism $\tau : KF \longrightarrow K\langle y_1, y_2\rangle[[\lambda_1, \lambda_2]]$ defined by*

$$\tau(x_i) = 1 + \lambda_i y_i,$$

$$\tau(x_i^{-1}) = \sum_{j=0}^{\infty}(-\lambda_i y_i)^j \quad (1 \leq i \leq 2),$$

is injective.

We also need to transfer a polynomial identity from the non-central elements to the entire algebra:

Lemma 8. *Let $\mathcal{A}$ be a non-commutative K-algebra. If a polynomial identity $P(x_1, x_2) = 0$ is satisfied by the non-central elements of $\mathcal{A}$ then $P = 0$ holds in $\mathcal{A}$.*

Proof. We may suppose that P is homogeneous in each variable, i.e. every monomial of P has degree m in x_1 and degree s in x_2 (see [15, p. 215]). Let $a, b, c, z \in \mathcal{A}$ such that a, b are non-central and c, z are central. Then $P(a + c, b + z) = 0$ and write $P(a + c, b + z) = \sum_{i,j} \Phi_{ij}(a, b)c^i z^j$, where $\Phi_{ij}(a, b)$ are polynomials in a and b with coefficients in K. We see that $\Phi_{m0}(a, b)c^m = P(c, b)$, $\Phi_{0s}(a, b)z^s = P(a, z)$ and $\Phi_{ms}(a, b)c^m z^s = P(c, z)$. Taking arbitrarily $c \in K, z \in K$ we see by the Vandermonde determinant argument that each $\Phi_{ij}(a, b)$ is zero. Hence, $P(c, b) = P(a, z) = P(c, z) = P(a, b) = 0$ and $P = 0$ holds in $\mathcal{A}$. □

Now we are ready to prove the theorem.

Proof of Theorem 5. Since the free group on two generators contains a free group on any finite number of generators, we may suppose that $\mathcal{U}(\mathcal{A})$ satisfies a two variable Laurent polynomial $0 \neq f(x_1, x_2) \in KF$. Fix $a, b \in \mathcal{A}$ and consider the elements $a_{\lambda_1}, b_{\lambda_2}, a_{\lambda_1}^{-1}, b_{\lambda_2}^{-1}$, which are contained in $\mathcal{A}[\lambda_1, \lambda_2]_S$ (see (3) and (4)).

We have the following commutative diagram of K-algebras and homomorphisms:

$$\begin{array}{ccccc}
0 & \longrightarrow & KF & \xrightarrow{\tau} & K\langle y_1, y_2\rangle[[\lambda_1, \lambda_2]] \\
 & & \downarrow \phi & & \downarrow \psi \\
0 & \longrightarrow & \mathcal{A}[\lambda_1, \lambda_2]_S & \xrightarrow{i} & \mathcal{A}[[\lambda_1, \lambda_2]] \\
 & & \downarrow \epsilon & & \\
 & & \mathcal{A} & &
\end{array}$$

Here ϕ is determined by $x_1 \to 1 + \lambda_1 a$, $x_2 \to 1 + \lambda_2 b$, ϵ is the evaluation given by $\lambda_1 \to \alpha_1$, $\lambda_2 \to \alpha_2$ for admissible $\alpha_1, \alpha_2 \in K$, ψ is defined by $y_1 \to a$, $y_2 \to b$, i denotes the inclusion of $\mathcal{A}[\lambda_1, \lambda_2]_S$ into $\mathcal{A}[[\lambda_1, \lambda_2]]$ and τ is defined in Lemma 7.

By Lemma 7, $0 \neq \tau(f) = \sum \lambda_1^i \lambda_2^j P_{ij}(y_1, y_2)$, where $P_{ij} \in K\langle y_1, y_2\rangle$, $deg_{y_1} P_{ij} = i$, $deg_{y_2} P_{ij} = j$, $P_{00} = f(1,1) = 0$. Take fixed indices i_0, j_0 such that $P_{i_0,j_0}(y_1, y_2)$ is a non-zero polynomial.

Looking at the diagram we see on one side, that $\phi(f)$ is an element of $\mathcal{A}[\lambda_1, \lambda_2]_S$, which by the inclusion i, can be seen also as an element of $\mathcal{A}[[\lambda_1, \lambda_2]]$. Since by the evaluation $\phi(f)$ goes to zero in $\mathcal{A}$, it follows by Lemma 6 that $\phi(f)$ is identically zero. On the other side, the commutativity of the diagram implies that $0 = (i \circ \phi)(f) = (\psi \circ \tau)(f)$. This shows that $P_{i_0,j_0}(a, b) = 0$. Since $a, b \in \mathcal{A}$ were arbitrary, it follows that $P_{i_0,j_0}(y_1, y_2) = 0$ is a polynomial identity for $\mathcal{A}$. This proves the first statement of the theorem.

Suppose now that $\mathcal{A}$ is non-commutative and the non-central units of $\mathcal{A}$ satisfy an LPI $g(\zeta_1, \zeta_2, \ldots, \zeta_k) = 0$. It is easily seen that $\zeta_i \to x_2^{-i} x_1 x_2^i$ determines a monomorphism of the free group F_k with free generators $\zeta_1, \zeta_2, \ldots, \zeta_k$ into the free group F of rank 2 freely generated by x_1, x_2. Thus $g(\zeta_1, \zeta_2, \ldots, \zeta_k)$ can be written as a Laurent polynomial $f(x_1, x_2)$ and $f = 0$ is satisfied by the non-central units of $\mathcal{A}$. The above considerations show then that the non-central elements of $\mathcal{A}$ satisfy a polynomial identity $P(x_1, x_2) = 0$. Lemma 8 implies now that $\mathcal{A}$ satisfies $P = 0$. □

Corollary 9. *If $\mathcal{A}$ is an algebraic algebra over an infinite field and $\mathcal{U}(\mathcal{A})$ satisfies a Laurent polynomial identity $f = 0$, then $\mathcal{A}$ is locally finite and satisfies the same identity $f = 0$. In particular, if a polynomial (semigroup) identity is satisfied by $\mathcal{U}(\mathcal{A})$ then the same identity holds on $\mathcal{A}$. If $\mathcal{A}$ is non-commutative, then these facts remains true when substituting $\mathcal{U}(\mathcal{A})$ by the set of the non-central units of $\mathcal{A}$.*

Proof. Indeed, by the positive answer to the Kurosh's Problem for P.I. algebras [12, Theorem 6.4.3], $\mathcal{A}$ is locally finite and Theorem 1 implies that $\mathcal{A}$ satisfies $f = 0$.
□

Corollary 10. *Let K be an infinite field.*

(i) The Kurosh Problem has a positive answer for K-algebras whose non-central units satisfy an LPI.

(ii) Hartley's Conjecture holds for algebraic algebras over K.

Remark 11. While this note was in preparation, an article by Chia-Hsin Liu [5] became electronically available. Among other results, the author proves Hartley's Conjecture for algebraic algebras over infinite fields. Thus, Theorem 5 can be considered as an extension of his result. We also observe that our methods are completely different.

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