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GENERIC SIMPLICITY FOR THE SOLUTIONS OF A NONLINEAR PLATE EQUATION

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Generic simplicity for the solutions of a nonlinear plate equation

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Abstract

In this work we show the solutions of the Dirichlet problem for a semilinear equation with the Bilaplacian as its linear part are generically simple in the set of C^4-regular regions.

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1 Introduction

Perturbation of the boundary for boundary value problems in PDEs have been investigated by several authors, from various points of view, since the pioneering works of Rayleigh ([11]) and Hadamard ([1]).

In particular, generic properties for solutions of boundary value problems have been considered in [4], [5], [6], [7], [8], [10], [12] and [14].

More recently several works appeared in a related topic, generally known as 'shape analysis' or 'shape optimization', on which the main issue is to determine conditions for a region to be optimal with respect to some cost functional. Among others, we mention [13] and [15].

Many problems of this kind have also been considered by D. Henry in [2] where a kind of differential calculus with the domain as the independent variable was developed. This approach allows the utilization of standard analytic tools such as Implicit Function Theorems and the Lyapunov-Schmidt method. In his work, Henry also formulated and proved a generalized form of the Transversality Theorem, which ia the main tool we use in our arguments.

We consider here the semilinear equation

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$$\begin{cases} \Delta^2 u(x) + f(x, u(x), \nabla u(x), \Delta u(x)) = 0 & x \in \Omega \\ u(x) = \frac{\partial u(x)}{\partial N} = 0 & x \in \partial\Omega \end{cases}$$

where $f(x, \lambda, y, \mu)$ is a C^4 real function in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ with $f(x, 0, 0, 0) \equiv 0$ for all $x \in \mathbb{R}^n$.

We show that, for a residual set of regions $\Omega \subset \mathbb{R}^n$ (in a suitable topology), the solutions u of (1) are all simple, that is , the linearisation

$$\begin{aligned} L(u) : \dot{u} \quad \rightarrow \quad & \Delta^2 \dot{u} + \frac{\partial f}{\partial \mu}(\cdot, u, \nabla u, \Delta u)\Delta \dot{u} \\ & + \frac{\partial f}{\partial y}(\cdot, u, \nabla u, \Delta u) \cdot \nabla \dot{u} + \frac{\partial f}{\partial \lambda}(\cdot, u, \nabla u, \Delta u)\dot{u} \end{aligned}$$

is an isomorphism.

Our results can be seen as an extension of similar results for reaction-diffusion equations obtained by Saut and Teman ([12]) and Henry ([2]).

This paper is organized as follows: in section 2 we collect some results we need from [2]. In section 3 we prove that the differential operator

$$L = \Delta^2 + a(x)\Delta + b(x) \cdot \nabla + c(x) \quad x \in \mathbb{R}^n$$

is, generically, an isomorphism in the set of C^4-regular regions $\Omega \subset \mathbb{R}^n$. This result is used in section 4 to prove our main result, the generic simplicity of solutions of (1). The most difficult point there is the proof that a certain (pseudo differential) operator is not of finite range. This was proved in a separate work ([9]).

2 Preliminaries

The results in this section were taken from the monograph of Henry [2], where full proofs can be found.

2.1 Differential Calculus of Boundary Perturbation

Given an open bounded, C^m region $\Omega_0 \subset \mathbb{R}^n$, consider the following open subset of $C^m(\Omega, \mathbb{R}^n)$

$$\text{Diff}^k(\Omega) = \{h \in C^k(\Omega, \mathbb{R}^n) \mid h \text{ is injective and } 1/|\det h'(x)| \text{ is bounded in } \Omega\}$$

We introduce a topology in the collection of all regions $\{h(\Omega) \mid h \in \text{Diff}^k(\Omega)\}$. by defining a (sub-basis of) the neighborhoods of a given Ω by

$$\{h(\Omega_0) \mid \|h - i_{\Omega_0}\|_{C^k(\Omega_0, \mathbb{R}^n)} < \epsilon, \ \epsilon \text{ sufficiently small}\}.$$

When $\|h - i_\Omega\|_{C^m(\Omega, \mathbb{R}^n)}$ is small, h is a C^m imbedding of Ω in $\mathbb{R}^n$, a C^m diffeomorphism to its image $h(\Omega)$. Micheletti shows in [4] that this topology is metrizable, and the set of regions C^m-diffeomorphic to Ω may be considered a separable metric space which we denote by $\mathcal{M}_m(\Omega)$, or simply $\mathcal{M}_m$. We say that a function F defined in the space $\mathcal{M}_m$ with values in a Banach space is C^m or analytic if $h \mapsto F(h(\Omega))$ is C^m or analytic as a map of Banach spaces (h near i_Ω in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces. More specifically, consider a formal non-linear differential operator $u \to v$

$$v(x) = f\Big(x, u(x), Lu(x)\Big),\ x \in \mathbb{R}^n$$

where

$$Lu(x) = \Big(u(x), \frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x), \frac{\partial^2 u}{\partial x_1^2}(x), \frac{\partial^2 u}{\partial x_1 \partial x_2}(x), \ldots\Big),\ x \in \mathbb{R}^n$$

More precisely, suppose $Lu(\cdot)$ has values in $\mathbb{R}^p$ and $f(x, \lambda)$ is defined for (x, λ) in some open set $O \subset \mathbb{R}^n \times \mathbb{R}^p$. For subsets $\Omega \subset \mathbb{R}^n$ define F_Ω by

$$F_\Omega(u)(x) = f(x, Lu(x)),\ x \in \Omega \tag{1}$$

for sufficiently smooth functions u in Ω such that $(x, Lu(x)) \in O$ for any $x \in \bar{\Omega}$.

Let $h : \Omega \longrightarrow \mathbb{R}^n$ be C^m imbedding. We define the composition map (or *pull-back*) h^* of h by

$$h^* u(x) = (u \circ h)(x) = u(h(x)),\ x \in \Omega$$

where u is a function defined in $h(\Omega)$. Then h^* is an isomorphism from $C^m(h(\Omega))$ to $C^m(\Omega)$ with inverse $h^{*-1} = (h^{-1})^*$. The same is true in other function spaces.

The differential operator

$$F_{h(\Omega)} : D_{F_{h(\Omega)}} \subset C^m(h(\Omega)) \longrightarrow C^0(h(\Omega))$$

given by (1) is called the *Eulerian* form of the formal operator $v \mapsto f(\cdot, Lv(\cdot))$, whereas

$$h^* F_{h(\Omega)} h^{*-1} : h^* D_{F_{h(\Omega)}} \subset C^m(\Omega) \longrightarrow C^0(\Omega)$$

is called the *Lagrangian* form of the same operator.

The Eulerian form is often simpler for computations, while the Lagrangian form is usually more convenient to prove theorems, since it acts in spaces of functions that do not depend on h, facilitating the use of standard tools such as the Implicit Function or the Transversality theorem. However, a new variable, h is introduced. We then need to study the differentiability properties of the map

$$(u, h) \mapsto h^* F_{h(\Omega)} h^{*-1} u \tag{2}$$

This has been done in [2] where it is shown that, if $(y, \lambda) \mapsto f(y, \lambda)$ is C^k or analytic then so is the map above, considered as a map from $Diff^m(\Omega) \times C^m(\Omega)$ to $C^0(\Omega)$ (other function spaces can be used instead of C^m). To compute the derivative we then need only compute the Gateaux derivative that is, the t-derivative along a smooth curve $t \mapsto (h(t,.), u(t,.)) \in Diff^m(\Omega) \times C^m(\Omega)$. For this purpose, its convenient to introduce the differential operator

$$D_t = \frac{\partial}{\partial t} - U(x,t)\frac{\partial}{\partial x}, \quad U(x,t) = (\frac{\partial h}{\partial x})^{-1}\frac{\partial h}{\partial t}$$

which is is called the *anti-convective derivative*. The results below (theorem 1) is are the main tools we use to compute derivatives.

Theorem 1 *Suppose $f(t,y,\lambda)$ is C^1 in an open set in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$, L is a constant-coefficient differential operator of order $\leq m$ with $Lv(y) \in \mathbb{R}^p$ (where defined). For open sets $Q \subset \mathbb{R}^n$ and C^m functions v on Q, let $F_Q(t)v$ be the function*

$$y \longrightarrow f(t,y,Lv(y)),\ y \in Q.$$

where defined.

Suppose $t \longrightarrow h(t,\cdot)$ is a curve of imbeddings of an open set $\Omega \subset \mathbb{R}^n$, $\Omega(t) = h(t,\Omega)$ and for $|j| \leq m$, $|k| \leq m+1$ $(t,x) \longrightarrow \partial_t\partial_x^j h(t,x)$, $\partial_x^k h(t,x)$, $\partial_x^k u(t,x)$ are continuous on $\mathbb{R} \times \Omega$ near $t=0$, and $h(t,\cdot)^{*^{-1}}u(t,\cdot)$ is in the domain of $F_{\Omega(t)}$. Then, at points of Ω

$$D_t(h^* F_{\Omega(t)}(t)h^{*^{-1}})(u) = (h^* \dot{F}_{\Omega(t)}(t)h^{*^{-1}})(u) + (h^* F'_{\Omega(t)}(t)h^{*^{-1}})(u) \cdot D_t u$$

where D_t is the anti-convective derivative defined above,

$$\dot{F}_Q(t)v(y) = \frac{\partial f}{\partial t}(t,y,Lv(y))$$

and

$$F'_Q(t)v \cdot w(y) = \frac{\partial f}{\partial \lambda}(t,y,Lv(y)) \cdot Lw(y),\ y \in Q$$

is the linearisation of $v \longrightarrow F_Q(t)v$.

Example. Let $f(x,\lambda,y,\mu)$ be a smooth function in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and consider the nonlinear differential operator

$$F_\Omega(v)(x) = \Delta^2 v(x) + f(x, v(x), \nabla v(x), \Delta v(x))$$

does not depending explicitly on t. Suppose also that $h(t,x) = x + tV(x) + o(t)$ in a neighborhood of $t=0$ and $x \in \Omega$.Then, since $\frac{\partial}{\partial t}\Big(F_\Omega(u)\Big) = 0$ and $F'_\Omega(u)\cdot w = L(u)w$, we have, by Theorem 1

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(h^* F_{h(\Omega)} h^{*^{-1}}(u)\Big) &= D_t\Big(h^* F_{h(\Omega)} h^{*^{-1}}(u)\Big)\Big|_{t=0} - h_x^{-1}h_t\nabla\Big(h^* F_{h(\Omega)} h^{*^{-1}}(u)\Big)\Big|_{t=0} \\
&= h^* F'_{h(\Omega)} h^{*^{-1}}(u)\cdot D_t(u)\Big|_{t=0} - h_x^{-1}h_t\nabla\Big(h^* F_{h(\Omega)} h^{*^{-1}}(u)\Big)\Big|_{t=0} \\
&= L(u)\Big(\frac{\partial u}{\partial t} - V\cdot\nabla u\Big) \\
&\quad -V\cdot\nabla\Big(\Delta^2 u + f(x,u(x),\nabla u(x),\Delta u(x))\Big)
\end{aligned}$$

where

$$L(u) = \Delta^2 + \frac{\partial f}{\partial \mu}(\cdot, u, \nabla u, \Delta u)\Delta + \frac{\partial f}{\partial y}(\cdot, u, \nabla u, \Delta u) \cdot \nabla + \frac{\partial f}{\partial \lambda}(\cdot, u, \nabla u, \Delta u).$$

2.2 Change of origin

We can always transfer the 'origin' or reference region from any $\Omega \subset \mathbb{R}^n$, to another diffeomorphic region. Indeed, if $\tilde{H} : \Omega \to \tilde{\Omega}$ is a diffeomorphism we define, for any imbedding $h : \Omega \to \mathbb{R}^n$, another imbedding $\tilde{h} = h \circ \tilde{H}^{-1} : \tilde{\Omega} \to \mathbb{R}^n$. If $\tilde{x} = \tilde{H}(x)$, $\tilde{u} = (\tilde{H}^*)^{-1}u$ $N_{\tilde{\Omega}}(\tilde{x}) = N_{\tilde{H}(\Omega)}(\tilde{H}(x)) = \frac{\tilde{H}^t_x N_\Omega(x)}{\|\tilde{H}^t_x N_\Omega(x)\|}$ then $h(\Omega) = \tilde{h}(\tilde{\Omega})$,

$$h^* F_{h(\Omega)} h^{*^{-1}} u(x) = \tilde{h}^* F_{\tilde{h}(\tilde{\Omega})} (\tilde{h}^*)^{-1} \tilde{u}(\tilde{x}),$$

$$h^* \mathcal{B}_{h(\Omega)} h^{*^{-1}} u(x) = \tilde{h}^* \mathcal{B}_{\tilde{h}(\tilde{\Omega})} (\tilde{h}^*)^{-1} \tilde{u}(\tilde{x}),$$

using the normal

$$\begin{aligned} N_{\tilde{h}(\tilde{\Omega})}(\tilde{h}(\tilde{x})) &= \frac{(\tilde{h}^{-1})^t_{\tilde{x}} N_{\tilde{\Omega}}(\tilde{x})}{\|(\tilde{h}^{-1})^t{}_{\tilde{x}} N_{\tilde{\Omega}}(\tilde{x})\|} \\ &= \frac{(h^{-1})^t_x N_\Omega(x)}{\|(h^{-1})^t{}_x N_\Omega(x)\|} \\ &= N_{h(\Omega)}(h(x)). \end{aligned}$$

This 'change of origin' will be frequently used in the sequel, as it allow us to compute derivatives with respect to h at $h = i_\Omega$, where the formulas are simpler.

2.3 The Transversality Theorem

A basic tool for our results will be the Transversality Theorem in the form below, due to D. Henry [2]. We first recall some definitions.

A map $T \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces is a *semi-Fredholm* map if the range of T is closed and at least one (or both, for Fredholm) of dim $\mathcal{N}(T)$, codim $\mathcal{R}(T)$ is finite; the *index* of T is then

$$index(T) = ind(T) = dim\mathcal{N}(T) - codim\mathcal{R}(T).$$

We say that a subset F of a topological space X is *rare* if its closure has empty interior and *meager* if it is contained in a countable union of rare subsets of X. We say that F is *residual* if its complement in X is meager. We also say that X is a *Baire space* if any residual subset of X is dense.

Let f be a $\mathcal{C}^k$ map between Banach spaces. We say that x is a *regular point* of f if the derivative $f'(x)$ is surjective and its kernel is finite-dimensional. Otherwise, x is called a *critical* point of f. A point is *critical* if it is the image of some critical point of f.

Let now X be a Baire space and $I = [0,1]$. For any closed or σ-closed $F \subset X$ and any nonnegative integer m we say that the codimension of F is greater or equal to m (*codim* $F \geq m$) if the subset $\{\phi \in \mathcal{C}(I^m, X) \mid \phi(I^m) \cap F$ is non-empty $\}$ is meager in $\mathcal{C}(I^m, X)$. We say *codim* $F = k$ if k is the largest integer satisfying *codim* $F \geq m$.

Theorem 2 *Suppose given positive numbers k and m; Banach manifolds X, Y, Z of class C^k; an open set $A \subset X \times Y$; a C^k map $f : A \to Z$ and a point $\xi \in Z$. Assume for each $(x,y) \in f^{-1}(\xi)$ that:*

1. $\frac{\partial f}{\partial x}(x,y) : T_x X \longrightarrow T_\xi Z$ *is semi-Fredholm with index* $< k$.
2. *(α)* $Df(x,y) : T_x X \times T_y Y \longrightarrow T_\xi Z$ *is surjective*

   or

   (β) dim $\left\{ \frac{\mathcal{R}(Df(x,y))}{\mathcal{R}(\frac{\partial f}{\partial x}(x,y))} \right\} \geq m + \dim \mathcal{N}(\frac{\partial f}{\partial x}(x,y))$.
3. $(x,y) \mapsto y : f^{-1}(\xi) \longrightarrow Y$ *is σ-proper, that is $f^{-1}(\xi)$ is a countable union of sets M_j such that $(x,y) \mapsto y : M_j \longrightarrow Y$ is a proper map for each j.[Given $(x_n, y_n) \in M_j$ such that $\{y_n\}$ converges in Y, there exists a subsequence (or subnet) with limit in M_j.]*

We note that 3 holds if $f^{-1}(\xi)$ is Lindelöf [every open cover has a countable subcover] or, more specifically, if $f^{-1}(\xi)$ is a separable metric space, or if X, Y are separable metric spaces.

Let $A_y = \{x | (x,y) \in A\}$ and

$$Y_{crit} = \{y \mid \xi \text{ is a critical value of } f(\cdot, y) : A_y \mapsto Z\}.$$

Then Y_{crit} is a meager set in Y and, if $(x,y) \mapsto y : f^{-1}(\xi) \mapsto Y$ is proper, Y_{crit} is also closed. If ind $\frac{\partial f}{\partial x} \leq -m < 0$ on $f^{-1}(\xi)$, then $(2(\alpha))$ implies $(2(\beta))$ and

$$Y_{crit} = \{y \mid \xi \in f(A_y, y)\}$$

has codimension $\geq m$ in Y.[Note Y_{crit} is meager iff codim $Y_{crit} \geq 1$].

3 Genericity of the isomorphism property for a class of linear differential operators

Let $a : \mathbb{R}^n \to \mathbb{R}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ be functions of class C^3 and consider the (formal) differential operator

$$L = \Delta^2 + a(x)\Delta + b(x) \cdot \nabla + c(x) \quad x \in \mathbb{R}^n.$$

We show in this section that the operator

$$\begin{aligned} L_\Omega : H^4 \cap H_0^2(\Omega) &\rightarrow L^2(\Omega) \\ u &\rightarrow Lu \end{aligned} \qquad (3)$$

is, generically, an isomorphism in the set of open, connected, bounded C^3-regular regions of $\mathbb{R}^n$. More precisely, we show that the set

$$\mathcal{I} = \{h \in \mathrm{Diff}^4(\Omega) \mid \text{the operator } h^* L_{h(\Omega)} h^{*-1} \text{ from } H^4 \cap H_0^2(\Omega) \text{ into } L^2(\Omega) \text{ is an isomorphism}\} \quad (4)$$

is an open dense set in $\mathrm{Diff}^4(\Omega)$. Observe that the operator $h^* L_{h(\Omega)} h^{*-1}$ is an isomorphism if, and only if the operator $L_{h(\Omega)}$ from $H^4 \cap H_0^2(h(\Omega))$ to $L^2(h(\Omega))$ is an isomorphism, since h^* and h^{*-1} are isomorphisms from $L^2(h(\Omega))$ to $L^2(\Omega)$ and $H^4 \cap H_0^2(\Omega)$ to $H^4 \cap H_0^2(h(\Omega))$ respectively. Consider the differentiable map

$$\begin{aligned} K : H^4 \cap H_0^2(\Omega) \times \mathrm{Diff}^4(\Omega) &\to L^2(\Omega) \\ (u, h) &\to h^* L_{h(\Omega)} h^{*-1} u. \end{aligned} \quad (5)$$

Proposition 3 *Let $a : \mathbb{R}^n \to \mathbb{R}$, $b : \mathbb{R}^n \to \mathbb{R}^n$, $c : \mathbb{R}^n \to \mathbb{R}$ be C^2 functions, $\Omega \subset \mathbb{R}^n$ an open, connected bounded C^4-regular region, and $h \in \mathrm{Diff}^4(\Omega)$. Then zero is a regular value of the map (application)*

$$\begin{aligned} K_h : H^4 \cap H_0^2(\Omega) &\longrightarrow L^2(\Omega) \\ u &\longrightarrow h^* L_{h(\Omega)} h^{*-1} u, \end{aligned}$$

if and only if $h^* L_{h(\Omega)} h^{*-1}$ is an isomorphism.

Proof. First observe that K_h is a Fredholm operator of index 0 since $L_{h(\Omega)}$ is Fredholm of index 0 and h^*, h^{*-1} are isomorphisms. If 0 is a regular value then the linearisation of K_h at 0, which is again K_h, must be surjective. Being of index 0 it is also injective and therefore an isomorphism by the Open Mapping Theorem. Reciprocally, if K_h is an isomorphism, it is surjective at any point. ∎

From 3 and the Implicit Function Theorem it follows that $\mathcal{I}$ is open . We thus only need to show density. For that we may work with more regular regions.

It would be very convenient for our purposes to have the following 'unique continuation' result.

If u is a solution of $L_\Omega u = 0$ with $\frac{\partial^2 u}{\partial N^2} = 0$ in a open set of $\partial\Omega$, then $u \equiv 0$.

Such a result is not available, to the best of our knowledge, but the following 'generic unique continuation result' will be sufficient for our needs. We will not prove it here since the argument is very similar to the one of 11 below.

Lemma 4 *Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded C^5-regular region with $n \geq 2$ and J an open nonempty subset of $\partial\Omega$. Consider the differentiable map*

$$G : B_M \times \mathrm{Diff}^5(\Omega) \to L^2(\Omega) \times H^{\frac{3}{2}}(J)$$

defined by

$$G(u, \lambda, h) = \left(h^* L_{h(\Omega)} h^{*-1} u, h^* \Delta h^{*-1} u \Big|_J \right)$$

where $B_M = \{u \in H^4 \cap H_0^2(\Omega) - \{0\} \mid \|u\| \le M\}$. *Then, the set*

$$C_M^J = \{h \in \mathrm{Diff}^5(\Omega) \mid (0,0) \in G(B_M, h)\}$$

is meager and closed in $\mathrm{Diff}^5(\Omega)$.

Theorem 5 *Let* $a : \mathbb{R}^n \to \mathbb{R}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ *and* $c : \mathbb{R}^n \to \mathbb{R}$ *be functions of class* C^3. *Then the operator* L_Ω *defined in* (3) *is generically an isomorphism in the set of open, connected* C^4*-regular regions* $\Omega \subset \mathbb{R}^n$, $n \ge 2$. *More precisely, if* $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ *is an open, connected* C^4*-regular region, than the set* $\mathcal{I}$ *defined in (4) is an open dense set in* $\mathrm{Diff}^4(\Omega)$.

Proof. By proposition 3, all we need to show is 0 is a regular value of K_h in a residual subset of $\mathrm{Diff}^4(\Omega)$. Since our spaces are separable and K_h is Fredholm of index 0, this would follow from the Transversality Theorem if we could prove that 0 is a regular value of K. Let us suppose that this is not true, that is, there exists a critical point $(u, h) \in K^{-1}(0)$ As explained in (2.2), we may suppose that $h = i_\Omega$. Since we only need to prove density, we may also suppose that Ω is C^5-regular. Then, there exists $v \in L^2(\Omega)$ such that

$$\int_\Omega v DK(u, i_\Omega)(\dot{u}, \dot{h}) = 0 \text{ for all } (\dot{u}, \dot{h}) \in H^4 \cap H_0^2(\Omega) \times C^5(\Omega, \mathbb{R}^n)$$

where $DK(u, i_\Omega)$ from $H^4 \cap H_0^2(\Omega) \times C^5(\Omega, \mathbb{R}^n)$ to $L^2(\Omega)$ is given by

$$DK(u, i_\Omega)(\dot{u}, \dot{h}) = L_\Omega(\dot{u} - \dot{h} \cdot \nabla u).$$

Choosing $\dot{h} = 0$ and varying $\dot{u}$ in $H^4 \cap H_0^2(\Omega)$, we obtain

$$\int_\Omega v L_\Omega \dot{u} = 0 \text{ for all } \dot{u} \in H^4 \cap H_0^2(\Omega)$$

and v is therefore a weak, hence strong, solution of

$$L_\Omega^* v = 0 \text{ in } \Omega \tag{6}$$

where L_Ω^* from $H^4 \cap H_0^2(\Omega)$ to $L^2(\Omega)$ is given by

$$L_\Omega^* v = \Delta^2 v + a\Delta v + (2\nabla a - b) \cdot \nabla v + (c + \Delta a - \operatorname{div} b)v.$$

By regularity of solutions of strongly elliptic equations, v is also a strong solution, that is, $v \in H^4 \cap H_0^2(\Omega) \cap C^{4,\alpha}(\Omega)$ for some $\alpha > 0$ and satisfies (6) [Note that $u \in H^5(\Omega)$, since Ω is C^5-regular.]

Choosing $\dot{u} = 0$ and varying $\dot{h}$ in $C^5(\Omega, \mathbb{R}^n)$, we obtain

$$0 = -\int_\Omega v L_\Omega(\dot{h} \cdot \nabla u) = \int_\Omega \{(\dot{h} \cdot \nabla u) L_\Omega^* v - v L_\Omega(\dot{h} \cdot \nabla u)\} = \int_{\partial\Omega} \dot{h} \cdot N \Delta v \Delta u,$$

for all $\dot{h} \in C^5(\Omega, \mathbb{R}^n)$ since $\Delta u|_{\partial\Omega} = \left.\frac{\partial^2 u}{\partial N^2}\right|_{\partial\Omega}$. Thus $\Delta v \Delta u \equiv 0$ in $\partial\Omega$. This is not a contradiction (or at least it is not clear that it is). We show now, however, that it is *a contradiction generically*; it can only happen in an 'exceptional' set of $\mathrm{Diff}^4(\Omega)$. The result then follows by reapplying the argument above outside this exceptional set. To be more precise, consider the map

$$H : B_M^2 \times \mathrm{Diff}^5(\Omega) \to L^2(\Omega)^2 \times L^1(\partial\Omega)$$

defined by

$$H(u,v,h) = (K(u,h), h^* L_{h(\Omega)}^* h^{*-1} v, h^*\Delta h^{*-1} u h^* \Delta h^{*-1} v|_{\partial\Omega})$$

where $B_M^2 = \{(u,v) \in H^4 \cap H_0^2(\Omega)^2 \mid \|u\|, \|v\| \leq M\}$. We show, using the Transversality Theorem that the set

$$\mathcal{H}_M = \{h \in \mathrm{Diff}^5(\Omega) \mid (0,0,0) \in H(B_M^2, h)\}$$

is meager and closed in $\mathrm{Diff}^5(\Omega)$ for all $M \in \mathbb{N}$. We may, by 'changing the origin' if necessary assume that the 'generic uniqueness property' stated in lemma 4 holds in Ω. We then apply the Transversality Theorem again for the map K, with h restricted to the complement of $\mathcal{H}_M$, obtaining another subset $\tilde{\mathcal{H}}_M$ of $\mathrm{Diff}^5(\Omega)$ such that 0 is a regular value of K_h for any $h \in \tilde{\mathcal{H}}_M$. Taking intersection for $M \in \mathbb{N}$, the desired result follows.

To show that $\mathcal{H}_M$ is a meager closed set we apply Henry's version of the Transversality Theorem for the map H. Since our spaces are separable and $\frac{\partial K}{\partial u}(u, i_{\partial\Omega})$ is Fredholm it remains only to prove that the map $(u,v,h) \mapsto h$: $H^{-1}(0,0,0) \to \mathrm{Diff}^5(\Omega)$ is proper and the hypothesis (2β). We first show that $(u,v,h) \to h : H^{-1}(0,0,0) \to \mathrm{Diff}^5(\Omega)$ is proper. Let $\{(u_n, v_n, h_n)\}_{n\in\mathbb{N}} \subset H^{-1}(0,0,0)$ be a sequence with $h_n \to i_\Omega$ in $\mathrm{Diff}^5(\Omega)$ (the general case is similar). Since $\{(u_n,v_n)\}_{n\in\mathbb{N}} \subset B_M^2$, we may assume, taking a subsequence that there exists $(u,v) \in H_0^2(\Omega)^2$ such that $u_n \to u$ and $v_n \to v$ in $H_0^2(\Omega)$. We have, for all $n \in \mathbb{N}$

$$h_n^*\Big(\Delta^2 + a\Delta + b\cdot\nabla + c\Big) h_n^{*-1} u_n = 0 \iff h_n^* \Delta^2 h_n^{*-1} u_n = -h_n^*\Big(a\Delta + b\cdot\nabla + c\Big) h_n^{*-1} u_n.$$

Since Δ^2 is an isomorphism, it follows that

$$u_n = -h_n^* (\Delta^2)^{-1}\Big(a\Delta + b\cdot\nabla + c\Big) h_n^{*-1} u_n. \tag{7}$$

By results on section 2.1, the right-hand side of (7) is analytic as an application from $H_0^2(\Omega) \times \mathrm{Diff}^5(\Omega)$ to $H^4 \cap H_0^2(\Omega)$. Taking the limit as $n \to +\infty$, we obtain that $u \in H^4 \cap H_0^2(\Omega)$ and satisfies $\Delta^2 u + a\Delta u + b\cdot\nabla u + cu = 0$. By lemma 10 of [10] we have, for $h \in \mathrm{Diff}^5(\Omega)$ and $v \in H^4 \cap H_0^2(\Omega)$

$$h^*\Delta^2 h^{*-1}(v) = \Delta^2(v) + L^h(v) \text{ with } \|L^h u\|_{L^2(\Omega)} \leq \epsilon(h)\|u\|_{H^4(\Omega)}$$

and $\epsilon(h) \to 0$ as $h \to i_\Omega$ in $C^4(\Omega, \mathbb{R}^n)$.
Since $u_n \to u$ in $H_0^2(\Omega)$ and $h_n \to i_\Omega$ in $\mathrm{Diff}^5(\Omega)$ as $n \to +\infty$, we obtain

$$\begin{aligned}\|\Delta^2(u_n-u)+L^{h_n}(u_n-u)\|_{L^2(\Omega)} &= \|h_n^*\Delta^2 h_n^{*-1}(u_n-u)\|_{L^2(\Omega)}\\ &= \|h_n^*\Delta^2 h_n^{*-1}u+h_n^*\Big(a\Delta+b\cdot\nabla+c\Big)h_n^{*-1}u_n\|_{L^2(\Omega)}\\ &\to \|\Delta^2 u+a\Delta u+b\cdot\nabla u+cu\|_{L^2(\Omega)}=0 \qquad (8)\end{aligned}$$

as $n\to+\infty$.

Since $\{(u_n,v_n)\}\subset B_M^2$ and $h_n\to i_\Omega$ in $\mathrm{Diff}^5(\Omega)$, we have

$$\|L^{h_n}(u_n-u)\|_{L^2(\Omega)}\le 2M\epsilon(h_n) \qquad (9)$$

It follows from (8) and (9) that $\|\Delta^2(u_n-u)\|_{L^2(\Omega)}\to 0$ as $n\to+\infty$. Since Δ^2 is an isomorphism from $H^4\cap H_0^2(\Omega)$ to $L^2(\Omega)$, we obtain $u_n\to u$ in $H^4\cap H_0^2(\Omega)$ and, therefore $\|u\|_{H^4\cap H_0^2(\Omega)}\le M$, for all $n\in\mathbb{N}$. Similarly, we prove that $v_n\to v$ in $H^4\cap H_0^2(\Omega)$ and $\|v\|_{H^4\cap H_0^2(\Omega)}\le M$ from which we conclude that the map $(u,v,h)\to h: H^{-1}(0,0,0)\to \mathrm{Diff}^5(\Omega)$ is proper.

It remains only to prove (2β), which we do by showing that

$$\dim\left\{\frac{\mathcal{R}(DH(u,v,h))}{\mathcal{R}\Big(\frac{\partial H}{\partial u}(u,v,h)\Big)}\right\}=\infty \text{ for alll } (u,v,h)\in H^{-1}(0,0,0).$$

Suppose this is not true for some $(u,v,h)\in H^{-1}(0,0,0)$. Assuming, as we may, that $h=i_\Omega$ it follows that there exist $\theta_1,\dots,\theta_m\in L^2(\Omega)^2\times L^1(\partial\Omega)$ such that, for all $\dot h\in\mathcal{C}^5(\Omega,\mathbb{R}^n)$ there exist $\dot u,\dot v\in H^4\cap H_0^2(\Omega)$ and scalars $c_1,\dots,c_m\in\mathbb{R}$ such that

$$DH(u,v,i_\Omega)(\dot u,\dot v,\dot h)=\sum_{i=1}^m c_i\theta_i,\qquad \theta_i=(\theta_i^1,\theta_i^2,\theta_i^3). \qquad (10)$$

Using theorem 1, we obtain

$$DH(u,v,i_\Omega)(\cdot) = \Big(DH_1(u,v,i_\Omega)(\cdot),DH_2(u,v,i_\Omega)(\cdot),DH_3(u,v,i_\Omega)(\cdot)\Big)$$

where

$$\begin{aligned}DH_1(u,v,i_\Omega)(\dot u,\dot v,\dot h) &= L_\Omega(\dot u-\dot h\cdot\nabla u)\\ DH_2(u,v,i_\Omega)(\dot u,\dot v,\dot h) &= L_\Omega^*(\dot v-\dot h\cdot\nabla v)\\ DH_3(u,v,i_\Omega)(\dot u,\dot v,\dot h) &= \Big\{\Delta v\Delta(\dot u-\dot h\cdot\nabla u)+\Delta u\Delta(\dot v-\dot h\cdot\nabla v)\\ &\quad +\dot h\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)\Big\}\Big|_{\partial\Omega}.\end{aligned}$$

It follows from (10) that

$$L_\Omega(\dot u-\dot h\cdot\nabla u) = \sum_{i=1}^m c_i\theta_i^1 \qquad (11)$$

$$L_\Omega^*(\dot v-\dot h\cdot\nabla v) = \sum_{i=1}^m c_i\theta_i^2 \qquad (12)$$

$$\sum_{i=1}^{m} c_i\theta_i^3 = \Big\{\Delta v\Delta(\dot u - \dot h\cdot\nabla u) + \Delta u\Delta(\dot v - \dot h\cdot\nabla v) + \dot h\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)\Big\}\Big|_{\partial\Omega}. \tag{13}$$

Let $\{u_1, ..., u_l\}$ be a basis for the kernel of L_Ω and consider the operators

$$\mathcal{A}_L : L^2(\Omega) \to H^4 \cap H_0^1(\Omega)$$

$$\mathcal{C}_L : H^{\frac{5}{2}}(\partial\Omega) \to H^4 \cap H_0^1(\Omega)$$

defined by

$$w = \mathcal{A}_L(z) + \mathcal{C}_L(g)$$

where $Lw - z$ belongs to a (fixed) complement of $\mathcal{R}(L_\Omega)$ in $L^2(\Omega)$, $\frac{\partial w}{\partial N} = g$ on $\partial\Omega$ and $\int_\Omega w\phi = 0$ for all $\phi \in \mathcal{N}(L_\Omega^*)$. Let also $\{v_1, ..., v_l\}$ be a basis for the kernel of L_Ω^* and consider the operators

$$\mathcal{A}_{L^*} : L^2(\Omega) \to H^4 \cap H_0^1(\Omega) \text{ and}$$

$$\mathcal{C}_{L^*} : H^{\frac{5}{2}}(\partial\Omega) \to H^4 \cap H_0^1(\Omega)$$

similarly defined. We have shown in [9] that these operators are well defined.

From equations (11) and (12), we obtain

$$\dot u - \dot h\cdot\nabla u = \sum_{i=1}^{l}\xi_i u_i + \sum_{i=1}^{m} c_i\mathcal{A}_L\theta_i^1 - \mathcal{C}_L(\dot h\cdot N\Delta u) \tag{14}$$

since $\frac{\partial}{\partial N}(\dot u - \dot h\cdot\nabla u)\Big|_{\partial\Omega} = -\dot h\cdot N\frac{\partial^2 u}{\partial N^2}\Big|_{\partial\Omega} = -\dot h\cdot N\Delta u|_{\partial\Omega}$ and

$$\dot v - \dot h\cdot\nabla v = \sum_{i=1}^{l}\eta_i v_i + \sum_{i=1}^{m} c_i\mathcal{A}_{L^*}\theta_i^1 - \mathcal{C}_{L^*}(\dot h\cdot N\Delta v) \tag{15}$$

since $\frac{\partial}{\partial N}(\dot v - \dot h\cdot\nabla v)\Big|_{\partial\Omega} = -\dot h\cdot N\frac{\partial^2 v}{\partial N^2}\Big|_{\partial\Omega} = -\dot h\cdot N\Delta v|_{\partial\Omega}$.

Substituting (14) and (15) in (13), we obtain that

$$\Big\{\dot h\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v) - \Big[\Delta u\Delta\mathcal{C}_{L^*}(\dot h\cdot N\Delta v) + \Delta v\Delta\mathcal{C}_L(\dot h\cdot N\Delta u)\Big]\Big\}\Big|_{\partial\Omega} \tag{16}$$

remains in a finite dimensional space when $\dot h$ varies in $\mathcal{C}^5(\Omega, \mathbb{R}^n)$.

The set $U = \{x \in \partial\Omega \mid \Delta u(x) \neq 0\}$ is nonempty since we have assumed that 'generic unique continuation' holds in Ω. Therefore, we must have $\Delta v|_U \equiv 0$. If $\dot h \equiv 0$ in $\partial\Omega - U$, then $\dot h\cdot N\Delta v \equiv 0$ in $\partial\Omega$ and, therefore

$$\Delta u\Delta\mathcal{C}_{L^*}(\dot h\cdot N\Delta v) = \Delta u\Delta\mathcal{C}_{L^*}(0)$$

belongs to the finite dimensional space $[\Delta u\Delta v_1, ..., \Delta u\Delta v_l]$ where $\{v_1, ..., v_l\}$ is a basis for the kernel of $\mathcal{N}(L_\Omega^*)$ It follows that

$$\left\{\dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)-\Delta v\Delta C_L(\dot{h}\cdot N\Delta u)\right\}\Big|_U = \dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)\Big|_U = \dot{h}\cdot N\Delta u\frac{\partial}{\partial N}(\Delta v)\Big|_U \quad (17)$$

remains in a finite dimensional space, when $\dot{h}$ varies in $C^5(\Omega,\mathbb{R}^n)$ with $\dot{h}\equiv 0$ in $\partial\Omega - U$. Since $\Delta u(x)\neq 0$ for any $x\in U$, this is only possible ($\dim\Omega\geq 2$) if $\frac{\partial\Delta v}{\partial N}\Big|_U\equiv 0$. But then $v\equiv 0$ in Ω by Theorem 6 below, and we reach a contradiction proving the result.

∎

During the proof of theorem 5 we have used the following uniqueness theorem, which is a direct consequence of Theorem 8.9.1 of [3].

Theorem 6 *Suppose $\Omega\subset\mathbb{R}^n$ is an open connected, bounded, C^4-regular domain and B is an open ball in $\mathbb{R}^n$ such that $B\cap\partial\Omega$ is a (nontrivial) C^4 hypersurface. Suppose also that $u\in H^4(\Omega)$ satisfies*

$$|\Delta^2 u|\leq C\Big(|\Delta u|+|\nabla u|+|u|\Big) \text{ a.e. in } \Omega$$

for some positive constant C and $u=\dfrac{\partial u}{\partial N}=\Delta u=\dfrac{\partial\Delta u}{\partial N}=0$ in $B\cap\partial\Omega$. Then u is identically null.

4 Generic simplicity of solutions

Let $f(x,\lambda,y,\mu)$ be a real function of class C^4 defined in $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ satisfying $f(x,0,0,0)\equiv 0$ for all $x\in\mathbb{R}^n$. We prove in this section our main result: generically in the set of connected, bounded C^4-regular regions $\Omega\subset\mathbb{R}^n$, $n\geq 2$, the solutions u of

$$\begin{cases}\Delta^2 u+f(\cdot,u,\nabla u,\Delta u)=0 & \text{in } \Omega\\ u=\frac{\partial u}{\partial N}=0 & \text{on } \partial\Omega\end{cases} \quad (18)$$

are all simple. We choose $p>\frac{n}{2}$, so that the continuous imbedding $W^{4,p}\cap W_0^{2,p}(\Omega)\hookrightarrow C^{2,\alpha}(\Omega)$ holds for some $\alpha>0$.

It follows then, from the Implicit Function Theorem, that the set of solutions is discrete in $W^{4,p}\cap W_0^{2,p}(\Omega)$ and, in particular finite if f is bounded.

Remark 7 *Since we have assumed $f(x,0,0,0)\equiv 0$ in $\mathbb{R}^n$, the null function $u\equiv 0$ is a solution of 18 for any $\Omega\subset\mathbb{R}^n$. It follows from theorem 5 that $u\equiv 0$ is simple for Ω in an open dense set of* $\mathrm{Diff}^4(\Omega)$. *We therefore concentrate in the proof of generic simplicity for the nontrivial solutions.*

Proposition 8 *Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, C^4-regular region and $f(x, \lambda, y, \mu)$ a C^2 real function defined in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$. Then, zero is a regular value of the map*

$$\begin{aligned} F_h : W^{4,p} \cap W_0^{2,p}(\Omega) &\longrightarrow L^p(\Omega) \\ u &\longrightarrow h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u), \end{aligned}$$

if and only if all solutions of (18) in $h(\Omega)$ are simple.

Proof. The proof is very similar to the one of proposition 3 and will be left to the reader. ∎

Proposition 9 *A function $u \in W^{4,p} \cap W_0^{2,p}(\Omega)$ is a solution (resp. a simple solution) of*

$$\begin{cases} h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u) = 0 & \text{in } \Omega, \\ u = \frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega \end{cases} \tag{19}$$

if and only if $v = h^{*-1}u$ is a solution (resp. simple solution) of (18) in $h(\Omega)$.

Proof. Let $u \in W^{4,p} \cap W_0^{2,p}(\Omega)$. Since h^* and h^{*-1} are isomorphisms, we have

$$\begin{aligned} &h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u) = 0 \\ \iff &\Delta^2 h^{*-1}u + f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u) = 0, \end{aligned}$$

It is clear that $u = 0$ in $\partial\Omega$ if and only if $v = 0$ in $\partial h(\Omega)$. Writing $y = h(x)$, we obtain

$$\begin{aligned} \frac{\partial v}{\partial N_{h(\Omega)}}(y) &= N_{h(\Omega)}(y) \cdot \nabla_y (u \circ h^{-1})(y) \\ &= N_{h(\Omega)}(y) \cdot (h^{-1})_y^t(y) \nabla_x u(x) \\ &= N_{h(\Omega)}(y) \cdot (h_x^{-1})^t(x) \nabla_x u(x) \\ &= N_{h(\Omega)}(y) \cdot ((h_x)^{-1})^t(x) \frac{\partial u}{\partial N}(x)\ N_\Omega(x) \\ &= \frac{\partial u}{\partial N}(x) \frac{1}{||(h_x^{-1})^t N_\Omega(x)||} ((h_x^{-1})^t(x) N_\Omega(x) \cdot (h_x^{-1})^t(x) N_\Omega(x)) \end{aligned}$$

where we have used that $u = 0$ in $\partial\Omega$. Since $h_x^{-1}(x)$ is non-singular it follows that $\frac{\partial v}{\partial N_{h(\Omega)}}(y) = 0$ if and only if $\frac{\partial u}{\partial N}(x) = 0$. Thus u is a solution of (19) if and only if $h^{*-1}u$ is a solution of (18) in $h(\Omega)$. Finally, since $h^* L(u) h^{*-1}$ is an isomorphism in $W^{4,p} \cap W_0^{2,p}(\Omega)$ if and only if $L(v)$ is an isomorphism in $W^{4,p} \cap W_0^{2,p}(h(\Omega))$ so the result follows. ∎

It follows from 9 and 8 that, in order to show generic simplicity of the solutions of (18) is enough to show that 0 is a regular value of F_h, generically in

$h \in \mathrm{Diff}^4(\Omega)$. We show, using the Transversality Theorem, that 0 is a regular value of

$$\begin{aligned} F_M : B_M \times V_M &\to L^p(\Omega) \\ (u,h) &\to h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u) \quad (20) \end{aligned}$$

where $B_M = \{u \in W^{4,p} \cap W_0^{2,p}(\Omega) - \{0\} \mid \|u\| \leq M\}$ and V_M is an open dense set in $\mathrm{Diff}^4(\Omega)$, for all $M \in \mathbb{N}$. Taking the intersection of V_M for $M \in \mathbb{N}$ we obtain the desired residual set.

Remark 10 *Applying the Implicit Function Theorem to the map F_M defined in (20) we obtain that the set*

$$\mathcal{F}_M = \{h \in \mathrm{Diff}^4(\Omega) \mid \text{ all solutions } u \text{ of } (19) \text{ with } \|u\|_{W^{4,p}\cap W_0^{2,p}(\Omega)} < M \text{ are simple }\}$$

is open in $\mathrm{Diff}^4(\Omega)$ for all $M \in \mathbb{N}$. To prove density, we may work with more regular (for example C^∞) regions.

If we try to apply the Transversality Theorem directly to the function F defined in $W^{4,p} \cap W_0^{2,p}(\Omega) \times \mathrm{Diff}^4(\Omega)$ by (20) we do not obtain a contradiction. What we do obtain is that the possible critical points must satisfy very special properties. The idea is then to show that these properties can only occur in a small (meager and closed) set and then restrict the problem to its complement. In our case the 'exceptional situation' is characterized by the existence of a solution u of (18) and a solution v of the problem

$$\begin{cases} L^*(u)v = 0 & \text{in } \Omega \\ v = \frac{\partial v}{\partial N} = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying the additional property $\Delta u \Delta v \equiv 0$ on $\partial\Omega$. We show in lemma 12 that this situation is really 'exceptional', that is, it can only happen if h is outside an open dense subset of $\mathrm{Diff}^4(\Omega)$ (for u and v restricted to a bounded set).

We will need the following 'generic unique continuation result'.

Lemma 11 *Let $\Omega \subset \mathbb{R}^n$ $n \geq 2$ be an open, connected, bounded, C^5-regular domain, J a nonempty open subset of Ω and $f(x,\lambda,y,\mu)$ a C^2 real function defined in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ with $f(\cdot,0,0,0) \equiv 0$. Consider the map*

$$G : A_M \times \mathrm{Diff}^5(\Omega) \to L^p(\Omega) \times W^{2-\frac{1}{p},p}(J)$$

defined by

$$G(u,h) = \left(h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u), h^*\Delta h^{*-1}u\Big|_J\right)$$

where $A_M = \{u \in W^{4,p} \cap W_0^{2,p}(\Omega) - \{0\} \mid \|u\| \leq M\}$, and $p > \frac{n}{2}$.

Then

$$C_M^J = \{h \in \mathrm{Diff}^5(\Omega) \mid (0,0) \in G(A_M, h)\}$$

is a closed meager subset of $\mathrm{Diff}^5(\Omega)$.

Proof. We apply the Transversality Theorem. Observe that G is differentiable. In fact it is analytic in h as observed in section 2.1, and the differentiability in u follows from the smoothness of f and the continuous immersion $W^{4,p} \cap W_0^{2,p}(\Omega) \subset C^{2,\alpha}(\Omega)$ for some $\alpha > 0$). We compute its differential using Theorem 1 (see example 2.1 of section 2.1).

$$\begin{aligned} DG(u, i_\Omega)(\dot{u}, \dot{h}) &= \Big(L(u)(\dot{u} - \dot{h} \cdot \nabla u), \\ &\qquad \Big\{ \Delta(\dot{u} - \dot{h} \cdot \nabla u) + \dot{h} \cdot N \frac{\partial \Delta u}{\partial N} \Big\}\Big|_J \Big). \end{aligned}$$

To verify (1) and (2), we proceed as in the proof of theorem 5. We prove that (2β) holds showing that

$$\dim \left\{ \frac{\mathcal{R}(DG(u,h))}{\mathcal{R}\left(\frac{\partial G}{\partial u}(u,h)\right)} \right\} = \infty \text{ for all } (u,h) \in G^{-1}(0,0).$$

Suppose, by contradiction this is not true for some $(u,h) \in G^{-1}(0,0)$. By 'changing the origin' we may suppose that $h = i_\Omega$. Then, there exist $\theta_1, \ldots, \theta_m \in L^p(\Omega) \times W^{2-\frac{1}{p},p}(J)$ for all $\dot{h} \in C^5(\Omega, \mathbb{R}^n)$ there exists $\dot{u} \in W^{4,p} \cap W_0^{2,p}(\Omega)$ and scalars $c_1, \ldots, c_m \in \mathbb{R}$ such that

$$DG(u, i_\Omega)(\dot{u}, \dot{h}) = \sum_{i=1}^{m} c_i \theta_i, \text{ that is,}$$

$$L(u)(\dot{u} - \dot{h} \cdot \nabla u) = \sum_{i=1}^{m} c_i \theta_i^1 \tag{21}$$

$$\Big\{ \Delta(\dot{u} - \dot{h} \cdot \nabla u) + \dot{h} \cdot N \frac{\partial \Delta u}{\partial N} \Big\}\Big|_J = \sum_{i=1}^{m} c_i \theta_i^2 \tag{22}$$

where

$$L(u) = \Delta^2 + \frac{\partial f}{\partial \mu}(\cdot, u, \nabla u, \Delta u)\Delta + \frac{\partial f}{\partial y}(\cdot, u, \nabla u, \Delta u) \cdot \nabla + \frac{\partial f}{\partial \lambda}(\cdot, u, \nabla u, \Delta u). \tag{23}$$

Let $\{u_1, \ldots, u_l\}$ be a basis for the kernel of $L_0(u) = L(u)\Big|_{W^{4,p} \cap W_0^{2,p}(\Omega)}$ and consider the operators

$$\mathcal{A}_{L(u)} : L^p(\Omega) \to W^{4,p} \cap W_0^{1,p}(\Omega)$$

$$\mathcal{C}_{L(u)} : W^{3-\frac{1}{p},p}(\partial\Omega) \to W^{4,p} \cap W_0^{1,p}(\Omega)$$

defined by

$$w = \mathcal{A}_{L(u)}(z) + \mathcal{C}_{L(u)}(g)$$

if $L(u)w - z$ belongs to a fixed complement of $\mathcal{R}(L_0(u))$ in $L^p(\Omega)$, $\frac{\partial w}{\partial N} = g$ on $\partial\Omega$ and $\int_\Omega w\phi = 0$ for all $\phi \in \mathcal{N}(L_0^*(u))$. (We proved these operators are well defined in [9]).

Choosing $\dot{h} \in C^5(\Omega, \mathbb{R}^n)$ such that $\dot{h} \equiv 0$ on $\partial\Omega - J$, we obtain from (21), that

$$\dot{u} - \dot{h} \cdot \nabla u = \sum_{i=1}^{l} \xi_i u_i + \sum_{i=1}^{m} c_i \mathcal{A}_{L(u)}(\theta_i^1) \tag{24}$$

since $\dot{u} - \dot{h} \cdot \nabla u \in W^{4,p} \cap W_0^{2,p}(\Omega)$.

Substituting (24) in (22), we obtain that $\dot{h} \cdot N \frac{\partial \Delta u}{\partial N}\Big|_J$ remains in a finite dimensional subspace when $\dot{h}$ varies in $C^5(\Omega, \mathbb{R}^n)$. Since $\dim \Omega \geq 2$ this is possible only if $\frac{\partial \Delta u}{\partial N} \equiv 0$ on J so u satisfies

$$\begin{cases} \Delta^2 u + f(\cdot, u, \nabla u, \Delta u) = 0 & \text{in } \Omega \\ u = \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega \\ \Delta u = \frac{\partial \Delta u}{\partial N} = 0 & \text{on} J. \end{cases} \tag{25}$$

We claim that u satisfies the hypotheses of Cauchy's Uniqueness Theorem (6). Indeed, since $u \in W^{4,p}(\Omega) \cap C^{2,\alpha}(\Omega)$ for some $\alpha > 0$ ($p > \frac{n}{2}$) and is a solution of the uniformly elliptic equation $\Delta^2 u + f(\cdot, u, \nabla u, \Delta u) = 0$ in Ω then $u \in W^{4,p}(\Omega) \cap C^{4,\alpha}(\Omega)$. Furthermore, $u = \frac{\partial u}{\partial N} = \Delta u = \frac{\partial \Delta u}{\partial N} = 0$ on $J \subset \partial\Omega$ and

$$\begin{aligned} |\Delta^2 u| &\leq |f(\cdot, u, \nabla u, \Delta u)| \\ &\leq |f(\cdot, u, \nabla u, \Delta u) - f(\cdot, 0, 0, 0)| \\ &\leq \max_{\bar{\Omega}}\{|Df(\cdot, u, \nabla u, \Delta u)|\}\Big(|u| + |\nabla u| + |\Delta u|\Big). \end{aligned}$$

We conclude that $u \equiv 0$, which gives the searched for contradiction. ▮

Lemma 12 *Let $\Omega \subset \mathbb{R}^n$ $n \geq 2$ be an open, connected, bounded, C^5-regular domain and $f(x, \lambda, y, \mu)$ a C^3 real function defined in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ with $f(\cdot, 0, 0, 0) \equiv 0$. Consider the map*

$$Q : A_{M,p} \times A_{M,q} \times D_M \to L^p(\Omega) \times L^q(\Omega) \times L^1(\partial\Omega)$$

defined by

$$\begin{aligned} Q(u, v, h) &= \Big(h^* \Delta^2 h^{*-1} u + h^* f(\cdot, h^{*-1} u, \nabla h^{*-1} u, \Delta h^{*-1} u), \\ &\quad h^* L^*(h^{*-1} u) h^{*-1} v, h^* \Delta h^{*-1} u h^* \Delta h^{*-1} v\Big|_{\partial\Omega}\Big) \end{aligned}$$

where $A_{M,p} = \{u \in W^{4,p} \cap W_0^{2,p}(\Omega) - \{0\} \mid \|u\| \leq M\}$ and $p^{-1} + q^{-1} = 1$ with $p > \frac{n}{2}$,
$A_{M,q} = \{u \in W^{4,q} \cap W_0^{2,q}(\Omega) - \{0\} \mid \|u\| \leq M\}$, $D_M = \mathrm{Diff}^5(\Omega) - C_M^{\partial\Omega}$, $C_M^{\partial\Omega}$

given by 11 and

$$
\begin{aligned}
L^*(w) &= \Delta^2 + \frac{\partial f}{\partial \mu}(\cdot, w, \nabla w, \Delta w)\Delta \\
&+\Big[2\nabla\Big(\frac{\partial f}{\partial \mu}(\cdot, w, \nabla w, \Delta w)\Big) - \frac{\partial f}{\partial y}(\cdot, w, \nabla w, \Delta w)\Big]\cdot\nabla \\
&+\Delta\Big[\frac{\partial f}{\partial \mu}(\cdot, w, \nabla w, \Delta w)\Big] - \ div\ \Big(\frac{\partial f}{\partial y}(\cdot, w, \nabla w, \Delta w)\Big) \\
&+\frac{\partial f}{\partial \lambda}(\cdot, w, \nabla w, \Delta w).
\end{aligned}
$$

Then

$$E_M = \{h \in D_M \mid (0,0,0) \in Q(A_M) \times A_{M,q}, h)\}$$

is a meager closed subset of $\mathrm{Diff}^5(\Omega)$.

(Observe that $L^*(w)$ is the formal adjoint of $L(w)$ defined by (23).)

Proof. We again apply the Transversality Theorem. The differentiability of Q is easy to establish, and its derivative can be computed using theorem 1 (see example 2.1)

$$
\begin{aligned}
DQ(u, v, i_\Omega)(\dot u, \dot v, \dot h) &= \Big(L(u)(\dot u - \dot h\cdot\nabla u), \\
& L^*(u)(\dot v - \dot h\cdot\nabla v) + \Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)(\dot u - \dot h\cdot\nabla u), \\
& \Big\{\Delta(\dot u - \dot h\cdot\nabla u)\Delta v + \Delta u\Delta(\dot v - \dot h\cdot\nabla v) + \dot h\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)\Big\}\Big|_{\partial\Omega}\Big)
\end{aligned}
$$

where $\frac{\partial L^*}{\partial w}(u)\cdot v$ is the second order differential operator given by

$$
\begin{aligned}
\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)z &= \Big(\frac{\partial^2 f}{\partial\lambda\partial\mu}v + \frac{\partial^2 f}{\partial y\partial\mu}\cdot\nabla v + \frac{\partial^2 f}{\partial\mu^2}\Delta v\Big)\Delta z \\
&+\Big[2\nabla\Big(\frac{\partial^2 f}{\partial\lambda\partial\mu}v + \frac{\partial^2 f}{\partial y\partial\mu}\cdot\nabla v + \frac{\partial^2 f}{\partial\mu^2}\Delta v\Big) \\
&-\Big(\frac{\partial^2 f}{\partial\lambda\partial y}v + \frac{\partial^2 f}{\partial y^2}\nabla v + \frac{\partial^2 f}{\partial\mu\partial y}\Delta v\Big)\Big]\cdot\nabla z \\
&\Big[\Big(\frac{\partial^2 f}{\partial\lambda^2}v + \frac{\partial^2 f}{\partial\lambda\partial y}\cdot\nabla v + \frac{\partial^2 f}{\partial\lambda\partial\mu}\Delta v\Big) \\
&+\Delta\Big(\frac{\partial^2 f}{\partial\lambda\partial\mu}v + \frac{\partial^2 f}{\partial y\partial\mu}\cdot\nabla v + \frac{\partial^2 f}{\partial\mu^2}\Delta v\Big) \\
&-\mathrm{div}\Big(\frac{\partial^2 f}{\partial\lambda\partial y}v + \frac{\partial^2 f}{\partial y^2}\nabla v + \frac{\partial^2 f}{\partial y\partial\mu}\Delta v\Big)\Big]z.
\end{aligned}
$$

(We have written f instead of $f(\cdot, u, \nabla u, \Delta u)$ to simplify the notation).

The hypotheses (1) and (3) of Transversality Theorem can be verified as in the proof of 5. We prove (2β) by showing that

$$\dim \left\{ \frac{\mathcal{R}(DQ(u,v,h))}{\mathcal{R}\left(\frac{\partial Q}{\partial(u,v)}(u,v,h)\right)} \right\} = \infty$$

for all $(u,v,h) \in Q^{-1}(0,0,0)$. Suppose this is not true for $(u,v,h) \in Q^{-1}(0,0,0)$. 'Changing the origin', we may assume that $h = i_\Omega$. Then, there exist $\theta_1, ..., \theta_m \in L^p(\Omega) \times L^q(\Omega) \times L^1(\partial\Omega)$ such that for all $\dot{h} \in \mathcal{C}^5(\Omega, \mathbb{R}^n)$ there exists $\dot{u} \in W^{4,p} \cap W_0^{2,p}(\Omega)$, $\dot{v} \in W^{4,q} \cap W_0^{2,q}(\Omega)$ and scalars $c_1, ..., c_m \in \mathbb{R}$ such that

$$DQ(u,v,i_\Omega)(\dot{u},\dot{v},\dot{h}) = \sum_{i=1}^{m} c_i\theta_i, \text{ that is,}$$

$$L(u)(\dot{u} - \dot{h}\cdot\nabla u) \;=\; \sum_{i=1}^{m} c_i\theta_i^1 \qquad (26)$$

$$L^*(u)(\dot{v} - \dot{h}\cdot\nabla v) + \left(\frac{\partial L^*}{\partial w}(u)\cdot v\right)(\dot{u} - \dot{h}\cdot\nabla u) \;=\; \sum_{i=1}^{m} c_i\theta_i^2 \qquad (27)$$

and

$$\sum_{i=1}^{m} c_i\theta_i^3 \;=\; \Big\{\Delta(\dot{u} - \dot{h}\cdot\nabla u)\Delta v + \Delta u\Delta(\dot{v} - \dot{h}\cdot\nabla v) + \dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v)\Big\}\Big|_{\partial\Omega}. \qquad (28)$$

Let $\{u_1, ..., u_l\}$ be a basis for the kernel of $L_0(u) = L(u)\Big|_{W^{4,p}\cap W_0^{2,p}(\Omega)}$, $\{v_1, ..., v_l\}$ a basis for the kernel of $L_0^*(u)$ and consider the operators

$$\mathcal{A}_{L(u)} : L^p(\Omega) \to W^{4,p} \cap W_0^{1,p}(\Omega)$$

$$\mathcal{C}_{L(u)} : W^{3-\frac{1}{p},p}(\partial\Omega) \to W^{4,p} \cap W_0^{1,p}(\Omega)$$

defined by

$$w = \mathcal{A}_{L(u)}(z) + \mathcal{C}_{L(u)}(g)$$

where $L(u)w - z$ belongs to a fixed complement of $\mathcal{R}(L_0(u))$ in $L^p(\Omega)$, $\frac{\partial w}{\partial N} = g$ on $\partial\Omega$, $\int_\Omega w\phi = 0$ for all $\phi \in \mathcal{N}(L_0^*(u))$ and

$$\mathcal{A}_{L^*(u)} : L^q(\Omega) \to W^{4,q} \cap W_0^{1,q}(\Omega)$$

$$\mathcal{C}_{L^*(u)} : W^{3-\frac{1}{q},q}(\partial\Omega) \to W^{4,q} \cap W_0^{1,q}(\Omega)$$

defined by

$$t = \mathcal{A}_{L^*(u)}(z) + \mathcal{C}_{L^*(u)}(g)$$

where $L^*(u)t - z$ belongs to a fixed complement of $\mathcal{R}(L_0^*(u))$ in $L^q(\Omega)$, $\frac{\partial t}{\partial N} = g$ on $\partial\Omega$ and $\int_\Omega t\varphi = 0$ for all $\varphi \in \mathcal{N}(L_0(u))$. (We proved these operators are well defined in [9]).

From (26) and (27) it follows that

$$\dot{u} - \dot{h}\cdot\nabla u = \sum_{i=1}^{l} \xi_i u_i + \sum_{i=1}^{m} c_i \mathcal{A}_{L(u)}(\theta_i^1) - \mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)$$

$$\begin{aligned}\dot{v} - \dot{h}\cdot\nabla v &= \sum_{i=1}^{s} \eta_i v_i + \sum_{i=1}^{m} c_i \mathcal{A}_{L^*(u)}(\theta_i^2) - \mathcal{C}_{L^*(u)}(\dot{h}\cdot N\Delta v)\\ &\quad -\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)(\dot{u} - \dot{h}\cdot\nabla u)\Big).\end{aligned}$$

Substituting in (28), we obtain that

$$\begin{aligned}&\Big\{\dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v) - \Delta v\Delta\Big(\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big)\\ &+\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big) - \mathcal{C}_{L^*(u)}(\dot{h}\cdot N\Delta v)\Big]\Big\}\Big|_{\partial\Omega}\end{aligned}$$

remains in a finite dimensional space when $\dot{h}$ varies in $\mathcal{C}^5(\Omega,\mathbb{R}^n)$, that is, the operator

$$\begin{aligned}\Upsilon(\dot{h}) &= \Big\{\dot{h}\cdot N\frac{\partial}{\partial N}(\Delta u\Delta v) - \Delta v\Delta\Big(\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big)\\ &\quad+\Delta u\Delta\Big[\mathcal{A}_{L^*(u)}\Big(\Big(\frac{\partial L^*}{\partial w}(u)\cdot v\Big)\mathcal{C}_{L(u)}(\dot{h}\cdot N\Delta u)\Big) - \mathcal{C}_{L^*(u)}(\dot{h}\cdot N\Delta v)\Big]\Big\}\Big|_{\partial\Omega}\end{aligned} \tag{29}$$

defined in $\mathcal{C}^5(\Omega,\mathbb{R}^n)$ is of finite range

We proved in [9] that, if $\dim\Omega \geq 2$, a necessary condition for Υ to be of finite range is

$$\frac{\partial}{\partial N}(\Delta u\Delta v) \equiv 0 \text{ on } \partial\Omega.$$

Thus the functions u, v must satisfy

$$\begin{cases}\Delta^2 u - f(\cdot,u,\nabla u,\Delta u) = 0 & \text{in } \Omega\\ u = \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega\end{cases}$$

$$\begin{cases}L^*(u)v = 0 & \text{in } \Omega\\ v = \frac{\partial v}{\partial N} = 0 & \text{on } \partial\Omega\end{cases}$$

and also

$$\Delta u\Delta v|_{\partial\Omega} = \frac{\partial}{\partial N}(\Delta u\Delta v)\Big|_{\partial\Omega} = 0. \tag{30}$$

Let $U = \{x \in \partial\Omega \mid \Delta u(x) \neq 0\}$. Observe that U is a nonempty, since $i_\Omega \in D_M^{\partial\Omega}$ (given by lemma 11).

By equation (30), we have $\Delta v|_U = \left.\frac{\partial \Delta v}{\partial N}\right|_U \equiv 0$. Therefore $v \in W^{4,q} \cap W_0^{2,q}(\Omega)$ satisfies the hypotheses of theorem 6. Thus $v \equiv 0$ in Ω and we reach a contradiction, proving the result.

∎

Theorem 13 *Generically in the set of open, connected, bounded C^4-regular regions of $\mathbb{R}^n$, $n \geq 2$, the solutions of (18) are all simple.*

Proof.

Consider the differentiable map

$$F : B_M \times U_M \to L^p(\Omega)$$

defined by

$$F(u,h) = h^*\Delta^2 h^{*-1}u + h^* f(\cdot, h^{*-1}u, \nabla h^{*-1}u, \Delta h^{*-1}u)$$

where $B_M = \{u \in W^{4,p} \cap W_0^{2,p}(\Omega) - \{0\} \mid \|u\| \leq M\}$, $p > \frac{n}{2}$, $U_M = D_M - E_M$, D_M is the complement of the meager closed set given by lemma 11 and E_M is the meager closed set given by 12. Observe that U_M an open dense subset of $\mathrm{Diff}^4(\Omega)$. We show, using the Transversality Theorem, that the set

$$\{h \in U_M \mid u \to F(u,h) \text{ has } 0 \text{ as a regular value }\}$$

is open and dense in U_M. Our result then follows by taking intersection with M varying in $\mathbb{N}$.

As observed in 10 we may suppose, wolog that our regions are C^5-regular. Also by 7, we only need to consider the nontrivial solutions.

As in the previous results, the verification of hypotheses (1) and (3) of the Transversality Theorem is simple, so we just show that (2α) holds.

Suppose, by contradiction, that there exists a critical point $(u,h) \in F^{-1}(0)$ and, wolog $h = i_\Omega$. Then, there exists $v \in L^q(\Omega)$ such that

$$\int_\Omega v \, DF(u, i_\Omega)(\dot{u}, \dot{h}) = 0 \tag{31}$$

for all $(\dot{u}, \dot{h}) \in W^{4,p} \cap W_0^{2,p}(\Omega) \times C^5(\Omega, \mathbb{R}^n)$ where $DF(u, i_\Omega) : W^{4,p} \cap W_0^{2,p}(\Omega) \times C^5(\Omega, \mathbb{R}^n) \to L^p(\Omega)$ is given by

$$DF(u, i_\Omega)(\dot{u}, \dot{h}) = L(u)(\dot{u} - \dot{h} \cdot \nabla u)$$

with $L(u) = \Delta^2 + \frac{\partial f}{\partial \mu}(\cdot, u, \nabla u, \Delta u)\Delta + \frac{\partial f}{\partial y}(\cdot, u, \nabla u, \Delta u) \cdot \nabla + \frac{\partial f}{\partial \lambda}(\cdot, u, \nabla u, \Delta u)$.

Taking $\dot{h} = 0$ in (31), we have

$$\int_\Omega v \, L(u) \, \dot{u} = 0 \quad \forall \dot{u} \in W^{4,p} \cap W_0^{2,p}(\Omega),$$

that is, $v \in \mathcal{N}(L^*(u))$. Since $\partial\Omega$ is C^5-regular and f is of class C^4, it follows by regularity results for uniformly elliptic equations that $v \in W^{5,q}(\Omega) \cap C^{4,\alpha}(\Omega)$ for $\alpha > 0$ and satisfy

$$\begin{cases} L^*(u)\, v = & 0 \text{ in } \Omega \\ v = \frac{\partial v}{\partial N} = & 0 \text{ on } \partial\Omega. \end{cases} \tag{32}$$

If $\dot{u} = 0$ and $\dot{h}$ varies in $C^5(\Omega, \mathbb{R}^n)$, we obtain

$$\begin{aligned} 0 &= -\int_\Omega v\, L(u)(\dot{h} \cdot \nabla u) \\ &= \int_\Omega \left\{ (\dot{h} \cdot \nabla u) L^*(u) v - v L(u)(\dot{h} \cdot \nabla u) \right\} \\ &= -\int_{\partial\Omega} \dot{h} \cdot N \Delta v \Delta u, \quad \forall \dot{h} \in C^5(\Omega, \mathbb{R}^n). \end{aligned}$$

Therefore, we have $\int_{\partial\Omega} \dot{h} \cdot N \Delta v \Delta u = 0 \quad \forall \dot{h} \in C^5(\Omega, \mathbb{R}^n)$ from which $\Delta v \Delta u \equiv 0$ on $\partial\Omega$. Since $i_\Omega \in U_M$, we reach a contradiction, proving the theorem. ∎

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