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The Borel Map for Compact Subanalytic Subsets of $\mathbb{C}^m$

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Abstract

We define, for a compact subset K of complex Euclidean space containing the origin, the so-called Borel map (at the origin). We discuss its properties in detail and state, in the case when K is subanalytic, two conjectures relating the injectivity and surjectivity of the Borel map with properties of the polynomial hull of K. We give strong evidence for the validity of the conjectures (e.g. the open mapping property) and show that they are true when K is convex.

Keywords Borel map · Open mapping property · Fréchet algebras of functions · Sub analytic sets · Convex sets

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1 Introduction

In this paper, we introduce the Borel map of a compact set $K \subset \mathbb{C}^m$ and discuss it in detail for convex compact subanalytic sets. The idea behind the Borel map is simple: While the Banach algebra $A(K)$ is a well known flexible tool it does not help us to recognize whether or not the polynomially convex hull $\widehat{K}$ has an analytic structure. A natural hope is that if one looks at the more restrictive Fréchet algebra $A^\infty(K)$ (consisting of limits of polynomials together with their derivatives of every order), then this algebra might encode more information. On the level of characters, the spectrum of $A^\infty(K)$ is again $\widehat{K}$, but the algebra naturally supports higher order derivations.

The Borel map $\mathfrak{b}_K$ is designed to make this idea precise: By mapping a uniform limit in all derivatives on K to the formal power series consisting of the evaluation of these derivatives at a point (which we assume to be the origin for ease of notation), it connects *local* information about the existence of varieties in K with *partial convergence* properties of these formal power series.

The subanalyticity assumption on K is designed to exclude pathological examples in the spirit of *tame topology*: We willingly exchange the myriad of distinct geometric, metric, and topological assumptions we would need to satisfy for the validity of our theorems with one class of nice sets which furthermore has the property that it is stable under projections (which will be crucial for our purposes).

We think that the Borel map is a powerful tool to help with the analysis of questions related not only to the existence of complex structure in certain types of hulls, but also to shed light on the connections between themes like open mapping, maximum modulus, and peaking, and we try to highlight its role for these questions throughout the paper.

The convexity assumption helps us considerably to simplify the geometry (in particular, of the boundary of $\widehat{K}$) with a certain stratification, while the subanalyticity assumption is crucial for the analysis we carry out. Restricting ourselves to a tame topological situation opens the door to the use of relative peaking functions associated with the stratification of the convex set we construct, and we are able to exploit the curvature of the boundary even at limit points of varieties contained in it.

Our first Theorem shows that a purely algebraic property of the Borel map implies an *Open Mapping Property*.

Theorem 1 *Let $K \subset \mathbb{C}^m$ be compact and connected, $0 \in K$, and assume that K is polynomially convex and subanalytic. Assume also that $\mathfrak{b}_K$ is injective. If $h \in \mathcal{O}(K) \setminus \{0\}$ vanishes at the origin then $h(K)$ is a neighborhood of the origin in the complex plane.*

In the presence of some weak curvature of K, injectivity of the Borel map happens *only* in the interior of K.

Theorem 2 *Let $K \subset \mathbb{C}^m$ be compact, connected and subanalytic and assume $0 \in K$. Assume that for every $p \in \partial K$ there exists an $f^{(p)} \in \mathcal{O}(K) \setminus \{0\}$ such that $\mathrm{Re}\, f^{(p)}|_K \leq 0$ and $f^{(p)}(p) = 0$. If $\mathfrak{b}_K$ is injective then K is a neighborhood of the origin. In particular, if $K \subset \mathbb{C}^m$ is compact, convex, subanalytic with nonempty interior, and if the Borel map of K is injective, then $0 \in K^\circ$.*

We then discuss in detail a model case, namely polyhedra in $\mathbb{C}^m$, where we are able to give a complete description of injectivity *and* surjectivity of the Borel map.

Theorem 3 *Let K be a polyhedron in $\mathbb{C}^m$ and assume that $0 \in K$. Then:*

(1) *If $\mathfrak{b}_K$ is injective then $0 \in K^\circ$;*
(2) *Assume that $0 \in \partial K$.*

  (a) *If 0 is a vertex of K then $\mathfrak{b}_K$ is surjective;*
  (b) *If 0 belongs to the interior of $\mathfrak{a}$, a q-face of K, $q \geq 1$, and if E is the real subspace of $\mathbb{C}^m$ which contains $\mathfrak{a}$ then $\mathfrak{b}_K$ is surjective if and only if E is totally real.*

Actually, injectivity and surjectivity of the Borel map are mutually exclusive as long as K is not a singleton. In our discussion of polyhedra, we are able to show substantially more, as the directions of the complex subspaces $E^c = E \cap iE$ in K are exactly those variables in which "partial" Borel maps are not surjective (but rather, injective). We refer the reader to the discussion in section 5 for the details.

For general convex subanalytic sets we obtain the following theorem characterizing surjectivity of the Borel map in terms of varieties in K passing through the origin. It requires us to restrict K to a compact neighbourhood of the origin, but this is a natural localization as its conclusion is purely local.

Theorem 4 *Assume that K is subanalytic and convex. Then there exists a (compact) neighbourhood U of the origin such that the Borel map $\mathfrak{b}_{K\cap U}$ is surjective if and only if K does not contain a germ of complex line passing through* 0.

This result gives some evidence that the following conjecture must be true (cf. also Proposition 4 below).

Conjecture 1 *Let K be a subanalytic compact subset in $\mathbb{C}^m$, with $0 \in K$. The (local) Borel map at the origin is surjective if and only if there is no nontrivial analytic disc through the origin and contained in $\widehat{K}$.*

Let us discuss our main motivation for studying the Borel map in this general context. We are moslty interested in *locally integrable structures* and their solutions. A locally integrable structure consists of a smooth manifold Ω together with a smooth subbundle $\mathcal{V} \subset \mathbb{C}T\Omega$ having the property that, if we denote by m the corank of $\mathcal{V}$, then $\mathcal{V}^\perp$ is locally generated, near each point of M, by the differentials of m smooth functions. We also recall that a *smooth solution* for $\mathcal{V}$ is a smooth function whose differential is a section of $\mathcal{V}^\perp$.

In order to state the Baouendi-Treves approximation theorem [1] we observe that, given any point $p \in \Omega$, we can find an open neighborhood U of p, smooth solutions $Z_1, \ldots, Z_m$ defined on U and pairwise, commuting complex vector fields $M_1, \ldots, M_m$ also defined on U such that (i) $\mathrm{d}Z_1, \ldots, \mathrm{d}Z_m$ span $\mathcal{V}^\perp|_U$ and (ii) $\mathrm{d}Z_j(M_k) = \delta_{jk}$, $j, k = 1 \ldots, m$.

We write $Z : U \to \mathbb{C}^m$, $Z = (Z_1, \ldots, Z_m)$, and there is no loss of generality assuming that $Z(p) = 0$. The Baouendi-Treves theorem gives the existence of a compact neighborhood $K \subset U$ of p for which the following holds: given a smooth solution u on U there is a sequence of holomorphic polynomials $\{p_j(z)\}$ in $\mathbb{C}^m$ such that $\{(\partial_z^\alpha p_j)(Z)\}$ converges uniformly to $M^\alpha u$ on K, for every $\alpha \in \mathbb{Z}_+^m$.

In particular it follows that there are $\mathfrak{u}_\alpha \in C(Z(K))$ such that $M^\alpha u = \mathfrak{u}_\alpha \circ (Z|_K)$ and that the formal power series $\mathfrak{u} = \sum_\alpha \mathfrak{u}_\alpha X^\alpha$ belongs to $A^\infty(Z(K))$. In other words, solutions in U can be represented, over K, by elements in $A^\infty(Z(K))$, and this association is an algebra homomorphism.

Now, one of the natural questions about solutions of such locally integrable structures is whether or not they are *uniquely determined*: Are two (germs of) smooth solutions which agree to infinite order at a point actually the same? At present, the only examples for which unique determination holds are *hypocomplex* structures introduced in [2]. We recall that, under the notation just established, $\mathcal{V}$ is hypocomplex at p if any distributional solution near p is of the form $H \circ Z$, where H is holomorphic near the origin in $\mathbb{C}^m$. The Borel map allows us to investigate the *converse* to this question, at least in the real-analytic case: If unique determination holds, is the structure hypocomplex? This is indeed so when $m = 1$ or when the structure is in so-called "tube form" (cf. [3] for more details). We also refer the reader to [4] for a discussion of the Borel map in locally integrable structures from a more intrinsic point of view. While the present work does not give a complete answer to this question, it highlights a number of interesting properties which point into this direction.

Being more precise, when the functions Z_j are real-analytic and K is subanalytic (which certainly we can assume) then $Z(K)$ is subanalytic, and Theorem 5.2 in [4] shows that our question has a positive answer if one can the prove following conjecture:

Conjecture 2 *Let K be a subanalytic compact subset in $\mathbb{C}^m$, with $0 \in K$. The Borel map at the origin is injective if and only if the origin belongs to the interior of $\widehat{K}$.*

Needless to add this conjecture is in accordance with the open mapping property and also that both conjectures 1 and 2 are verified when K is a polyhedron (cf. Theorem 3).

The paper is organized as follows. We give definitions and explore some structural properties of our algebras in Sect. 2. We then discuss the open mapping property in Sect. 3. Surjectivity of the Borel map is characterized in functional analytic terms in Sect. 4, followed by the proofs of Theorem 3 in Sect. 5 and of Theorem 4 in Sect. 6.

2 Preliminaries

2.1 The algebra $A^\infty(K; L)$

In what follows we denote by $\mathcal{F}_m$ the space $\mathbb{C}[\![X_1, \dots, X_m]\!]$ of formal power series in m variables endowed with its natural structure of a (local) Fréchet algebra with topology defined by the semi-norms

$$\mathfrak{S} = \sum_{\alpha \in \mathbb{Z}_+^m} a_\alpha X^\alpha \mapsto \mathsf{q}_n(\mathfrak{S}) = \sum_{|\alpha| \le n} |a_\alpha|.$$

Let $K \subset \mathbb{C}^m$ be a nonempty compact subset of $\mathbb{C}^m$, and denote by $C(K; \mathcal{F}_m) = C(K)[\![X_1, \dots, X_m]\!]$ the Fréchet algebra whose topology is defined by the semi-norms

$$\mathsf{p}_n(\mathbf{u}) = \sum_{|\alpha|\le n} \sup_K |u_\alpha|, \quad n \in \mathbb{Z}_+, \quad \mathbf{u} = \sum_{\alpha\in\mathbb{Z}_+^m} u_\alpha X^\alpha \in C(K;\mathcal{F}_m).$$

We let L be another compact subset of $\mathbb{C}^m$, $K \subset L$, and denote by $A^\infty(K;L)$ the closure in $C(K;\mathcal{F}_m)$ of the subalgebra

$$\mathcal{H}(K;L) = \left\{ \mathbf{f} : \mathbf{f} = \sum_{\alpha\in\mathbb{Z}_+^m} \frac{(\partial^\alpha f)|_K}{\alpha!} X^\alpha \in C(K;\mathcal{F}_m), \ f \in \mathcal{O}(L) \right\}.$$

We note that the map $\gamma_{K,L} : \mathcal{O}(L) \to \mathcal{H}(K;L)$, $\gamma(f) = \mathbf{f}$, is an algebra isomorphism if L is connected and also that $A^\infty(K;L)$ = (closure of the range of $\gamma_{K,L}$) is a closed subalgebra of $C(K;\mathcal{F}_m)$ and hence a Fréchet algebra itself.

Example 1 We recall that the algebras $A^\infty(K)$ and $M^\infty(K)$ are defined by

$$A^\infty(K) = \overline{\mathbb{C}[X]|_K}, \quad M^\infty(K) = \overline{\left\{ f|_K : f = \frac{p}{q} \in R(X), q \neq 0 \text{ on } K \right\}},$$

the closures taken in $C(K;\mathcal{F}_m)$. These algebras appear in the present context as follows: we have $A^\infty(K;\widehat{K}) = A^\infty(K)$ and $A^\infty(K;\tilde{K}) = M^\infty(K)$, where $\widehat{K}$ (resp. $\tilde{K}$) denotes the polynomial (resp. rational) hull of K. In order to prove both statements we rely on Stolzenberg [5] (based on the fundamental work [6] by K. Oka). Noticing that $\widehat{\widehat{K}} = \widehat{K}$ and $\tilde{\tilde{K}} = \tilde{K}$ statements (A.2) and (A.3) in [5, p. 283] give the following property:

- There is a fundamental system $\{U_j\}$ of Stein open neighborhoods of $\widehat{K}$ (resp. $\tilde{K}$) such that the following is true: given $j \in \mathbb{Z}_+$ and $f \in \mathcal{O}(U_j)$ there is a sequence $\{f_k\}$ of holomorphic polynomials (resp. rational functions) such that $f_k \to f$ uniformly over the compact subsets of U_j.

Cauchy's estimates show that the convergence is actually uniform in all derivatives in this case.

2.2 The Borel map

For $p \in K$, We shall refer to the continuous algebra homomorphism

$$\mathfrak{b}_{K,L;p} : A^\infty(K;L) \to \mathcal{F}_m, \quad \mathfrak{b}_{K,L;p}(\mathbf{u}) = \mathbf{u}(p)$$

as the Borel map of K relative to L at p; in what follows we shall assume that K properly contains the origin and write $\mathfrak{b}_{K,L}$ for the Borel map of K relative to L at the origin. When $L = \widehat{K}$ we shall simply write $\mathfrak{b}_{K,\widehat{K}} = \mathfrak{b}_K$. We will show below in 2.3 that the maximal ideal space of $A^\infty(K;L)$ is not a unitary set and hence it follows that $A^\infty(K;L)$ is never a local algebra (for a local algebra has a unitary set as its spectrum). Since $\mathcal{F}_m$ is a local algebra we conclude that the Borel map for K relative to L at the origin is never bijective. If we say that an element $\mathbf{u}$ of $A^\infty(K;L)$

is *flat* at p if $\mathfrak{b}_{K,L;p}(\mathbf{u}) = 0$, then this means that if the Borel map at p is surjective, then necessarily there have to be elements of $A^\infty(K; L)$ which are flat at p; on the other hand, if there are no nontrivial elements of $A^\infty(K; L)$ which are flat at p, then the Borel map at p cannot be surjective.

Example 2 Consider $K = \{0\}$ and L any compact containing K. Then $A^\infty(K; L) = \mathcal{F}_m$. Indeed, we can identify $C(K; \mathcal{F}_m) \simeq \mathcal{F}_m$ and, moreover, if $\mathfrak{s}(X) = \sum_\alpha \mathfrak{s}_\alpha X^\alpha \in \mathcal{F}_m$, then $\mathfrak{s}(X) = \lim_{n\to\infty} \sum_{|\alpha|\le n} \mathfrak{s}_\alpha X^\alpha$, which shows that $\mathsf{s} \in A^\infty(K; L)$.

Example 3 Let Δ denote the unit circle in $\mathbb{C}$ and let $K = L = \overline{\Delta} = \{|z| \le 1\} \subset \mathbb{C}$. Then $\mathfrak{b}(A^\infty(K))$ consists of the series $\sum_j \mathfrak{s}_j X^j$ satisfying

$$\sum_j |\mathfrak{s}_j| j^k < \infty$$

for every $k \in \mathbb{Z}_+$.

Example 4 Consider $K = \overline{\Delta} \times [0, 1] \subset \mathbb{C}^2_{(z,w)}$. Then $\mathfrak{b}(A^\infty K))$ consists of the series $\sum_{j,k} \mathfrak{s}_{j,k} z^j w^k$ where $\sum_j |\mathfrak{s}_{j,k}| j^\ell < \infty$ for every $k, \ell \in \mathbb{Z}_+$.

Example 5 Let U be a bounded, open subset of $\mathbb{C}^m$. We denote by $\mathcal{O}_\bullet(\bar{U})$ the space of all holomorphic functions h in U such that h and all its derivatives extend continuously to $\bar{U}$. Notice that $\mathcal{O}_\bullet(\bar{U})$ carries a natural structure of a Fréchet algebra and that $A^\infty(\bar{U})$ can be naturally embedded as a subalgebra of $\mathcal{O}_\bullet(\bar{U})$. Notice that equality occurs if the homomorphism $\mathcal{O}(\mathbb{C}^m) \to \mathcal{O}_\bullet(\bar{U})$ induced by restriction has dense image.

2.3 The Maximal Ideal Space

We shall now determine the maximal ideal space of $A^\infty(K; L)$. Let us consider $\mu : A^\infty(K; L) \to \mathbb{C}$ a continuous multiplicative functional. If U is an open neighborhood of L we obtain a continuous multiplicative functional μ_U on the Fréchet algebra $\mathcal{O}(U)$ by means of the composition

$$\mu_U := \left(\mathcal{O}(U) \to \mathcal{O}(L) \xrightarrow{\gamma_{K,L}} A^\infty(K; L) \xrightarrow{\mu} \mathbb{C}\right).$$

Since μ is continuous on $A^\infty(K; L)$ there are constants $C > 0$, $N \in \mathbb{Z}_+$ such that

$$|\mu_U(h)| \le C \sum_{|\alpha|\le N} \sup_K |\partial^\alpha h|, \quad h \in \mathcal{O}(U).$$

By Cauchy's inequalities we obtain

$$|\mu_U(h)| \le C_\varepsilon \sup_{K_\varepsilon} |h|, \quad h \in \mathcal{O}(U),\ \varepsilon > 0,$$

where K_ε denotes the set of all $z \in \mathbb{C}^m$ such that $\operatorname{dist}(z, K) \le \varepsilon$. Replacing h by the powers h^n and letting $n \to \infty$ we obtain the preceding inequality with $C_\varepsilon = 1$.

Letting also $\varepsilon \to 0$ we finally obtain

$$|\mu_U(h)| \leq \mathsf{p}_0(h) = \sup_K |h|, \quad h \in \mathcal{O}(U).$$

Following [7] we shall denote by $A(K; U)$ the Banach algebra obtained as the completion of the normed algebra $(\mathcal{O}(U)/\{h : h|_K = 0\}, \mathsf{p}_0)$. Thus μ_U defines an element in $\sigma(K; U)$, the spectrum of $A(K; U)$.

We now take another open neighborhood $V \subset U$ of L. The restriction map induces a Banach algebra homomorphism $A(K; U) \to A(K; V)$ and hence by transposition a map

$$\Theta_{V,U} : \sigma(K; V) \longrightarrow \sigma(K; U). \tag{1}$$

We shall denote by $\sigma(K; L)$ the projective limit of the system (1). For the convenience of the reader, we recall that this means that

$$\sigma(K; L) \subset \prod_U \sigma(K; U)$$

consists of all elements $(f_U) \in \prod_U \sigma(K; U)$ satisfying $\Theta_{V,U}(f_V) = f_U$. It is clear that $\sigma(K; L)$ is compact. Furthermore according to our construction we have

$$\mu_U = \Theta_{V,U}(\mu_V).$$

This shows that the system $\{\mu_U\}$ defines an element in $\mu_* \in \sigma(K; L)$. An elementary argument allows one to show the following result:

Proposition 1 *The map $\mu \mapsto \mu_*$ defines an homeomorphism between the maximal ideal space of $A^\infty(K; L)$ and $\sigma(K; L)$.*

In some circumstances it is possible to embed the maximal ideal space of $A^\infty(K; L)$ into $\mathbb{C}^m$ and identify it with a suitable hull. We therefore introduce the following notation:

$$\widehat{K}_L := \{z \in L : |h(z)| \leq \sup_K |h|, \ h \in \mathcal{O}(L)\} = \bigcap_{L \subset \Omega} \widehat{K}_\Omega.$$

The map $\widehat{K}_L \ni z \mapsto (\mathcal{O}(L) \ni h \mapsto h(z))$ allows us to inject $\widehat{K}_L$ into $\sigma(K; L)$.

Proposition 2 *If L has a fundamental system of Stein neighborhoods then $\sigma(K; L) = \widehat{K}_L$.*

Indeed for each $U \supset K$ open and Stein we have $\sigma(K; U) = \widehat{K}_U$. □

Example 6 If we consider $L = \widehat{K}$ (resp. $L = \tilde{K}$) then $\sigma(K; L) = \widehat{K}_L = \widehat{K}$ (resp. $\sigma(K; L) = \widehat{K}_L = \tilde{K}$).

2.4 Strong Unique Continuation

If we endow the image $\mathfrak{b}_{K,L}(A^\infty(K,L)) \subset \mathcal{F}_m$ with the topology induced by the homomorphism $A^\infty(K;L)/\ker \mathfrak{b}_{K,L} \hookrightarrow \mathcal{F}_m$, that is, the Fréchet topology generated by the semi-norms given by

$$\dot{\mathsf{p}}_n(\mathbf{u} + \ker \mathfrak{b}_{K,L}) = \inf\{\mathsf{p}_n(\mathbf{u}+\mathbf{v}) : \mathbf{v} \in \ker \mathfrak{b}_{K,L}\},$$

then it can be thought of as a Fréchet algebra of power series. Notice that $\dot{\mathsf{p}}_n \leq \dot{\mathsf{p}}_{n+1}$ for all $n \geq 0$ since the same is true for p_n. We will say that $E \subset \hat{K}_L$ is a strong unique continuation set for p if every $\mathbf{u} \in A^\infty(K;L)$ with $\mathbf{u}(p) = 0$ satisfies $u_0(z) = 0$ for every $z \in E$. Note that the union of strong unique continuation sets is again a strong unique continuation set for p, and if E is a strong unique continuation set, then $\overline{E}$ is also a strong unique continuation set. There is a natural maximal (strong) unique continuation set defined by

$$\begin{aligned} M(\mathfrak{b}_{K,L}(A^\infty(K,L))) &= \left\{\mu \in \sigma(K;L) : \mu(\mathbf{u}) = 0, \forall \mathbf{u} \in \ker \mathfrak{b}_{K,L}\right\} \\ &\supset \bigcap_{\mathbf{u} \in \ker \mathfrak{b}_{K,L}} \left\{z \in \widehat{K}_L : u_0(z) = 0\right\}, \end{aligned} \tag{1}$$

i.e. if E is a strong unique continuation set, then $E \subset M(\mathfrak{b}_{K,L}(A^\infty(K,L)))$. In particular if L has a fundamental system of Stein neighborhoods then

$$M(\mathfrak{b}_{K,L}(A^\infty(K,L))) = \bigcap_{\mathbf{u} \in \ker \mathfrak{b}_{K,L}} \{z \in \widehat{K}_L : u_0(z) = 0\}. \tag{2}$$

We finally say that K has the strong unique continuation property (relative to L) if K is a strong unique continuation set for every point $p \in K \setminus \partial K$ (the boundary with respect to the relative topology).

2.5 The Arens-Michael Representation

In some particular but nevertheless especially important cases we can show that the locally convex topology of $A^\infty(K;L)$ is actually Fréchet-Schwartz (that is, a projective limit of Fréchet spaces linked by compact linear maps). We shall show that this is indeed so if we assume that p_0 (and hence p_n for all n) is a norm. For this we study the Arens-Michael representation of $A^\infty(K;L)$. Let $A_n(K;L)$ be the completion of $(A^\infty(K;L), \mathsf{p}_n)$. Then $A_n(K;L)$ is a Banach algebra which can be identified with the closure in $C(K;\mathcal{P}_n)$ of the space

$$\left\{ \sum_{|\alpha| \leq n} \frac{h^{(\alpha)}|_K}{\alpha!} X^\alpha : h \in \mathcal{O}(L) \right\}.$$

It is then easy to see that

$$A^{\infty}(K;L) = \varprojlim A_n(K;L).$$

where the maps $J_n : A_{n+1}(K;L) \to \mathcal{O}_n(K;L)$ are induced by the natural projections $C(K;\mathcal{P}_n) \to C(K;\mathcal{P}_{n-1})$. We shall show that under some favorable assumptions, these maps are actually *compact*.

Proposition 3 *If* p_0 *is a norm and if* K *is subanalytic then for each* $n \geq 0$ *the homomorphism* J_{n+1} *is compact.*

Proof There is no loss of generality if we assume that both K and L are connected. We must show that $\{\mathbf{u} \in A_{n+1}(K;L) : \mathsf{p}_{n+1}(\mathbf{u}) \leq 1\}$ is relatively compact in $A_n(K)$ or, which is the same, in $C(K;\mathcal{P}_{n+1})$. By the Arzela-Ascoli theorem it suffices to show that

(1) $\mathcal{B} = \{\mathbf{u} \in A_{n+1}(K) : \mathsf{p}_{n+1}(\mathbf{u}) \leq 1\}$ is equicontinuous in $C(K;\mathcal{P}_n)$;
(2) For each $z_0 \in K$ the set $\{\mathbf{u}(z_0) \in \mathcal{P}_n : \mathbf{u} \in A_{n+1}(K),\ \mathsf{p}_{n+1}(\mathbf{u}) \leq 1\}$ is bounded in $\mathcal{P}_n$.

Since K is subanalytic and compact there are constants $C, \sigma > 0$ such that given any two points $z, w \in K$ there is a rectifiable curve joining z and w and of length $\leq C|z-w|^{\sigma}$ (cf. [8, Theorem 6.10]). Hence given $z, w \in K$ and $h \in \mathcal{O}(L)$ we obtain, for any $\alpha \in \mathbb{Z}_+^m$,

$$|h^{(\alpha)}(z) - h^{(\alpha)}(w)| \leq C|z-w|^{\sigma} \sum_{|\beta|=|\alpha|+1} \sup_K |h^{(\beta)}|\,.$$

Taking limits we derive, if $\mathbf{u} \in A_{n+1}(K;L)$ and $\mathsf{p}_{n+1}(\mathbf{u}) \leq 1$,

$$\mathsf{q}_n\left(\mathbf{u}(z) - \mathbf{u}(w)\right) \leq C|z-w|^{\sigma}\mathsf{p}_{n+1}(\mathbf{u}) \leq C|z-w|^{\sigma},\ \ z, w \in K$$

which shows that $\mathcal{B}$ is equicontinuous. For (2) it suffices to notice that, if $\mathbf{u} \in A^{\infty}(K;L)$ then $\mathsf{q}_n(\mathbf{u}(z_0)) \leq \mathsf{p}_n(\mathbf{u})$. The proof is complete. □

By the Arens-Michael representation it follows that $A^{\infty}(K;L)$ equals the projective limit of a compact sequence. According to Komatsu [9], $A^{\infty}(K;L)$ is then a separable Fréchet-Schwartz, Montel space. Moreover, its strong dual $A^{\infty}(K;L)'$ is then the inductive limit of a compact sequence

$$A^{\infty}(K;L)' = \varinjlim A_n(K;L)'.$$

and thus a **(DFS)** space.

3 Injectivity of the Borel Map and Open Mapping

3.1 The Open Mapping Property

We say that a compact and connected set L has the open mapping property at $p \in L$ if for every $h \in \mathcal{O}(L)$ which is not constant then the image $h(L)$ is a neighborhood of $h(p)$. There is a subtle relationship between unique continuation, open mapping, and maximum principles. Indeed, note that if the open mapping property holds, then the maximum principle holds at p in the sense that if $h \in \mathcal{O}(L)$ attains its maximum modulus over L at p, then h is necessarily constant.

Regarding the next result, stated concisely it says that if the strong unique continuation principle holds, then the same is true for the open mapping property.

Theorem 5 *Let $K \subset L \subset \mathbb{C}^m$, with K and L compact, $0 \in K$, and assume that L is connected and subanalytic. Assume also that $\mathfrak{b}_{K;L}$ is injective. If $h \in \mathcal{O}(L)$ vanishes at the origin and if L is not contained in the zero set of h then $h(L)$ is a neighborhood of the origin in the complex plane.*

Proof Suppose that $h(L)$ is not a neighborhood of the origin in $\mathbb{C}$. Since $h(L)$ is subanalytic we can apply the results of [10], which shows that the Borel map of $h(L)$ is surjective (and therefore, not injective). This shows that we can find a sequence $\{g_n\} \subset \mathbb{C}(z)$ of rational functions in $\mathbb{C}$, with poles outside $h(L)$, and a sequence $\{v_k\} \subset C(h(L))$ such that

$$g_n^{(k)}(0) \to 0, \quad \sup_{h(L)} |g_n^{(k)} - v_k| \to 0 \;\; \text{when} n \to \infty, \quad \forall k \in \mathbb{Z}_+$$

and $v_{k_0} \not\equiv 0$ for some $k_0 \in \mathbb{Z}_+$. We can assume $v_k \equiv 0$ if $0 \leq k < k_0$.

Since their poles lie outside $h(L)$, each g_n is an element in $\mathcal{O}(h(L))$. We then let $\{g_n \circ h\} \subset \mathcal{O}(L)$. Since $h(0) = 0$ the Faà di Bruno formula gives

$$\partial^\alpha (g_n \circ h)(0) \to 0, \forall \alpha \in \mathbb{Z}_+^m,$$

$$\partial^\alpha (g_n \circ h) \longrightarrow u_\alpha \text{ uniformly on } L, \; \forall \alpha \in \mathbb{Z}_+^m,$$

where $\mathbf{u} = \sum_\alpha u_\alpha X^\alpha \in C(L)[[X_1, \ldots, X_m]]$.

For any $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| = k_0$ we have

$$\partial^\alpha (g_n \circ h)|_L = \left[g_n^{(k_0)} \circ (h|_L) \right] (\nabla h)^\alpha |_L + \cdots$$

where the dots indicate terms that only contains derivatives of g_n of order $< k_0$. Taking the limit as $n \to \infty$ gives

$$\partial^\alpha (g_n \circ h)|_L \to \left[v_{k_0} \circ (h|_L) \right] (\nabla h)^\alpha |_L.$$

Now ∇h does not vanish identically in L for otherwise h would also do so, which contradicts our hypothesis (here we have used the connectedness and subanalyticity of

L). It follows that there is $j \in \{1, \ldots, m\}$ such that $\partial h/\partial z_j$ does not vanish identically in L. If we choose $\alpha = (0, \ldots, k_0, \ldots, 0)$, with k_0 in the position j, it follows that

$$u_\alpha = \left[v_{k_0} \circ (h|_L)\right] (\partial h/\partial z_j)^{k_0}|_L \not\equiv 0.$$

Hence $\mathbf{u} \neq 0$ and $\mathbf{u} \in \ker \mathfrak{b}_{K,L}$, which concludes the proof. □

Needless to add Theorem 1 is an immediate consequence of Theorem 5 .

3.2 The Maximum Principle

As pointed out before, the open mapping property implies in particular a version of the maximum principle.

Corollary 1 *Let $K \subset L \subset \mathbb{C}^m$, with K and L compact, $0 \in K$, and assume that L is connected and subanalytic. Assume also that $\mathfrak{b}_{K;L}$ is injective. If $h \in \mathcal{O}(L)$ attains its maximum at $0 \in K$, then h is constant on L.*

This version of the maximum principle is a direct consequence of Theorem 5.

4 Surjectivity of the Borel Map

4.1 Characterizing Surjectivity of the Borel Map

The non surjectivity of the Borel map at a point is characterized by the fact that a form of generalized (weak) Cauchy estimates hold for the elements of $A^\infty(K; L)$.

Theorem 6 *$\mathfrak{b}_{K,L}$ is not surjective if and only there are a sequence $\{P_j\} \subset \mathbb{C}[X_1, \ldots, X_m]$ and $M \in \mathbb{Z}_+$ such that degree $P_j \longrightarrow \infty$ and*

$$|(P_j(\partial)h)(0)| \leq \sum_{|\alpha| \leq M} \sup_K |\partial^\alpha h|,$$

for $h \in \mathcal{O}(L)$ and $j \in \mathbb{Z}_+$.

Proof Suppose that $\mathfrak{b}_{K,L}$ is not surjective. Since the image of $\mathfrak{b}_{K,L}$ contains $\mathbb{C}[X_1, \ldots, X_m]$, it is dense in $\mathbb{C}[\![X_1, \ldots, X_m]\!]$ and hence $\mathfrak{b}_{K,L}$ is not a homomorphism. The homomorphism theorem [11, p.18] therefore implies that the image of the transpose

$${}^t\mathfrak{b}_{K,L} : \mathbb{C}[X_1, \ldots, X_m] \longrightarrow A^\infty(K; L)'$$

is not strongly sequentially closed. It follows that there is a sequence $\{Q_k\} \subset \mathbb{C}[X_1, \ldots, X_m]$ such that $\Gamma = \{{}^t\mathfrak{b}_{K,L}(Q_k) : k \geq 1\}$ is strongly bounded and hence equicontinuous but $\{Q_k\}$ has no convergent subsequence.

Since $\mathbb{C}[X_1, \ldots, X_m]$ is a Montel space (with its usual LF-topology) it follows that $\{Q_j\}$ is not bounded. Notice that there is no $\ell \geq 0$ such that $\{Q_k\} \subset \mathbb{C}_{(\ell)}[X_1, \ldots, X_m]$, the space of polynomials of degree $\leq \ell$. Indeed since the latter is finite dimensional, ${}^t\mathfrak{b}_{K,L} : \mathbb{C}_{(\ell)}[X_1, \ldots, X_m] \to A^\infty(K, L)'$ is a homomorphism and the fact that Γ is strongly bounded would imply that $\{Q_k\}$ is bounded, a contradiction. In conclusion we can extract a subsequence $k_j < k_{j+1}$ such that the degree of Q_{k_j} is $> j$ for every j. Setting $P_j = Q_{k_j}$ proves the existence of the sequence $\{P_j\}$ with the required properties.

Conversely the existence of the sequence $\{P_j\}$ gives the existence of a weakly unbounded sequence in the dual of $\mathbb{C}[\![X_1, \ldots, X_m]\!]$ whose image by ${}^t\mathfrak{b}_{K,L}$ is a weakly bounded set in the dual of $A^\infty(K, L)$. Since ${}^t\mathfrak{b}_{K,L}$ is injective the map

$$\left({}^t\mathfrak{b}_{K,L}\right)^{-1} :{}^t \mathfrak{b}_{K,L}(\mathbb{C}[\![X_1, \ldots, X_m]\!]) \longrightarrow \mathbb{C}[\![X_1, \ldots, X_m]\!]$$

is well defined and not continuous for the weak topologies. Then $\mathfrak{b}_{K,L}$ is not a homomorphism and hence it is not surjective. □

4.2 Discs in the Hull

Denote by $z = (z_1, \ldots, z_m)$ the coordinates in $\mathbb{C}^m$. Of course, the coordinate functions $z_j \in A^\infty(K, L)$ for every $j = 1, \ldots, m$ and every L, and therefore (with the obvious abuse of notation) $z_j \in \mathfrak{b}_{K,L}(A^\infty(K, L))$. We recall that any homomorphism $\hat{\varphi}\colon \mathcal{F}_m \to \mathcal{F}_1$ can be regarded as a formal disc $\zeta \mapsto (\gamma_1(\zeta), \ldots, \gamma_m(\zeta))$, where $\gamma_j(\zeta) = \varphi(z_j) \in \mathbb{C}[\![\zeta]\!]$. Any formal disc γ therefore also induces a homomorphism $\varphi_\gamma\colon A^\infty(K, L) \to \mathcal{F}_1$, which factors as $\varphi_\gamma = \hat{\varphi}_\gamma \circ \mathfrak{b}_{K,L}$. Failure of surjectivity of $\mathfrak{b}_{K,L}$ can be detected by the presence of an *analytic* disc. We prove:

Proposition 4 *If there is a (non constant) analytic disc $\sigma : \Delta \to \mathbb{C}^m$ through the origin and contained in $\widehat{K}_L$ then $\mathfrak{b}_{K;L}$ is not surjective.*

Proof According to (1) it suffices to prove that $\sigma(\Delta)$ is a strong unique continuation set, and therefore contained in the intersection in (1), because then $\mathfrak{b}_{K,L}(A^\infty(K, L))$ is not a local algebra and hence $\mathfrak{b}_{K,L}(A^\infty(K, L)) \neq \mathcal{F}_m$. In order to prove our claim we observe that if $\mathbf{u} \in \ker \mathfrak{b}_{K;L}$ then there is a sequence $\{f_n\} \subset \mathcal{O}(L)$ such that $f_n \to u_0$ uniformly in $\widehat{K}_L$ and $(\partial^\alpha f_n)(0) \to 0$ for all α. In particular $(f_n \circ \sigma) \to u_0 \circ \sigma$ uniformly in Δ. It follows that $u_0 \circ \sigma$ is a holomorphic function in Δ which vanishes to infinite order at the origin. Then $u_0 \circ \sigma$ vanishes identically, and this concludes the proof. □

Example 7 If we consider the algebra $A^\infty(K)$ for $K = \overline{\Delta} \times [0, 1] \subset \mathbb{C}^2_{(z,w)}$, then K is polynomially convex, i.e. $K = \widehat{K}$. The disc $\zeta \mapsto (\zeta, 0)$ is contained in K, and the Borel map $\mathfrak{b}_K$ is not surjective; indeed, one can show that

$$\mathfrak{b}_K(A^\infty(K)) = A^\infty(\overline{\Delta})\widehat{\otimes}\mathbb{C}[\![w]\!].$$

Let $K \subset L \subset \mathbb{C}^m$, with both K and L compact and connected, as usual, and assume that the origin is contained in K. We then define the real convex cone

$$\mathcal{N}(K, L)_0 := \left\{f'(0)\colon f \in \mathcal{O}(L) \setminus \{0\}, \operatorname{Re} f|_K \leq 0\right\}.$$

We also define the notion of a *bad polynomial*:

Definition 1 Let $Q \in \mathbb{C}[z]$ be a homogeneous polynomial of degree N. We say that the polynomial Q is *bad* for the Borel map $\mathfrak{b}_{K,L}$ if the operator $h \mapsto (Q(\partial)h)(0)$ is bounded with respect to the norm p_{N-1}, i.e. if writing $Q(z) = \sum_{|\alpha|=N} Q_\alpha z^\alpha$, there exists a constant C such that

$$\left| \sum_{|\alpha|=N} Q_\alpha h^{(\alpha)}(0) \right| \leq C \mathsf{p}_{N-1}(h) = C \sum_{|\beta| \leq N-1} \sup_K |h^{(\beta)}|, \quad h \in \mathcal{O}(L). \tag{3}$$

Bad polynomials are restricted by directions in $\mathcal{N}(K, L)_0$:

Proposition 5 *If Q is a homogeneous polynomial which is bad for the Borel map, then $Q(v) = 0$ for every $v \in \mathcal{N}(K, L)_0$.*

Proof Since Q is bad, there are constants $N \in \mathbb{Z}_+$, $C > 0$ such that 3 holds. Let $v \in \mathcal{N}(K, L)_0$. Taking $\rho \geq 1$ and applying this inequality to $h(z) = e^{\rho f(z)}$, where $f \in \mathcal{O}(L)$ is chosen so that $f'(0) = v$, we obtain, for some constant $\tilde{C} > 0$,

$$|Q(\partial)h(0)| \leq \tilde{C} \rho^{N-1} \sup_{z \in K} e^{\rho \operatorname{Re} f(z)}.$$

Since

$$\partial_z^\alpha \left\{ e^{\rho f} \right\} = \rho^{|\alpha|} (f'(z))^\alpha + \mathrm{O}(\rho^{|\alpha|-1})$$

we obtain, for a new constant $A > 0$,

$$|Q(v)| \leq A/\rho, \quad \rho \geq 1.$$

The proof is complete. □

We observe that the same kind of result holds with starting terms of functions vanishing to degree k if the right hand side is controlled by p_{N-k}.

Corollary 2 *If $\mathcal{N}(K, L)_0$ contains a uniqueness set for holomorphic functions, then the Borel map $\mathfrak{b}_{K,L}$ is surjective. In particular, if L is convex and has more than m supporting real hyperplanes in general complex position at the origin, then $\mathfrak{b}_{K,L}$ is surjective.*

Proof By Theorem 6, if $\mathfrak{b}_{K,L}$ is not surjective, there exists a bad polynomial Q. By proposition 5, we have $Q(v) = 0$ for all $v \in \mathcal{N}(K, L)_0$. It follows that $\mathcal{N}(K, L)_0$ cannot be a uniqueness set, because $Q \neq 0$.

The second claim follows because the closed convex cone

$$\mathcal{C}(L)_0 = \{\zeta \in \mathbb{C}^m : \operatorname{Re}\langle z, \zeta \rangle \leq 0, \ \forall z \in L\}$$

of outer normals to supporting real hyperplanes of L at the origin satisfies $\mathcal{C}(L)_0 \subset \mathcal{N}(K, L)_0$, and contains a uniqueness set if and only if it contains an open piece of a maximally totally real subspace. □

The notion of bad polynomials connects nicely with discs in the hull of K. So, if $\gamma\colon \Delta \to \widehat{K}_L$ is a holomorphic disc (extending continuously to the closure), then by Cauchy estimates

$$|(f\circ\gamma)^{(k)}(0)| \le Ck!\|f\|_K \quad \forall f\in\mathcal{O}(L).$$

In particular, if we define the set

$$\mathcal{B}_j := \{P\colon P \text{ bad polynomial for the Borel map of degree } j\},$$

and the set of derivatives of holomorphic discs valued in L as above as

$$\mathcal{T}_1 := \left\{\gamma'(0)\colon \gamma \text{ holomorphic disc }, \gamma(\overline{\Delta})\subset L\right\},$$

then obviously $\mathcal{T}_1\subset\mathcal{B}_1$ (where we identify linear maps with vectors as usual). However, that is not all.

For any linear form $\ell(z)=\sum_{j=1}^m a_j z_j$ and $k\in\mathbb{N}$ we recall that its k-th symmetric power $\ell^{[k]}$ is defined as

$$\ell^{[k]}(z) := \sum_{|\alpha|=k}\alpha! a^\alpha z^\alpha.$$

Noting that

$$(f\circ\gamma)^{(k)}(0) = (\gamma'(0))^{[k]}(\partial)f(0)+\ldots,$$

with the dots denoting derivatives of lower order, we have

Lemma 1 *If γ is an analytic disc in L, then $(\gamma'(0))^{[k]}\in\mathcal{B}_k$ for every k.*

5 Polyhedra in $\mathbb{C}^m$

5.1 Preliminaries

In this section we assume that K is a (convex) polyhedron in $\mathbb{C}^m=\mathbb{R}^{2m}$ that is, K is the convex hull of a finite set of points in $\mathbb{C}^m$. We assume that the origin belongs to K. Recall that the dimension of K is the smallest number $p\le 2m$ such that K is contained in a real subspace $F\subset\mathbb{R}^{2m}$ of dimension p. The boundary ∂K (resp. interior $\operatorname{int}(K)$ of K) is by definition the boundary (resp. interior) of K as a topological subspace of F.

A vertex of K is a boundary point $A\in\partial K$ for which the intersection of all supporting hyperplanes for K through A reduces to $\{A\}$ (this is the standard definition of a vertex for a general convex set). If K is a polyhedron the vertices are precisely the extremal points of K (this property is not true for a general compact convex set). In particular by the Krein-Milman theorem a polyhedron is given by the convex hull of its vertices. We recall the following important result [12, p.18], which is valid for any compact convex set:

Proposition 6 *If* 0 *is a vertex of* K *then there are* $2m$ *linearly independent supporting hyperplanes planes for* K *through* 0.

As an easy consequence of this result we obtain:

Proposition 7 *If* 0 *is a vertex of* K *then* $\mathcal{C}(K)_0$ *has non empty interior in* $\mathbb{R}^{2m}$.

Proof By Proposition 1 there is a basis $\{\zeta_1, \ldots, \zeta_{2m}\}$ of $\mathbb{R}^{2m}$ such that $\mathfrak{H}(\zeta_j)$ are supporting hyperplanes for K at the origin, $j = 1, \ldots, 2m$. Multiplying each ζ_j by -1 if necessary we can assume $\zeta_j \in \mathcal{C}(K)_0$ for all j. But then the real convex cone spanned by $\{\zeta_1, \ldots, \zeta_{2m}\}$ is contained in $\mathcal{C}(K)_0$. Since the interior of this cone in $\mathbb{R}^{2m}$ is non empty we reach our conclusion. □

5.2 Faces

We now define the q-faces of K ($q \geq 1$) by descending induction on $q = p-1, \ldots, 1$. A $(p-1)$-face is a $(p-1)$-dimensional polyhedron $\mathfrak{a}$ contained in ∂K which is the convex hull of a set of vertices and which is maximal with respect to this property: any other $(p-1)$-dimensional polyhedron which is contained in ∂K, is equal to the convex hull of a finite set of vertices and if it contains $\mathfrak{a}$ must be equal to $\mathfrak{a}$. If $q \leq p-2$ a q-face of K is a q-face of a $q+1$-face of K. If $\mathfrak{a}$ is a q-face of K and if $E \subset \mathbb{R}^{2m}$ is the affine subspace of dimension q which contains $\mathfrak{a}$ then the *interior of* $\mathfrak{a}$ is the topological interior of $\mathfrak{a}$ in E.

A final remark which will be important in what follows is the following:

Remark 1 Suppose that the origin belongs to the interior of $\mathfrak{a}$, a q-face of K, $q \geq 1$, and let E be the real subspace of $\mathbb{C}^m$ which contains $\mathfrak{a}$. Then E is contained in any supporting hyperplane for K at the origin. In particular if $\mathfrak{H}(\zeta)$ is a supporting plane for K at the origin then a fortiori $\zeta \in E^{\perp}$

Proof Suppose that for some supporting hyperplane $\mathfrak{H}(\zeta)$ for K at the origin we have $E \not\subset \mathfrak{K}(\zeta)$. There is an open ball $B \subset E$ centered at the origin and contained in $\mathfrak{a}$. It follows that there is $w \in B$, $w \notin \mathfrak{H}(\zeta)$. Hence $\mathrm{Re}\langle w, \zeta\rangle \neq 0$. Changing w for $-w$ if necessary we can achieve $\mathrm{Re}\langle w, \zeta\rangle < 0 < \mathrm{Re}\langle -w, \zeta\rangle$, a contradiction for $\mathfrak{H}(\zeta)$ was assumed to be a supporting hyperplane for K at 0 and $w, -w \in \mathfrak{a} \subset K$. □

Proposition 4. *Assume that the origin belongs to the interior of* $\mathfrak{a}$, *a* q*-face of* K, $q \geq 1$, *and let* E *be the real subspace of* $\mathbb{C}^m$ *which contains* $\mathfrak{a}$. *Then* $\mathcal{C}(K)_0$ *is contained in* $E^{\perp}$ *and it has non empty interior in* $E^{\perp}$.

Proof That $\mathcal{C}(K)_0$ is contained in $E^{\perp}$ follows from the remark that precedes the statement. For the other statement we let $\Pi : \mathbb{C}^m \to E^{\perp}$ denote the orthogonal projection. It follows that $\Pi(K)$ is a polyhedron contained in $E^{\perp}$ for which the origin is now a vertex. By Proposition 3 it follows that

$$\{\zeta \in E^{\perp} : \mathrm{Re}\langle \Pi(z), \zeta\rangle \leq 0, \ \forall z \in K\} \tag{4}$$

has non empty interior in $E^\perp$. Since $\mathcal{C}(K)_0$ is contained in $E^\perp$ the set in (4) equals $\mathcal{C}(K)_0$ and the proof is complete. □

5.3 Surjectivity of the Borel Map on Polyhedra

We will now show that our conjectures 1 and 2 stated in the Introduction are verified for polyhedra.

Proof of Theorem 3 All the statements have already been proved with the exception of (2b). Notice firstly that if E is not totally real then $\mathfrak{a}$ will contain a complex line through the origin and thus $\mathfrak{b}_K$ cannot be surjective. On the other hand assume that E is totally real. Then we can assume that

$$E = \{z \in \mathbb{C}^m : \operatorname{Re} z_j = 0, \operatorname{Im} z_k = 0,\ j = q+1, \dots m,\ k = 1, \dots, m\}.$$

It follows that

$$E^\perp = \{z \in \mathbb{C}^m : \operatorname{Re} z_j = 0,\ j = 1, \dots, q\}$$

and hence the sets $\{w\}+i\mathbb{R}^m$, $w \in \mathbb{R}^{m-r}$, which are uniqueness sets, foliate $E^\perp$. Since $\mathcal{C}(K)_0$ has non empty interior in $E^\perp$, it contains an uniqueness set which, according to Corollary 1, completes the proof. □

5.4 The Image of the Borel Map on Polyhedra

Even if $\mathfrak{b}_K$ is not surjective, the methods above can be employed to find the image of $\mathfrak{b}_K$, or at least the image of $\mathfrak{b}_{K\cap U}$ for some small enough neighborhood U of 0 in $\mathbb{C}^m$. To that end, we will need the following lemma:

Lemma 2 *Assume that $0 \in K$ belongs to the interior of $\mathfrak{a}$, a q-face of K, $q \geq 1$, and that E is the real subspace of $\mathbb{C}^m$ which contains $\mathfrak{a}$. Let $F \subset \mathbb{C}^m$ be a subspace containing $E^\perp$, and denote by $\pi : \mathbb{C}^m \to F$ the associated orthogonal projection. Then $\pi(0)$ belongs to the interior of a face $\mathfrak{b}$ of the polyhedron $\widetilde{K} = \pi(K) \subset F$ whose tangent space is $E \cap F$.*

Proof Let $k = \dim_{\mathbb{R}} E$ and $\ell = \dim_{\mathbb{R}} F$. Since $E^\perp \subset F$, we have that $F^\perp \subset E$ and so $\pi(E) = E \cap F$ is a subspace of real dimension $d = 2m - (\ell + k)$. Furthermore, $\pi^{-1}(\pi(E)) = E$, and since $\mathfrak{a} = K \cap E$, we have $(\pi|_K)^{-1}(\pi(\mathfrak{a})) = \mathfrak{a}$.

Since by assumption $\mathfrak{a}$ contains a neighborhood U of 0 in E, $\pi(\mathfrak{a})$ contains a neighborhood of $\pi(0)$ in $E \cap F$. It follows that $\pi(0)$ is contained in the interior of a (uniquely determined) face $\mathfrak{b}$ of $\widetilde{K} = \pi(K)$ of dimension at least d.

The conclusion will follow from the fact that the dimension of $\mathfrak{b}$ is exactly d, i.e. $E\cap F$ contains a neighborhood of $\pi(0)$ in $\mathfrak{b}$. Otherwise, there would exist $p_1', p_2' \in \mathfrak{b}\setminus E$ such that $\pi(0)$ belongs to the interior of the segment $I = [p_1', p_2']$. Choose $p_1, p_2 \in K$ such that $\pi(p_1) = p_1', \pi(p_2) = p_2'$, and let $J = [p_1, p_2]$: then $\pi(J) = I$, and there exists q in the interior of J such that $\pi(q) = \pi(0)$. Using the fact that

$(\pi|_K)^{-1}(\pi(\mathfrak{a})) = \mathfrak{a}$ we deduce that $q \in \mathfrak{a}$, but then the segment $J \subset K$ meets the face $\mathfrak{a}$ at an interior point, which implies that $J \subset \mathfrak{a}$. It follows that $p_1, p_2 \in \mathfrak{a}$ and $\pi(p_1), \pi(p_2) \in \pi(\mathfrak{a}) \subset E$, contradicting the choice of p_1', p_2'. □

Again, let $K, 0 \in \partial K, E$ be as above, and let $E^c = E \cap iE$ be the largest complex subspace of E, $\dim_{\mathbb{C}} E^c = \ell > 0$. Up to a complex linear change of coordinates we can assume that E^c is spanned by $\frac{\partial}{\partial z_j}$, $1 \le j \le \ell$, so that $(E^c)^\perp$ is spanned by $\frac{\partial}{\partial z_k}$, $\ell + 1 \le j \le m$.

We now define the set $A_\infty^{(2)}(K)$ as the preimage of $\mathbb{C}[\![z_{\ell+1}, \dots, z_m]\!] \subset \mathbb{C}[\![z_1, \dots, z_m]\!]$ by $\mathfrak{b}_K$, and denote the restriction of $\mathfrak{b}_K$ to $A_\infty^{(2)}(K)$ by $\mathfrak{b}_K^{(2)}$ (in agreement with the definition of partial Borel maps given in [4, Section 4]).

Proposition 8 *The partial Borel map* $\mathfrak{b}_K^{(2)} : A_\infty^{(2)}(K) \to \mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!]$ *is surjective.*

Proof Let $\pi : \mathbb{C}^m \to (E^c)^\perp$ denote the orthogonal projection. Then $\widetilde{K} = \pi(K)$ is a polyhedron contained in $(E^c)^\perp \cong \mathbb{C}^{m-\ell}$. Since $(E^c)^\perp$ contains the orthogonal space $E^\perp$ to the face $\mathfrak{a}$, by Lemma 2 the tangent space to the face $\mathfrak{b}$ of $\widetilde{K}$ containing $\pi(0)$ in its interior coincides with $E \cap (E^c)^\perp$. By construction $E \cap (E^c)^\perp$ is totally real, hence by Theorem 3 the Borel map $\mathfrak{b}_{\widetilde{K},\widetilde{K}} : A^\infty(\widetilde{K}) \to \mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!]$ is surjective.

On the other hand, the (complex linear) projection π induces the pull-back $\pi^* : A^\infty(\widetilde{K}) \to A^\infty(K)$ by $\pi^*(f) = f \circ \pi$, which commutes with the Borel maps:

$$\pi^* \mathfrak{b}_{\widetilde{K}} f = \mathfrak{b}_K \pi^* f = \mathfrak{b}_K^{(2)} \pi^* f$$

(where $\pi^* : \mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!] \to \mathbb{C}[\![z_1, \dots, z_m]\!]$ is just the inclusion map). Since $\mathfrak{b}_{\widetilde{K}} f$ is surjective, it follows that the restriction of $\mathfrak{b}_K^{(2)}$ to the subspace $\pi^*(A^\infty(\widetilde{K})) \subset A^\infty(K)$ is surjective. □

For any $R > 0$, let K_R be the intersection of K with the polydisc $P(0, R)$. We define the subspaces $A_\infty^{(1)}(K_R)$, $A_\infty^{(2)}(K_R)$ and the maps $\mathfrak{b}_{K_R}^{(1)}$, $\mathfrak{b}_{K_R}^{(2)}$ in the same way as before.

Corollary 3 *If* $R > 0$ *is small enough, then*

$$\mathfrak{b}(A^\infty(K_R)) = (\mathbb{C}_R\{z_1, \dots, z_\ell\}) \widehat{\otimes} (\mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!])$$

where $\mathbb{C}_R\{z_1, \dots, z_\ell\}$ *denotes the space of power series convergent on the polydisc* $P(0, R) \cap \mathbb{C}^\ell$.

Proof By proposition 8 (passing through the restriction map $A_\infty^{(2)}(K) \to A_\infty^{(2)}(K_R)$) we have that $\mathfrak{b}_{K_R}^{(2)} : A_\infty^{(2)}(K_R) \to \mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!]$ is surjective. Furthermore $\mathfrak{b}_{K_R}^{(1)} : A_\infty^{(1)}(K_R) \to \mathbb{C}_R\{z_1, \dots, z_\ell\}$ is surjective by construction (any holomorphic function of $P(0, R) \cap \mathbb{C}^\ell$ can be extended to $P(0, R)$ by composing with the projection π). By [13, Theorems 6.5,6.6] we conclude that $\mathfrak{b}(A^\infty(K_R))$ contains $(\mathbb{C}_R\{z_1, \dots, z_\ell\}) \widehat{\otimes} (\mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!])$. Since the opposite inclusion is clear, we conclude that $\mathfrak{b}(A^\infty(K_R)) = (\mathbb{C}_R\{z_1, \dots, z_\ell\}) \widehat{\otimes} (\mathbb{C}[\![z_{\ell+1}, \dots, \mathbb{C}_m]\!])$. □

6 Convex Subanalytic Sets

Let $K \subset \mathbb{C}^n$ be a convex subanalytic set, and let $0 \in bK$. We denote by $\mathfrak{b}_0$ the Borel map defined on *germs at* 0 of elements of $A^\infty(K)$.

By proposition 4, it is clear that $\mathfrak{b}_0$ is not surjective in case K contains a germ of complex line through 0. Our main task in this section is to prove the opposite implication: we will prove the surjectivity of $\mathfrak{b}_0$ rather directly, constructing a germ in $A^\infty(K)$ realizing a given power series by "elementary" means.

For a large, even integer $d > 0$, and $C > 0$ small, we will use repeatedly the following function $\ell_{d,C} : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$:

$$\ell_{d,C}(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ Ct^d & \text{if } t < 0 \end{cases}.$$

Note that with this definition, we have that

$$\left\{(s,t) \in \mathbb{R}^2 : s \geq \ell_{d,C}(t)\right\} = \left\{(s,t) \in \mathbb{R}^2 : s \geq 0, t \geq -\sqrt[d]{\frac{s}{C}}\right\}.$$

We summarize some facts about the function $\ell_{d,C}(t)$ in the following lemma:

Lemma 3 *The function $\ell_{d,C}$ satisfies the following properties:*

(i) *For every $d \leq d'$, and $C > 0$, $R > 0$, there exists $C' > 0$ such that $\ell_{d',C'} \leq \ell_{d,C}$ if $|t| < R$.*
(ii) *$\ell_{d,C}(-\ell_{d',C'}(t)) = \ell_{C'C^d,dd'}(t)$, in particular $\ell_{d,C}(-\ell_{d,C}(t)) = \ell_{C^{d+1},d^2}(t)$;*
(iii) *for any fixed $a \geq 0$ we have*

$$\ell_{d,C}(a - \ell_{d,C}(t)) = \begin{cases} C(a - Ct^d)^d & \text{for } t \leq \sqrt[d]{\frac{a}{C}} \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *for any fixed $a \leq 0$ we have*

$$\ell_{d,C}(a - \ell_{d,C}(t)) = C(a - \ell_{d,C}(t))^d.$$

The following lemma will later be used iteratively in the proof of Theorem 4. In order to formulate it, for any $c > 0$, as well as $a_1, \ldots, a_j \in \mathbb{R}$, and an even integer $d \geq 4$ we define

$$\mathscr{L}(t) = \ell_{d,C}(a_j - \ell_{d,C}(a_{j-1} - \ldots - \ell_{d,C}(a_1 - t)\ldots)).$$

Lemma 4 *There exists a constant A_j (only depending on j) with the following property. Whenever $|a_1|, \ldots, |a_j|, |\delta|, |c|, |C| \leq A_j$, then the upper level set*

$$\{s \geq \mathscr{L}(\ell_{d,C}(t - \delta)) - c\}$$

is contained in the complement of the disc of center $(s,t)=(-1,0)$ *and radius*

$$\frac{2}{(2-\mathscr{L}(C(\delta/2)^d))(1+2c)}.$$

Proof First of all, we observe that for $|a_1|,\ldots,|a_j|,|C|$ bounded by a constant, $\mathscr{L}$ can be seen as the composition of functions which are uniformly Lipschitz on a common neighborhood of 0, hence we can write $|\mathscr{L}((\delta/2)^d)-\mathscr{L}(0)|\le B_j(\delta/2)^d$ for a constant $B_j>0$ depending only on j and small δ. Now, since the function $\mathscr{L}(\ell_{d,C}(t-\delta))-c$ is decreasing we have

$$\mathscr{L}(\ell_{d,C}(t-\delta))-c\ge\mathscr{L}(\ell_{d,C}(-\delta/2))-c=\mathscr{L}(C(\delta/2)^d)-c \quad\text{for all}\quad t\le\delta/2,$$

and since $\mathscr{L}(\ell_{d,C}(t-\delta))$ is non negative this implies

$$\mathscr{L}(\ell_{d,C}(t-\delta))-c\ge\mathscr{L}(C(\delta/2)^d)\chi_{[-\frac{\delta}{2},\frac{\delta}{2}]}-c$$

for all $t\in\mathbb{R}$ (where χ indicates the characteristic function). This means that

$$\left\{s\ge\mathscr{L}(\ell_{d,C}(t-\delta))-c\right\}\subset\left\{(s,t)\colon s\ge\mathscr{L}\left(C\left(\frac{\delta}{2}\right)^d\right)-c,\,|t|<\frac{\delta}{2}\right\}$$
$$\cup\left\{(s,t)\colon s\ge -c,\,|t|\ge\frac{\delta}{2}\right\}.$$

Next, we note that the intersection of the disc of center $(-1,0)$ and radius $(1+\mathscr{L}(C(\delta/2)^d))/(1+2c)$ with $\{s\ge\mathscr{L}(0)-c\}$ is contained in the rectangle $[-\frac{\delta}{2},\frac{\delta}{2}]\times[\mathscr{L}(0)-c,\mathscr{L}(C(\delta/2)^d)-c]$. This follows because the distance of $(\frac{\delta}{2},\mathscr{L}(0)-c)$ to $(0,-1)$ is larger than $(1+\mathscr{L}(C(\delta/2)^d))/(1+2c)$:

$$\begin{aligned}
(1+2c)^2\cdot\ &\left(\left(\frac{\delta}{2}\right)^2+(\mathscr{L}(0)+1-c)^2\right)\\
&=\left(\left(\frac{\delta}{2}\right)^2+(\mathscr{L}(0)+1)^2\right)(1+2c)^2+(c^2-2(1+\mathscr{L}(0))c)(1+2c)^2\\
&>\left(\left(\frac{\delta}{2}\right)^2+(\mathscr{L}(0)+1)^2\right)+c(4-3(1+\mathscr{L}(0))\\
&>\left(\frac{\delta}{2}\right)^2+(\mathscr{L}(0)+1)^2\\
&>\left(\frac{\delta}{2}\right)^2+\left(\mathscr{L}(C(\delta/2)^d)+1-B_j\left(\frac{\delta}{2}\right)^d\right)^2\\
&>\left(\frac{\delta}{2}\right)^2-2B_j\left(\mathscr{L}(C(\delta/2)^d)+1\right)\left(\frac{\delta}{2}\right)^d+\left(\mathscr{L}(C(\delta/2)^d)+1\right)^2
\end{aligned}$$

$$> \left(\mathscr{L}(C(\delta/2)^d) + 1\right)^2,$$

where in the 4th line we used that $0 \leq \mathscr{L}(0) < \frac{1}{3}$ if A_j is chosen small enough. Finally, the claim follows from the fact that $2/(2 - \mathscr{L}(C(\delta/2)^d)) < 1 + \mathscr{L}(C(\delta/2)^d)$ for δ small enough. □

Corollary 4 *With the same notation as in Lemma 4, the upper level set*

$$\{s \geq \mathscr{L}(\ell_{d,C}(t - \delta))\}$$

is also contained in the complement of the disc of center $(0, 0)$ *and radius* $\mathscr{L}(C(\delta/2)^d)$*, provided* $|a_1|, \ldots, |a_j|, |\delta|$ *are bounded by a (possibly smaller) constant only depending on* j.

Proof Following the argument in the previous lemma, we have the upper level set considered in the claim is contained in the complement of the following set:

$$\left(\left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \times [0, \mathscr{L}(C(\delta/2)^d)]\right) \cup (\mathbb{R} \times (0, \mathscr{L}(0))).$$

For $|a_1|, \ldots, |a_j|, |\delta|, |C|$ small enough we can write

$$\left(\frac{\delta}{2}\right)^2 \geq 2B_j\mathscr{L}(0)C\left(\frac{\delta}{2}\right)^d + B_j^2C^2\left(\frac{\delta}{2}\right)^{2d}$$

therefore

$$\mathscr{L}(0)^2 + (\delta/2)^2 \geq (\mathscr{L}(0) + B_jC(\delta/2)^d)^2 > \mathscr{L}(C(\delta/2)^d)^2$$

and thus the distance of the origin from the point $(\delta/2, \mathscr{L}(0))$ is larger than $\mathscr{L}(C(\delta/2)^d)$. □

We will use the subanalyticity and convexity of the sets in the next construction via the following lemma, which is a consequence of the Lojasiewicz inequality:

Lemma 5 *Let* A *be a compact subanalytic subset of* $\mathbb{R}^{m+1}$ *(on which we take coordinates* (t, s)*,* $s \in \mathbb{R}$*,* $t \in \mathbb{R}^m$*). Suppose that* $A \subset \{s \geq 0\}$*, and furthermore* $A \cap \{s = 0\} \subset \{f(t) \geq 0\}$*, where* f *is a continuous subanalytic function on* $\mathbb{R}^m$*. Then there exist* $d, C > 0$*, with* d *a (large) even integer and* C *a (small) real constant such that*

$$A \subset \{s \geq \ell_{d,C}(f(t))\}.$$

Proof We follow the argument given in [8, Remark 3.11]. Define the function $g(t) = \min_{(t,r)\in A} |r|$, and let B be the set

$$B = \{(t, r, s) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} : (t, r) \in A,\ s \geq |r|\}.$$

Define π by $\pi(t, r, s) = (t, s)$. Then the set B is subanalytic (as the intersection of subanalytic sets) and $\pi|_B$ is proper, and so the projection $\pi(B) \subset \mathbb{R}^m \times \mathbb{R}$ is also subanalytic. By definition $\pi(B) = \{(s, t) \in \mathbb{R}^m \times \mathbb{R} : s \geq g(t)\}$. Applying Theorem 3.10 in [8] it follows that $\{s < g(t)\}$, and hence $\{s \leq g(t)\}$ are subanalytic sets. By intersection, the graph $\{s = g(t)\} \subset \mathbb{R}^m \times \mathbb{R}$ of g is a subanalytic set, which means that g is a subanalytic function.

The assumption on A implies that $g(t) > 0$ whenever $t \in \{f > 0\}$. By the Lojasiewicz inequality and since A is compact, for a large enough exponent d and a small enough constant C it follows that $g(t) \geq f(t)^d$ whenever $f(t) > 0$, that is, $g(t) \geq \max\{Cf(t)^d, 0\} = \ell_{d,C}(f(t))$. The claim of the Lemma then follows from the fact that, by construction, $A \subset \{s \geq g(t)\}$. □

In order to use the main geometric assumption of Theorem 4 (namely, that K does not contain complex lines through 0) we need to state the following elementary fact of complex linear algebra. We use $(z_1, \ldots, z_k)$ as variables in $\mathbb{C}^k$ and write $z_j = x_j + i y_j$, with $x_j, y_j \in \mathbb{R}$ the real and imaginary part of z_j, respectively.

Lemma 6 *For fixed $\ell < k \in \mathbb{N}$, let $H \subset \mathbb{C}^k$ be the generic real subspace defined as*

$$H = \{(z_1, \ldots, z_k) \in \mathbb{C}^k : x_{\ell+1} = \ldots = x_k = 0\},$$

with real dimension $k + \ell$ and CR dimension ℓ. Let $H' \subset H$ be a real hyperplane in H. Then one and only one of the following cases holds:

Case 1: the CR dimension of H' is again ℓ, and after a complex linear change of variables in $\mathbb{C}^k$ the planes H, H' can be written as

$$H = \{x_{\ell+1} = \ldots = x_k = 0\},\ \ H' = \{x_{\ell+1} = \ldots = x_k = y_k = 0\},$$

and H' is a generic subspace of $\mathbb{C}^{k-1}_{(z_1,\ldots,z_{k-1})}$.

Case 2: the CR dimension of H' is $\ell - 1$, and after a complex linear change of variables in $\mathbb{C}^k$ the planes H, H' can be written as

$$H = \{x_{\ell+1} = \ldots = x_k = 0\},\ \ H' = \{x_\ell = x_{\ell+1} = \ldots = x_k = 0\}$$

and H' is again a generic subspace of $\mathbb{C}^k$.

Lemma 7 *Let $K \subset \mathbb{C}^n$ be convex, $0 \in K$. If K does not contain any germ of complex line passing through 0, then there exists $1 \leq j_0 \leq n$, complex linear coordinates $(z_1, \ldots, z_n)$, and a decreasing sequence of real subspaces $H_j \subset \mathbb{C}^n$ (where $j_0 \leq j \leq 2n - 1$) with the following properties:*

(a) *H_j is j-dimensional and $H_{j-1} \subset H_j$ for all j;*
(b) *For every j, there exist $1 \leq m_1^j \leq m_2^j \leq n$, such that $m_1^j + m_2^j = 2n - j$ and*

$$H_j = \left\{x_k = y_\ell = 0,\quad m_1^j \leq k \leq n, m_2^j < \ell \leq n\right\}$$

(c) *H_{j-1} is a support hyperplane at 0 for the convex set $K \cap H_j$. More precisely, we have either $K \cap H_j \subset \{x_{m_1^j - 1} \geq 0\}$ or $K \cap H_j \subset \{y_{m_2^j} \geq 0\}$;*

(d) *H_{j_0} is a totally real subspace, spanned by $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{j_0}}$.*

Proof By assumption, 0 does not belong to the interior of K, hence there exists a supporting hyperplane $H_{2n-1} \subset \mathbb{C}^n$ for K at 0. We set $m_1^{2n-1} = n-1$ and $m_2^{2n-1} = n$ and choose the z_n coordinate so that $H_{2n-1} = \{x_n = 0\}$ and $K \subset \{x_n \geq 0\}$.

Now, suppose inductively that a sequence of subspaces H_j has been constructed for $j' \leq j \leq 2n-1$, satisfying conditions (a), (b), (c) above; in particular,

$$H_{j'} = \{x_k = y_\ell = 0 : m_1^{j'} \leq k \leq n,\ m_2^{j'} < \ell \leq n\} \tag{5}$$

for certain $1 \leq m_1^{j'} \leq m_2^{j'} \leq n$. If $H_{j'}$ is totally real (which is equivalent to $m_1 = 1$) then (d) is also satisfied and we are done. Otherwise, $H_{j'}$ can be seen as a generic subspace of $\mathbb{C}^{m_2^{j'}}$,

$$H_{j'} = H_{j'} \cap \mathbb{C}^{m_2} = \left\{x_k = 0 : m_1^{j'} \leq k \leq m_2^{j'}\right\}$$

of positive CR dimension $m_1^{j'} - 1$ (possibly $H_{j'} = \mathbb{C}^{m_2 - 1}$, in case $m_1 = m_2$). Thus 0 does not belong to the (relative) interior of the convex set $K \cap H_{j'} \subset H_{j'}$, or K would contain a germ of complex space of positive dimension passing through 0. Hence there exists a supporting hyperplane $H_{j'-1} \subset H_{j'}$ for $K \cap H_{j'}$ at 0. We can apply Lemma 6 to $H_{j'}, H_{j'-1} \subset \mathbb{C}^{m_2}$ and conclude that, after a linear change of variables involving $z_1, \ldots, z_{m_2}$, either

$$H_{j'-1} = \{x_k = y_\ell = 0 : m_1^{j'} - 1 \leq k \leq n,\ m_2^{j'} < \ell \leq n\}$$

or

$$H_{j'-1} = \{x_k = y_\ell = 0 : m_1^{j'} \leq i_1 \leq n,\ m_2^{j'} - 1 < i_2 \leq n\}$$

(note that if $m_1^{j'} - 1 = m_2^{j'}$ the former happens, since then we are necessarily in Case 2 of Lemma 6). Since $H_{j'-1}$ is again of the form given in (5), we can iterate the procedure until we find a totally real space. □

Based on the previous lemma, we can fix a sequence of variables $w_{2n-1}, \ldots, w_{j_0}$ such that the nested supporting hyperplanes of the lemma are defined by $H_j = \{w_{2n-1} = \ldots = w_j = 0\}$. Furthermore each w_j is one of the coordinate variables x_k, y_ℓ (listed in descending order), with the conditions that all the x_ℓ variables appear in the list, and furthermore each subsequence $w_{2n-1}, \ldots, w_j$ contains at least as many x_k as y_ℓ.

Example 1 Possible allowed sequences for convex sets in $\mathbb{C}^4$ are $(x_4, x_3, x_2, y_4, y_3, y_2, x_1)$ and $(x_4, x_3, y_4, x_2, y_3, y_2, x_1)$, but not $(x_4, y_4, y_3, x_3, x_2, x_1)$.

Using Lemma 5 iteratively we now obtain that K is contained in a model " enveloping set"

Lemma 8 *For all sufficiently large d and small enough C, it holds that*

$$K \subset \left\{w_{2n-1} \geq \ell_{d,C}(w_{2n-2} - \ell_{d,C}(\ldots - \ell_{d,C}(w_{j_0+1} - \ell_{d,C}(w_{j_0}))))\right\}.$$

We discuss this result with the help of our examples in the context of $\mathbb{C}^4$.

Example 2 With the sequences of the previous examples we would have

$$K \subset \left\{x_4 \geq \ell_{d,C}(x_3 - \ell_{d,C}(x_2 - \ell_{d,C}(y_4 - \ell_{d,C}(y_3 - \ell_{d,C}(y_2 - \ell_{d,C}(x_1))))))\right\}$$

respectively

$$K \subset \left\{x_4 \geq \ell_{d,C}(x_3 - \ell_{d,C}(y_4 - \ell_{d,C}(x_2 - \ell_{d,C}(y_3 - \ell_{d,C}(y_2 - \ell_{d,C}(x_1))))))\right\}.$$

Note that Lemma 3 ii) shows that the intersection of the enveloping set with $\mathbb{C}^k$ ($k \leq n$) is of the form

$$\begin{aligned} M \cap \mathbb{C}^k &= \left\{x_k \geq \ell_{d_{j_k},C_{j_k}}(w_{j_k} - \ell_{d_{j_k-1},C_{j_k-1}}(\ldots \ell_d(w_{j_0})))\right\} \\ &= \left\{x_k \geq \mathscr{L}_k(w_{j_k}, \ldots, w_{j_0})\right\} \end{aligned}$$

where the variable y_k may or may not appear in the sequence, and the integers $d_{j_k}, d_{j_k-1}, \ldots$ are bounded by d^n (from above), and the constants C_{j_k} by a certain power of C from below; furthermore, for any of the intersections, one actually needs to adjust the sequence only if one sets the y_k equal to 0. The following examples illustrate that fact.

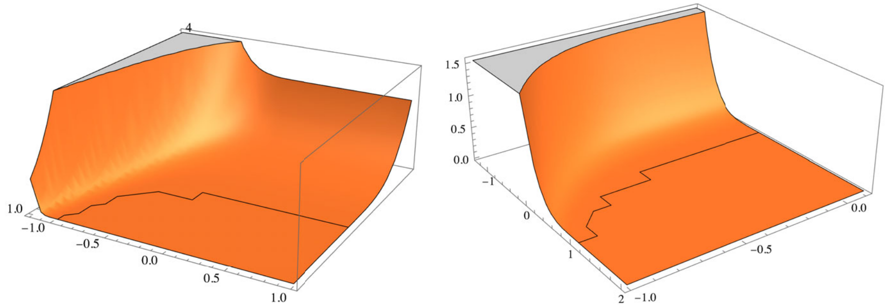
Example 3 In the case of the sequence $(x_4, x_3, x_2, y_4, y_3, y_2, x_1)$ from the previous example we find

$$\begin{aligned} K \cap \mathbb{C}^3 &= \{x_3 \geq \ell_d(x_2 - \ell_{d^2,C^{d+1}}(y_3 - \ell_{d,C}(y_2 - \ell_{d,C}(x_1))))\} \\ K \cap \mathbb{C}^2 &= \{x_2 \geq \ell_{d^3,C^{d^2+d+1}}(y_2 - \ell_{d,C}(x_1))\} \\ K \cap \mathbb{C} &= \{x_1 \geq 0\}. \end{aligned}$$

For the other sequence, $(x_4, x_3, y_4, x_2, y_3, y_2, x_1)$, we have

$$\begin{aligned} K \cap \mathbb{C}^3 &= \{x_3 \geq \ell_{d^2,C^{d+1}}(x_2 - \ell_{d,C}(y_3 - \ell_{d,C}(y_2 - \ell_{d,C}(x_1))))\} \\ K \cap \mathbb{C}^2 &= \{x_2 \geq \ell_{d^2,C^{d+1}}(y_2 - \ell_{d,C}(x_1))\} \\ K \cap \mathbb{C} &= \{x_1 \geq 0\}. \end{aligned}$$

Example 8 Intuitively, the graphs of $\mathscr{L}$ are obtained by sliding the basic graph of $\ell_{d,C}$ along figures obtained iteratively in exactly that way. As an illustration, here are plots of the function $\ell_{2,1}(x - \ell_{2,1}(y))$.



For $n \geq k \geq 1$, we will define further

$$\mathcal{K}_k = \ell_{(d_k')^2,C_k}(w_{j_k} - \ell_{(d_k')^2,C_k}(\ldots - \ell_{(d_k')^2,C_k}(0 - \ell_{(d_k')^2,C_k}(\ldots - \ell_{(d_k')^2,C_k}(w_{j_0}))))))$$

for a sequence of integers d_k' satisfying $d_n' \geq d$ and $d_k' \geq (d_{k+1}')^2$, $k = 1, \ldots, n-1$ and an appropriate choice of C_k's; here the 0 might appear depending on whether y_k is part of the sequence $(w_{j_k}, \ldots, w_{j_0})$ or not.

Example 4 In the examples above, this would give

$$\begin{aligned} \mathcal{K}_3 &= \ell_{(d_3')^2,C_3}(x_2 - \ell_{(d_3')^2,C_3}(y_2 - \ell_{(d_3')^2,C_3}(x_1))) \\ \mathcal{K}_2 &= \ell_{(d_2')^2,C_2}(x_1) \\ \mathcal{K}_1 &= 0 \end{aligned}$$

for both of the sequences.

We will prove the surjectivity of the Borel map via an explicit construction, based on the following sequence of auxiliary sets $\Omega_0^k, \Omega_1^k \subset \mathbb{C}^k$, $\Delta^{k+1} \subset \mathbb{C}^{k+1}$:

$$\begin{aligned} \Omega_0^k &= \{(z_1, \ldots, z_k) \in \mathbb{C}^k : x_k \geq -1, |z_k + 1| \geq \mathcal{K}_k\}, \\ \Omega_1^k &= \{(z_1, \ldots, z_k) \in \mathbb{C}^k : |z_k - 1| \leq 2 - \mathcal{K}_k\}, \\ \Delta^{k+1} &= \{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : x_{k+1} \geq 0, |z_k - 1| \leq (1 + \sqrt[d']{|z_{k+1}|})(2 - \mathcal{K}_k)\}. \end{aligned}$$

The main motivation behind the definition of Ω_1^k, Δ^{k+1} lies in the following extension lemma:

Lemma 9 *There exists a compact polydisc U centered at $(0, 1, 0)$ such that the restriction map $A^\infty(\Delta^{k+1} \cap U) \to A^\infty(\Omega_1^k \cap U)$ is surjective. More precisely, for all $f \in A^\infty(\Omega_1^k \cap U)$ there exists $F \in A^\infty(\Delta^{k+1} \cap U)$ such that $F|_{\Omega_1^k \cap U} = f$ and $\frac{\partial^j F}{\partial z_{k+1}^j}|_{\Omega_1^k \cap U} = 0$ for all $j \geq 1$.*

Proof By applying a complex affine transformation in the z_k variable, we will assume that Ω_1^k and Δ^{k+1} are expressed more conveniently as follows (so that the sections with complex lines in the z_k-direction are discs centered at 0)

$$\Omega_1^k = \{(z_1, \ldots, z_k) \in \mathbb{C}^k : |z_k| \le 1 - \mathcal{K}_k\},$$
$$\Delta^{k+1} = \{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : x_{k+1} \ge 0, |z_k| \le (1 + \sqrt[d]{|z_{k+1}|})(1 - \mathcal{K}_k)\}.$$

We will write $z' = (z_1, \ldots, z_{k-1})$ for notational convenience and leave out the intersections with U from the notation for readability.

Let $f \in A^\infty(\Omega_1^k)$. According to Example 5 in Sect. 2.A we can interpret f as an element in $\mathcal{O}_\bullet(\Omega_1^k)$ which, furthermore, belongs to the closure of the image of the homomorphism $\mathcal{O}(\mathbb{C}^k) \to \mathcal{O}_\bullet(\Omega_1^k)$ induced by restriction. Our first step is to express f as a power series in z_k with coefficients holomorphic in z',

$$f(z', z_k) = \sum_{j \ge 0} a_j(z') z_k^j$$

where the series is convergent for $|z_k| < 1 - \mathcal{K}_k$. Since f is smooth up to the boundary of Ω_1^k, for any fixed $s, r \in \mathbb{N}$ there exists a constant $A_{s,r} > 0$ such that, for all multiindices α with $|\alpha| \le r$,

$$\left| \frac{\partial^\alpha a_j(z')}{\partial z'^\alpha} \right| \le \frac{A_{s,r}}{j^s} \frac{1}{(1 - \mathcal{K}_k)^j}.$$

Consider now the sequence of functions $\varphi_j(z) = \varphi_j(z_1, z)$ constructed in the proof of [4, Proposition 8.17] (note that the functions constructed there were independent of z_1). This sequence depends only on one parameter, called k_0 in [4]: we will take $k_0 = d$. By [4, Lemma 8.21], there exists $\tau' > 0$ such that for all $m \ge 1$ there is $N_m > 0$ such that

$$\left| \frac{\partial^m \varphi_j}{\partial z^m}(z) \right| \le N_m j^{3dm} \frac{1}{(1 + \sqrt[d]{|z|})^j}$$

for all $x \ge 0$, $|z| \le \tau'$ and all $j \in \mathbb{N}$. We define a function $F(z', z_k, z_{k+1})$ as

$$F(z', z_k, z_{k+1}) = \sum_{j \ge 0} a_j(z') z_k^j \varphi_j(z_{k+1})$$

where the series is convergent for $|z_k| < (1 - \mathcal{K}_k)(1 + \sqrt[d]{|z_{k+1}|})$, and actually converges uniformly on compact sets of Δ^{k+1}. This defines F as a holomorphic function in the interior of Δ^{k+1}. The same proof as in [4] now shows that for all fixed $m \ge 1$, $s \in \mathbb{N}$ and multiindex α, we have that

$$\sup_{\Omega_c} \left| \frac{\partial^{|\alpha|+s+m} F}{\partial z'^\alpha \partial z_k^s \partial z_{k+1}^m} \right| \to 0$$

as $c \to 0$, where $\Omega_c = \Omega \cap \{z_{k+1} = c\}$. This implies that F extends smoothly up to Ω_1^k and is flat with respect to z_{k+1}. Furthermore F belongs to the closure of the image of the homomorphism $\mathcal{O}(\mathbb{C}^{k+1}) \to \mathcal{O}_\bullet(\Delta^{k+1})$ since the analogous property holds for f and the φ_j can be approximated by polynomials by Mergelyan's theorem.

We can restrict ourselves in the rest of the proof to $|\alpha| = 0$, $s = 0$ since we can treat the higher order derivatives by replacing F with $\frac{\partial^{|\alpha|+s} F}{\partial z'^\alpha \partial z_k^s}$. So let us fix $1 \le m \in \mathbb{N}$. We choose $A_{3dm+2} > 0$ such that

$$|a_j(z')| \le \frac{A_{3dm+2}}{j^{3dm+2}} \frac{1}{(1 - \mathcal{L}_d(z'))^j}$$

for all $j \in \mathbb{N}$. Given $\epsilon > 0$, let $j_0 \in \mathbb{N}$ such that $N_m A_{3dm+2} \sum_{j>j_0} \frac{1}{j^2} < \epsilon$. For any $p = (z', z_k, z_{k+1}) \in \Delta^{k+1}$ we get

$$\begin{aligned}
\left|\frac{\partial^m F}{\partial z_{k+1}^m}(p)\right| &= \left|\sum_{j\le j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1}) + \sum_{j>j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})\right| \\
&\le \left|\sum_{j\le j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})\right| + \sum_{j>j_0} |a_j(z')||z_k|^j \left|\frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})\right| \\
&\le \left|\sum_{j\le j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})\right| \\
&\quad + \sum_{j>j_0} \frac{A_{3dm+2}}{j^{3dm+2}} N_m j^{3dm} \left(\frac{|z_k|}{(1-\mathcal{K}_k)(1+\sqrt[d]{|z_{k+1}|})}\right)^j \\
&\le \left|\sum_{j\le j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})\right| + \epsilon
\end{aligned}$$

where we used the fact that $|z_k| \le (1 - \mathcal{K}_k)(1 + \sqrt[d]{|z_{k+1}|})$ on Δ^{k+1}. Since $\sum_{j\le j_0} a_j(z') z_k^j \frac{\partial^m \varphi_j}{\partial z_{k+1}^m}(z_{k+1})$ is a finite sum and $\partial^m \varphi_j / \partial z^m$ is flat at 0 for all $j \in \mathbb{N}$, $m \ge 1$, we conclude that $\left|\frac{\partial^m F}{\partial z_{k+1}^m}(p)\right| < 2\epsilon$ for $|z_{k+1}|$ small enough. □

It will also be useful to introduce the following sets, obtained from Ω_1^k, Δ^{k+1} through a Cayley transform in the z_k variable:

$$\widetilde{\Omega}_1^k = \left\{(z_1, \ldots, z_k) \in \mathbb{C}^k : |z_k + 1| \ge \frac{2}{2 - \mathcal{K}_k}\right\},$$

$$\widetilde{\Delta}^{k+1} = \left\{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : x_{k+1} \ge 0, |z_k + 1| \ge \frac{2}{(2 - \mathcal{K}_k)(1 + \sqrt[d']{|z_{k+1}|})}\right\}.$$

Proof of Theorem 4 We use the results on partial Borel maps proved in [4]: we will show that, for any $j = 1, \dots, n$, there exists an element of $A^\infty(K)$ which has an arbitrarily prescribed expansion in the z_j variable and is flat at 0 in the other variables. Thus, the partial Borel maps are all surjective at 0, which in turn implies that $\mathfrak{b}_K$ is surjective. We will need to restrict ourselves to (small, compact convex subanalytic) neighbourhoods $K \cap U$ of the origin finitely many times, and tacitly denote the shrunken $K \cap U$ by K again.

Observe first that, for all $1 \le j \le n$, the set $K \cap \mathbb{C}^j$ is contained in the half-space $\{x_j \ge 0\} \subset \mathbb{C}^j$. For any fixed power series in the variable z_j, we can find a function in $A^\infty(\{x_j \ge 0\})$ realizing that power series, and constant with respect to the variables $z_1, \dots, z_{j-1}$. By restriction, we have a function with the same properties in $A^\infty(K \cap \mathbb{C}^j)$. The claim about the partial Borel maps is verified if we show that this function can be extended to an element of $A^\infty(K)$ with a flat expansion in the variables $z_{j+1}, \dots, z_n$. In order to achieve the extension we will work with the sequence of sets Ω_1^k defined above. We will carry out the construction explicitly for $j = 1$, as the same process can be used for $j > 1$ (by starting directly with Ω_1^j instead of Ω_1^1).

We begin with an arbitrary formal power series at $z_1 = 0$. By using the Cayley transform

$$\Phi(z_1) = \frac{z_1 - i}{z_1 + i}$$

we can define a function $f_1 \in A^\infty(\Omega_1^1) = A^\infty(\{|z_1 - 1| \le 2\})$, realizing the corresponding Taylor expansion expansion (the original one, composed with the Cayley transform) at $z_1 = -1$. By Lemma 9, we can extend f_1 to an element $f_2 \in A^\infty(\Delta^2)$ (flat in the z_2 variable) and composing with the inverse of the Cayley transform $\Phi(z_1)$ we can pull back f_2 to an element (still denoted as f_2) of $A^\infty(\widetilde{\Delta}^2)$. Next, we use Lemma 4 with $2c = \sqrt[d_1']{|z_2|}$ (and without the variables a_j) to deduce that, for small enough $\varepsilon > 0$, the set $\widetilde{\Delta}^2 \cap \{|x_1| < \varepsilon\}$ contains the set

$$\begin{aligned}\left\{x_2 \ge 0, x_1 \ge -\sqrt[d_1']{|z_2|}/2\right\} \cap \{|x_1| < \varepsilon\} &= \left\{x_2 \ge 0, |z_2| \ge \ell_{d_1', C_1}(2x_1)\right\} \cap \{|x_1| < \varepsilon\} \\ &\supset \left\{x_2 \ge 0, |z_2| \ge \ell_{(d_2')^2, C_2}(x_1), |x_1| < \varepsilon\right\}\end{aligned}$$

and thus we can restrict f_2 to an element of the latter. After another translation in the z_2 plane we can actually assume that f_2 is defined on the set

$$\begin{aligned}&\left\{(z_1, z_2) \in \mathbb{C}^2 : x_2 \ge -1, |z_2 + 1| \ge \ell_{(d_2')^2, C_2}(x_1), |x_1| < \varepsilon\right\} \\ &\qquad \supset \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_2 - 1| = 2 - \ell_{(d_2')^2}(x_1), |x_1| < \varepsilon\right\} = \Omega_1^2\end{aligned}$$

(note that $\mathcal{K}_2 = \ell_{(d_2')^2}(x_1)$). We have thus achieved an extension (modulo some change of variables and restrictions) of $f_1 \in A^\infty(\Omega_1^1)$ to $f_2 \in A^\infty(\Omega_1^2)$.

Assume now by induction that the extension to a function $f_k \in \mathcal{A}^\infty(\Omega_1^k)$ has been obtained for a certain $k < n$. Again using Lemma 9, we extend f_k to an element

$f_{k+1} \in A^\infty(\Delta^{k+1})$ (flat in the z_{k+1} variable) which can be identified (via a pull-back by the Cayley transform $\Phi(z_k)$) with an element $f_{k+1} \in A^\infty(\widetilde{\Delta}^{k+1})$. We then use Lemma 4 with $(s,t) = (x_k, y_k)$, $d = d'_k$, and $2c = \sqrt[d'_k]{|z_{k+1}|}$, with the a_j corresponding to the subset of the variables $w_{j_k}, w_{j_k-1}, \ldots$ appearing to the left of y_k in the expression of $\mathcal{L}_k$ (taken to be the empty set if y_k does not appear) and

$$\left(\frac{\delta}{2}\right)^{d'_k} = \ell_{(d'_k)^2, C_k}(w_{j'_k} - \ldots - \ell_{(d'_k)^2, C_k}(w_{j_0}))$$

where $w_{j'_k}$ is the first variable appearing to the right of y_k in $\mathcal{L}_k$ (we take $j'_k = j_k$ in case y_k does not appear). Let us write, for the sake of notational simplicity,

$$\mathcal{K}_k\left(w_{j_k}, \ldots y_k, w_{j'_k}, \ldots, w_{j_0}\right) = \hat{\mathcal{K}}_k\left(w_{j_k}, \ldots y_k - \tilde{\mathcal{K}}_k\left(w_{j'_k}, \ldots, w_{j_0}\right)\right).$$

With that notation we deduce that, for small enough $|z_1|, \ldots, |z_k|$, the set $\widetilde{\Delta}^{k+1}$ is contained in

$$\begin{aligned} &\left\{x_{k+1} \geq 0, x_k \geq \hat{\mathcal{K}}_k\left(w_{j_k}, \ldots, y_k - 2\tilde{\mathcal{K}}_k\left(w_{j'_k}, \ldots, w_{j_0}\right)\right) - \frac{(|z_{k+1}|)^{\frac{1}{d'_k}}}{2}\right\} \\ &= \left\{x_{k+1} \geq 0, |z_{k+1}| \geq 2^{d'_k} \ell_{d'_k, C_k}\left(x_k - \hat{\mathcal{K}}_k\left(w_{j_k}, \ldots, y_k - 2\tilde{\mathcal{K}}_k\left(w_{j'_k}, \ldots, w_{j_0}\right)\right)\right)\right\} \\ &\supset \left\{x_{k+1} \geq 0, |z_{k+1}| \geq \mathcal{K}_{k+1}(x_k, \ldots, w_{j_0})\right\}. \end{aligned}$$

Just as before, composing with a translation in the z_{k+1} variable we can take f_{k+1} to be defined on

$$\begin{aligned} \Omega_0^{k+1} &= \{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : x_{k+1} \geq -1, |z_{k+1}| \geq \mathcal{K}_{k+1}\} \\ &\supset \{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : |z_{k+1}| \leq 2 - \mathcal{K}_{k+1}\} \\ &\supset \Omega_1^{k+1} \end{aligned}$$

which concludes the inductive step. Finally, we use Corollary 4 in the same way as Lemma 4 to show that f_n defines an element of

$$A^\infty(\{w_{2n-1} \geq \ell_d(w_{2n-2} - \ell_d(\ldots - \ell_d(w_{j_0+1} - \ell_d(w_{j_0}))))\})$$

and by our choices and Lemma 8 thus an element of $A^\infty(K)$. □

Example 5 We will now follow explicitly the steps of the previous proof, as applied to the examples introduced before. For simplicity, we assume that throughout the constructions, we can choose $C = 1$ and drop it from the notation (this happens if the sets involved are "small", anyway). In the first step the sets Ω_1, $\widetilde{\Omega}_1^1$, Δ^2, $\widetilde{\Delta}^2$ are described as follows, independently of the model:

$$\Omega_1^1 = \{z_1 \in \mathbb{C} : |z_1 - 1| \leq 2\}, \quad \widetilde{\Omega}_1^1 = \{z_1 \in \mathbb{C} : |z_1 + 1| \geq 1\}$$

$$\Delta^2 = \{(z_1, z_2) \in \mathbb{C}^2 : x_2 \geq 0, |z_1 - 1| \leq 2(1 + \sqrt[d_1']{|z_{k+1}|})\},$$

$$\widetilde{\Delta}^2 = \left\{(z_1, z_2) \in \mathbb{C}^2 : x_2 \geq 0, |z_1 + 1| \geq \frac{1}{(1 + \sqrt[d_1']{|z_2|})}\right\}.$$

In the first step the function $f_1 \in A^\infty(\Omega_1^1)$ is extended to a function f_2 defined on Δ^2 and, after a Cayley transform, on $\widetilde{\Delta}^2$. We can then restrict f_2 to the following sets

$$\{x_2 \geq 0, x_1 \geq -\sqrt[d_1']{|z_2|}/2\} = \{x_2 \geq 0, |z_2| \geq \ell_{d_1'}(2x_1)\} \supset \{x_2 \geq 0, |z_2| \geq \ell_{(d_2')^2}(x_1)\}$$

and after a translation, to

$$\Omega_0^2 = \{x_2 \geq -1, |z_2+1| \geq \ell_{(d_2')^2}(x_1)\} \supset \{|z_2 - 1| \leq 2 - \mathcal{K}_2 = 2 - \ell_{(d_2')^2}(x_1)\} = \Omega_1^2.$$

Next, we apply the same procedure to extend the function $f_2 \in A^\infty(\Omega_1^2)$ to a function f_3 defined on

$$\widetilde{\Delta}^3 = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, |z_2 + 1| \geq \frac{2}{(2 - \ell_{(d_2')^2}(x_1))(1 + \sqrt[d_2']{|z_3|})}\right\}.$$

We use Lemma 4 with $(s, t) = (x_2, y_2)$, $2c = \sqrt[d_2']{|z_3|}$, $(k/2)^{d_2'} = \ell_{(d_2')^2}(x_1)$ (hence $k = 2\ell_{d_2'}(x_1)$) to obtain that $\widetilde{\Delta}^3$ includes

$$\{x_3 \geq 0, x_2 \geq \ell_{d_2'}(y_2 - 2\ell_{d_2'}(x_1)) - \sqrt[d_2']{|z_3|}/2\} =$$

$$= \{x_3 \geq 0, |z_3| \geq 2^{d_2'}\ell_{d_2'}(x_2 - \ell_{d_2'}(y_2 - 2\ell_{d_2'}(x_1)))\} \supset$$

$$\supset \{x_3 \geq 0, |z_3| \geq \ell_{(d_3')^2}(x_2 - \ell_{(d_3')^2}(y_2 - \ell_{(d_3')^2}(x_1)))\}$$

again, for the last inclusion to be valid we require $|z_1|, |z_2|$ (but not $|z_3|$) to be small enough. The last set is equivalent Ω_0^3, up to a translation in the z_3 plane. We can thus restrict f_3 to

$$\begin{aligned}\Omega_1^3 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_3 - 1| \leq 2 - \mathcal{K}_3 \\ &= 2 - \ell_{(d_3')^2}(x_2 - \ell_{(d_3')^2}(y_2 - \ell_{(d_3')^2}(x_1)))\},\end{aligned}$$

and then, repeating the process, extend it to a function f_4 defined on

$$\begin{aligned}\widetilde{\Delta}^4 &= \Big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : x_4 \geq 0, |z_3 + 1| \\ &\geq \frac{2}{(2 - \ell_{(d_3')^2}(x_2 - \ell_{(d_3')^2}(y_2 - \ell_{(d_3')^2}(x_1))))(1 + \sqrt[d_3']{|z_4|})}\Big\}.\end{aligned}$$

We use Lemma 4 with $(s,t)=(x_3,y_3)$, $2c=\sqrt[d_3']{|z_4|}$, $a_1=x_2$, $(k/2)^{d_3'}=\ell_{(d_3')^2}(y_2-\ell_{(d_3')^2}(x_1))$ (hence $k=2\ell_{d_3'}(y_2-\ell_{(d_3')^2}(x_1))$) to obtain that $\widetilde{\Delta}^4$ includes

$$\{x_4\geq 0, x_3\geq \ell_{(d_3')^2}(x_2-\ell_{(d_3')^2}(y_3-2\ell_{d_3'}(y_2-\ell_{(d_3')^2}(x_1))))-\sqrt[d_3']{|z_4|}/2\}=$$

$$=\{x_4\geq 0, |z_4|\geq 2^{d_3'}\ell_{d_3'}(x_3-\ell_{(d_3')^2}(x_2-\ell_{(d_3')^2}(y_3-2\ell_{d_3'}(y_2-\ell_{(d_3')^2}(x_1)))))\}\supset$$

$$\supset\{x_4\geq 0, |z_4|\geq \ell_{(d_4')^2}(x_3-\ell_{(d_4')^2}(x_2-\ell_{(d_4')^2}(y_3-\ell_{(d_4')^2}(y_2-\ell_{(d_4')^2}(x_1)))))\}.$$

As a last step we apply Corollary 4 with $(s,t)=(x_4,y_4)$, and different choices of the a_j (in one case, $a_2=x_3$, $a_1=x_2$, in the other case just $a_1=x_3$) to conclude that f_4 can be locally restricted to the enveloping models in Example 2.

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