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PERIODIC SOLUTIONS OF A PLANAR DIFFERENTIAL EQUATION WITH TWO DELAYS

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ABSTRACT. In this paper we re-state for two delays, the results of [12]. An existence theorem is obtained for nonconstant periodic solutions for the system of delay equations:

$$\begin{aligned} \dot{x}_1(t) &= -\alpha x_1(t) + \alpha F_1(x(t-1)) \\ \dot{x}_2(t) &= -\alpha x_2(t) + \alpha F_2(x(t-r)) \end{aligned} \tag{1}$$

for $\alpha > 0$ and $0 < r < 1$. We use a fixed point theorem for an operator defined by the flow in the phase space $C([-1,0], \mathbf{R}^2)$, in order to state the existence of periodic solutions. As a consequence of a study of the characteristic equations of the linear part of (1) a sequence of values of α, $\alpha_0 < \alpha_1 < \cdots$ where a Hopf bifurcation happens, was achieved.

1. INTRODUCTION

In this paper we deal with the system of two delay-differential equations

$$\begin{aligned} \dot{y}_1(t) &= -\beta y_1(t) + \beta G_1(y(t-\rho_1)) \\ \dot{y}_2(t) &= -\beta y_2(t) + \beta G_2(y(t-\rho_2)) \end{aligned} \tag{1.0}$$

with $\rho_1, \rho_2 > 0$, $y = \text{col}\,(y_1, y_2) \in \mathbb{R}^2$, $\beta > 0$ a parameter and $G = \text{col}\,(G_1, G_2)$ a map of $\mathbb{R}^2$ into itself. After a normalization through a rescaling of time this system can be written as:

$$\begin{aligned} \dot{x}_1(t) &= -\alpha x_1(t) + \alpha F_1(x(t-1)) \\ \dot{x}_2(t) &= -\alpha x_2(t) + \alpha F_2(x(t-r)) \end{aligned} \tag{1.1}$$

where $\alpha > 0$ is a parameter, $F = \text{col}\,(F_1, F_2)$ is a continuous map from $\mathbb{R}^2$ into itself, and $0 < r < 1$. Therefore we shall concentrate our study on the system (1.1).

All over this paper we assume there exists $M > 0$ such that $||F(x)|| \leq M$ and F is differentiable up to order two at the origin. The original motivation was the study of the scalar version of (1.1) under the so called *negative feedback condition* on F. See [9] and [5]. The equation (1.1) is studied in [12] and [1] with only a one delay, i.e., $r = 1$. Indeed, our main objective is to re-state their results for two delays.

Now, we need to fix some notation: $\mathcal{C} = C([-1,0], \mathbb{R}^2)$ with the sup norm $||.||$; $x(t) = x(t, \varphi)$ is the unique solution of equation (1.1) such that $x\big|_{[-1,0]} = \varphi$. For each continuous function $x : [\sigma - 1, \sigma + a] \to \mathbb{R}^2$, $a > 0$, $x_t \in \mathcal{C}$, $\sigma \leq t < \sigma + a$, is given by $x_t(\theta) := x(t+\theta)$, $-1 \leq \theta \leq 0$.

Let us denote $\dfrac{\partial F_1(0)}{\partial x_2} = \delta_1$, $\dfrac{\partial F_2(0)}{\partial x_1} = -\delta_2$; $\quad \delta_1, \delta_2 > 0$.

From now on, we assume the hypothesis

$$x_2 F_1(x) > 0, \quad x_2 \neq 0, \quad x_1 F_2(x) < 0, \quad x_1 \neq 0. \tag{H$_1$}$$

In this manner, we have that

$$JF(0) = \begin{pmatrix} 0 & \delta_1 \\ -\delta_2 & 0 \end{pmatrix}.$$

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2. The Characteristic Equation

The linearization of (1.1) near the origin gives

$$\dot{x}(t) + \alpha x(t) = B_1(\alpha)\, x(t-1) + B_2(\alpha)\, x(t-r) \tag{2.1}$$

where $B_1(\alpha) = \begin{pmatrix} 0 & \alpha\delta_1 \\ 0 & 0 \end{pmatrix}$ and $B_2(\alpha) = \begin{pmatrix} 0 & 0 \\ -\alpha\delta_2 & 0 \end{pmatrix}$.

A necessary and sufficient condition for the existence of a nontrivial solution of the form $x(t) = e^{\lambda t}u$, $u \in \mathbb{R}^2$, $\lambda \in \mathbb{C}$, is that u is a nonzero solution of $\Delta(\lambda)\, u = 0$; $\Delta(\lambda)\, u := (\lambda+\alpha)I - B_1(\alpha)e^{-\lambda} - B_2(\alpha)e^{-\lambda r}$, which leads to the characteristic equation

$$(\lambda+\alpha)^2 e^{\lambda(1+r)} = -\alpha^2\delta^2, \quad \delta = \sqrt{\delta_1\,\delta_2}. \tag{2.2}$$

Notice that $\lambda = 0$ is the unique real root of (2.2) and it occurs when $\alpha = 0$, and if $\lambda = \lambda(\alpha)$ is a root of (2.2) its complex conjugate is too; therefore, we can restrict our study to the upper semi-plane $\Im(\lambda) > 0$.

Lemma 2.1. *There is no root of (2.2) in the lines $\lambda = a + bi$, with $b = \pm\frac{(2k+1)}{1+r}\pi$, $k = 0, 1, 2, \ldots$.*

Proof: If $\lambda = a + bi$, the equation (2.2) becomes

$$(a+\alpha+bi)^2 e^{[a(1+r)+ib(1+r)]} = -\alpha^2 b^2.$$

Since $\pm b = \frac{(2k+1)\pi}{1+r}$ implies $\cos b(1+r) = -1$ and $\sin b(1+r) = 0$, one obtains

$$\begin{cases} (a+\alpha)^2 - b^2 = \alpha^2\delta^2 e^{-a(1+r)} \\ 2(a+\alpha)b = 0 \end{cases}$$

which is incompatible. □

The Lemma (2.1) says that the roots of (2.2) lie in the strips

$$S_k = \{\lambda = a + bi\,;\, \frac{(2k-1)\pi}{1+r} < b < \frac{(2k+1)\pi}{1+r}, b > 0\},\ k = 0, \pm 1, \pm 2, \ldots.$$

Lemma 2.2. *If $\lambda \in S_k$, $k = 0, 1, 2, \ldots$, the equation (2.2) is equivalent to the equation*

$$(\lambda+\alpha)\exp\left(\lambda\frac{1+r}{2}\right) = (-1)^k i\delta\alpha. \tag{2.3}$$

Proof. : If $\lambda = a + bi \in S_k$ and satisfies (2.2) we have that

$$(a+\alpha+bi)\exp\left[a\left(\frac{1+r}{2}\right) + b\left(\frac{1+r}{2}\right)i\right] = \pm i\delta\alpha$$

which is equivalent to the system

$$\begin{cases} (a+\alpha)\cos b\left(\frac{1+r}{2}\right) - b\sin b\left(\frac{1+r}{2}\right) = 0 \\ \\ (a+\alpha)\sin b\left(\frac{1+r}{2}\right) - b\cos b\left(\frac{1+r}{2}\right) = \pm\delta\alpha\exp\left[-a\left(\frac{1+r}{2}\right)\right]. \end{cases}$$

Multiplying the first equation by $b\sin[b(1+r)/2]$, we obtain

$$(a+\alpha)b\cos\left[b\left(\frac{1+r}{2}\right)\right]\sin\left[b\left(\frac{1+r}{2}\right)\right] = b^2\sin^2\left[b\left(\frac{1+r}{2}\right)\right] \geq 0$$

and $(a+\alpha)\sin[b(1+r)/2]$ and $b\cos[b(1+r)/2]$ have the same sign. Furthermore, in S_k, the sign of $b\cos[b(1+r)/2]$ is $(-1)^k$.

Then, $(\lambda+\alpha)\exp[\lambda(1+r)/2] = (-1^k)i\delta\alpha$. □

Lemma 2.3. *There exists a sequence λ_k of characteristic roots such that $\lambda_k = b_k i \in S_k$, $\delta \sin[b_k(\frac{1+k}{2})] = (-1)^k$ and $\alpha_k = b_k/\sqrt{\delta^2-1}$, $k = 0,1,2,\dots$, for $\delta > 1$.*

Proof. : If $\lambda = b_k i$, $k = 0,1,2,\dots$ in (2.3) we obtain

$$\begin{cases} \alpha_k = b_k \tan\left[b_k(\frac{1+r}{2})\right] \\ \\ \alpha_k \sin\left[b_k(\frac{1+r}{2})\right] + b_k \cos\left[b_k(\frac{1+r}{2})\right] = (-1)^k \delta \alpha_k. \end{cases} \tag{2.4}$$

The insertion of the first equation into the second one, leads to

$$\sin\left[b_k\left(\frac{1+r}{2}\right)\right] = (-1)^k \frac{1}{\delta},$$

which gives, with the second equation in (2.4),

$$\cos\left[b_k\left(\frac{1+r}{2}\right)\right] = (-1)^k \alpha_k\left(\frac{\delta^2-1}{\delta}\right), \quad k = 0,1,2,\dots. \quad \square$$

Remark. : Lemma 2.3 shows that the first value $\alpha = \alpha_o$ where a root λ crosses the imaginary axis is

$$\alpha_o = \frac{b_o}{\sqrt{\delta^2-1}}, \quad 0 < b_o = \frac{2}{1+r}\arcsin\frac{1}{\delta} < \frac{\pi}{1+r} \tag{2.5}$$

that corresponds to the characteristic root $\lambda_0 = b_0 i \in S_0$. Moreover, $0 < \alpha_0 < \alpha_1 < \cdots \longrightarrow +\infty$, as $k \to +\infty$, because $b_k = b_0 + 2k\pi/(1+r)$, $k \geq 1$. If $c_k = b_k(1+r)/2$, we have that $(2k-1)\frac{\pi}{2} < c_k < (2k+1)\frac{\pi}{2}$, and $c_k = c_0 + k\pi$.

Lemma 2.4. *Each strip S_k, $k \geq 0$ contains precisely one root of (2.2) for each $\alpha \neq 0$ and it is simple.*

Proof. : Putting $\lambda = a + bi \in S_k$, $k \geq 0$, in (2.3), it follows that

$$\begin{cases} a + \alpha = b\tan\left[b\left(\frac{1+r}{2}\right)\right] \\ \\ (a+\alpha)\sin\left[b\left(\frac{1+r}{2}\right)\right] + b\cos\left[b\left(\frac{1+r}{2}\right)\right] = (-1)^k\delta\alpha\exp\left[-a\left(\frac{1+r}{2}\right)\right] \end{cases} \tag{2.6}$$

and, defining $c = b(1+r)/2$, $b\tan c\ \sin c + b\cos c = (-1)^k\delta\alpha\exp[-a(1+r)/2]$ which combined with (2.5) gives b as a function of α. Defining $\beta = \delta\alpha\exp[\alpha(1+r)/2]$, it follows that $\frac{d\beta}{d\alpha} = \delta(1+\alpha(1+r)/2)\exp[\alpha(1+r)/2] > 0$ for $\alpha > a$. Furthermore,

$$\beta = \Lambda(b) = (-1)^k b\exp\left[\frac{c\tan c}{\cos c}\right] = (-1)^k \sec c\exp[\alpha(1+r)/2] \tag{2.7}$$

and

$$\frac{d\beta}{db} = \frac{d\Lambda}{db} = (-1)^k e^{c\tan c}\sec c[(1+c\tan c)^2 + c^2] > 0. \tag{2.8}$$

Therefore, $\Lambda(b) \to 0$ as $b \to 0^+$ or $b \to \frac{(2k-1)}{1+r}\pi^+$ and $\Lambda(b) \to +\infty$ as $b \to \frac{(2k+1)}{1+r}\pi^-$, $k = 1, 2, \ldots$. Then, in each strip S_k, $b = \Im(\lambda)$ is a strictly increasing function of $\alpha \in (0, \infty)$, into $(0, \frac{\pi}{1+r})$, if $k = 0$, and $\left(\frac{(2k-1)}{1+r}\pi, \frac{(2k+1)}{1+r}\pi\right)$ if $k \geq 1$.

In this manner, it follows from the first equation of (2.6) that $a = \Re(\lambda)$ is also a function of α in each strip S_k. Moreover, λ is differentiable with respect to α.

Let λ_o be a root of (2.2). Then

$$\frac{d}{d\lambda}[(\lambda+\alpha)^2 e^{\lambda(1+r)} + \alpha^2\delta^2]_{\lambda=\lambda_0} = (\lambda_0+\alpha)e^{\lambda_o(1+r)}[\alpha + (\lambda_0+\alpha)(1+r)] \neq 0,$$

therefore, λ_0 is simple. □

Lemma 2.5. *If $\delta > 1$ and $\lambda = \lambda(\alpha)$ is a root of (2.2), then $\Re(\lambda) < (2\ln\delta)/(1+r)$ and $\lim_{\alpha\to+\infty} \lambda(\alpha) = \frac{2\ln\delta}{1+r} + i\frac{(2k+1)}{1+r}\pi$, in each strip S_k, $k = 0, 1, 2, \ldots$.*

Proof. : Recalling that $c = b(1+r)/2$, the equation (2.6) implies $(1 + a/\alpha)\exp[a(1+r)/2] = (-1)^k\delta\sin c$. By Lemma 2.4, $\lim_{\alpha\to+\infty}(-1)^k\delta\sin c = \delta$ and, because $(1+a/\alpha)\exp[a(1+r)/2] > \exp[a(1+r)/2]$, it follows that $a = a(\alpha)$ is bounded as $\alpha \to +\infty$ and $\delta = \lim_{\alpha\to+\infty}\exp[a(1+r)/2]$.

Then $a = \Re(\lambda) \to (2\ln\delta)/(1+r)$, as $\alpha \to +\infty$. From the proof of Lemma 2.4, one obtains $b = \Im(\lambda) \to \frac{(2k+1)}{1+r}\pi$ as $\alpha \to= +\infty$. □

Lemma 2.6. *If $\lambda_k = a_k + b_k i \in S_k$, $k = 0, 1, 2, \ldots$ are roots of (2.2) for some $\alpha > 0$, then $a_0 > a_1 > \cdots \to -\infty$.*

Proof. : From (2.2) we obtain

$$|(\lambda+\alpha)^2 e^{\lambda(1+r)}| = \gamma^2, \quad \gamma^2 = \alpha^2\delta^2.$$

If $\lambda = a + bi$, then

$$[(a+\alpha)^2 + b^2]e^{(a+\alpha)(1+r)} = \gamma^2 \tag{2.9}$$

and this, for $\gamma = 2\pi/(1+r)$, defines $b > 0$ as a function of a in the interval $-\infty < a \leq 0$. Such a function is strictly decreasing because

$$\frac{db}{da} = -\frac{1}{2b}\left[\frac{4\pi^2}{1+r}e^{-(a+r)(1+r)} + 2(a+\alpha)\right] < 0, \quad \text{if} \quad a \leq 0.$$

Then, $a_0 = 0 > a_1 > a_2 > \cdots > a_k > \cdots$.

Let us suppose this is false for some value of γ. Since $\Re(\lambda)$ is a continuous function of γ in each strip S_k, there exists some γ for which $a_k = a_\ell$, $k > \ell$. But, by (2.9) we have

$$[(a+\alpha)^2 + b_k^2]e^{(a+\alpha)(1+r)} = [(a+\alpha)^2 + b_\ell^2]e^{(a+\alpha)(1+r)}$$

and this implies $b_k = b_\ell$, which is a contradiction. □

Lemma 2.7. *If $-1 < \delta < 1$, then each solution of (2.2) satisfies $\Re(\lambda) < 0$.*

The proof follows from (2.2).

The Lemma 2.5 shows that $\delta > 1$ implies λ crosses the imaginary axis, in each strip S_k, $k = 0, 1, 2, \ldots$. According to the proof of Lemma 2.4 one has that this crossing is transvesal. Lemmas 2.1 - 6, provide a complete description of the behavior of the characteristic roots in the complex plane.

We observe that for $0 < \Im(\lambda) < \pi/(1+r)$, the origin and $\lambda_0 = b_0 i$ defined in (2.5) are characteristic roots and since $\frac{d}{d\alpha}\Re[\lambda(\alpha)]_{\alpha=0} < 0$, lead to $\Re\lambda(\alpha) < 0$ for $\alpha \in (0, \alpha_0)$.

When $\delta > 1$ and the parameter α increases and crosses a certain value α_0, a pair of conjugate roots $\lambda_0 = \lambda(\alpha_0)$ and $\overline{\lambda}_0$ crosses transversally the imaginary axis from left to right. From this, two complex conjugate roots with positive real parts appear. When the parameter α assumes a new value α_1, a new par $\lambda_1 = \lambda(\alpha_1)$ and $\overline{\lambda}_1$ crosses the imaginary axis, and so on. These results and the Hopf Bifurcation Theorem in [6] lead us to the following result:

Theorem 2.1. *For $\delta > 1$, there exists a sequence of parameters α, so that $0 < \alpha_0 < \alpha_1 < \cdots < +\infty$, for which the equation (1.1) has a local Hopf bifurcation at $\alpha = \alpha_k$, $k = 0, 1, 2, \ldots$. For every $\alpha > \alpha_k$, $k = 0, 1, 2, \ldots$, the equation (1.1) has a nontrivial periodic solution $x(\cdot;\varphi)$ with period ω near $\frac{2\pi}{b_k}$, b_k defined by $b_k = \frac{\pi}{1+r}\arcsin(-1)^k\frac{1}{\delta} \in \left(\frac{(2k-1)}{1+r}\pi, \frac{(2k+1)}{1+r}\pi\right) \cap (0, \infty)$, $k = 0, 1, 2, \ldots$.*

The locus of the eigenvalues λ of (2.2) is shown in the figure below, with $\delta = 1.4$ and $r = 0.5$.

3. Existence of Periodic Solutions

The main objective now is to prove the following Theorem:

Theorem 3.1. *Consider the equation (1.1). Suppose there exists a constant $M > 0$ so that $||F(x)|| \leq M$, for all $x \in \mathbb{R}^2$ and (H_1) holds. Let α_0 be like in Lemma 2.3. Then, for every $\alpha > \alpha_0$, equation (1.1) has a nonconstant periodic solution.*

We denote by $x(\cdot;\alpha,\varphi)$ the solution of (1.1) such that $x\big|_{[-1,0]} = \varphi$ with $\varphi \in C$. We define the closed and convex subset

$$K_\alpha^r = \{\varphi = (\varphi_1, \varphi_2) \in C,\ \varphi_1(-1) = 0,\ \varphi_1(\theta) \geq 0,\ \varphi_2(-1) \geq 0,\ e^{\alpha\theta}\varphi_1(\theta) \text{ increasing for } \theta \in [-1, 0],\ e^{\alpha\theta}\varphi_2(\theta) \text{ increasing for } \theta \in [-1, -1+r] \text{ and decreasing for } \theta \in [-1+r, 0]\}$$

and $K_\alpha^r(M) = K_\alpha^r \cap B_M$, where $B_M = \{\varphi \in C; ||\varphi|| \leq M\}$. Let $A_\alpha : K_\alpha^r(M) \to K_\alpha^r(M)$ given by $A\alpha(\varphi) = x_\tau(\cdot;\alpha,\varphi)$ for some $\tau = \tau(\varphi) > 0$. We will prove that A_α has a nontrivial fixed point, which corresponds to a nontrivial periodic solution of (1.1). For this we need the following concept of ejective point due to Browder [2]

Definition 3.1. Let X be a Banach convex set, U a subset of X and $x \in U$. The point x is said to be an ejective point of a map $A : U \setminus \{x\} \to X$ if there is an open neighbourhood $G \subset X$ of x such that if $y \in G \cap U$, $y \neq x$, there is an integer $m = m(y) > 0$ such that $A^m y \notin G \cap U$.

The following theorem is due to Nussbaum

Theorem 3.2. *If K is a closed, bounded, convex, infinite dimensional set in X, $A : K \setminus \{x_0\} \to K$ is completely continuous, and $x_0 \in K$ is an ejective point of A, then there is a fixed point of A in $K \setminus \{x_0\}$. If K is finite dimensional and x_0 is an extreme point of K, then the same conclusion holds.*

Remark. : A set K be of finite dimension means to be a subset of a finite dimension manifold.

Firstly we need to prove that the origin is an ejective point with respect to the operator $A_\alpha : K_\alpha^r(M) \to K_\alpha^r(M)$. This follows from:

Lemma 3.1. *Suppose that the following conditions are fulfilled:*
(i) There is a characteristic root λ of (2.2) satisfying $\Re\lambda > 0$.
(ii) Given $G \subset C$ open, $0 \in G$, there is a neighborhood V of 0 such that $x_t(\cdot;\alpha,\varphi) \in G$ if $\varphi \in V \cap K$, $\varphi \neq 0$ and $0 \le t \le \tau(\varphi)$.
(iii) There is a closed convex subset K of C, $0 \in K$, and a continuous function $\tau : K \setminus \{0\} \to [a,\infty)$, $a > 0$, such that the map $A : K \to C$ given by

$$A\varphi = \begin{cases} x_{\tau(\varphi)}(\cdot;\alpha,\varphi) & ,\varphi \neq 0 \\ 0 & ,\varphi = 0 \end{cases}$$

is completely continuous and $AK \subset K$.
(iv) $\inf\{||\pi_\lambda x_t|| : x_t = x_t(\cdot;\alpha,\varphi), \varphi \in K, ||\varphi|| = a, 0 \le t \le \tau(\varphi)\} > 0$, where $\pi_\lambda(\varphi) = \varphi^{P_\lambda}$ and P_λ is given by the decomposition of $C = P_\lambda \oplus Q_\lambda$.
Then, 0 is an ejective point of A.

The proof of this lemma is a combination of the Lemmas 3.2-10 given below.

Lemma 3.2. *If F is a bounded map, any solution $x(t) = x(t;\alpha,\varphi)$, $\varphi \in C$, of equation (1.1) is bounded.*

Proof. : Suppose $||F(x)|| \le M$, for all $x \in \mathbb{R}^2$ and $N > M$. From equation (1.1) we have

$$\frac{d}{dt}|x(t)|^2 \le -\alpha|x(t)| + \alpha M|x(t)| = -\alpha|x(t)|[x(t) - M].$$

Therefore, if $|x(t)| > N$, it follows that $\frac{d}{dt}|x(t)|^2 = -\alpha N(N - M) < 0$. Then, $\frac{d}{dt}|x(t)|^2 < 0$, if $x(t) \notin \overline{B}_M$, that is, if $\varphi \in \overline{B}_M$, $x_t(\cdot;\alpha,\varphi) \in B_M$, $t \ge 0$. □

Let us define $Q_1 = \{x \in \mathbb{R}^2, x_1 \ge 0, x_2 \ge 0\}$ and $Q_2 = \{x \in \mathbb{R}^2, x_1 \ge 0, x_2 \le 0\}$.

Lemma 3.3. *Suppose Hypothesis* (H_1) *is satisfied, with F bounded. Then, for any $\alpha > 0$, there exists a continuous function $t_{1\alpha} : K_\alpha^r(M) \setminus \{0\} \to [0,\infty)$ such that $x_1(t_{1\alpha}(\varphi);\alpha,\varphi) = 0$, $x_2(t_{1\alpha}(\varphi);\alpha,\varphi) < 0$ and $x_1(t;\alpha,\varphi) > 0$ if $0 \le t < t_{1\alpha}$.*

Proof. : Let $\varphi(0) \in Q_2$. We need to consider three cases: (i) $\varphi_1(0) > 0$ and $\varphi_2(0) < 0$; (ii) $\varphi_1(0) = 0$ and $\varphi_2(0) < 0$; (iii) $\varphi_1(0) > 0$ and $\varphi_2(0) = 0$.
i) There exists $t^* \in [-1,0]$ with $\varphi_2(t^*) = 0$, because $\varphi_1(-1) \ge 0$. We will prove that there exists $t_1 = t_{1\alpha}(\varphi)$ such that $x_1(t_{1\alpha};\alpha,\varphi) = 0$, $x_2(t_{1\alpha}a,\varphi) < 0$ and $x(t;\alpha,\varphi) \in Q_2$ if $t^* \le t < t_{1\alpha}(\varphi)$. Suppose, for a moment, that for some $\alpha > 0$ and some $\varphi \in K_\alpha^r(M)\setminus\{0\}$ there is no such a $t_1 \ge 0$ so that $x(t) \in Q_2$, $0 \le t \le t_1$, $x_1(t_1) = 0$, $x_2(t_1) < 0$.

Equation (1.1) and (H_1) imply the following conditions:
a) $x_2(t) \le \varphi_2(0)e^{-\alpha t} < 0$ and this, together with the nonexistence of t_1, gives that $x(t)$ remains in Q_2.
b) $\dot{x}_1(t) \le -\alpha x_1(t) \le 0$, therefore, $x_1(t)$ is decreasing for $x(t) \in Q_2$ and $\lim_{t\to\infty} x_1(t) = \eta \ge 0$.

If $\eta > 0$, there exists $\bar{t} \ge 0$ such that $x_1(t) > \frac{\eta}{2}$ for all $t \ge \bar{t}$ and, by (b), it follows that $x_1(t) \le x_1(\bar{t}) - \alpha\frac{\eta}{2}(t - \bar{t})$, $t \ge \bar{t}$. Then $x_1(t)$ becomes negative at some finite time, and this is a contradiction because $x(t) \in Q_2$, $0 \le t < \infty$. Thus, we must have $\lim_{t\to\infty} x_1(t) = 0$.

We claim that

$$\lim_{t\to\infty} x_2(t) = 0. \tag{3.1}$$

If $\lim_{t\to\infty} x_2(t) = \xi < 0$, the equation (1.1) together with the fact that $F_2(x_1, x_2)$ converges to $F_2(0,\xi) = 0$, as $t \to \infty$, gives $\lim_{t\to\infty} x_2(t) = -\alpha\,\xi > 0$, which is a contradiction.

Suppose that $\lim_{t\to\infty} x_2(t)$ does not exist. Then, there exist $\xi \in \mathbb{R}$ such that

$$0 \geq \lim_{t\to\infty} \sup x_2(t) > \xi > \lim_{t\to\infty} \inf x_2(t) \geq -M$$

and a sequence $t_n \to \infty$, with $x_2(t_n) = \xi$, $\dot{x}_2(t_n) \leq 0$.

The equation (1.1) implies $\dot{x}_2(t_n) \to -\alpha\xi > 0$, as $t_n \to \infty$, which contradicts the choice of t_n. Then (3.1) holds whenever $x(t)$ remains in Q_2, for all $t \geq 0$. Thus,

$$\lim_{t\to\infty} x(t) = 0. \tag{3.2}$$

Since $F_1(x_1, 0) = 0$, the Taylor's expansion of $F_1(x)$ near the origin is

$$F_1(x) = \delta_1 x_2 + x_2[a x_1 + b x_2 + o(|x|)],$$

and this implies there exist a number $0 < \eta < \delta_1$ and a neighborhood V of 0 :

$$F_1(x) < (\delta_1 + \eta)x_2, \quad \text{if } x_2 < 0 \quad \text{and} \quad x \in V.$$

There is no loss in assuming that $x(t) \in V \cap Q_2$, $-1 \leq t \leq \infty$. So, it follows that

$$\begin{aligned} x_1(n+1) &= x_1(0)e^{-\alpha(n+1)} + \alpha \int_0^{n+1} e^{\alpha(s-n-1)} F_1(x(s-1))\, ds \\ &< x_1(0)e^{-\alpha(n+1)} + \alpha e^{-\alpha n}(\delta_1 + \eta) \int_0^{n+1} e^{\alpha(s-1)} x_2(s-1)\, ds. \end{aligned}$$

Since $e^{\alpha t} x_2(t)$ is a decreasing function for $t \geq 0$, $x(t) \in Q_2$, we can write

$$\begin{aligned} x_1(n+1) &< x_1(0)e^{-\alpha(n+1)} + \alpha e^{-\alpha n}(\delta_1 + \eta) \int_0^{n+1} e^{-\alpha} x_2(-1)\, ds \\ &= x_1(0)e^{-\alpha(n+1)} + \alpha e^{-\alpha(n+1)}(\delta_1 + \eta)x_2(-1)(n+1) \\ &= e^{-\alpha(n+1)}[x_1(0) + \alpha(n+1)(\delta_1 + \eta)x_2(-1)] < 0 \end{aligned}$$

and as the expression in the brackets is negative for large n, $x_1(t)$ becomes negative in a finite time. That is, there exists a finite $t_1 \geq 0$ such that $x(t) \in Q_2$, $0 \leq t \leq t_1$, $x_1(t_1) = 0$, $x_2(t_1) < 0$.
ii) We define $t_{1\alpha}(\varphi) = 0$.
iii) $x(t) \in Q_2 \setminus Q_1$ for all $t \in (0, t_{1\alpha}]$ where $t_{1\alpha}$ is defined in (i).

The continuity of $t_{1\alpha}(\varphi)$ follows from the transversality of $x(t; \alpha, \varphi)$ and the x_2 axis, and the continuity with respect to the initial conditions.

Let $\varphi(0) \in Q_1$. With adaptations of the above case, we prove there exists a $t_0 \geq 0$ such that $x(t; \alpha, \varphi) \in Q_1$, $0 \leq t \leq t_0$, $x_1(t_0) > 0$ and $x_2(t_0) = 0$. Since $\dot{x}_2(t_0) < 0$, we can define $t_{1\alpha}$ as in (a). □

Lemma 3.4. *Let $A \subset C$ be open, $0 \in A$. There is a neighborhood V of zero in C such that $x_t(\cdot : \alpha, \varphi) \in A$, $0 \leq t \leq t_{1\alpha}(\varphi) + r$ and for all $\alpha > 0$, $\varphi \in V \cap K_\alpha^r(M)$, $\varphi \neq 0$.*

Proof. : We will prove that given an open subset $B \subset \mathbb{R}^2$, $0 \in B$ there exists a neighborhood V of zero in C so that if $\alpha > 0$ and $\varphi \in V \cap K_\alpha^r(M)$, $\varphi \neq 0$, then $x(t; \alpha, \varphi) \in B$ for $-1 \leq t \leq t_{1\alpha} + r$. Notice that if $x(t; \alpha, \varphi) \in B$, for $-1 \leq t \leq t_{1\alpha}$, $\varphi \in V$, then $x(t; \alpha, \varphi) \in B$, for $t_{1\alpha} \leq t \leq t_{1\alpha} + r$. Suppose that the assertion is not valid for $t \in [-1, t_{1\alpha}(\varphi)]$. Then we can find an open rectangle $R = \{(x_1, x_2); -\ell < x_1 < \ell, -k < x_2 < k\} \ni (0,0)$ such that, for some sequences (α_n) and $\varphi_n \in K_\alpha^r(M) \setminus \{0\}$, $\varphi_n \to 0$, as $n \to \infty$, there exists a t_n, $-1 \leq t_n \leq t_{1\alpha_n}(\varphi_n)$, with $x(t_n; \alpha_n, \varphi_n) \in \partial R$, $n = 1, 2, \ldots$.

It is possible that either: (a) $\varphi_{n_j}(0) \in Q_2$ for a subsequence φ_{n_j} or, (b) $\varphi_{n_k}(0) \in Q_1$ for a subsequence φ_{n_k}. We will do the proof only for the case (a), denoting $\varphi_{n_j} = \varphi_n$. The case (b) can be done analogously.

Let $x_n(t) = (x_{n1}(t), x_{n2}(t)) := x(t; \alpha_n, \varphi_n)$. Then $x_{n1}(t) \geq 0$, $x_{n2}(t) \leq 0$ as long as $x_n(t)$ remains in Q_2. Note that $x_{n1}(t)$ is decreasing if $\varphi_{n1}(0) > 0$ or $x_{n1}(t) = 0$, $-1 \leq t \leq t_{1\alpha}$ if $\varphi_{n1}(0) = 0$. Then $x_n(t) \in [0, q_n] \times [-k, k]$, $-1 \leq t \leq t_n$ where $q_n = ||\varphi|| \to 0$, as $n \to \infty$. So, $x_{n2}(t_n) = -k$, $n = 1, 2, \ldots$.

Since $F_2(0, x_2) = 0$, the continuity of F_2 implies that $\lim_{x_1 \to 0} F(x_1, x_2) = 0$, uniformly in x_2, $-k \leq x_2 \leq k$. Then, as long as $x_n(t) \in [0, q_n] \times [-k, k]$, we can write

$$
\begin{aligned}
|x_{n2}(t)| &= |x_{n2}(0)e^{-\alpha_n t} + \alpha_n e^{-\alpha_n t} \int_0^t e^{\alpha_n s} F_2(x_n(s-r))\, ds| \\
&\leq \varphi_{n2}(0)e^{-\alpha_n t} + \alpha_n e^{-\alpha_n t} \int_0^t e^{\alpha_n s} M_n\, ds \\
&\leq q_n e^{-\alpha_n t} + M_n[1 - e^{-\alpha_n t}],
\end{aligned}
$$

where $M_n = \sup\{|F_2(x)|; x \in [0, q_n] \times [-k, k]\}$.

Then, $|x_{n2}(t_n)| \leq q_n + M_n < k$, for n sufficiently large. Therefore $x_n(t_n) \notin \partial R$, which is a contradiction. □

Now we consider the rotation

$$
\mathcal{R} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
$$

and the map $A_{1\alpha} : K_\alpha^r(M) \to \mathcal{R}K_\alpha^r(M) = -K_\alpha^r(M)$ defined by

$$
A_{1\alpha}(\varphi) = \begin{cases} x_{t_{1\alpha}+1}(\cdot, \varphi) & , \varphi \neq 0 \\ 0 & , \varphi = 0. \end{cases}
$$

Lemma 3.5. *The map $A_{1\alpha} : K_\alpha^r(M) \to -K_\alpha^r(M)$ is completely continuous for any $\alpha > 0$.*

Proof. : The continuity in $K_\alpha^r(M) \backslash \{0\}$ follows from the continuity of $t_{1\alpha}$ and from the continuity of the solutions of (1.1) with respect to the initial conditions. The continuity at $\varphi = 0$ follows from Lemma 3.4. Let $L = \overline{D}_\mu \cap K_\alpha^r(M)$, where $\overline{D}_\mu = \{\varphi \in C, ||\varphi|| \leq \mu\}$ and $A_{1\alpha}(L) = \{x_{t_{1\alpha}(\varphi)+1}(\cdot; \alpha, \varphi), \varphi \in L \backslash \{0\}\} \cup \{0\}$. By the same arguments as in the proof of Lemma 3.2, we can show that $A_{1\alpha}(L)$ is bounded by $\max\{M, \mu\}$. It follows from Ascoli-Arzela Theorem that $A_{1\alpha}$ is a completely continuous map. □

We define now the completely continuous map $A_{2\alpha} : -K_\alpha^r(M) \to K_\alpha^r(M)$ by $A_{2\alpha}(\varphi) = -A_{1\alpha}(-\varphi) = (-A_{1\alpha} \circ (-I))(\varphi)$. Our return map $A_\alpha : K_\alpha^r(M) \to K_\alpha^r(M)$ is defined by $A_\alpha = A_{2\alpha} \circ A_{1\alpha}$. So, if for any $\varphi \in K_\alpha^r(M) \setminus \{0\}$, we call $\tau_\alpha(\varphi)$ the time for which $A_\alpha(\varphi) = x_{\tau_\alpha(\varphi)}(\cdot; \alpha, \varphi)$, it follows that $\tau_\alpha(\varphi) = t_{1\alpha}(\varphi) + t_{2\alpha}(\varphi) + 2$. So, τ_α is a continuous function from $K_\alpha^r(M) \setminus \{0\}$ into $[2, \infty]$.

We need now the decomposition of the phase space as a direct sum, $C = \mathrm{P}_\lambda \oplus \mathrm{Q}_\lambda$, associated to the eigenvalue λ of (2.2). See [6].

Let $L(\varphi) := -\varphi(0) + B_1\varphi(-1) + B_2\varphi(-r)$ be the transformation given by the linearization (2.1) of (1.1). Let $\eta(\theta)$, $\theta \in [-1, 0]$, be the matrix

$$
\eta(\theta) := \begin{cases} 0 & , \theta = -1 \\ B_1 & , -1 < \theta < -r \\ B_1 + B_2 & , -r < \theta < 0 \\ -I + B_1 + B_2 & , \theta = 0. \end{cases}
$$

Then for every $\varphi \in C$, $L\varphi = \int_{-1}^{0} [d\eta(\theta)]\varphi(\theta)$.

Let G_α the infinitesimal generator of the semigroup $\{T_\alpha(t), t \geq 0\}$ defined by the solutions of (2.1), $T_\alpha(t)\varphi = x_t(.,\alpha,\varphi)$, $\varphi \in C$, $t \geq 0$.

We take $\alpha_0 \in (0,\infty)$ given by (2.5). For each $\alpha > \alpha_0$, G_α has a unique simple eigenvalue $\lambda = \lambda(\alpha) \in S_0 = \{\lambda \in C, 0 < \Im(\lambda) < \frac{\pi}{2}\}$, with $\Re(\lambda(\alpha)) > 0$. Then the generalized eigenspace $M_\lambda(G_\alpha)$ is one-dimensional. If G_α^* is the formal adjoint operator of G_α, λ is also an eigenvalue of G^* and its generalized eigenspace $M_\lambda(G_\alpha^*)$ is one-dimensional. If $C^* = C([0,1], \mathbb{R})$, where $\mathbb{R}^{2*}$ is the plane of the row 2-vectors, the bilinear form of $C^* \times C$, given by

$$(3.3) \qquad \langle\psi,\varphi\rangle := \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi-\theta)[d\,\eta(\theta)]\varphi(\xi)\,d\xi\,,$$

$\psi \in C^*$, $\varphi \in C$, appears in a natural way in the decomposition of C.

If μ is a nontrivial solution of $\Delta_\alpha(\lambda)\mu = 0$, with $\Delta_\alpha(\lambda) := (\lambda+\alpha)I - \alpha B_1 e^{-\lambda} - \alpha B_2 e^{-\lambda r}$, and the row vector v is a nontrivial solution of $v\,\Delta_\alpha(\lambda) = 0$, the functions $\rho_\alpha(\theta) = e^{\lambda\theta}\mu$, $\theta \in [-1,0]$, and $\sigma_\alpha(s) = ve^{-\lambda s}$, $s \in [0,1]$, span $M_\lambda(G_\alpha)$ and $M_\lambda(G_\alpha^*)$, respectively. The decomposition $C = P_\lambda \oplus Q_\lambda$ splits any $\varphi \in C$ as $\varphi = \varphi^{P_\lambda} \oplus \varphi^{Q_\lambda}$ and φ^{P_λ} is given by

$$\pi_\alpha(\varphi) = \varphi^{P_\lambda} = \langle\sigma,\varphi\rangle\rho.$$

Lemma 3.6. *For any compact set $J_0 \subset (\alpha_0,\infty)$ and $a \in \mathbb{R}$, $0 < a < M$*

$$\gamma = \inf\{||\pi_\alpha x_t||\,;\, x_t = x_t(\cdot,\alpha,\varphi), \varphi \in K_\alpha^r(M), ||\varphi|| = a, 0 \leq t \leq \tau_\alpha(\varphi)\} > 0.$$

For the proof, we will need the three following Lemmas, who describe the behavior of the solution $x(\cdot;\alpha,\varphi)$ of (1.1) with $\alpha \in J_0$, $\varphi = (\varphi_1,\varphi_2) \in K_\alpha^r(M)$, $||\varphi_1|| \geq d$. We call $\alpha_1 = \min J_0$, $\alpha_2 = \max J_0$ and we note that if $\alpha \in J_0$ and $\varphi \in K_\alpha^r(M)$, then $\varphi_1(\theta) \leq e^{\alpha_2}\varphi_1(s)$, $-1 \leq \theta \leq s \leq 0$.

Lemma 3.7. *Let $d > 0$ and $J_0 \subset (0,\infty)$ a compact set. There exists $\tau_1 = \tau_1(\alpha,\varphi) > 0$ such that $x_2(\tau_1,\alpha,\varphi) < -\sigma$ and $x_1(t,\alpha,\varphi) \geq k$, $t \in [0,\tau_1]$, for some $k > 0$, $\sigma > 0$ and for every $\alpha \in J_0$ and $\varphi \in K_\alpha^r(M)$, $||\varphi_1|| \geq d$.*

Proof. : We need to consider two cases: $\varphi(0) \in Q_1$ or $\varphi(0) \in Q_2$.

a) Suppose that $\varphi(0) \in Q_1$. From Lemma 3.3, there exists $t_0 = t_0(\varphi)$ such that $x_1(t_0) > 0$ and $x_2(t_0) = 0$. From (H$_1$) the Taylor's expansion of $F_1(x)$ near the origin is

$$F_1(x) = \delta_1 x_2 + x_2[ax_1 + bx_2 + o(|x|)], \quad \text{as} \quad x \to 0.$$

This gives the existence of a number η, $0 < \eta < \delta_1$, such that, for all x in some neighborhood V of zero,

$$(3.4) \qquad F_1(x) < (\delta_1 + \eta)x_2, \quad x_2 < 0.$$

Let ξ be chosen in such a way that the rectangle $R = \{(x_1,x_2)\,;-\xi < x_1 < \xi, -\xi < x_2 < \xi\} \subset V$. We define

$$(3.5) \qquad k := \min\left\{\xi, d/4e^{\alpha_2(t_0+3)}\right\}.$$

The Hypothesis (H$_1$) gives

$$(3.6) \qquad \exists\sigma' > 0\,;\,(x_1,x_2) \in Q_2, x_1 > k \Longrightarrow F_2(x_1,x_2) < -\min\left\{\sigma', \frac{\sigma'}{\alpha_1}\right\}.$$

Recalling that $\lim_{x_2 \to 0} F_1(x) = 0$, we have, for any $\varepsilon > 0$, $\varepsilon < d/2e^{\alpha_2(t_0+3)}$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(3.7) \qquad (x_1, x_2) \in Q_2,\ 0 < x_2 < \delta(\varepsilon) \Longrightarrow -\varepsilon < F_1(x_1, x_2) < 0.$$

We take

$$(3.8) \qquad \sigma := \min\left\{ \frac{\delta(\varepsilon)}{e^{\alpha_2}}, \frac{\sigma'}{2}, k \right\}.$$

From (3.7) and (1.1) it follows that

$$x_1(t;\alpha,\varphi) > [\varphi_1(0) + \varepsilon]e^{-\alpha t} - \varepsilon, \qquad t \in [0, t_0 + 1].$$

The hypothesis $\|\varphi_1\| \geq d$ implies $|\varphi_1(0)| \geq de^{-\alpha_2}$ and so $x_1(t) > [de^{-\alpha_2} + \varepsilon]e^{-\alpha t} - \varepsilon$, $t \in [0, t_0 + 1]$.

The function $E(t) = [de^{-\alpha_2} + \varepsilon]e^{-\alpha t} - \varepsilon$ is decreasing in $[0, t_0 + 1]$ and $E(t_0 + 1) = [de^{\alpha_2} + \varepsilon]e^{-\alpha(t_0+1)} - \varepsilon > k$. The Lemma 3.3 gives the existence of a first $\hat{t}$ such that $x_1(\hat{t}, \alpha, \varphi) = k$ and $x_1(t;\alpha,\varphi) > k$ for $t < \hat{t}$.

Let us suppose, for a moment, that $x_2(t) \geq -\sigma$, $t \in [0, \hat{t}]$. Since $x_1(t) \geq E(t)$, $t \in [0, \hat{t}]$ and $E(t_0 + 2) > d/2e^{\alpha_2(t_0+3)} > k$, one sees that $\hat{t} > t_0 + 2$.

From (1.1) and (3.6) it follows that, for $t \in [t_0 + 1, t_0 + 2]$,

$$\dot{x}_2(t) = -\alpha x_2(t) + \alpha F_2(x(t - r)) < \alpha\sigma - \min\{\sigma', \sigma'/\alpha_1\} < \alpha\sigma - \min\{2\alpha\sigma, 2\sigma\} \leq -\sigma,$$

therefore, there exists $t \in (t_0 + 1, t_0 + 2)$ such that $x_2(t) < -\sigma$, which is a contradiction. This assures the existence of $\tau_1 \in (0, \hat{t})$.

b) With obvious adaptations, the proof in this case is analogous to the last part of the proof in the case (a). □

Lemma 3.8. *Under the hypothesis of Lemma 3.7, consider k and σ given by (3.5) and (3.8), respectively. If, for $\alpha \in J_0$, $\varphi \in K_\alpha^r(M)$, the solution $x(t;\alpha,\varphi)$ of the equation (1.1) satisfies $x_2(\tau_1;\alpha,\varphi) < -\sigma$, $x_1(\tau_1;\alpha,\varphi) \geq k$, for some $\tau_1 \geq 0$, then there exists $\tau_2 = \tau_2(\alpha,\varphi) > 0$ such that $x_2(t;\alpha,\varphi) < -\sigma e^{-\alpha_2}$ for $t \in [\tau_1, \tau_2]$, $0 < x_1(\tau_2;\alpha,\varphi) < k$.*

Proof. : Notice that $x_2(\tau_1;\alpha,\varphi) < -\sigma$ and (1.1) imply that $x_2(t;\alpha,\varphi) < -\sigma e^{-\alpha_2}$, $t \in [\tau_1, \tau_1 + r]$. If $0 < x_1(t_0;\alpha,\varphi) < k$, for any $t_0 \in [\tau_1, \tau_1 + r]$, we define $\tau_2 = t_0$. Let us now suppose $x_1(t;\alpha,\varphi) > k$ for $t \in [\tau_1, \tau_1 + r]$. Let $\tilde{t} = \tilde{t}(\alpha,\varphi) > \tau_1 + r$ be the instant t when $x_1(\tilde{t};\alpha,\varphi) = k$ and $x_1(\tilde{t} + \varepsilon;\alpha,\varphi) < k$, for all $\varepsilon > 0$. The equation (1.1) together with (3.6) and (3.8) implies that $x_2(t) < -\sigma e^{-\alpha_2}$, for all $t \in [\tau_1 + r, \tilde{t} + r]$. Consequently the existence of τ_2 is assured. □

Lemma 3.9. *Under the hypothesis of Lemma 3.7, let k and σ be defined by (3.5) and (3.8), respectively, $\alpha \in J_0$, $\varphi \in K_\alpha^r$, $\|\varphi_1\| \geq d$. Let $x(t;\alpha,\varphi)$ be the solution of (1.1) satisfying $x_2(\tau_2;\alpha,\varphi) < -\sigma e^{-\alpha_2}$, $0 < x_1(\tau_2;\alpha,\varphi) < k$, for some $\tau_2 \geq 0$. Then, there exists $c > 0$, such that $x_2(t;\alpha,\varphi) \leq -c$, $t \in [t_1, t_1 + r]$, with $t_1 = t_{1\alpha}(\varphi) \geq 0$ defined in Lemma 3.3.*

Proof. : If $x_2(t;\alpha,\varphi) < -\sigma e^{-\alpha_2}$ for t in some interval $(t_1 - \epsilon, t_1)$, we define $c = \sigma e^{-\alpha_2}$.

Suppose, for a moment, this is not true. Let $\bar{t} = \bar{t}(\alpha,\varphi)$, $\tau_2 < \bar{t} < t_1$, the last time t for which $x_2(t;\alpha,\varphi) = -\sigma e^{-\alpha_2}$ and $(e^{\alpha t}x_2(t))' < 0$, if $t \in [\bar{t}, t_1 + r]$. Then $x_2(t)$ is a decreasing function for $t \in [\bar{t}, t_1 + r]$. Therefore, $x_2(t) < -\sigma e^{-\alpha_2}$, for all $t \in [t_1, t_1 + r]$. □

Lemma 3.10. *For any real constant d, $0 < d < M$ and any compact $J_0 \in (0,\infty)$, if*

$$m = \inf\{\langle\sigma_\alpha, x_t\rangle : x_t = x_t(\cdot;\alpha,\varphi), \varphi \in K_\alpha^r(M), d \le ||\varphi_1|| \le M, \alpha \in J_0, 0 \le t \le t_{1\alpha}(\varphi) + r\},$$

then $m > 0$

Proof. : Choosing $\sigma_\alpha(s) = (1,-i)e^{-\lambda s}$, it follows from (3.3) that

$$\langle\sigma_\alpha, x_t\rangle = x_1(t) - ix_2(t) + \alpha\delta_1 \int_{-1}^{0} e^{-\lambda(\xi+1)} x_2(t+\xi)d\xi + i\delta_2 \int_{-r}^{0} e^{-\lambda(\xi+r)} x_1(t+\xi)\, d\xi.$$

Taking $\lambda = a + bi$, $a > 0$ and $0 < b < \frac{\pi}{2}$, we obtain

$$\begin{aligned}\langle\sigma_\alpha, x_t\rangle = x_1(t) &+ \alpha\delta_1 \int_{-1}^{0} e^{-a(\xi+1)} x_2(t+\xi)\cos b(\xi+1)\, d\xi \\ &+ \alpha\delta_2 \int_{-r}^{0} e^{-a(\xi+r)} x_1(t+\xi)\sin b(\xi+r)\, d\xi \\ &+ i\{-x_2(t) - \alpha\delta_1 \int_{-1}^{0} e^{-a(\xi+1)} x_2(t+\xi)\sin b(\xi+1)\, d\xi \\ &+ \alpha\delta_2 \int_{-r}^{0} e^{-a(\xi+r)} x_1(t+\xi)\cos b(\xi+r)\, d\xi\}.\end{aligned}$$

Suppose that $m = 0$. Then, there exist sequences $\alpha_n \in J_0$, $\varphi_n \in K_{\alpha_n}^r(M)$, $d \le ||\varphi_{1n}|| \le M$, $t_n \in [0, t_{1\alpha_n}(\varphi_n) + r]$, $n = 1, 2, \ldots$, such that if $x(t;\alpha_n,\varphi_n) = x_n(t) = (x_{1n}(t), x_{2n}(t))$, $\lambda_n = \lambda(\alpha_n)$, $\langle\sigma_{\alpha_n}, x_{nt_n}\rangle \to 0$, $n \to \infty$. It follows from $d \le ||\varphi_{1n}|| \le M$ and from the proof of Lemma 3.2 that $||x_n(t)|| \le M$, for $t \ge -1$, $n \in \mathbb{N}$. Then, $x_n(t_n)$ is convergent. Taking a subsequence if necessary, we can assume $\alpha_n \to \alpha$, because J_0 is compact. Let $(L_1, L_2) = \lim_{n\to\infty}(x_{1n}(t_n), x_{2n}(t_n))$. It follows that $(L_1, L_2) \ne (0,0)$ from Lemmas 3.6-8. There are the following cases to examine: $L_1 > 0$, $L_2 \ge 0$; $L_1 = 0$, $L_2 > 0$; $L_1 > 0$, $L_2 < 0$; $L_1 < 0$, $L_2 \ge 0$; $L_1 = 0$, $L_2 < 0$; $L_1 < 0$, $L_2 < 0$. In any one it can be seeing that either $|\Re\langle\sigma_{\alpha_n}, x_{nt_n}\rangle| > 0$ or $|\Im\langle\sigma_{\alpha_n}, x_{nt_n}\rangle| > 0$ and bounded away from zero.

Then $\langle\sigma_{\alpha_n}, (x_n)_{t_n}\rangle \not\to 0$,as $n \to \infty$ which is a contradiction. Therefore, $m > 0$. □

Remark. : If we replace $K_\alpha^r(M)$ by $-K_\alpha^r(M)$ and $[0, t_{1\alpha}(\varphi) + r]$ by $[t_{1\alpha}(\varphi) + r, t_{2\alpha}(\varphi) + 1]$, the conclusion of Lemma 3.10 remains valid.

Proof of Lemma 3.6. : Let $0 < a < M$. Lemma 3.10 implies that

$$\inf\{|\langle\sigma_\alpha, x_t\rangle| : x_t = x_t(\cdot : \alpha,\varphi), \alpha \in J_0, \varphi \in K_\alpha^r(M), ||\varphi|| = a, 0 \le t \le t_{1\alpha}(\varphi) + r\} > 0.$$

From Lemma 3.9 there exists $c > 0$ such that $||x_{t_{1\alpha}(\varphi)+1}(\cdot;\alpha,\varphi)|| \ge c$, for all $\alpha \in J_0$, $\varphi \in K_\alpha^r(M)$, $||\varphi|| = a$. Lemma 3.1 implies

$$||x_{t_{1\alpha}(\varphi)+1}(\cdot;\alpha,\varphi)|| \le M. \alpha \in J_0, \varphi \in K_\alpha^r(M), ||\varphi|| = a.$$

Then, by Lemma 3.6,

$$\inf\{|\langle\sigma_\alpha, x_t\rangle| : x_t = x_t(\cdot : \alpha,\varphi), \alpha \in J_0, \varphi \in K_\alpha^r(M), ||\varphi|| = a, 0 \le t \le \tau_\alpha(\varphi) + r\} > 0. \quad \square$$

The hypothesis (ii), (iii) and (iv) follow from the Lemmas 3.7, 3.8 and 3.9, respectively. Theorem 3.1 follows now from Theorem 3.2.

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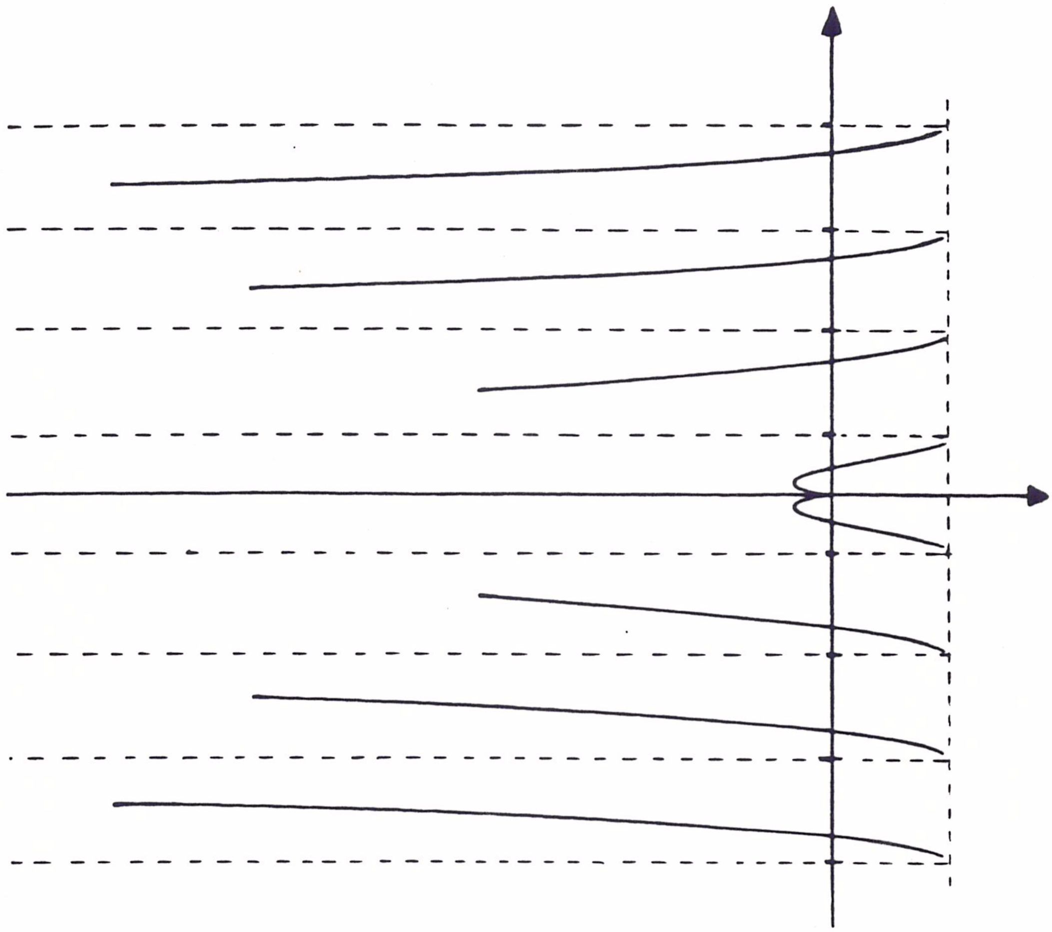

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