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Let A be an artin algebra. Denote by mod$A$ the category of finitely generated left A-modules and by ind$A$ the subcategory of mod$A$ with one representative of each isoclass of indecomposable A-module. For an A-module X denote by pd$X$ its projective dimension. Also, the global and the finitistic projective dimensions of A are defined, respectively, as

$$\text{gl.dim}A = \max \{ \text{pd}X : X \in \text{ind}A\} \text{ and}$$

$$\text{fpd}A = \max\{ \text{pd}X : X \in \text{ind}A \text{ and pd}X < \infty\}$$

If fpd$A = 0$, then clearly the only modules with finite projective dimension are the projective. This is the case, for instance, of selfinjective algebras, or local algebras. If we assume, in addition, that A is representation-infinite (that is, such that there are infinitely many nonisomorphic indecomposable A-modules), then we infer that there are infinitely many nonisomorphic indecomposable A-modules with infinite projective dimension. One could ask whether it is possible for an algebra of finitistic projective dimension greater than zero to have only finitely many nonisomorphism indecomposable modules of infinite projective dimension. The main purpose of these notes is to show the following theorem.

Theorem. *Let A be a representation-infinite artin algebra of infinite global dimension. If there are only finitely many nonisomorphic indecomposable A-modules of infinite projective dimension, then $2 \leq fpdA < \infty$.*

At the end of this note we shall exhibit examples showing that for each $2 \leq n < \infty$, there are algebras of finitistic projective dimension equal to n containing only finitely many nonisomorphic indecomposable A-modules of infinite projective dimension.

1 Preliminaries

We keep the notations from the introduction. We assume that all algebras are associative, with identity, basic and connected artin algebras over a commutative artinian ring R. We shall use basic results and notions on the representation theory of artin algebras and refer to [5, 12] for details. We shall use the notation Γ_A for the Auslander-Reiten quiver of A and we shall not distinguish between a vertex in Γ_A and the corresponding indecomposable A-module. All components are assumed to be connected.

In order to prove our main theorem, we will first recall some results from [7, 8]. We start with some definitions. We say that a module $Y \in \text{ind}A$ is a *successor* of an indecomposable A-module X if there exists a chain of irreducible maps from X to Y. Also, we say that a property holds for *almost all modules* if it holds for all but finitely many nonisomorphic indecomposable modules.

Definition. A component Γ of Γ_A is called an *ι-component* provided: (i) almost all modules in Γ lie in the DTr-orbit of an injective; and (ii) there are at most finitely many indecomposable A-modules in Γ belonging to oriented cycles.

The notion of ι-component was introduced to characterize those components of Γ_A which contain only preinjective modules in the sense of Auslander-Smalø [6]. The next results summarize the important informations on ι-components we shall need here.

Theorem 1.1 [7, 8] The following are equivalent for an artin algebra A:

(i) The projective dimension of almost all indecomposable A-modules is at most one;

(ii) Γ_A has a component Γ containing all the injective modules and such that any module $X \in \Gamma$ has only finitely many successors;

(iii) Γ_A has an ι-component containing all the injective modules.

(iv) $\text{rad}^\infty(DA, -) = 0$.

For tilting theory we refer the reader to [1, 10]. In particular, we recall that tilted algebras are characterized by the existence of complete slices in a component of their Auslander-Reiten quiver, called *connecting component* ([12](4.2)). We shall also use here the fact that tilted algebras have global dimension at most 2. For the next result recall that a component Γ of Γ_A is called *preinjective* provided it has no oriented cycles and each module on it lie in a DTr-orbit of an injective. Clearly, preinjective components are ι-components.

Lemma 1.2 Let A be an algebra satisfying $\text{rad}^\infty(DA, -) = 0$ and let Γ be the ι-component of Γ_A containing all the injective modules. If Γ has no projective modules, then Γ is a preinjective component with a complete slice. In particular, A is tilted.

Proof: In order to show that Γ is preinjective, it is enough to prove that it has no oriented cycles. We first show that Γ has no DTr-periodic modules. In fact, if Γ has a DTr-periodic module, then there exists an irreducible map $X \longrightarrow Y$ such that X is DTr-periodic and Y is not DTr-periodic. By [3](6.2), there exists an n such that $(\mathrm{DTr})^n Y$ is projective, a contradiction to our hypothesis on Γ. Suppose that

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t$$

is an oriented cycle in Γ. Since Γ has no projective modules, then for each $j \geq 0$ and $i = 1, \cdots, t$, $(\mathrm{DTr})^j X_i \neq 0$ and

$$(\mathrm{DTr})^j X_0 \longrightarrow (\mathrm{DTr})^j X_1 \longrightarrow \cdots \longrightarrow (\mathrm{DTr})^j X_t$$

is an oriented cycle. Since Γ has no DTr-periodic modules, we infer that the set $\{ (\mathrm{DTr})^j X_i : j \geq 0 \text{ and } i = 1, \cdots, t\}$ is infinite, a contradiction to the fact that Γ is an ι-component. Therefore, Γ is preinjective. The fact that Γ contains a complete slice follows now from [2](2.8). □

2 Main result

Following [6], a subcategory $\mathcal{C} \subset \mathrm{ind}A$ is said to be *contravariantly finite* if for each $X \in \mathrm{ind}A$, there exists a morphism $f_X: Y \longrightarrow X$, with $Y \in \mathrm{add}\mathcal{C}$ such that for each morphism $g: Z \longrightarrow X$, with $Z \in \mathrm{add}\mathcal{C}$, the induced morphism

$$\mathrm{Hom}_A(C, g): \mathrm{Hom}_A(C, Y) \longrightarrow \mathrm{Hom}_A(C, X)$$

is surjective for all $C \in \mathrm{add}\mathcal{C}$. In [4], Auslander and Reiten have shown that if the subcategory $\mathcal{P}^{<\infty}(A)$ of $\mathrm{ind}A$ consisting of the modules of finite projective dimension is contravariantly finite, then $\mathrm{fpd}A$ is finite. We shall now prove our main result.

Theorem 2.1 Let A be a representation-infinite artin algebra of infinite global dimension. If there are only finitely many nonisomorphic indecomposable A-modules of infinite projective dimension, then $2 \leq fpdA < \infty$.

Proof: Since there are at most finitely many nonisomorphic indecomposable A-modules with infinite projective dimension, we have that the category $\mathcal{P}^{<\infty}(A)$ is cofinite in ind$A$. By [6] (see also [8]), we infer that $\mathcal{P}^{<\infty}(A)$ is contravariantly finite and then, by the above remark, we have that fpd$A < \infty$, and part of our theorem is proven.

Suppose now fpd$A \leq 1$. Since, by hypothesis, there are only finitely many indecomposable A-modules of infinite projective dimension, we infer that pd$X \leq 1$ for almost all $X \in$ ind$A$. By 1.1, there exists an ι-component Γ of Γ_A containing all the injective modules. We claim that Γ has projective modules. Assume the contrary. Then by 1.2, we infer that Γ is a preinjective component containing a complete slice. Therefore, A is a tilted algebra and Γ is a connecting component. In particular, gldim$A \leq 2$ which contradicts our hypothesis on A. This proves the claim.

Observe also that there are projective modules belonging to components other than Γ. In fact, if all the indecomposable projective A-modules belong to Γ, then by 1.1, we infer that there are only finitely many nonisomorphic indecomposable successors of projective modules. Hence A is representation-finite, a contradiction.

Let $\{e_1, \cdots, e_m, \cdots, e_n\}$ be a complete set of primitive orthogonal idempotents of A ordered in such a way that Ae_i belongs to Γ if and only if $m \leq i \leq n$. By the above remarks, clearly, $m < n$. Put $e = e_1 + \cdots + e_m$ and let $B =$ End(Ae). By 1.1, the support of the functor Hom$_A(A(1-e), -)$ is finite, and then ind$B$ is cofinite in ind$A$.

We now claim that B is a product of tilted algebras. The argument is very similar to that used in the proof of [2](Theorem3.2). For the convenience of the reader we shall however repeat it here. Let Γ' be the translation subquiver of Γ_B consisting of the indecomposable B-modules which, when considered as

A-modules, belong to Γ. By the dual of [7](7.4), Γ' is a (finite) union of ι-components $\Gamma_1, \cdots, \Gamma_t$. Moreover, by construction, none of the Γ_i contains a projective module. Consequently, by 1.2, $\Gamma_1, \cdots, \Gamma_t$ are preinjective components in Γ_B. For each i, let Σ_i be a maximal subsection in Γ_i chosen so that it embeds fully in Γ and has no predecessor which is an injective A-module. We shall denote by B_i the support algebra of Σ_i. Clearly, Σ_i is faithful in $\mathrm{mod} B_i$. Since preinjective components are standard (by [12](2.4)(11)p.80), we have that $\mathrm{Hom}_{B_i}(U, \tau V) = 0$ for all $U, V \in \Sigma_i$. By [11, 13], Σ_i is a complete slice in Γ_{B_i} and B_i is a tilted algebra. The claim is now proven. In particular, $\mathrm{gldim} B \leq 2$. On the other hand, since B is a factor of A, we infer that $\mathrm{fpd} B \leq 1$. Therefore, $\mathrm{gldim} B \leq 1$ and the B_i's are, in fact, hereditary algebras. Moreover, by construction, each B_i is representation-infinite.

We now claim that there exists a nonprojective module $X \in \mathrm{ind} B$ and a nonzero map from X to an indecomposable projective module in Γ. In fact, since A is connected, there exist indecomposable modules P and P' with $P \in \Gamma$, $P' \notin \Gamma$ and $\mathrm{Hom}_A(P', P) \neq 0$. Let f be a nonzero morphism in $\mathrm{Hom}_A(P', P)$. Since $P' \notin \Gamma$, we get that $P' \in \mathrm{ind} B_i$, for some i. Without loss of generality put $i = 1$. Observe that then $\mathrm{Im} f \in \mathrm{mod} B_1$ (because it is a quotient of a B_1-module). If $\mathrm{Im} f$ has an indecomposable summand which is nonprojective, then our claim is proven. Suppose $\mathrm{Im} f$ is projective. Therefore $\mathrm{Im} f \cong P'$. Now, since $f \in \mathrm{rad}^\infty(\mathrm{mod} A)$, we infer that for each $t \geq 1$, there exists a chain of irreducible maps

$$X_t \xrightarrow{f_t} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = P$$

and a morphism $g_t : P' \longrightarrow X_t$ such that $f = f_1 \cdots f_t g_t$. Since there are only finitely many nonisomorphic indecomposable A-modules which are not B-modules, there exists an r such that X_r has an indecomposable summand $X \in \mathrm{ind} B_1$ such that $\mathrm{Hom}_A(X, P) \neq 0$, and our claim is proven. Therefore, P

has an indecomposable nonprojective submodule $N \in \text{ind}B$. Now if

$$0 \longrightarrow P_1(N) \longrightarrow P_0(N) \longrightarrow N \longrightarrow 0$$

is a projective resolution of N, then

$$0 \longrightarrow P_1(N) \longrightarrow P_0(N) \longrightarrow P \longrightarrow P/N \longrightarrow 0$$

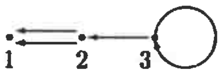
is a projective resolution of P/N, and then $\text{pd}_A P/N = 2$, which contradicts our assumption. The result is proven. □

The following corollary generalises a result proven in [9].

Corollary 2.2 Let A be an algebra with $\text{fpd}A \leq 1$. If A is representation-infinite and not hereditary, then there are infinitely many nonisomorphic indecomposable A-modules of infinite projective dimension.

We finish these notes by showing some examples of nonhereditary representation-infinite algebras A with $\text{fpd}A > 1$ such that almost all of its indecomposable A-modules have finite projective dimension.

Examples. (a) Let k be a field and A be the radical square zero k-algebra given by the following quiver



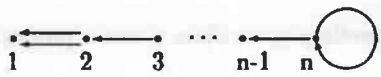
This is, clearly, a right glued algebra in the sense [2] of the (Kronecker) algebra B by the representation-finite algebra C, where B and C are the algebras given, respectively, by the quivers

 and  with $\alpha^2 = 0$.

The only indecomposable A-modules with support in the vertex 3 are: $S(3)$,

$P(3)$, $P(3)/S(3)$ and $P(3)/S(2)$, which have projective dimensions equal to $\infty, 0, \infty$ and 2, respectively. All the other indecomposable A-modules can be identified to modules over the Kronecker algebra and therefore, of projective dimension at most one. Therefore $\text{fpd}A = 2$ and almost all indecomposable A-modules have finite projective dimension.

(b) More generally, for each $n \geq 3$, consider the radical square zero algebra A_n given by the quiver



Then the projective dimension of $S(n)$ is infinite, $\text{fpd}A_n = n - 1$ and almost all indecomposable A_n-modules have finite projective dimension.

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