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A NOTE ON THE MINIMAL IMMERSIONS OF THE TWO-SPHERE

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§1. *Introduction*

Let (S^2, g) be the 2-sphere S^2 with a Riemannian metric g of Gaussian curvature K and let $S^N(1) \subset \mathbf{R}^{N+1}$ be the standard N-dimensional unit sphere. In [KS], M.Kozlowski and U.Simon proved the following theorem:

Theorem A. Let the curvature K of (S^2, g) satisfy $1/6 \leq K \leq 1/3$. If $f: S^2 \to S^N(1)$ is an isometric minimal immersion, then either $K \equiv 1/3$ and $f = \Psi_{2,2}$, or $K \equiv 1/6$ and $f = \Psi_{2,3}$.

Here $\Psi_{2,s}$ is the unique, modulo congruences, full minimal immersion of the constantly curved 2-sphere $S^2(K(s))$ into $S^{2s}(1)$, where $K(s) = 2/s(s+1)$ (cf. [Ca] and [doCW]). Theorem A is an affirmative answer, for s=2, to the following general conjecture of U.Simon:

Conjecture. Suppose that the curvature K of (S^2, g) satisfies $K(s+1) \leq K \leq K(s)$. If $f: S^2 \to S^N(1)$ is an isometric minimal immersion, then either $K \equiv K(s)$ and $f = \Psi_{2,s}$, or $K \equiv K(s+1)$ and $f = \Psi_{2,s+1}$.

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The proof of this conjecture for s=1 is not difficult and it is known since 1979; see the Introduction of [KS].

The purpose of this note is to use the ideas and methods developed by S.S.Chern in [Ch] to give an alternative proof of Theorem A and a proof of the above conjecture for general s, but with an extra hypothesis when $s \geq 3$. Once one has some familiarity with Chern's paper, our proof becomes elementary. Before going to it, we will derive in §2 below a few straightforward, but interesting formulas that can be viewed as a complement to [Ch]. Most of the proof also works when $S^N(1)$ is replaced by any Riemannian manifold $\bar{M}^N_c$ of constant positive curvature c, so we can state our result as:

Theorem B. Suppose that the curvature K of (S^2, g) satisfies $2c/(s+1)(s+2) \leq K \leq 2c/s(s+1)$ and let $f: S^2 \to \bar{M}^N_c$ be a full isometric minimal immersion. We have

(a) If s=2, then either $K \equiv c/3$ and N=4, or $K \equiv c/6$ and N=6.

(b) If s>2, assume further that f is (s-1)-generic. Then either $K \equiv 2c/s(s+1)$ and N=2s, or $K \equiv 2c/(s+1)(s+2)$ and N=2+2s.

The hypothesis "f is (s-1)-generic" means that $\dim N^m(p)=2$ for every p in S^2 and $0 \leq m \leq s-1$, where $N^m(p)$ is the m^{th} normal space of S^2 at p. As we shall see in §3, this extra hypothesis is automatically satisfied when s=2. Obviously Theorem A follows from part (a) of Theorem B.

§2. *Review of Chern's results.*

For a general setting, we consider an isometric immersion $f: M \to \tilde{M}^N_c$ of a connected 2-dimensional Riemannian manifold M into $\tilde{M}^N_c$. To each $p \in M$, let N_pM denote the normal space of M at p, that is, N_pM is the orthogonal complement of the tangent space T_pM in $T_p\tilde{M}$. The second fundamental form of M at p is the symmetric bilinear tensor B^1_p: $T_pM \times T_pM \to N_pM$ given by

$$(2.1) \qquad B^1_p(X_1,X_2) = (\tilde{\nabla}_{\tilde{X}_1}\tilde{X}_2)^{\perp}(p),$$

where $\tilde{\nabla}$ is the Riemannian connection of $\tilde{M}$, $\tilde{X}_j$ are local fields which extend X_j and $\perp$ means projection on N_pM. Let us call $\bar{B}^1 = B^1$. Suppose $n > 1$ and $\bar{B}^k$ already defined for $k = 1,\dots,n-1$. Then we define

$$(2.2) \qquad \bar{B}^n(X_1,\dots,X_{n+1}) = (\nabla_{X_{n+1}}\bar{B}^{n-1})(X_1,\dots,X_n)$$
$$= \nabla^{\perp}_{X_{n+1}}\bar{B}^{n-1}(X_1,\dots,X_n) - \sum_{k=1}^{n}\bar{B}^{n-1}(X_1,\dots,\nabla_{X_{n+1}}X_k,\dots,X_n),$$

where ∇ is the Riemannian connection of M and $\nabla^{\perp}$ is the connection of the normal bundle of M.

Let $Osc^m M(p) \subset T_p\tilde{M}$ denote the <u>m^{th} osculating space of M at</u> p ([S], p.240). We know that $Osc^1 M(p) = T_pM$ and $Osc^m M(p) \subseteq Osc^{m+1}M(p)$, $m \geq 1$. We now define the <u>m^{th} normal space</u> $N^m(p)$ <u>of</u> M <u>at</u> p by the relation

$$(2.3) \qquad Osc^{m+1}M(p) = Osc^m M(p) \oplus N^m(p).$$

If we agree that $Osc^0 M(p) = \{0\}$, then $N^0(p) = T_pM$. The <u>$(m+1)^{th}$ fundamental form</u> B^m_p <u>of</u> M <u>at</u> p is given by

$$(2.4) \qquad B^m_p: T_pM \times \dots \times T_pM \longrightarrow (Osc^m M(p))^{\perp} \subset N_pM$$
$$(X_1,\dots,X_{m+1}) \longmapsto T^m(\bar{B}^m(X_1,\dots,X_{m+1})),$$

where $T^m: T_p\bar{M} \to (Osc^m M(p))^\perp$ is the orthogonal projection. It is well known that each $N^m(p)$ is spanned by the image of B_p^m and that B_p^m is a symmetric (m+1)-linear tensor ([S], pp. 242-244).

Assume now that the immersion is minimal. Then, given a local orthonormal frame (e_1,e_2) on M, it is easy to see that

(2.5) $$B^m(e_{i_1},\dots,e_{i_{m+1}}) = \pm B^m(e_1,\dots,e_1) \text{ or } \pm B^m(e_1,\dots,e_1,e_2),$$

for any $m \geq 1$ and $i_1,\dots,i_{m+1} = 1,2$. Thus $\dim N^m(p) \leq 2$ for every p in M. We say that p is a <u>s-generic point</u> if $\dim N^m(p) = 2$ for $m = 1,\dots,s$; we say that the <u>immersion is s-generic</u> if every point of M is s-generic. Also if $\{X,Y\}$ is any orthonormal tangent basis with, say, $X = \cos t\, e_1 + \operatorname{sen} t\, e_2$, then by induction on m we have

(2.6) $$B^m(X,\dots,X) = \cos(m+1)t\; B^m(e_1,\dots,e_1) + \sin(m+1)t\; B^m(e_1,\dots,e_1,e_2),$$

$$B^m(X,\dots,X,Y) = -\sin(m+1)t\; B^m(e_1,\dots,e_1) + \cos(m+1)t\; B^m(e_1,\dots,e_1,e_2).$$

Suppose from now on that f is minimal and that M is (homeomorphic to) the 2-sphere S^2. Let $z = x_1 + ix_2$ be local isothermal coordinates on S^2, with coordinate vectors $X_j = \partial/\partial x_j$, j=1,2. It follows from [Ch] that

(2.7) $$\Phi^m(z) = [(\|B^m(X_1,\dots,X_1)\|^2 - \|B^m(X_1,\dots,X_1,X_2)\|^2)$$

$m \geq 1$, is a globally well defined Abelian form of degree $2(m+1)$ on S^2. Thus $\phi^m(z) \equiv 0$ on S^2. From this and (2.6) we easily get that

$$\begin{aligned} &\| B^m(e_1,\dots,e_1) \| = \| B^m(e_1,\dots,e_1,e_2) \| = r_m \geq 0, \\ &\langle B^m(e_1,\dots,e_1),\ B^m(e_1,\dots,e_1,e_2) \rangle = 0 \end{aligned} \tag{2.8}$$

holds for any local orthonormal frame (e_1,e_2) on M and r_m does not depend on the frame. Then $p \in S^2$ is a s-generic point iff $r_m(p) > 0$ for $m=1,\dots,s$. Chern also defines certain local invariants given by quantities $k_m \geq 0$, $m \geq 1$, which satisfy $k_m \equiv 0$ or $k_m = 0$ only at isolated points of S^2. An inspection on Chern's paper shows that at the regular points we have

$$k_m = r_m/r_{m-1}, \quad m \geq 1, \tag{2.9}$$

where $r_o = 1$ is the ratius of the unit tangent circle. Thus

(2.10) For each r_m, we have $r_m \equiv 0$ or $r_m = 0$ only at isolated points of S^2.

If $r_m \equiv 0$ for some m, then clearly $r_n \equiv 0$ for all $n \geq m$ and, using well known results on the reduction of codimension (see §1 of [D]), it follows that $f(S^2) \subset \tilde{M}_c^{2m}$ for some totally geodesic submanifold $\tilde{M}_c^{2m}$ of $\tilde{M}_c^N$. Thus there exists a maximal $n \geq 0$ for which $r_n \not\equiv 0$ and $r_{n+1} \equiv 0$. In this case, $f(S^2) \subset \tilde{M}_c^{2+2n}$ and the (minimal) immersion $f: S^2 \to M_c^{2+2n}$ is full, that is, we cannot reduce the codimension to a totally geodesic or totally umbilical submanifold of $\tilde{M}_c^{2+2n}$.

(2.11) Let m be such that $r_m \neq 0$. Then, even at the isolated points p of S^2 where $r_m(p) = 0$, the "normal space of order m" is well defined and varies continously with the point. This property of r_m is an easy consequence of an analogous property of k_m; see [Ch], p. 35.

Now we restric to full minimal immersions $f: S^2 \longrightarrow \bar{M}^N_C$. Then N must be even, say $N = 2+2n$, where $n \geq 0$. By (2.10) the set Z of singular points of r_m, $1 \leq m \leq n$, is at most isolated and $S^2 - Z$ is open. Observe that k_m, k_m^2, r_m may be differentiable only on $S^2 - Z$ while r_m^2 is everywhere differentiable.

Let us recall the following formula of [Ch], p.38, which holds on $S^2 - Z$:

(2.12) $$\Delta \log(k_1 \ldots k_s) = (s+1)K - 2k_s^2 + 2k_{s+1}^2 \,,\quad 1 \leq s \leq n,$$

where $k_{n+1} \equiv 0$. By bringing (2.9) into (2.12), we get on S^2-Z:

(2.13) $$\Delta \log(r_s) = (s+1)K - 2r_s^2/r_{s-1}^2 + 2r_{s+1}^2/r_s^2 \,,\quad 1 \leq s \leq n.$$

Then on S^2- Z we have

(2.14) $$\Delta r_s = r_s[(s+1)K - 2r_s^2/r_{s-1}^2 + 2r_{s+1}^2/r_s^2] + \frac{1}{r_s} \| \operatorname{grad} r_s \|^2$$

Using the well known formula for the Laplacian

(2.15) $$\Delta(gh) = g\Delta h + h\Delta g + 2 < \operatorname{grad} g, \operatorname{grad} h >,$$

it follows easily from (2.14) that

$$(2.16) \qquad \Delta(r_s^2) = 4\|\text{grad } r_s\|^2 + 2r_s^2[(s+1)K - 2r_s^2/r_{s-1}^2] + 4r_{s+1}^2.$$

By the repeated use of (2.15), (2.16) and the Gauss Equation $K = c - 2r_1^2$, we obtain <u>on S^2</u>:

$$(2.17) \qquad \Delta(r_1^2 \dots r_s^2) = 4\|\text{grad}(r_1 \dots r_s)\|^2 + 2r_1^2 \dots r_s^2 \left[\frac{(s+1)(s+2)}{2}K - c\right] + 2r_1^2 \dots r_{s-1}^2 r_{s+1}^2 .$$

We will need one more formula, which follows directly from (2.13) and holds only on $S^2 - Z$:

$$(2.18) \qquad \Delta \log(r_1 \dots r_s) = \left[\frac{(s+1)(s+2)}{2} K - c\right] + 2r_{s+1}^2/r_s^2.$$

§3. *Proof of Theorem B.*

Assume that $s=2$ and $c/6 \leq K \leq c/3$. By the Gauss Equation we get $2c/3 \leq 2r_1^2 \leq 5c/6$. In particular $r_1 > 0$ on S^2 and the immersion is at least 1 - generic, that is, the immersion is automatically $(s-1)$-generic if $s=2$. With this in mind, we will assume in the following that $s \geq 2$ and f is $(s-1)$-generic, and we will proceed to the proof of parts (a) and (b) of Theorem B simultaneously.

Since $K \geq 2c/(s+1)(s+2)$, we have $\Delta(r_1^2 \dots r_s^2) \geq 0$ on S^2, by (2.17). Then $\Delta(r_1^2 \dots r_s^2) \equiv 0$ so each summand of the right hand side of (2.17) must be $\equiv 0$. We also must have $r_j > 0$ on S^2 for $1 \leq j \leq s-1$, by the $(s-1)$-genericity of the immersion. Then $2r_1^2 \dots r_{s-1}^2 r_{s+1}^2 \equiv 0$ implies $r_{s+1} \equiv 0$, so $2s \leq N \leq 2+2s$ by

(2.10). If $N = 2+2s$, again by (2.10) we cannot have $r_s \equiv 0$. Then $r_1, \ldots, r_{s-1}$ are all positive, $r_s = 0$ only at isolated points and

$$2r_1^2 \ldots r_s^2 \left[\frac{(s+1)(s+2)}{2} K-c\right] \equiv 0.$$

Thus $K \equiv 2c/(s+1)(s+2)$ in case $N = 2+2s$. Now if $N = 2s$ then $r_s \equiv 0$ and what we have is a $(s-1)$-generic minimal immersion with $K \leq 2c/s(s+1)$. Then $Z = \emptyset$ and by (2.18),

$$\Delta \log (r_1 \ldots r_{s-1}) = \left[\frac{s(s+1)}{2} K-c\right] \leq 0$$

holds on S^2. By integration $\Delta \log(r_1 \ldots r_{s-1}) \equiv 0$. that is, $K \equiv 2c/s(s+1)$ if $N = 2s$. This completes the proof of Theorem B. □

A consequence of the above proof is

(3.1) *Corollary*. <u>There is no minimal</u> S^2 <u>into</u> $\tilde{M}_c^4$ <u>with</u> $K < c/3$ <u>everywhere</u>.

For if there exists such a S^2, we would have $r_1 > 0$ on S^2 and then $\Delta \log(r_1) = 3K-c < 0$ everywhere, which is impossible.

§4. *Final Comments*.

(4.1) Suppose that we are under the hypothesis of Theorem B with $s=3$, that is, with $c/10 \leq K \leq c/6$, but this time <u>we do not assume that f is 2-generic</u>. Then $r_1 > 0$ and $r_2 \not\equiv 0$ by Corollary (3.1). Using (2.17) with $s=3$, we obtain $r_4 \equiv 0$ so that $6 \leq N \leq 8$. If $N=8$ we conclude, as in §3, that $K \equiv c/10$. If $N=6$ what remains is a 1-generic full minimal S^2 into $\tilde{M}_c^6$ with $c/10 \leq K \leq c/6$. On $S^2 - Z$ we still have

$$\Delta\log(r_1r_2) = 6K - c \leq 0$$

but we cannot integrate $\Delta\log(r_1r_2)$ on S^2 to conclude that $6K - c \equiv 0$ because Z may happen to be nonempty, that is, r_2 may be zero somewhere. This is why our proof fails in this case.

(4.2) For a full minimal S^2 into $\tilde{M}_c^{2+2n}$ we have that the area A of S^2 satisfies $A \geq 2\pi(n+1)(n+2)/c$, with equality if and only if $Z = \emptyset$; this is proved in [A], for instance. Let us restrict to the case $\tilde{M}_c^N = S^N(1)$. Thus for a full minimal S^2 into $S^6(1)$ with $1/10 \leq K \leq 1/6$ we have $24\pi \leq A \leq 40\pi$ and $A = 24\pi$ iff $r_2 > 0$ everywhere. So a counterexample to Simon's conjecture for s=3 would have $24\pi < A \leq 40\pi$. For a minimal S^2 into $S^{2+2n}(1)$, Barbosa [B] proved that A is an integer multiple of 4π and constructed examples with $A = 4\pi k$, for nay pair k,n of integers satisfying $4\pi k \geq 2\pi(n+1)(n+2)$. Then there do exist examples of minimal S^2 into $S^6(1)$ with $24\pi < A \leq 40\pi$, more precisely, with $A = 28\pi, 32\pi, 36\pi, 40\pi$. And for each such S^2 we must have $r_2 = 0$ somewhere. The question is: Does any of these examples satisfy $1/10 \leq K \leq 1/6$? If yes, we then have a counter-example to Simon's conjecture. We were not able to estimate the curvature of Barbosa's examples but we suspect that no one of them satisfy $1/10 \leq K \leq 1/6$, when $n=2$ and $A > 24\pi$.

(4.3) In [A] we make a more extensive study of the compact generic minimal M^2 into $\tilde{M}_c^N$. Our main conclusions are: (a) If $N = 2 + 2n$ $(n \geq 0)$ is even, then M^2 is homeomorphic to a 2-sphere and Area $(M^2) = 2\pi(n+1)(n+2)/c$; (b) If $N = 2n + 1$ $(n \geq 1)$ is odd, then M^2 is homeomorphic to a torus. In particular, there are no compact generic minimal M^2 into $\bar{M}_c^N$ with $K \leq 0$ and $K \not\equiv 0$ (cf. with Problem 101, p. 692 of [Y]).

(4.4) When completing the manuscript of this note we learnt that T.Ogata [O] studied minimal 2-spheres in $S^N(1)$ with $1/6 \leq K \leq 1$ (and Theorem A in particular) by methods quite similar to ours.

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