UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2569

HYPOTHESES TESTING ON A MULTIVARIATE NULL INTERCEPT ERRORS-IN-VARIABLES MODEL

CIBELE M. RUSSO
REIKO AOKI
DORIVAL LEÃO

Nº 330

RELATÓRIOS TÉCNICOS

São Carlos – SP
Set./2008

Hypotheses testing on a multivariate null intercept errors-in-variables model

Cibele M. Russo, Reiko Aoki, Dorival Leão

Departamento de Matemática Aplicada e Estatística,

ICMC, Universidade de São Paulo - São Carlos,

Caixa Postal: 668, CEP: 13560-970, São Carlos, SP, Brazil.

Abstract

Considering the Wald, score and the likelihood ratio asymptotic test statistics, we analyze a multivariate null intercept errors-in-variables regression model, where the explanatory and the response variables are subject to measurement errors, and a possible structure of dependency between the measurements taken within the same individual are incorporated, representing a longitudinal structure. This model was proposed by Aoki et al. (2003b) and analyzed under the bayesian approach. In this paper, considering the classical approach, we analyze asymptotic test statistics and present a simulation study to compare the behavior of the three test statistics for different sample sizes, parameter values and nominal levels of the test. Also, closed form expressions for the score function and the Fisher information matrix are presented. We consider two real numerical illustrations, the odontological data set from Hadgu & Koch (1999) and a quality control data set.

Key words: null intercept errors-in-variables models; score statistic; likelihood ratio; Wald statistic; EM algorithm.

1 Introduction

The errors-in-variables models are usually considered in the regression analysis when beyond the random errors involved in the model, measurement errors due to the instrument or process of measurement must be incorporated to the structure of the model. This kind of errors are usually found in industrial or biological problems. The errors-in-variables models can be constructed considering different assumptions and it can be classified as a functional, structural or ultrastructural measurement error model. The distinction between the functional and the structural models were first clearly described by Kendall (1951, 1952), while the ultrastructural model were defined in Dolby (1976). Also, a wide bibliography may be found in Kendall & Stuart (1961), Moran (1971) and Fuller (1987). The simplest structure for a measurement errors regression model can be defined as follows. Suppose the pairs $(\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)$, satisfy the linear relation, $\eta_i = \alpha + \beta\xi_i$, $i = 1, \ldots, n$, where (ξ_i, η_i) can not be observed directly, but with errors, through (x_i, y_i), that is,

$$x_i = \xi_i + u_i,$$
$$y_i = \eta_i + \varepsilon_i, i = 1, \ldots, n.$$

The errors u_i and ε_i are random variables with mean 0 and finite variance σ^2 and σ_ε^2, respectively, independently distributed for $i, i = 1, \ldots, n$. The functional model assumes that ξ_i are unknown constants, the structural model supposes that ξ_i are random variables with the same distribution (with $\mathrm{E}(\xi_i) = \mu$ and $\mathrm{Var}(\xi_i) = \sigma_x^2$) and the ultrastructural model assumes that ξ_i are random variables with different means and variances. It is common to assume that all the random variables in the structural measurement error model are jointly normal. However, it is well known that such a model is not identifiable and some extra assumption must be made to bypass this inconvenience, for instance the knowledge of the errors variance ratio ($\lambda = \sigma_\varepsilon^2/\sigma^2$), the knowledge of one of the errors variance (σ_ε^2 or σ^2), both of the variance (σ_ε^2 and σ^2) known, the attenuation factor ($k_x = \sigma^2/(\sigma^2+\sigma_x^2)$) known or the intercept α known (and $\mu \neq 0$). Each of these assumptions make the model identifiable. Models assuming the knowledge of the intercept can be found, for instance, in Chan & Mak (1979), Aoki et al. (2001, 2002, 2003a,b), Labra et al. (2005) and will be considered here.

Aoki et al. (2003b) proposed the following multivariate null intercept errors-in-variables model

$$\begin{cases} \mathbf{x}_i = \boldsymbol{\xi}_i + \boldsymbol{\delta}_i, \\ \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \mathbf{e}_i, \quad i = 1, ..., p, \end{cases} \tag{1}$$

to analyze a pretest/posttest data considering the bayesian approach. The observed vectors $\mathbf{x}_i = (x_{i_1}, \ldots, x_{i_{n_i}})^T$ and $\mathbf{y}_i = (\mathbf{y}_{1i}^T, \mathbf{y}_{2i}^T)^T$, with $\mathbf{y}_{1i} = (y_{1i_1}, \ldots, y_{1i_{n_i}})^T$ and $\mathbf{y}_{2i} = (y_{2i_1}, \ldots, y_{2i_{n_i}})^T$, are taken longitudinally for the n_i experimental units of the group i, $i = 1, \ldots, p$. The vector $\boldsymbol{\xi}_i = (\xi_{i_1}, \ldots, \xi_{i_{n_i}})^T$ represents the real values of the explanatory variables, which can not be observed directly, but with errors $\boldsymbol{\delta}_i = (\delta_{i_1}, \ldots, \delta_{i_{n_i}})^T$. The matrix $\mathbf{X}_i$ is given by $\mathbf{X}_i = \begin{pmatrix} \boldsymbol{\xi}_i & 0 \\ 0 & \boldsymbol{\xi}_i \end{pmatrix}$ and $\mathbf{e}_i = (\mathbf{e}_{1i}^T, \mathbf{e}_{2i}^T)^T$, where $\mathbf{e}_{1i} = (e_{1i_1}, \ldots, e_{1i_{n_i}})^T$ and $\mathbf{e}_{2i} = (e_{2i_1}, \ldots, e_{2i_{n_i}})^T$, represent the measurement errors of the response variables and $\boldsymbol{\beta}_i = (\beta_{1i}, \beta_{2i})^T$ represents the parameters of interest. We suppose that $\delta_{i_j} \overset{ind.}{\sim} N(0, \sigma^2)$, $e_{1i_j} \overset{ind.}{\sim} N(0, \sigma_{e_{1i}}^2)$, $e_{2i_j} \overset{ind.}{\sim} N(0, \sigma_{e_{2i}}^2)$ and $\xi_{i_j} \overset{ind.}{\sim} N(\mu, \sigma_x^2)$, $i = 1, ..., p; j = 1, ..., n_i$. We assume that the experimental units are randomized in the beginning of the study, so that the p groups are homogeneous, thus we consider the same variance (σ^2) for the initial measurement errors for the p groups. However, we suppose that the long-term behavior of the variables may not be the same, thus we assume $\boldsymbol{\sigma}_{e_i}^2 = (\sigma_{e_{1i}}^2, \sigma_{e_{2i}}^2)^T$, $i = 1, ..., p$, allowing different values for the errors variance after the beginning of the study for each group.

To analyze the model defined in (1) under the classical approach, this paper unfolds as follows. Section 2 is dedicated to the obtention of the maximum likelihood estimates via the EM algorithm, since it is not possible to explicit expressions for the maximum likelihood estimator. In Section 3 we discuss hypotheses testing considering the Wald, score and the likelihood ratio test statistics, as well as the restricted estimation under the null hypotheses of interest. Section 4 brings two numerical illustrations, one of them related to odontological data set and the other related to a quality control experiment. In Section 5 we present the results of the simulation study, comparing the behavior of those test statistics and finally, in Section 6 we discuss the general obtained results.

2 EM algorithm

In this section we propose the use of the EM algorithm for the obtention of the maximum likelihood estimates of the parameters of the model. Considering the model defined in (1), let $\mathbf{z}_{i_j}$ denote the vector of observations for the jth individual from the ith group. We have

$$\mathbf{z}_{i_j} = \begin{pmatrix} x_{i_j} \\ y_{1i_j} \\ y_{2i_j} \end{pmatrix} \sim N_3\left(\begin{pmatrix} \mu \\ \beta_{1i}\mu \\ \beta_{2i}\mu \end{pmatrix}, \begin{pmatrix} \sigma_x^2+\sigma^2 & \beta_{1i}\sigma_x^2 & \beta_{2i}\sigma_x^2 \\ \beta_{1i}\sigma_x^2 & \beta_{1i}^2\sigma_x^2+\sigma_{e_{1i}}^2 & \beta_{1i}\beta_{2i}\sigma_x^2 \\ \beta_{2i}\sigma_x^2 & \beta_{1i}\beta_{2i}\sigma_x^2 & \beta_{2i}^2\sigma_x^2+\sigma_{e_{2i}}^2 \end{pmatrix}\right). \qquad (2)$$

It can be easily shown that the correlations between the measurements taken in the same individual, x_{i_j}, y_{1i_j} and y_{2i_j} are given by

$$\rho(x_{i_j}, y_{ki_j}) = \frac{\beta_{ki}\sigma_x^2}{\sqrt{(\sigma_x^2+\sigma^2)(\beta_{ki}^2\sigma_x^2+\sigma_{e_{ki}}^2)}}, k = 1, 2,$$
$$\rho(y_{1i_j}, y_{2i_j}) = \frac{\beta_{1i}\beta_{2i}\sigma_x^2}{\sqrt{(\beta_{1i}^2\sigma_x^2+\sigma_{e_{1i}}^2)(\beta_{2i}^2\sigma_x^2+\sigma_{e_{2i}}^2)}}, i = 1, \ldots, p; j = 1, \ldots n_i.$$

Defining $\mathrm{E}(\mathbf{z}_{i_j}) = \mathbf{m}_i$ and $\mathrm{Var}(\mathbf{z}_{i_j}) = \mathbf{V}_i$, the joint probability function for the experimental units from the group i is given by

$$f_i(\mathbf{z}_i, \boldsymbol{\theta}) = (2\pi)^{-\frac{3n_i}{2}}|\mathbf{V}_i|^{-\frac{n_i}{2}} \exp\{-\frac{1}{2}\sum_{j=1}^{n_i}(\mathbf{z}_{i_j}-\mathbf{m}_i)^T\mathbf{V}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)\}, \qquad (3)$$

where $\mathbf{z}_i = (z_{i_1}, \ldots, z_{i_{n_i}})^T$, so that the log-likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\theta}/\mathbf{z}) = &-\frac{3N}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{p} n_i \log\nu_i - \frac{1}{2}\sum_{i=1}^{p}\frac{\sigma_{e_{1i}}^2\sigma_{e_{2i}}^2 + \sigma_x^2(\beta_{1i}^2\sigma_{e_{2i}}^2 + \beta_{2i}\sigma_{e_{1i}}^2)}{\nu_i}\sum_{j=1}^{n_i}(x_{i_j}-\mu)^2 + \\ &\sum_{i=1}^{p}\frac{\beta_{1i}\sigma_x^2\sigma_{e_{2i}}^2}{\nu_i}\sum_{j=1}^{n_i}(x_{i_j}-\mu)(y_{1i_j}-\beta_{1i}\mu) + \sum_{i=1}^{p}\frac{\beta_{2i}\sigma_x^2\sigma_{e_{1i}}^2}{\nu_i}\sum_{j=1}^{n_i}(x_{i_j}-\mu)(y_{2i_j}-\beta_{2i}\mu) - \\ &\sum_{i=1}^{p}\frac{(\sigma^2+\sigma_x^2)\sigma_{e_{2i}}^2 + \sigma_x^2\sigma^2\beta_{2i}^2}{2\nu_i}\sum_{j=1}^{n_i}(y_{1i_j}-\beta_{1i}\mu)^2 - \sum_{i=1}^{p}\frac{(\sigma^2+\sigma_x^2)\sigma_{e_{1i}}^2 + \sigma_x^2\sigma^2\beta_{1i}^2}{2\nu_i}\sum_{j=1}^{n_i}(y_{2i_j}-\beta_{2i}\mu)^2 + \\ &\sum_{i=1}^{p}\frac{1}{\nu_i}(\beta_{1i}\beta_{2i}\sigma_x^2\sigma^2)\sum_{j=1}^{n_i}(y_{1i_j}-\beta_{1i}\mu)(y_{2i_j}-\beta_{2i}\mu), \end{aligned} \qquad (4)$$

where $\nu_i = (\sigma_x^2+\sigma^2)(\sigma_{e_{1i}}^2\sigma_{e_{2i}}^2) + \sigma_x^2\sigma^2(\beta_{1i}^2\sigma_{e_{2i}}^2 + \beta_{2i}^2\sigma_{e_{1i}}^2)$, $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_p)^T$ and $N = \sum_{i=1}^{p} n_i$.

The elements of the score function (Appendix A) present nonlinearity and it is not feasible to obtain the explicit expressions for the estimators. The alternative is to obtain numerical estimates considering, for example, the EM algorithm (Dempster et al. 1977). In general, for measurement errors models, one way to simplify the complete data log-likelihood function is to introduce the latent variables $\xi_{i_j}, i = 1, \ldots, p, j = 1, \ldots, n_i$, which correspond to the true value of the unobserved explanatory variables, as the missing data. Let $\mathbf{w}_{i_j} = (\xi_{i_j}, x_{i_j}, y_{1i_j}, y_{2i_j})^T = (\xi_{i_j}, \mathbf{z}_{i_j}^T)^T$ be the vector of observations for the complete data, with $\mathbf{z}_{i_j}$ as defined in (2). Thus, the log-likelihood function for the complete data is given by

$$
\begin{aligned}
L(\mathbf{w}, \boldsymbol{\theta}) = & -2N\log(2\pi) - \frac{N}{2}\log\sigma^2 - \frac{N}{2}\log\sigma_x^2 - \frac{1}{2}\sum_{i=1}^{p} n_i \log\sigma_{e_{1i}}^2 - \frac{1}{2}\sum_{i=1}^{p} n_i \log\sigma_{e_{2i}}^2 \\
& -\frac{1}{2\sigma_x^2}\sum_{i=1}^{p}\sum_{j=1}^{n_i}(\xi_{i_j} - \mu)^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{p}\sum_{j=1}^{n_i}(x_{i_j} - \xi_{i_j})^2 - \frac{1}{2}\sum_{i=1}^{p}\frac{1}{\sigma_{e_{1i}}^2}\sum_{j=1}^{n_i}(y_{1i_j} - \beta_{1i}\xi_{i_j})^2 \\
& -\frac{1}{2}\sum_{i=1}^{p}\frac{1}{\sigma_{e_{2i}}^2}\sum_{j=1}^{n_i}(y_{2i_j} - \beta_{2i}\xi_{i_j})^2,
\end{aligned}
$$

which is much simpler than (4). The steps of the EM algorithm can be summarized as

E step (expectation): Obtain

$$
\begin{aligned}
\xi_{i_j}^{(r+1)} &= E[\xi_{i_j}/\mathbf{z}, \boldsymbol{\theta}] = \frac{\left(\frac{\mu^{(r)}}{\sigma_x^{2(r)}} + \frac{x_{i_j}}{\sigma^{2(r)}} + \frac{y_{1i_j}\beta_{1i}^{(r)}}{\sigma_{e_{1i}}^{2(r)}} + \frac{y_{i_j}\beta_{2i}^{(r)}}{\sigma_{e_{2i}}^{2(r)}}\right)}{\left(\frac{1}{\sigma^{2(r)}} + \frac{1}{\sigma_x^{2(r)}} + \frac{(\beta_{1i}^{(r)})^2}{\sigma_{e_{1i}}^{2(r)}} + \frac{(\beta_{2i}^{(r)})^2}{\sigma_{e_{2i}}^{2(r)}}\right)} \text{ and} \\
(\xi_{i_j}^2)^{(r+1)} &= E[\xi_{i_j}^2/\mathbf{z}, \boldsymbol{\theta}] = (\xi_{i_j}^{(r+1)})^2 + \frac{1}{\left(\frac{1}{\sigma^{2(r)}} + \frac{1}{\sigma_x^{2(r)}} + \frac{(\beta_{1i}^{(r)})^2}{\sigma_{e_{1i}}^{2(r)}} + \frac{(\beta_{2i}^{(r)})^2}{\sigma_{e_{2i}}^{2(r)}}\right)}.
\end{aligned} \tag{5}
$$

M step (maximization): Update $\widehat{\boldsymbol{\theta}}$ by maximizing $\mathrm{E}(L(\mathbf{w}, \boldsymbol{\theta})/\mathbf{z}, \widehat{\boldsymbol{\theta}}^{(r+1)})$

$$
\begin{aligned}
\mu^{(r+2)} &= \frac{1}{N}\sum_{i=1}^{p}\sum_{j=1}^{n_i}\xi_{i_j}^{(r+1)}, \sigma^{2(r+2)} = \frac{1}{N}\sum_{i=1}^{p}\sum_{j=1}^{n_i}[x_{i_j}^2 - 2x_{i_j}\xi_{i_j}^{(r+1)} + (\xi_{i_j}^2)^{(r+1)}], \\
\sigma_x^{2(r+2)} &= \frac{1}{N}\sum_{i=1}^{p}\sum_{j=1}^{n_i}[(\xi_{i_j}^2)^{(r+1)} - 2\mu^{(r+2)}\xi_{i_j}^{(r+1)} + (\mu^{(r+2)})^2], \\
\beta_{1i}^{(r+2)} &= \frac{\sum_{j=1}^{n_i} y_{1i_j}\xi_{i_j}^{(r+1)}}{\sum_{j=1}^{n_i}(\xi_{i_j}^2)^{(r+1)}}, \beta_{2i}^{(r+2)} = \frac{\sum_{j=1}^{n_i} y_{2i_j}\xi_{i_j}^{(r+1)}}{\sum_{j=1}^{n_i}(\xi_{i_j}^2)^{(r+1)}}, \\
\sigma^{2(r+2)}_{e_{1i}} &= \frac{1}{n_i}\sum_{i=1}^{p}\sum_{j=1}^{n_i}[y_{1i_j}^2 - 2\beta_{1i}^{(r+2)}y_{1i_j}\xi_{i_j}^{(r+1)} + \beta_{1i}^2(\xi_{i_j}^2)^{(r+1)}], \\
\sigma^{2(r+2)}_{e_{2i}} &= \frac{1}{n_i}\sum_{i=1}^{p}\sum_{j=1}^{n_i}[y_{2i_j}^2 - 2\beta_{2i}^{(r+2)}y_{2i_j}\xi_{i_j}^{(r+1)} + \beta_{2i}^2(\xi_{i_j}^2)^{(r+1)}],
\end{aligned} \tag{6}
$$

where the upper index indicate the iteration. For the initial values we can use, for instance, the method of moments estimates. Notice that all the expressions in the EM algorithm are given in closed form expressions, which makes the algorithm extremely simple and computationally inexpensive. In the next section we discuss the hypotheses testing of interest.

3 Hypotheses testing

Let us consider the following tests of hypotheses about the model parameters:

- The inter-groups tests, for the measurements taken in the two periods (k) of interest:

$$\begin{cases} H_0 : \beta_{kl} = \beta_{km} \text{ versus} \\ H_1 : \beta_{kl} \neq \beta_{km}, \quad \text{for} \quad k = 1, 2; m, l = 1, \ldots, p; \; l \neq m, \end{cases} \tag{7}$$

- The intra-group tests, to compare the long-term behavior of each group:

$$\begin{cases} H_0 : \beta_{1l} = \beta_{2l} \text{ versus} \\ H_1 : \beta_{1l} \neq \beta_{2l}, \quad \text{for} \quad l = 1, \ldots, p. \end{cases} \tag{8}$$

- The test considering specific value for the parameter

$$\begin{cases} H_0 : \beta_{kl} = c \text{ versus} \\ H_1 : \beta_{kl} \neq c, \quad \text{for} \quad k = 1, 2; l = 1, \ldots p; c \in \mathbb{R} \end{cases} \tag{9}$$

For testing these hypotheses, it is necessary to obtain the asymptotic distribution of the maximum likelihood estimators. For the model defined in (1), the vector of parameters is given by $\boldsymbol{\theta}_{(4p+3)\times 1} = (\boldsymbol{\beta}_1^T, \ldots, \boldsymbol{\beta}_p^T, \mu, \sigma^2, \sigma_x^2, \boldsymbol{\sigma}_{e_1}^{2^T}, \ldots, \boldsymbol{\sigma}_{e_p}^{2^T})^T$, where $\boldsymbol{\beta}_i = (\beta_{1i}, \beta_{2i})^T$ and $\boldsymbol{\sigma}_{e_i}^2 = (\sigma_{e_{1i}}^2, \sigma_{e_{2i}}^2)^T, i = 1, \ldots, p$.

In this case, the observations are not identically distributed for the p groups. There are some parameters that are specific for each group, β_{1i} for example is related only to the group i, and there are also common parameters to all groups (μ, σ_x^2 and σ^2). Bradley & Gart (1962) classify these groups as associated populations. Under regularity conditions, they have shown that

$$\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{\mathcal{D}} \mathcal{N}_q(\mathbf{0}, J^{-1}(\boldsymbol{\theta}_0)) \text{ when } N \to \infty, \text{ where}$$

$$J(\boldsymbol{\theta}) = \sum_{i=1}^{p} \frac{n_i}{N} I_i(\boldsymbol{\theta}), \text{ with } I_i(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \log f_i}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T}\right] \text{ and } N = \sum_{i=1}^{p} n_i,$$

with q representing the dimension of $\boldsymbol{\theta}$ ($q = 4p + 3$), $\widehat{\boldsymbol{\theta}}$ the maximum likelihood estimator of $\boldsymbol{\theta}$ and $I_i(\boldsymbol{\theta})$ is the Fisher information matrix for each group, with f_i given by (3). Using also Dolby (1976), for each

$i = 1, \ldots, p$, with $L(\mathbf{z}_i, \boldsymbol{\theta}) = \log f_i$, we have

$$I_i(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 L(\mathbf{z}_i, \boldsymbol{\theta})}{\partial\theta_u \partial\theta_v}\right] = \left\{\frac{1}{2} tr(\mathbf{V}_i^{-1}\mathbf{V}_{i_{\theta_u}}\mathbf{V}_i^{-1}\mathbf{V}_{i_{\theta_v}}) + \mathbf{d}_{i_{\theta_u}}\mathbf{V}_i^{-1}\mathbf{d}_{i_{\theta_v}}\right\}, \tag{10}$$

where $\boldsymbol{V}_{i_{\theta_u}}$ and $\mathbf{d}_{i_{\theta_u}}$ are respectively the matrix and the vector of derivatives element by element (in relation to the parameter θ_u) of the variance-covariance matrix ($\mathbf{V}_i$) and of the deviances mean ($\mathbf{d}_i$), $i = 1, \ldots, p$, that is,

$$\mathbf{V}_i = \begin{pmatrix} \sigma_x^2 + \sigma^2 & \beta_{1i}\sigma_x^2 & \beta_{2i}\sigma_x^2 \\ \beta_{1i}\sigma_x^2 & \beta_{1i}^2\sigma_x^2 + \sigma_{e_{1i}}^2 & \beta_{1i}\beta_{2i}\sigma_x^2 \\ \beta_{2i}\sigma_x^2 & \beta_{1i}\beta_{2i}\sigma_x^2 & \beta_{2i}^2\sigma_x^2 + \sigma_{e_{2i}}^2 \end{pmatrix} \text{ and } \mathbf{d}_i = \begin{pmatrix} \frac{1}{n_i}\sum_{j=1}^{n_i} x_{i_j} - \mu \\ \frac{1}{n_i}\sum_{j=1}^{n_i} y_{1i_j} - \beta_{1i}\mu \\ \frac{1}{n_i}\sum_{j=1}^{n_i} y_{2i_j} - \beta_{2i}\mu \end{pmatrix}.$$

For instance,

$$V_{i_{\sigma_x^2}} = \begin{pmatrix} 1 & \beta_{1i} & \beta_{2i} \\ \beta_{1i} & \beta_{1i}^2 & \beta_{1i}\beta_{2i} \\ \beta_{2i} & \beta_{1i}\beta_{2i} & \beta_{2i}^2 \end{pmatrix} \quad \text{and} \quad \mathbf{d}_{i_\mu} = \begin{pmatrix} -1 \\ -\beta_{1i} \\ -\beta_{2i} \end{pmatrix}, \quad i = 1, \ldots, p.$$

The elements of $J(\boldsymbol{\theta})$ can be found in Appendix B.

Let us consider the general test of hypotheses:

$$\begin{cases} H_0 : \mathbf{h}(\boldsymbol{\theta}) = \mathbf{0} \textit{ versus} \\ H_1 : \mathbf{h}(\boldsymbol{\theta}) \neq \mathbf{0}, \end{cases}$$

where $\mathbf{h}(\boldsymbol{\theta}) = (h_1, \ldots, h_w)^T : \mathbb{R}^q \to \mathbb{R}^w$, with the usual regularity conditions (see Bradley & Gart 1962), let $\mathbf{H}(\boldsymbol{\theta}) = \frac{\partial \mathbf{h}(\boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}}$ be a $(q \times w)$ matrix, continuous in $\boldsymbol{\theta}$ with rank w, $\widehat{\boldsymbol{\theta}}$ the maximum likelihood estimator of $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ the maximum likelihood estimator restricted to H_0, that is, such that $\mathbf{h}(\overline{\boldsymbol{\theta}}) = 0$, then we have

(a) $Q_W = N\mathbf{h}(\widehat{\boldsymbol{\theta}})^T[\mathbf{H}^T(\widehat{\boldsymbol{\theta}})J^{-1}(\widehat{\boldsymbol{\theta}})\mathbf{H}(\widehat{\boldsymbol{\theta}})]^{-1}\mathbf{h}(\widehat{\boldsymbol{\theta}})$ (**Wald test**),

(b) $Q_L = -2\log\lambda = 2(L(\mathbf{z}, \widehat{\boldsymbol{\theta}}) - L(\mathbf{z}, \overline{\boldsymbol{\theta}}))$, where $\lambda = \frac{\sup_{\{\boldsymbol{\theta} \in \Theta : \mathbf{h}(\boldsymbol{\theta}) = \mathbf{0}\}} L(\mathbf{z}, \boldsymbol{\theta})}{\sup_{\{\boldsymbol{\theta} \in \Theta\}} L(\mathbf{z}, \boldsymbol{\theta})}$ (**likelihood ratio test**),

(c) $Q_S = N^{-1}U^T(\overline{\boldsymbol{\theta}})J^{-1}(\overline{\boldsymbol{\theta}})U(\overline{\boldsymbol{\theta}})$, where $U(\overline{\boldsymbol{\theta}}) = \sum_{i=1}^{p}\sum_{j=1}^{n_i} \left.\frac{\partial \log f_i(\mathbf{z}_{i_j}, \boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}=\overline{\boldsymbol{\theta}}}$ (**score test**).

Under regularity conditions and under the null hypothesis, the three test statistics Q_W, Q_L and Q_S follow asymptotically a chi-square distribution with degrees of freedom w (χ_w^2).

Consider the hypotheses of interest given by (7) and (8) and $p = 3$. In this case, we have $\boldsymbol{\beta} = (\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}, \beta_{13}, \beta_{23})^T$ and if our interest is to test an inter-group test of hypotheses defined in (7), as for example

$$\begin{cases} H_0 : \beta_{11} - \beta_{12} = 0 \text{ } versus \\ H_1 : \beta_{11} - \beta_{12} \neq 0, \end{cases} \tag{11}$$

or an intra-group test of hypotheses defined in (8) as, for instance

$$\begin{cases} H_0 : \beta_{11} - \beta_{21} = 0 \text{ } versus \\ H_1 : \beta_{11} - \beta_{21} \neq 0, \end{cases} \tag{12}$$

these hypotheses can be written as

$$\begin{cases} H_0 : A\boldsymbol{\beta} = 0 \text{ } versus \\ H_1 : A\boldsymbol{\beta} \neq 0, \end{cases}$$

with $A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$ for (11) and $A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$ for (12). The Wald test statistic can be written in the form

$$Q_W = N\widehat{\boldsymbol{\beta}}^T A^T [A\Omega_{\beta\beta}A^T]^{-1} A\widehat{\boldsymbol{\beta}},$$

where $\Omega_{\beta\beta}$ is given in

$$[J(\boldsymbol{\theta})]^{-1} = \begin{pmatrix} \Omega_{\beta\beta} & \Omega_{\beta\theta^\star} \\ \Omega_{\theta^\star\beta} & \Omega_{\theta^\star\theta^\star} \end{pmatrix},$$

in which $\boldsymbol{\theta}$ is partitioned as $\boldsymbol{\theta} = (\boldsymbol{\beta}^T, \boldsymbol{\theta}^{\star T})^T$, and $\boldsymbol{\theta}^\star = (\mu, \sigma^2, \sigma_x^2, \boldsymbol{\sigma}_{e_1}^{2^T}, \ldots, \boldsymbol{\sigma}_{e_p}^{2^T})^T$.

In order to use the likelihood ratio and score test statistics, it is necessary to obtain the restricted maximum likelihood estimates. To obtain those estimates we are going to consider the ECM algorithm (Meng & Rubin 1993), which is an extension of the EM algorithm for cases in which the expressions of the M step are not simple and require some extra computational effort. Let $Q = Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{m-1})$ denote the expected values of the complete data log-likelihood function with respect to the unknown data, given the observed data and the current parameter estimates. The ECM algorithm replaces each M step of the EM algorithm by a sequence of S conditional maximization steps, which is called the CM steps, each of which maximizes Q over $\boldsymbol{\theta}$ but with some vector function of $\boldsymbol{\theta}$, $g_s(\boldsymbol{\theta})$, $s = 1, \ldots, S$, fixed at its previous value under the null hypothesis. For instance, the ECM algorithm for the estimation restricted to the null hypothesis in (7) ($\beta_{kl} = \beta_{km} = \beta$) may be summarized as:

E step (expectation): Given $\theta^{(r)} = \widehat{\theta}^{(r)}$, estimate $\xi_{i_j}^{(r+1)}$ and $(\xi_{i_j}^2)^{(r+1)}$, for $i \neq l, m$ and $j = 1, \ldots, n_i$, simply considering the expressions given in (5). For $i = l$ or m and $j = 1, \ldots, n_i$, compute $\xi_{i_j}^{(r+1)}$ and $(\xi_{i_j}^2)^{(r+1)}$ considering the expressions in (5) replacing $\beta_{kl}^{(r)}$ and $\beta_{km}^{(r)}$ by $\beta^{(r)}$.

CM step (conditional maximization): Update $\mu^{(r+2)}$, $\sigma^{2(r+2)}$, $\sigma_x^{2(r+2)}$ considering the expressions in the M step from the unrestricted EM algorithm (6). For $i \neq l, m$ and $g \neq k$, update $\beta_{gi}^{(r+2)}$ and $\sigma^{2(r+2)}_{e_{gi}}$ considering their expressions in (6). Compute $\sigma^{2(r+2)}_{e_{kl}}$ and $\sigma^{2(r+2)}_{e_{km}}$ using their respective expressions in (6) replacing $\beta_{kl}^{(r+2)}$ and $\beta_{km}^{(r+2)}$ by $\beta^{(r+1)}$ and compute $\beta^{(r+2)}$ as follows:

$$\beta^{(r+2)} = \frac{\dfrac{1}{\sigma^{2(r+2)}_{e_{kl}}} \sum_{j=1}^{n_l} y_{kl_j} \xi_{l_j}^{(r+1)} + \dfrac{1}{\sigma^{2(r+2)}_{e_{km}}} \sum_{j=1}^{n_l} y_{km_j} \xi_{m_j}^{(r+1)}}{\dfrac{1}{\sigma^{2(r+2)}_{e_{kl}}} \sum_{j=1}^{n_l} (\xi_{l_j}^2)^{(r+1)} + \dfrac{1}{\sigma^{2(r+2)}_{e_{km}}} \sum_{j=1}^{n_l} (\xi_{m_j}^2)^{(r+1)}}.$$

Similarly, the ECM algorithm to obtain the maximum likelihood estimates restricted to the null hypothesis in (8) ($\beta_{1l} = \beta_{2l} = \beta$) starts computing E step as (5) adequately according to the null hypothesis, that is, for $i = l$ and $j = 1, \ldots, n_i$, compute $(\xi_{i_j})^{(r+1)}$ and $(\xi_{i_j}^2)^{(r+1)}$ replacing $\beta_{1l}^{(r)}$ and $\beta_{2l}^{(r)}$ by $\beta^{(r)}$ and for $i = 1, \ldots, p$, $i \neq l$ use exactly the expressions in (5). In CM step, firstly update $\mu^{(r+2)}$, $\sigma^{2(r+2)}$, $\sigma_x^{2(r+2)}$ considering the expressions in the M step from the unrestricted EM algorithm (6). Moreover, for $i = 1, \ldots, p$, $i \neq l$ update $\beta_{1i}^{(r+2)}$, $\beta_{2i}^{(r+2)}$, $\sigma^{2(r+2)}_{e_{1i}}$, $\sigma^{2(r+2)}_{e_{2i}}$ exactly as in (6). For $i = l$, compute $\sigma^{2(r+2)}_{e_{1l}}$ and $\sigma^{2(r+2)}_{e_{2l}}$ considering the expressions in (6) replacing $\beta_{1l}^{(r+2)}$ and $\beta_{2l}^{(r+2)}$ by $\beta^{(r+1)}$. Finally, compute

$$\beta^{(r+2)} = \frac{\dfrac{1}{\sigma^{2(r+2)}_{e_{1l}}} \sum_{j=1}^{n_l} y_{1l_j} \xi_{l_j}^{(r+1)} + \dfrac{1}{\sigma^{2(r+2)}_{e_{2l}}} \sum_{j=1}^{n_l} y_{2l_j} \xi_{l_j}^{(r+1)}}{\left(\dfrac{1}{\sigma^{2(r+2)}_{e_{1l}}} + \dfrac{1}{\sigma^{2(r+2)}_{e_{2l}}}\right) \sum_{j=1}^{n_l} (\xi_{l_j}^2)^{(r+1)}}.$$

For the maximum likelihood estimates restricted to the null hypothesis given in (9), in which $\beta_{kl} = c$, $k = 1, 2$; $l = 1, \ldots p$; $c \in \mathbb{R}$, it is straightforward to adapt the EM algorithm presented in Section 2.

In the next section, these results will be applied to the two numerical data sets, which motivated the simulation studies that is described in Section 5.

4 Numerical Illustration

The multivariate null intercept errors-in-variables model defined in (1) can widely be applied to many experiments where the explanatory and responses variables are subject to measurement errors. In this section we will present two numerical applications related to real data sets, one of which is about the odontological data presented in Hadgu & Koch (1999) and analyzed by Aoki et al. (2003b) under the bayesian approach. Another numerical illustration is related to a quality control data set.

Odontological data set

In the odontological data set presented in Hadgu & Koch (1999), 109 volunteers were randomized to three groups in order to test two experimental mouth rinses in the prevention of the dental plaque. In that study, each individual used one of the three mouth rinses - the control mouth rinse (i=1) or the experimental mouth rinses A (i=2) or B (i=3). The dental plaque indexes, which are subject to measurement errors, were taken in the beginning of the study ($\mathbf{x}_i$) and after the use of the respective mouth rinse i, $i = 1, 2, 3$, at three ($\mathbf{y}_{1i}$) and six ($\mathbf{y}_{2i}$) months from the baseline. The main interests of the study were to find out if the two experimental mouth rinses were more effective than the control mouth rinse in inhibiting the development of the dental plaque, if one of the experimental mouth rinses is more effective than the other and if each of the two experimental mouth rinses have long-lasting effect. In this case, we have $p = 3$ mouth rinses ($n_1 = 36, n_2 = 33$ and $n_3 = 36$) and the parameters of interest β_{1i} (β_{2i}), $i = 1, 2, 3$ represent the remaining mean percentage plaque after three (six) months from the baseline for the n_i experimental units who used the ith mouth rinse. $\boldsymbol{\xi}_i$ represents the real values of the dental plaque index in the beginning of the study (without measurement error) and the vectors $\boldsymbol{\delta}_i$ and $\mathbf{e}_i = (\mathbf{e}_{1i}^T, \mathbf{e}_{2i}^T)^T$, represent the measurement errors in the beginning of the study and after three and six months from the baseline, respectively.

First, the maximum likelihood estimates of the parameters were obtained considering the EM algorithm and they are presented in Table 1 with the standard deviations between parentheses, obtained using the Fisher information matrix presented in the Appendix B. The questions of interest of the researcher related to the odontological data set were:

- If two different mouth rinses have the same efficiency after three months of use (inter-groups tests (the test given by (7) with $k = 1$)),
- If two different mouth rinses have the same efficiency after six months of use (inter-groups tests (the test given by (7) with $k = 2$)),
- If each mouth rinse keep preventing the dental plaque after three months of use (intra-group tests (the test given by (8)).

Considering the Wald, likelihood ratio and score asymptotic test statistics, we have obtained the results described in Table 2.

Analyzing Table 2, we notice that some of the conclusions depend on the value of the significance level, which motivated a simulation study presented in the next section. The conclusions are summarized as follows:

1. Considering any of the three tests, it is concluded that the experimental mouth rinses A and B are more efficient than the control mouth rinse after three months ($H_0 : \beta_{11} = \beta_{12}$ and $H_0 : \beta_{11} = \beta_{13}$,

Table 1: Maximum likelihood estimates for the parameters (with standard deviations).

Parameter	β_{11}	β_{21}	β_{12}	β_{22}	β_{13}	β_{23}	μ	σ^2
Estimate	0.703	0.687	0.525	0.502	0.508	0.414	2.535	0.010
(SD)	(0.037)	(0.032)	(0.045)	(0.045)	(0.033)	(0.029)	(0.033)	(0.022)

Parameter	σ_x^2	$\sigma_{e_{11}}^2$	$\sigma_{e_{21}}^2$	$\sigma_{e_{12}}^2$	$\sigma_{e_{22}}^2$	$\sigma_{e_{13}}^2$	$\sigma_{e_{23}}^2$
Estimate	0.103	0.312	0.234	0.430	0.431	0.255	0.192
(SD)	(0.026)	(0.075)	(0.057)	(0.107)	(0.107)	(0.061)	(0.046)

Table 2: Wald, likelihood ratio and score test statistics and the corresponding p-values for the odontological data set.

H_0	$\beta_{11} = \beta_{12}$	$\beta_{11} = \beta_{13}$	$\beta_{12} = \beta_{13}$	$\beta_{21} = \beta_{22}$	$\beta_{21} = \beta_{23}$
Q_W	9.524	15.429	0.072	11.294	40.114
(p-value)	0.002	0.000	0.788	0.001	0.000
Q_L	9.095	13.824	0.072	10.587	31.525
(p-value)	0.003	0.000	0.788	0.001	0.000
Q_S	11.446	16.119	3.207	13.004	30.565
(p-value)	0.001	0.000	0.073	0.000	0.000

H_0	$\beta_{22} = \beta_{23}$	$\beta_{11} = \beta_{21}$	$\beta_{12} = \beta_{22}$	$\beta_{13} = \beta_{23}$
Q_W	2.714	0.113	0.124	4.687
(p-value)	0.099	0.737	0.724	0.030
Q_L	2.669	0.115	0.128	4.335
(p-value)	0.102	0.735	0.721	0.037
Q_S	5.672	3.163	3.220	7.152
(p-value)	0.017	0.075	0.073	0.007

respectively), as well as, after six months of use ($H_0 : \beta_{21} = \beta_{22}$ and $H_0 : \beta_{21} = \beta_{23}$, respectively). Considering the simulation study presented in the next section, all of the three asymptotic test statistics presented approximately the same behaviour in this case, even for samples of moderate size. Also, Aoki et al. (2003b) and Hadgu & Koch (1999) obtained the same conclusion.

2. The conclusion between the experimental mouth rinses A and B after three months from the baseline ($H_0 : \beta_{12} = \beta_{13}$) depends on the considered significance level. For the significance level $\alpha = 1\%$ and $\alpha = 5\%$ the three tests concluded that the two experimental mouth rinses are equivalent after three months from the baseline. However, considering $\alpha = 10\%$ the score test concluded that the mouth rinses A and B are not equivalent after that period. Nevertheless, according to simulation results which will be presented in the next section, we have observed that the score test is more liberal in this case (Table 9 with $\sigma^2 = 0.01$ and $\sigma_x^2 = 0.1$), that is, it rejects more than the nominal level when σ^2 is small ($\sigma^2 = 0.01$). Aoki et al. (2003b) and Hadgu & Koch (1999) concluded that there is no difference between the two experimental mouth rinses in the first three months of use.

3. Comparing the experimental mouth rinses A and B after six months of use, the conclusion also depends on the considered significance level. For $\alpha = 1\%$ the three test statistics concluded that

both mouth rinses are equivalent. For $\alpha = 5\%$ the score test concluded that these two rinses are not equivalent, while the other two test statistics concluded the contrary. For $\alpha = 10\%$ the Wald and score test statistics concluded that the rinses are not equivalent, while the likelihood ratio test concluded the contrary. Aoki et al. (2003b) and Hadgu & Koch (1999) concluded that the reduction rate of the plaque index after six months from the beginning of the study considering the mouth rinses A and B is equivalent. Also, in the simulation study, we have observed that the score test is liberal in this case (Table 10 with $\sigma^2 = 0.01$ and $\sigma_x^2 = 0.1$).

4. Testing the long-term efficiency of the control and experimental mouth rinse A, under the significance level $\alpha = 1\%$ and $\alpha = 5\%$, the three test statistics concluded that they are not long-lasting. However, considering $\alpha = 10\%$, the score test rejects this null hypothesis while the Wald and likelihood ratio tests do not reject it. Testing the long-term efficiency of mouth rinse B and considering the significance level $\alpha = 1\%$, the score test rejects the null hypothesis, but the Wald and likelihood ratio test statistics do not. Considering $\alpha = 5\%$ and $\alpha = 10\%$, the three test statistics reject the null hypothesis and conclude that B keeps preventing the dental plaque after three months of use. In Aoki et al. (2003b) and Hadgu & Koch (1999), it was concluded that B is the only mouth rinse that is long lasting.

Motivated by the different conclusions obtained for the three asymptotic test statistics, we have conducted a simulation study considering the three test statistics and different sample sizes, parameter values and nominal levels.

Quality control data set

In order to analyze the dimensional characteristics of pistons in an industrial quality control procedure, the KS Pistons developed an appropriate measurement system to evaluate the diameter of the pieces. However, after the production, the pistons are washed at a temperature around 70° C, which increases the temperature of the pieces and changes their dimensional characteristics. As the specifications of the pieces were defined for the standard temperature of 20° C, they are enclosed in a climatized room after the washing process in order to stabilize the temperature. It is known that 6 hours are enough to stabilize the temperature of the pistons around 20° C and, consequently they can compare the measurements of the pistons with the specification. However, to minimize costs and improve productivity, the Six Sigma team of the KS Pistons needs to find out if it is possible to reduce this period of 6 hours. In order to answer this question, it was realized an experiment where the diameters of the pistons were measured right after the washing process and after 4 and 6 hours. The measurements are presented in Appendix C. This experiment was carried out in two different days considering different pistons in each day, which induces the independency of the two groups. To account for the possible structure of dependency in the measurements taken in the same pistons in different hours, it is natural to fit the model defined in (1). The data were linearly transformed to consider only the numbers from the second decimal places on, in millimeters, which represents the significance variation of the pistons. The observed measurements, which are subject to errors, are represented in the following way: $\mathbf{x}_i$ is the vector composed by the observed measurements taken after 6 hours from the washing process (supposed to be stabilized), $\mathbf{y}_{1i}$ is the observed second measurements taken after 4 hours from the washing process and $\mathbf{y}_{2i}$ is the observed first measurements taken right after the washing process. In this case, we have $p = 2$ groups of pistons, each group measured in a different day, with $n_1 = n_2 = 80$. The parameters of interest β_{1i} (β_{2i}), $i = 1, 2$ represent the reducing rate of the piston size, that is, the average reduction of size after four (zero) hours from the washing time, with respect to their real size at 20°C after 6 hours, for the n_i pistons from the group i. It is natural to fit an errors-in-variables model because the measurements are subject to measurement errors, induced by the measurement system used in the process. The vector $\boldsymbol{\xi}_i$ represents the real size of the pistons from the group i, $i = 1, 2$ at 20°C (without the measurement error), $\boldsymbol{\delta}_i$ represents the measurement errors in the final measurement and $\mathbf{e}_i = (\mathbf{e}_{1i}^T, \mathbf{e}_{2i}^T)^T$, represent the measurement errors after four and zero hours from the washing time, respectively.

Table 3: Maximum likelihood estimates for the parameters (with standard deviations).

Parameter	β_{11}	β_{21}	β_{12}	β_{22}	μ	σ^2	σ_x^2
Estimate	1.0038	1.0651	1.0034	1.0571	47.3109	0.1004	7.9161
(SD)	(0.0014)	(0.0015)	(0.0012)	(0.0019)	(0.3165)	(0.0228)	(1.2549)

Parameter	$\sigma_{e_{11}}^2$	$\sigma_{e_{21}}^2$	$\sigma_{e_{12}}^2$	$\sigma_{e_{22}}^2$
Estimate	0.0686	0.0792	0.0192	0.2173
(SD)	(0.0264)	(0.0300)	(0.0225)	(0.0553)

Table 3 presents the maximum likelihood estimates of the parameters obtained considering the EM algorithm with the standard deviations between parentheses, obtained using the asymptotic distribution. The questions of interest of the researcher about the quality control data set were:

- If the process is influenced by external factors, for instance variation in the washing machine temperature and the efficiency of the climatized room. In other words, if the day in which the process was carried out had influence on the stability of the pistons size (test given by (7) with $k = 1$ and $k = 2$).
- If each group of pistons had already achieved the stability after 4 hours from the washing process (the test given by (9) with $k = 1$, $l = 1, 2$ and $c = 1$) to answer to the question about the possibility of reducing the waiting time from 6 to 4 hours to improve the production process.

The results of the Wald, likelihood ratio and score asymptotic test statistics for the pistons data set are described in Tables 4. The conclusions are summarized as follows:

Table 4: Wald, likelihood ratio and score test statistics and the corresponding p-values for the quality control data set.

H_0	$\beta_{11}=\beta_{12}$	$\beta_{21}=\beta_{22}$	$\beta_{11}=1$	$\beta_{12}=1$
Q_W	0.0682	22.1205	15.0942	16.7658
(p-value)	0.7940	0	0.0001	0
Q_L	0.0631	20.4951	13.7212	16.8766
(p-value)	0.8017	0	0.0002	0
Q_S	0.1647	19.8716	12.9789	15.5865
(p-value)	0.6849	0	0.0003	0.0001

Table 5: 95% confidence intervals for the expected values of the variables of interest in the quality control problem.

Difference of interest	lower limit	upper limit
$E(y_{11}-x_1)$	0.05267	0.30689
$E(y_{12}-x_2)$	0.02762	0.29408

1. If we consider the whole period of the experiment (the six hours), we conclude that the climatized room did not have the same efficiency in each day, as the three test statistics reject $H_0:\beta_{21}=\beta_{22}$.
2. Considering any little significance level α, the three test statistics lead us to conclude that 4 hours are not enough to stabilize the temperature of the pistons around 20^{o}C (rejection of $H_0:\beta_{11}=1$ and $H_0:\beta_{12}=1$).

Considering the Delta method, we obtained a 95% confidence intervals for the expected values of the differences of interest. These intervals, which are shown in Table 5, have confirmed the results obtained by the tests of hypotheses for the null hypothesis $H_0:\beta_{11}=1$ and $H_0:\beta_{12}=1$, that is, none of the groups of pistons have achieved the stability after four hours from the washing time.

5 Simulation study

In this section we present the simulation study in order to compare the behaviour of the Wald, likelihood ratio and score test statistics related to the sample sizes, parameter values and nominal levels. For the implementation of the simulations we considered the software Ox (Doornik 2002). We have sampled from two ($p=2$) and three ($p=3$) populations considering different parameter values for σ^2, σ_x^2 and μ. We have considered all the combinations of the values 1, 2.5, 5 for μ, 0.01, 0.05, 0.1 and 0.5 for σ^2, 0.01, 0.1 and 0.5 for σ_x^2, restricted to the cases in which σ^2 is smaller than σ_x^2. For the parameters β_{ki}, $k=1,2$; $i=1,2,3$, some sets of values were considered to simulate different situations, but the values for these parameters did not influence the conclusions of the results of the simulations. So, although we have conducted the simulation study for all the cases described earlier, we are going to summarize the general conclusions and show just some tables, considering only the values β_{11} =0.7, β_{12} =0.53, β_{13} =0.51, β_{21} =0.69, β_{22} =0.5, β_{23} =0.41, which are values close to the maximum likelihood estimates for the odontological data set presented in Hadgu & Koch (1999). Similarly, for the variance of the measurement errors the values considered were $\sigma^2_{e_{11}} = 0.31$, $\sigma^2_{e_{21}} = 0.23$, $\sigma^2_{e_{12}} = 0.43$, $\sigma^2_{e_{22}} = 0.43$, $\sigma^2_{e_{13}} = 0.25$ and $\sigma^2_{e_{23}} = 0.19$. Also, we have considered simulations involving just two populations, taking values close to the maximum likelihood estimates of the parameters of the quality control data set and four different sample sizes. The obtained results followed the same patterns as the cases for three populations with $\sigma^2 = 0.1$, thus the results for two populations will be omitted here.

For each combination of the parameter values described earlier, considering three populations, it were generated 10000 samples of sizes $(n_1 = 17, n_2 = 14, n_3 = 20)$, $(n_1 = 35, n_2 = 28, n_3 = 36)$, $(n_1 = 50, n_2 = 46, n_3 = 55)$ and $(n_1 = 101, n_2 = 95, n_3 = 105)$ of random vectors $(x_{ij}, y_{1ij}, y_{2ij})$, $i = 1,2,3$, $j = 1, \ldots, n_i$ according to the model defined in (1). Considering the nominal significance levels $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$ we obtained the corresponding empirical significance levels as the ratio between the number of samples for which the corresponding test statistics was greater than the $\chi^2_{(1)}$ and the total of samples. The aim of these simulation studies is to compare the behavior of the test statistic size for different values of the parameters, sample sizes and nominal levels. The conclusions for the simulation study are given as follows:

Table 6: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{11} = \beta_{12}$, with μ =2.5 ($\alpha = 0.01$).

Test	(n_1, n_2, n_3)	σ_x^2	0.01	0.1			0.5			
		σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)		0.020	0.020	0.022	0.020	0.019	0.022	0.023	0.023
L	(17,14,20)		0.013	0.013	0.015	0.013	0.012	0.014	0.015	0.015
S	(17,14,20)		0.020	0.013	0.009	0.007	0.017	0.011	0.011	0.006
W	(35, 28, 36)		0.014	0.016	0.015	0.016	0.014	0.015	0.015	0.014
L	(35, 28, 36)		0.011	0.013	0.012	0.012	0.010	0.012	0.012	0.011
S	(35, 28, 36)		0.016	0.013	0.009	0.011	0.016	0.009	0.009	0.009
W	(50, 46, 55)		0.013	0.013	0.013	0.013	0.011	0.012	0.014	0.014
L	(50, 46, 55)		0.011	0.012	0.011	0.012	0.010	0.010	0.012	0.011
S	(50, 46, 55)		0.015	0.013	0.010	0.011	0.016	0.009	0.011	0.011
W	(101, 95, 105)		0.011	0.010	0.010	0.011	0.010	0.009	0.013	0.010
L	(101, 95, 105)		0.010	0.010	0.009	0.011	0.009	0.009	0.012	0.010
S	(101, 95, 105)		0.014	0.011	0.009	0.011	0.015	0.009	0.011	0.009

- Depending on the situation, the three test statistics have approximately the same behavior even for small and moderate sample sizes. For instance, consider the following tests presented in Table 2: $H_0 : \beta_{11} = \beta_{12}$, $H_0 : \beta_{11} = \beta_{13}$, $H_0 : \beta_{21} = \beta_{22}$, $H_0 : \beta_{21} = \beta_{23}$, which are the comparisons between the control mouth rinse with one of the experimental mouth rinses A or B. We summarize these cases in Tables 6, 7 and 8, respectively for $H_0 : \beta_{11} = \beta_{12}$ (whith $\alpha = 1\%$), $H_0 : \beta_{11} = \beta_{13}$ (with $\alpha = 5\%$) and for $H_0 : \beta_{21} = \beta_{22}$ (with $\alpha = 10\%$). Notice that the Wald statistics seems to be a little more liberal, except in some cases when $\sigma^2 = 0.01$, in these cases the score test tends to be more liberal. In general (except when $\sigma^2 = 0.01$), the score test seems to be closer to the nominal

Table 7: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{11} = \beta_{13}$, with μ =2.5 ($\alpha = 0.05$).

Test	(n_1, n_2, n_3)	σ_x^2 0.01	0.1			0.5			
	σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)	0.065	0.065	0.069	0.067	0.069	0.070	0.072	0.065
L	(17,14,20)	0.057	0.059	0.060	0.057	0.058	0.060	0.062	0.057
S	(17,14,20)	0.074	0.081	0.056	0.051	0.085	0.065	0.054	0.048
W	(35, 28, 36)	0.058	0.064	0.061	0.060	0.056	0.059	0.059	0.058
L	(35, 28, 36)	0.054	0.059	0.056	0.056	0.052	0.054	0.055	0.054
S	(35, 28, 36)	0.067	0.081	0.053	0.051	0.079	0.056	0.052	0.052
W	(50, 46, 55)	0.053	0.059	0.054	0.057	0.055	0.055	0.055	0.057
L	(50, 46, 55)	0.051	0.056	0.053	0.053	0.051	0.052	0.051	0.053
S	(50, 46, 55)	0.057	0.073	0.049	0.051	0.077	0.052	0.048	0.052
W	(101, 95, 105)	0.053	0.049	0.054	0.052	0.052	0.051	0.051	0.052
L	(101, 95, 105)	0.051	0.049	0.053	0.050	0.050	0.051	0.049	0.051
S	(101, 95, 105)	0.055	0.065	0.051	0.049	0.072	0.050	0.048	0.049

Table 8: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{21} = \beta_{22}$, with μ =2.5 ($\alpha = 0.1$).

Test	(n_1, n_2, n_3)	σ_x^2 0.01	0.1			0.5			
	σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)	0.122	0.130	0.132	0.126	0.126	0.122	0.132	0.135
L	(17,14,20)	0.111	0.116	0.119	0.116	0.114	0.111	0.120	0.120
S	(17,14,20)	0.181	0.154	0.116	0.106	0.178	0.126	0.115	0.105
W	(35,28,36)	0.109	0.110	0.117	0.117	0.109	0.109	0.117	0.115
L	(35,28,36)	0.106	0.106	0.111	0.110	0.103	0.104	0.111	0.107
S	(35,28,36)	0.147	0.139	0.106	0.106	0.175	0.111	0.107	0.099
W	(50,46,55)	0.107	0.108	0.103	0.105	0.100	0.106	0.109	0.108
L	(50,46,55)	0.104	0.104	0.101	0.102	0.097	0.102	0.105	0.106
S	(50,46,55)	0.133	0.129	0.099	0.103	0.164	0.103	0.101	0.101
W	(101,95,105)	0.107	0.101	0.100	0.104	0.100	0.105	0.104	0.102
L	(101,95,105)	0.104	0.099	0.098	0.102	0.097	0.102	0.101	0.100
S	(101,95,105)	0.120	0.123	0.097	0.102	0.164	0.102	0.099	0.098

level for small and moderated samples sizes than Wald and likelihood ratio statistics.

- Table 9 shows the results for the test of hypotheses $H_0 : \beta_{12} = \beta_{13}$, which represents the comparison of the experimental mouth rinses A and B after three months of use. In this case, we observed in Table 2 that the score statistic rejected H_0 for $\alpha = 10\%$ while Wald and likelihood ratio statistics did not. It can be noticed, by simulation, that the score test is more liberal when $\sigma^2 = 0.01$ and it can be the cause for the difference in the conclusions for the odontological data set. However, for larger values of σ^2, the score test is closer to the nominal level than the other statistics, and also it is closer to the significance level in the majority of the cases, except for the small value of σ^2 (σ^2=0.01) as can be seen in Tables 6, 7 and 8.

- Similarly, for the test in which $H_0 : \beta_{22} = \beta_{23}$, which is related to the test that compares the mouth rinses A and B after six months of use, the score test rejects H_0 for $\alpha > 1.7\%$ while the Wald test

Table 9: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{12} = \beta_{13}$. with μ =2.5 ($\alpha = 0.1$).

Test	(n_1, n_2, n_3)	σ_x^2	0.01	0.1			0.5			
		σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)		0.126	0.127	0.119	0.130	0.117	0.130	0.127	0.138
L	(17,14,20)		0.117	0.116	0.115	0.117	0.107	0.117	0.115	0.122
S	(17,14,20)		0.168	0.149	0.108	0.105	0.169	0.125	0.105	0.108
W	(35,28,36)		0.118	0.111	0.118	0.117	0.113	0.116	0.112	0.113
L	(35,28,36)		0.112	0.105	0.113	0.111	0.107	0.110	0.106	0.106
S	(35,28,36)		0.142	0.136	0.107	0.105	0.168	0.112	0.101	0.099
W	(50,46,55)		0.102	0.107	0.109	0.109	0.109	0.108	0.104	0.105
L	(50,46,55)		0.100	0.104	0.106	0.106	0.105	0.105	0.101	0.101
S	(50,46,55)		0.122	0.125	0.103	0.102	0.168	0.107	0.099	0.099
W	(101,95,105)		0.102	0.107	0.103	0.106	0.102	0.103	0.103	0.101
L	(101,95,105)		0.100	0.103	0.101	0.105	0.100	0.101	0.101	0.099
S	(101,95,105)		0.117	0.119	0.099	0.102	0.146	0.100	0.101	0.097

Table 10: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{22} = \beta_{23}$, with μ =2.5 ($\alpha = 0.05$).

Test	(n_1, n_2, n_3)	σ_x^2	0.01	0.1			0.5			
		σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)		0.069	0.071	0.071	0.071	0.049	0.073	0.078	0.078
L	(17,14,20)		0.059	0.059	0.059	0.059	0.048	0.061	0.064	0.064
S	(17,14,20)		0.091	0.076	0.052	0.048	0.076	0.061	0.055	0.049
W	(35, 28, 36)		0.064	0.061	0.059	0.060	0.049	0.060	0.066	0.066
L	(35, 28, 36)		0.058	0.056	0.057	0.054	0.048	0.056	0.060	0.059
S	(35, 28, 36)		0.079	0.075	0.048	0.050	0.076	0.056	0.055	0.051
W	(50, 46, 55)		0.057	0.056	0.057	0.059	0.051	0.056	0.056	0.054
L	(50, 46, 55)		0.054	0.053	0.056	0.053	0.052	0.052	0.053	0.051
S	(50, 46, 55)		0.070	0.068	0.052	0.050	0.075	0.052	0.049	0.048
W	(101, 95, 105)		0.051	0.047	0.055	0.052	0.051	0.049	0.054	0.054
L	(101, 95, 105)		0.049	0.047	0.053	0.051	0.052	0.048	0.052	0.051
S	(101, 95, 105)		0.060	0.057	0.052	0.050	0.073	0.048	0.051	0.051

rejects it for $\alpha > 9.9\%$ and the likelihood ratio test, for $\alpha > 10.2\%$ (Table 2). We present the simulation results for $\alpha = 5\%$ in Table 10, which conducts to the same conclusion as the previous cases, that is, the score test tends to be more liberal when $\sigma^2 = 0.01$ but it is closer to the nominal level for small and moderate sample sizes than Wald and likelihood ratio tests in the other cases.

- For the intragroup tests, a similar conclusion can be noticed in Table 11 ($H_0 : \beta_{12} = \beta_{22}$ and $\alpha = 10\%$). The score test, as it was expected in the results shown in Table 2, tends to reject more than the Wald and likelihood ratio tests when $\sigma^2 = 0.01$, and seems better than the other two tests in the majority of other cases. The obtained results for $H_0 : \beta_{11} = \beta_{21}$ are very similar to the ones shown in Table 11 and they will be omitted.

- Finally, in Table 12, we show the results for the test of hypotheses $H_0 : \beta_{13} = \beta_{23}$ with $\alpha = 1\%$. The three test statistics in this case are very close, which can also be noticed in the results for the

Table 11: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{12} = \beta_{22}$, with μ =2.5 ($\alpha = 0.1$).

Test	(n_1, n_2, n_3)	σ_x^2	0.01	0.1			0.5			
		σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)		0.121	0.124	0.132	0.126	0.133	0.126	0.127	0.134
L	(17,14,20)		0.114	0.114	0.123	0.115	0.122	0.112	0.117	0.118
S	(17,14,20)		0.173	0.163	0.121	0.106	0.191	0.130	0.111	0.102
W	(35,28,36)		0.112	0.106	0.109	0.112	0.110	0.114	0.119	0.115
L	(35,28,36)		0.109	0.102	0.104	0.106	0.104	0.107	0.113	0.108
S	(35,28,36)		0.150	0.136	0.102	0.103	0.177	0.112	0.108	0.101
W	(50,46,55)		0.110	0.106	0.107	0.107	0.102	0.106	0.105	0.108
L	(50,46,55)		0.108	0.102	0.103	0.104	0.100	0.103	0.101	0.105
S	(50,46,55)		0.136	0.136	0.101	0.102	0.167	0.107	0.105	0.100
W	(101,95,105)		0.103	0.102	0.104	0.097	0.099	0.104	0.103	0.107
L	(101,95,105)		0.103	0.099	0.102	0.097	0.099	0.103	0.101	0.105
S	(101,95,105)		0.118	0.121	0.101	0.098	0.150	0.102	0.102	0.100

odontological data set, in Table 2.

Table 12: Empirical sizes for the Wald, likelihood ratio and score test statistics for the test $H_0 : \beta_{13} = \beta_{23}$, with μ =2.5 ($\alpha = 0.01$).

Test	(n_1, n_2, n_3)	σ_x^2	0.01	0.1			0.5			
		σ^2	0.01	0.01	0.05	0.1	0.01	0.05	0.1	0.5
W	(17,14,20)		0.019	0.017	0.019	0.018	0.016	0.017	0.017	0.019
L	(17,14,20)		0.015	0.011	0.015	0.012	0.011	0.012	0.012	0.014
S	(17,14,20)		0.016	0.015	0.010	0.009	0.018	0.012	0.009	0.009
W	(35, 28, 36)		0.015	0.013	0.015	0.015	0.013	0.016	0.015	0.015
L	(35, 28, 36)		0.013	0.011	0.013	0.012	0.011	0.012	0.012	0.013
S	(35, 28, 36)		0.015	0.014	0.010	0.010	0.018	0.012	0.009	0.009
W	(50, 46, 55)		0.013	0.013	0.014	0.013	0.012	0.012	0.012	0.012
L	(50, 46, 55)		0.012	0.011	0.012	0.011	0.011	0.011	0.010	0.010
S	(50, 46, 55)		0.013	0.014	0.010	0.010	0.017	0.010	0.009	0.009
W	(101, 95, 105)		0.011	0.011	0.011	0.010	0.012	0.012	0.010	0.012
L	(101, 95, 105)		0.010	0.011	0.011	0.010	0.011	0.011	0.010	0.010
S	(101, 95, 105)		0.010	0.014	0.010	0.010	0.016	0.010	0.010	0.010

Also, to compare the power of the three test statistics, 10000 samples were generated, considering the same sample sizes as before, denoted by $N_1 = (17, 14, 20)$, $N_2 = (35, 28, 36)$, $N_3 = (50, 46, 55)$ and $N_4 = (101, 95, 105)$. The percentages of the observed values of the test statistics which were greater than the quantiles of 1%, 5% and 10% of a Chi-squared distribution with 1 degree of freedom were obtained, while the distance of the parameter values with respect to the values of the null hypothesis were increased by 0.05 until the distance was given by 3.0. The aim of these simulation stuy is to evaluate the behavior of the test statistics for different parameter values and sample sizes, according to the distance of the alternative hypothesis to the null hypothesis. Considering the power of the test for different null hypotheses and parameter values, the three test statistics presented similar behavior. We present the results for the test $H_0 : \beta_{12} = \beta_{13}$ against $H_0 : \beta_{12} \neq \beta_{13}$ in Tables 13 and 14, for μ =2.5

Table 13: Simulated power for $H_0 : \beta_{12} = \beta_{13}$, with μ =2.5.

	W	L	S	W	L	S	W	L	S	W	L	S
	$\sigma_x^2 = \sigma^2 = 0.01$			$\sigma_x^2 = 0.1, \sigma^2 = 0.01$			$\sigma_x^2 = 0.1, \sigma^2 = 0.05$			$\sigma_x^2 = \sigma^2 = 0.1$		
N_1	0.111	0.098	0.138	0.117	0.103	0.133	0.117	0.101	0.100	0.126	0.110	0.102
	0.258	0.233	0.273	0.268	0.244	0.267	0.254	0.230	0.211	0.257	0.234	0.212
	0.473	0.444	0.471	0.474	0.444	0.451	0.460	0.426	0.398	0.447	0.416	0.387
	0.685	0.651	0.669	0.692	0.660	0.662	0.678	0.646	0.616	0.680	0.649	0.612
	0.860	0.835	0.843	0.858	0.837	0.832	0.845	0.823	0.803	0.840	0.816	0.791
	0.953	0.941	0.943	0.950	0.941	0.935	1.000	0.999	0.998	0.939	0.923	0.909
N_2	0.143	0.136	0.180	0.152	0.142	0.176	0.150	0.141	0.147	0.149	0.140	0.145
	0.401	0.386	0.424	0.404	0.389	0.415	0.398	0.383	0.378	0.395	0.381	0.376
	0.718	0.702	0.728	0.721	0.706	0.719	0.708	0.692	0.683	0.695	0.681	0.671
	0.917	0.910	0.919	0.919	0.911	0.915	0.907	0.898	0.892	0.905	0.898	0.891
	0.988	0.986	0.987	0.985	0.984	0.985	0.985	0.982	0.981	0.980	0.978	0.977
	0.999	0.998	0.998	0.999	0.998	0.999	1.000	1.000	0.998	0.998	0.998	0.997
N_3	0.190	0.184	0.236	0.197	0.192	0.223	0.192	0.187	0.193	0.189	0.182	0.196
	0.572	0.563	0.600	0.575	0.567	0.586	0.559	0.547	0.546	0.559	0.549	0.547
	0.884	0.878	0.895	0.892	0.886	0.895	0.876	0.871	0.867	0.872	0.866	0.865
	0.989	0.988	0.990	0.990	0.989	0.990	0.986	0.984	0.984	0.981	0.981	0.980
	1.000	1.000	1.000	1.000	0.999	1.000	0.999	0.999	0.999	0.998	0.998	0.998
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
N_4	0.326	0.317	0.401	0.339	0.335	0.367	0.327	0.324	0.328	0.312	0.308	0.320
	0.849	0.842	0.870	0.854	0.851	0.863	0.850	0.847	0.848	0.838	0.833	0.835
	0.993	0.992	0.994	0.995	0.994	0.995	0.992	0.991	0.991	0.992	0.991	0.991
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

and different values of σ_x^2 and σ^2. For fixed values of these parameters and sample sizes, we represent in the columns, from top to bottom, the percentage of the test statistics greater than the quantile 5% of a Chi-square distribution with 1 degree of freedom while increasing the distance from the null hypothesis in 0.05 until it achieves 3.0. It can be noticed in these tables that for small sample size, the Wald test presents a little greater power related to the likelihood ratio test and score test, but notice that the Wald test presented greater percentages of rejection for small sample size for the majority of the cases considered (Tables 6 through 12). For moderate sample sizes, the differences between the three test statistics practically vanish, and for greater sample sizes, the three tests present approximately the same behavior, and none of them can be considered better with respect to the power of the test.

To illustrate the results obtained by these simulation studies, we present some quantile-quantile graphics, which compares the observed values of each test statistic to the quantiles of a Chi-square distribution with 1 degree of freedom. To interpret this graphic, it can be observed how close the points are to the identity line, and as closer the points are to this line, better are the approximation of the test statistic distribution to the Chi-square distribution. Comparing these plots for different null hypotheses, the results were very similar, so we will present only one case, the test $H_0 : \beta_{12} = \beta_{13}$ versus $H_0 : \beta_{12} \neq \beta_{13}$ with parameter values $\sigma^2 = 0.01$, $\sigma_x^2 = 0.1$ and $\mu = 2.5$ (Figure 1).

Table 14: Simulated power for $H_0 : \beta_{12} = \beta_{13}$, with μ =2.5.

	W	L	S	W	L	S	W	L	S	[illegible]		
	$\sigma_x^2 = 0.5, \sigma^2 = 0.01$			$\sigma_x^2 = 0.5, \sigma^2 = 0.05$			$\sigma_x^2 = 0.5, \sigma^2 = 0.1$			$\sigma_x^2 = \sigma^2 = 0.5$		
N_1	0.125	0.115	0.140	0.124	0.112	0.108	0.122	0.113	0.101	0.116	0.106	0.089
	0.275	0.249	0.274	0.262	0.242	0.229	0.261	0.242	0.217	0.225	0.210	0.187
	0.492	0.457	0.466	0.487	0.456	0.433	0.454	0.425	0.393	0.400	0.373	0.335
	0.717	0.691	0.689	0.698	0.668	0.645	0.681	0.649	0.610	0.598	0.573	0.530
	0.878	0.858	0.847	0.857	0.833	0.808	0.841	0.816	0.784	0.754	0.726	0.686
	0.959	0.950	0.942	0.947	0.936	0.921	1.000	1.000	0.999	1.000	1.000	0.995
N_2	0.161	0.156	0.184	0.149	0.148	0.144	0.143	0.140	0.130	0.125	0.128	0.121
	0.423	0.414	0.443	0.418	0.407	0.399	0.392	0.381	0.365	0.339	0.336	0.322
	0.738	0.727	0.742	0.725	0.714	0.705	0.700	0.689	0.671	0.619	0.613	0.597
	0.934	0.927	0.931	0.917	0.911	0.904	0.904	0.898	0.889	0.851	0.845	0.832
	0.990	0.989	0.989	0.985	0.984	0.982	0.983	0.980	0.977	0.949	0.945	0.938
	0.999	0.999	0.999	0.999	0.998	0.998	1.000	1.000	1.000	1.000	1.000	1.000
N_3	0.211	0.208	0.247	0.205	0.204	0.202	0.197	0.196	0.187	0.167	0.172	0.166
	0.600	0.592	0.625	0.583	0.578	0.569	0.557	0.554	0.544	0.489	0.493	0.480
	0.909	0.906	0.913	0.897	0.893	0.889	0.878	0.876	0.869	0.801	0.800	0.791
	0.990	0.990	0.990	0.988	0.987	0.985	0.985	0.984	0.983	0.961	0.961	0.959
	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999	0.996	0.995	0.994
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
N_4	0.343	0.343	0.380	0.332	0.334	0.330	0.328	0.331	0.325	0.281	0.290	0.282
	0.869	0.869	0.883	0.859	0.858	0.856	0.844	0.842	0.838	0.771	0.777	0.768
	0.995	0.995	0.996	0.995	0.995	0.995	0.993	0.993	0.992	0.979	0.980	0.977
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

6 Discussion

The errors-in-variables regression models are used in problems in which the covariates cannot be measured directly, which means that there exist measurement errors beyond the random errors commonly involved in the model. Considering a multivariate null intercept errors-in-variables regression model, we have obtained the score function and the Fisher information matrix in closed form expressions and presented the EM algorithm to obtain the maximum likelihood estimates of the full parameter vector and the maximum likelihood estimates under the hypothesis of interest. We have applied this model to two real data sets, one of them considering the odontological data set presented in Hadgu & Koch (1999) and the other a quality control data set. Also, a simulation study was performed, which showed that the behavior of the test statistics can vary with the magnitude of the variance parameters. In particular, when the variance of the measurement errors are small, the best results were obtained considering the likelihood ratio test. Otherwise, the score showed to be the best test to use in the situations considered in this simulation study, as their size were in general closer to the nominal levels than the likelihood ratio and Wald test statistics. Considering the numerical illustration with respect to the odontological data set we concluded that the experimental mouth rinses A and B are more efficient than the control mouth rinse, but only the experimental mouth rinse B is long-lasting, results which confirm the conclusions obtained in the original paper involving the data and the paper written by Aoki et al. (2003b). Considering the quality control data set, we have concluded that the results of the experiment were influenced by

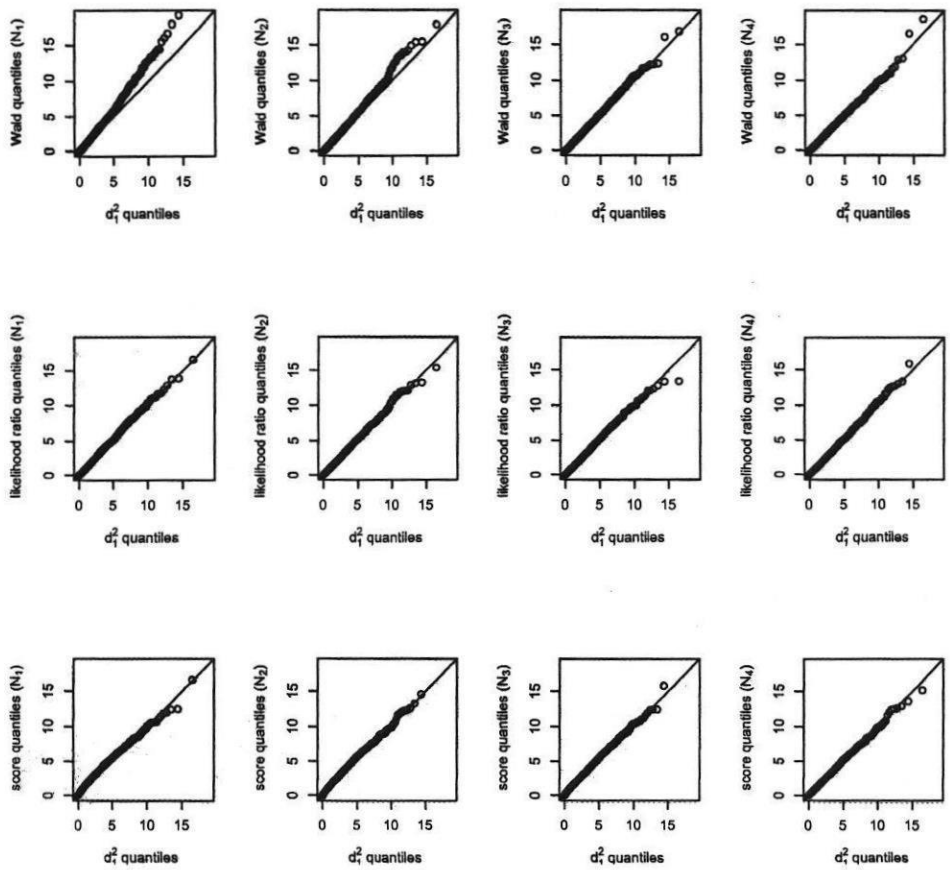

Figure 1: Quantile-quantile plots for $\sigma^2 = 0.01$ and $\sigma_x^2 = 0.1$ considering the test $H_0 : \beta_{12} = \beta_{13}$ versus $H_0 : \beta_{12} \neq \beta_{13}$ with $\mu = 2.5$.

external factors, as the variation in the washing machine temperature and the efficiency of the climatized room, and also that 4 hours are not enough to the process of stabilization of the pistons after the washing process, results which were confirmed by the confidence intervals for the expected values of the differences of the variables of interest. This information was important to the Six Sigma team of the KS Pist If they want to reduce the time of stabilization of the pieces, it is necessary to reduce variation due to external factors.

Acknowledgements

This research was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil. We are grateful to KS Pistons for kindly providing the data set and all the information used in the quality control illustration.

References

Aoki, R., Achcar, J. A., Bolfarine, H., Singer, J. M. (2003a). Bayesian analysis of null intercept errors-in-variables regression for pretest/postest data. Journal of Applied Statistics 30: 5–14.

Aoki, R., Bolfarine, H., Achcar, J. A., Pinto Junior, D. L. (2003b). Bayesian Analysis of a Multivariate Null Intercept Errors-in-Variables Regression Model. Journal of Biopharmaceutical Statistics 13: 767–775.

Aoki, R., Bolfarine, H., Singer, J. M. (2001). Null Intercept Measurement Error Regression Models. *Test* 10: 441–457.

Aoki, R., Bolfarine, H., Singer, J. M. (2002). Asymptotic efficiency of methods of moments estimators under null intercept measurement error regression models, Brazilian Journal of Probability and Statistics 16: 157–166.

Bradley, J. J., Gart, R. A. (1962). The assymptotic properties of ml estimators when sampling from associated populations. Biometrika 49: 205–214.

Chan, L. K., Mak, T. K. (1979). On the maximum likelihood estimation of a linear structural relashionship when the intercept is known. Journal of Multivariate Analysis 9: 304–313.

Dempster, A. P., Laird, N. M., Rubin, D. B. (1977). Maximum likelihood from incomplete data via the em algorithm (with discussion). Journal of the Royal Statistical Society - Series B 39: 1–38.

Dolby, G. R. (1976). The ultra structural relation: A synthesis of the functional and structural relations. Biometrika 63: 39–50.

Doornik, J. A. (2002). *Object-Oriented Matrix Programming Using Ox*, London: Timberlake Consultants Press and Oxford, http://www.doornik.com.

Fuller, A. (1987). *Measurement Error Models*, 1st edn, Wiley.

Hadgu, A., Koch, G. (1999). Application of generalized estimating equations to a dental randomized clinical trial. Journal of Biopharmaceutical Statistics 9(1): 161–178.

Kendall, M. G. (1951). Structure and functional relationship i. Biometrika 38: 11–15.

Kendall, M. G. (1952). Structure and functional relationship ii. Biometrika 39: 96–108.

Kendall, M. G., Stuart, A. (1961). *The Advanced Theory of Statistics.* London: Griffin.

Labra, F. V., Aoki, R., Bolfarine, H. (2005). Local influence in null intercept measurement error regression under a student-t model. Journal of Applied Statistics 32: 723–740.

Meng, X. L., Rubin, D. B. (1993). Maximum likelihood estimation via the ECM algorithm: A general framework. Biometrika 80: 267–278.

Moran, P. A. P. (1971). Estimating structural and functional relationships. Journal of Multivariate Analysis 1: 232–255.

A Score function

The score function of the model is given by the first order derivatives of the likelihood function $L(\mathbf{z}, \boldsymbol{\theta})$ in relation to $\boldsymbol{\theta}$, and after algebraic manipulations it can be shown that:

$$
\begin{aligned}
\frac{\partial L(\mathbf{z},\boldsymbol{\theta})}{\partial \boldsymbol{\beta}_i} =& \sum_{j=1}^{n_i}\{-\sigma_x^2 b_i^{-1}\left\{1+\sigma_x^2 b_i^{-1}[(\mathbf{z}_{i_j}-\mathbf{m}_i)^T\mathbf{B}_i(\mathbf{z}_{i_j}-\mathbf{m}_i)]+\mu[\mathbf{a}_i^T\mathbf{A}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)]\right\}\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i \\
&+\left\{\mu+\sigma_x^2 b_i^{-1}[\mathbf{a}_i^T\mathbf{A}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)]\right\}\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})(\mathbf{y}_{i_j}-\boldsymbol{\beta}_i\mu)\},i=1,\ldots,p \\
\frac{\partial L(\mathbf{z},\boldsymbol{\theta})}{\partial \mu} =& \sum_{i=1}^{p}\sum_{j=1}^{n_i} b_i^{-1}[\mathbf{a}_i^T\mathbf{A}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)] \\
\frac{\partial L(\mathbf{z},\boldsymbol{\theta})}{\partial \sigma^2} =& -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n_i}\{\sigma^{-2}[1-\sigma_x^2\sigma^{-2}b_i^{-1}-\sigma^{-2}(x_{i_j}-\mu)^2] \\
&+\sigma_x^2\sigma^{-4}b_i^{-1}\left\{2(x_{i_j}-\mu)[\mathbf{a}_i^T\mathbf{A}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)]-\sigma_x^2 b_i^{-1}[(\mathbf{z}_{i_j}-\mathbf{m}_i)^T\mathbf{B}_i(\mathbf{z}_{i_j}-\mathbf{m}_i)]\right\}\} \\
\frac{\partial L(\mathbf{z},\boldsymbol{\theta})}{\partial \sigma_x^2} =& -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n_i}\{\sigma_x^{-2}(1-b_i^{-1})-b_i^{-2}[(\mathbf{z}_{i_j}-\mathbf{m}_i)^T\mathbf{B}_i(\mathbf{z}_{i_j}-\mathbf{m}_i)]\} \\
\frac{\partial L(\mathbf{z},\boldsymbol{\theta})}{\partial \boldsymbol{\sigma}^2_{\mathbf{e}_i}} =& \sum_{j=1}^{n_i}\{\frac{1}{2}\mathbf{D}(\mathbf{y}_{i_j}-\boldsymbol{\beta}_i\mu)\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})(\mathbf{y}_{i_j}-\boldsymbol{\beta}_i\mu) \\
&+\frac{1}{2}\sigma_x^2 b_i^{-1}\left\{\sigma_x^2 b_i^{-1}[(\mathbf{z}_{i_j}-\mathbf{m}_i)^T\mathbf{B}_i(\mathbf{z}_{i_j}-\mathbf{m}_i)]+1\right\}\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i \\
&-\sigma_x^2 b_i^{-1}[\mathbf{a}_i^T\mathbf{A}_i^{-1}(\mathbf{z}_{i_j}-\mathbf{m}_i)]\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})(\mathbf{y}_{i_j}-\boldsymbol{\beta}_i\mu)-\tfrac{1}{2}\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{1}_2\},i=1,\ldots,p,
\end{aligned}
$$

where $\boldsymbol{a}_i$, $\mathbf{A}_i$, b_i and $\mathbf{B}_i$ are given by

$$\mathbf{a}_i=(1,\boldsymbol{\beta}_i^T)^T=(1,\beta_{1i},\beta_{2i})^T \text{ and } \mathbf{A}_i=\mathbf{D}\left(\sigma^2,\boldsymbol{\sigma}^2_{e_i}\right)^T=\mathbf{D}\left(\sigma^2,\sigma^2_{e_{1i}},\sigma^2_{e_{2i}}\right)^T,$$

where $\mathbf{D}(\mathbf{v})$ represents a diagonal matrix with the elements of $\mathbf{v}$.

$$\mathbf{V}_i^{-1}=\mathbf{A}_i^{-1}-\sigma_x^2 b_i^{-1}\mathbf{B}_i \text{ e } |\mathbf{V}_i|=b_i|\mathbf{A}_i|, \text{ onde}$$

$$b_i=1+\sigma_x^2\mathbf{a}_i^T\mathbf{A}_i^{-1}\mathbf{a}_i=1+\sigma_x^2[\sigma^{-2}+\boldsymbol{\beta}_i^T\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{e_i})\boldsymbol{\beta}_i] \quad \text{e} \quad \mathbf{B}_i=\mathbf{A}_i^{-1}\mathbf{a}_i\mathbf{a}_i^T\mathbf{A}_i^{-1}. \tag{13}$$

B The information matrix

Let us denote the elements of the expected information matrix by

$$J_{\boldsymbol{\theta}_u,\boldsymbol{\theta}_v^T} = -\sum_{i=1}^{p}\frac{n_i}{N}E\left[\frac{\partial^2 L(\mathbf{z}_i,\theta)}{\partial\boldsymbol{\theta}_u\partial\boldsymbol{\theta}_v^T}\right], u,v=1,\ldots,2p+3.$$

After algebraic manipulations, the expressions of $J(\boldsymbol{\theta})$ can be written as follows.

$$J_{\boldsymbol{\beta}_i,\boldsymbol{\beta}_i{}^T} = \frac{n_i}{N}\left\{[2\sigma_x^4 b_i^{-2}+\sigma_x^2 b_i^{-1}\mu^2]\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i\boldsymbol{\beta}_i{}^T\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i}) + (\sigma_x^2-\sigma_x^2 b_i^{-1}+\mu^2)\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i}) -(\sigma_x^2+2\mu^2)[b_i\sigma_x^2\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{1}\mathbf{1}^T\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i)]\right\}, i=1,\ldots,p$$

$$J_{\boldsymbol{\beta}_i,\boldsymbol{\beta}_j{}^T} = 0,\ \ i\neq j; i,j=1,\ldots,p$$

$J_{\boldsymbol{\beta}_i,\mu} = \frac{n_i}{N}b_i^{-1}\mu\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i,\ J_{\boldsymbol{\beta}_i,\sigma^2} = -\frac{n_i}{N}\sigma_x^4 b_i^{-2}\sigma^{-4}\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i,\ J_{\boldsymbol{\beta}_i,\sigma_x^2} = \frac{n_i}{N}(b_i^{-1}-b_i^{-2})\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i, i=1,\ldots,p$

$$J_{\boldsymbol{\beta}_i,\boldsymbol{\sigma}^{2T}_{\mathbf{e}_i}} = -\frac{n_i}{N}\left[\sigma_x^4 b_i^{-2}\mathbf{D}^{-1}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i\boldsymbol{\beta}_i{}^T\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i) - \sigma_x^2 b_i^{-1}\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i)\right], i=1,\ldots,p$$

$$J_{\boldsymbol{\beta}_i,\boldsymbol{\sigma}^{2T}_{\mathbf{e}_j}} = 0,\ \ i\neq j, i; j=1,\ldots,p$$

$J_{\mu,\mu} = \sum_{i=1}^{p}\frac{n_i}{N}(1-b_i^{-1})\sigma_x^{-2},\ \ J_{\mu,\sigma^2}=0,\ \ J_{\mu,\sigma_x^2}=0,\ \ J_{\mu,\boldsymbol{\sigma}^{2T}_{\mathbf{e}_i}} = \mathbf{0}, i=1,\ldots,p$

$$J_{\sigma^2,\sigma^2} = \sum_{i=1}^{p}\frac{n_i}{2N}\frac{(\sigma^2 b_i-\sigma_x^2)^2}{\sigma^8 b_i^2},$$

$$J_{\sigma^2,\sigma_x^2} = \sum_{i=1}^{p}\frac{n_i}{2N}\sigma^{-4}b_i^{-2},$$

$$J_{\sigma^2,\boldsymbol{\sigma}^{2T}_{\mathbf{e}_i}} = \frac{n_i}{2N}\sigma_x^4 b_i^{-2}\sigma^{-4}\boldsymbol{\beta}_i{}^T\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i}), i=1,\ldots,p$$

$$J_{\sigma_x^2,\sigma_x^2} = \sum_{i=1}^{p}\frac{n_i}{2N}\sigma_x^{-4}(b_i^{-2}-2b_i^{-1}+1),$$

$$J_{\sigma_x^2,\boldsymbol{\sigma}^{2T}_{\mathbf{e}_i}} = \frac{n_i}{2N}b_i^{-2}\boldsymbol{\beta}_i{}^T\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i), i=1,\ldots,p$$

$$J_{\boldsymbol{\sigma}^2_{\mathbf{e}_i},\boldsymbol{\sigma}^{2T}_{\mathbf{e}_i}} = -\frac{n_i}{2N}\left[-\sigma_x^4 b_i^{-2}\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\boldsymbol{\beta}_i\boldsymbol{\beta}_i{}^T\mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i) + 2\sigma_x^2 b_i^{-1}\mathbf{D}(\boldsymbol{\beta}_i)\mathbf{D}^{-3}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\mathbf{D}(\boldsymbol{\beta}_i) - \mathbf{D}^{-2}(\boldsymbol{\sigma}^2_{\mathbf{e}_i})\right], i=1,\ldots,p$$

$$J_{\boldsymbol{\sigma}^2_{\mathbf{e}_i},\boldsymbol{\sigma}^{2T}_{\mathbf{e}_j}} = 0, i\neq j, i,j=1,\ldots,p\ .$$

C Quality control data set

We present here the original quality control data set, which contains the measurements of the piston diameters, in millimeters.

Table 15: Measurements of diameters of the pistons on the first day (group 1).

piston	0 hours	4 hours	6 hours	piston	0 hours	4 hours	6 hours
1	48.84936	48.84642	48.84633	41	48.85156	48.84877	48.84877
2	48.85700	48.85377	48.85373	42	48.84798	48.84509	48.84503
3	48.84428	48.84229	48.84123	43	48.85327	48.85032	48.85020
4	48.84868	48.84563	48.84557	44	48.85003	48.84694	48.84688
5	48.84613	48.84299	48.84293	45	48.85290	48.84983	48.84977
6	48.85146	48.84966	48.84954	46	48.85604	48.85297	48.85287
7	48.85336	48.85034	48.85027	47	48.84986	48.84703	48.84700
8	48.84722	48.84418	48.84436	48	48.84686	48.84375	48.84368
9	48.85188	48.84999	48.84885	49	48.85297	48.85000	48.84995
10	48.85309	48.85016	48.85108	50	48.84513	48.84211	48.84202
11	48.84879	48.84588	48.84580	51	48.84924	48.84628	48.84621
12	48.84847	48.84556	48.84547	52	48.85205	48.84894	48.84869
13	48.85107	48.84762	48.84696	53	48.84698	48.84388	48.84380
14	48.85005	48.84718	48.84704	54	48.84677	48.84379	48.84374
15	48.85097	48.84797	48.84893	55	48.84659	48.84350	48.84341
16	48.85468	48.85176	48.85174	56	48.84703	48.84400	48.84291
17	48.85102	48.84878	48.84786	57	48.85235	48.84933	48.84924
18	48.84694	48.84388	48.84380	58	48.84508	48.84218	48.84199
19	48.84797	48.84476	48.84475	59	48.84917	48.84617	48.84702
20	48.85264	48.84965	48.84959	60	48.85170	48.84853	48.84844
21	48.85483	48.85196	48.85192	61	48.85310	48.85098	48.84990
22	48.84731	48.84437	48.84432	62	48.84882	48.84571	48.84458
23	48.85255	48.84943	48.84935	63	48.84916	48.84607	48.84603
24	48.85489	48.85181	48.85173	64	48.85119	48.84812	48.84807
25	48.84896	48.84586	48.84583	65	48.85313	48.85004	48.84990
26	48.84826	48.84535	48.84524	66	48.84838	48.84536	48.84535
27	48.84908	48.84614	48.84603	67	48.84681	48.84378	48.84373
28	48.85263	48.84946	48.84958	68	48.85236	48.84941	48.84934
29	48.85177	48.84888	48.84883	69	48.85048	48.84758	48.84754
30	48.85176	48.84972	48.84867	70	48.84865	48.84573	48.84564
31	48.85254	48.84964	48.84960	71	48.84950	48.84634	48.84625
32	48.84616	48.84310	48.84292	72	48.85247	48.84951	48.84945
33	48.85018	48.84715	48.84713	73	48.85043	48.84841	48.84731
34	48.85092	48.84794	48.84783	74	48.85024	48.84733	48.84689
35	48.84635	48.84337	48.84333	75	48.85358	48.85169	48.85136
36	48.85230	48.84940	48.84928	76	48.85160	48.84866	48.84816
37	48.84797	48.84495	48.84488	77	48.84683	48.84481	48.84392
38	48.84941	48.84641	48.84636	78	48.84887	48.84596	48.84589
39	48.84731	48.84429	48.84430	79	48.85195	48.84884	48.84802
40	48.84959	48.84664	48.84660	80	48.85195	48.84897	48.84803

Table 16: Measurements of diameters of the pistons on the second day (group 2).

piston	0 hours	4 hours	6 hours	piston	0 hours	4 hours	6 hours
1	48.84884	48.84658	48.84686	41	48.84881	48.84666	48.84646
2	48.85518	48.85276	48.85309	42	48.85159	48.84997	48.84902
3	48.85321	48.84937	48.84883	43	48.85504	48.85222	48.85237
4	48.85033	48.84783	48.84815	44	48.84914	48.84676	48.84675
5	48.85288	48.85074	48.85106	45	48.85248	48.85005	48.84995
6	48.84774	48.84398	48.84368	46	48.85032	48.84774	48.84735
7	48.84508	48.84364	48.84316	47	48.85273	48.85019	48.85010
8	48.85012	48.84761	48.84831	48	48.84890	48.84666	48.84638
9	48.85149	48.84803	48.84759	49	48.85090	48.84843	48.84855
10	48.84878	48.84627	48.84615	50	48.84803	48.84584	48.84576
11	48.85194	48.84942	48.84986	51	48.85013	48.84768	48.84748
12	48.84553	48.84185	48.84174	52	48.84457	48.84222	48.84191
13	48.85605	48.85353	48.85365	53	48.84819	48.84547	48.84509
14	48.85246	48.84971	48.85004	54	48.85661	48.85420	48.85407
15	48.85003	48.84720	48.84710	55	48.85145	48.84888	48.84840
16	48.85451	48.85200	48.85135	56	48.85069	48.84778	48.84796
17	48.84830	48.84526	48.84523	57	48.84973	48.84796	48.84708
18	48.85723	48.85496	48.85484	58	48.85335	48.85045	48.85030
19	48.85496	48.85239	48.85200	59	48.84874	48.84710	48.84652
20	48.84848	48.84583	48.84495	60	48.85052	48.84789	48.84763
21	48.84878	48.84499	48.84490	61	48.84127	48.83881	48.83881
22	48.84968	48.84700	48.84694	62	48.84898	48.84628	48.84588
23	48.84998	48.84754	48.84783	63	48.85230	48.85010	48.84974
24	48.85282	48.85052	48.85062	64	48.84517	48.84373	48.84354
25	48.84755	48.84480	48.84420	65	48.85389	48.85124	48.85100
26	48.85268	48.84998	48.84972	66	48.84526	48.84302	48.84239
27	48.84547	48.84299	48.84299	67	48.84704	48.84442	48.84418
28	48.84945	48.84655	48.84675	68	48.84957	48.84717	48.84725
29	48.84880	48.84604	48.84537	69	48.85150	48.84811	48.84811
30	48.84697	48.84505	48.84464	70	48.85031	48.84795	48.84757
31	48.85105	48.84864	48.84844	71	48.84691	48.84436	48.84434
32	48.84921	48.84678	48.84660	72	48.84600	48.84365	48.84363
33	48.85114	48.84840	48.84855	73	48.85545	48.85266	48.85319
34	48.85015	48.84794	48.84741	74	48.85545	48.85281	48.85252
35	48.84793	48.84546	48.84544	75	48.85050	48.84823	48.84778
36	48.85208	48.84929	48.84986	76	48.85307	48.85060	48.85012
37	48.85140	48.84875	48.84863	77	48.84887	48.84610	48.84628
38	48.85024	48.84764	48.84732	78	48.85149	48.84894	48.84838
39	48.84757	48.84512	48.84456	79	48.85123	48.84879	48.84909
40	48.84947	48.84696	48.84636	80	48.84869	48.84507	48.84477