EPJ Web of Conferences **274**, 06005 (2022) https://doi.org/10.1051/epjconf/202227406005
XVth Quark Confinement and the Hadron Spectrum

Reconciling the FOPT and CIPT Predictions for τ Hadronic Spectral Function Moments

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Abstract. Recently it has been clarified by Hoang and Regner that the long-standing discrepancy between the CIPT and FOPT expansion approaches in α_s determinations from the τ hadronic spectral function moments has been caused by an inconsistency of CIPT with the standard OPE approach. This inconsistency arises in the presence of IR renormalons in the underlying Adler function and is numerically dominated by the dimension-4 gluon condensate renormalon. In this talk we report on an approach to reconcile the CIPT based on a perturbative definition of a renormalon-free and scale-invariant gluon condensate scheme, called RF GC scheme. The scheme implies perturbative subtractions which eliminate the CIPT inconsistency for all practical applications of the τ hadronic spectral function moments. The scheme depends on the gluon condensate renormalon norm N_g as an independent input and on an IR subtraction scale R. We discuss three different approaches to determine N_g which yield consistent results and we apply the RF GC scheme in two full-fledged phenomenological α_s determinations based on the truncated OPE and the duality violation model approach. In the RF GC scheme the long-standing CIPT-FOPT discrepancy problem is gone and the CIPT and FOPT α_s determinations can be consistently combined.
UWThPh 2022-16

1 Introduction

The comparison of weighted finite-energy sum rule integrals (spectral function moments) over the experimental inclusive hadronic τ decay invariant mass spectral functions with the corresponding theory predictions represents one of the most precise methods to determine the strong coupling α_s. The method constrains the strong coupling at the low scale $\mu^2 \sim m_\tau^2$ with a precision of about 3-5%, which turns into a precision of 1% or below at the scale of

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the Z-boson mass. In the limit of massless quarks the theoretical expressions for the spectral function moments can be dissected into the following individual contributions,

$$R_{V/A}^{(w)}(s_0) \;=\; \frac{N_c}{2}\, S_{\rm ew}\, |V_{ud}|^2 \Big[\, \delta_w^{\rm tree} + \delta_w^{(0)}(s_0) + \sum_{d\geq 4} \delta_{w,V/A}^{(d)}(s_0) + \delta_{w,V/A}^{(\rm DV)}(s_0) \Big], \tag{1}$$

where $S_{\rm ew}$ denotes electroweak corrections and s_0 is the upper bound of the weighted spectral function squared invariant mass integrations. The term $\delta_w^{\rm tree}$ corresponds to the tree level contribution and $\delta_w^{(0)}(s_0)$ represents the perturbative QCD corrections. The index w refers to the weight function $w(x)$ which typically is a polynomial in x and where $x \equiv s/s_0$ is the fraction of the squared invariant mass with respect to its upper bound. The inclusive hadronic τ decay rate, is the particular spectral function moments (called kinematic moment) where the polynomial reads $w_{\rm kin} = (1-x)^2(1+2x) = 1 - 3x^2 + 2x^3$ and where we have $s_0 = m_\tau^2$. The term $\delta_w^{(d)}(s_0)$ stands for non-perturbative higher dimensional corrections in the operator product expansion (OPE). The last term in Eq. (1), $\delta_w^{(\rm DV)}(s_0)$, represents so-called duality violation (DV) contributions which quantify non-perturbative effects that cannot be captured by the OPE series in the vicinity of the positive real hadronic invariant mass axis. The DV effects are suppressed for ('pinched') weight functions with $w(1) = 0$ because the integration endpoint is the only part of the contour integration that cannot be deformed away from the real axis. The treatment of DV effects is based on Regge-theory inspired models since a first principles treatment is currently not available [1, 2].

The QCD corrections are given by contour integrals of the form $[W(x) = 2\int_x^1 {\rm d}z\, w(z)]$

$$\delta_w^{(0)}(s_0) \;=\; \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{{\rm d}s}{s}\, W(\tfrac{s}{s_0})\, \hat{D}(s), \tag{2}$$

involving the (reduced) partonic Adler function $\hat{D}(s)$, which is related to the vacuum polarization function $\Pi(s)$ by the relation $\frac{1}{4\pi^2}[1 + D(s)] = -s\frac{\rm d}{{\rm d}s}\Pi(s)$,[1]

$$\hat{D}(s) \;=\; \sum_{n=1}^{\infty} c_{n,1}\, (\tfrac{\alpha_s(-s)}{\pi})^n\,, \tag{3}$$

$$\;=\; \sum_{n=1}^{\infty} (\tfrac{\alpha_s(s_0)}{\pi})^n \sum_{k=1}^{n} k\, c_{n,k}\, \ln^{k-1}(\tfrac{-s}{s_0})\,. \tag{4}$$

The contour integral is analytically related to an integration along the real positive s-axis over the observable spectral function $\rho(s) = \frac{1}{\pi}{\rm Im}\,\Pi(s + i0)$. The deformation away from the real axis is the basis for the perturbative theoretical treatment for the spectral function moments. The nonperturbative OPE terms $\delta_w^{(d)}(s_0)$ are obtained from an analogous integral over the Adler function's OPE corrections $[a(\mu^2) = \alpha_s(\mu^2)\beta_0/4\pi]$

$$D^{\rm OPE}(s) \;=\; \frac{2\pi^2}{3}\, \frac{1 - \frac{22}{81}\, a(-s) + \ldots}{s^2}\, \langle \bar{G}^2 \rangle + \sum_{d=6}^{\infty} \frac{1}{(-s)^{d/2}} \sum_i C_{d,i}(\alpha_s(-s)) \langle \bar{\mathcal{O}}_{d,\gamma_i} \rangle\,, \tag{5}$$

which represent an expansion in powers of $\Lambda_{\rm QCD}^2/s \ll 1$. The leading $d = 4$ OPE correction involves the renormalization scheme-invariant gluon condensate (GC) matrix element $\langle \bar{G}^2 \rangle = \langle \Omega | (\frac{\alpha_s}{\pi} + \ldots) G^{\mu\nu} G_{\mu\nu} | \Omega \rangle$. We call this commonly used form of the OPE the $\overline{\rm MS}$ scheme OPE. For spectral function moments with a weight function without a linear x term, such as $w_{\rm kin}$,

[1] The coefficients up to 5 loops, $c_{1,1}, \ldots, c_{4,1}$ are known exactly [3], and for $c_{5,1}$ it is common practice to use some estimate.

the GC correction becomes strongly suppressed (and even vanishing when the higher order corrections in its Wilson coefficient are neglected) upon the contour integration. Because the value of the GC is – still – known only roughly [4] and otherwise represents a sizeable source of uncertainty, state-of-the-art strong coupling determinations rely on GC-suppressed moments.

For the calculation of the perturbative contribution $\delta_w^{(0)}(s_0)$ two different kinds of renormalization scale settings for the strong coupling are used. In the fixed-order (FOPT) approach $\delta_W^{(0)}(s_0)$ is treated in an expansion in powers of $\alpha_s(s_0)$ and the coefficients arise from integrals over the logarithmic polynomials in Eq. (4). For the contour-improved (CIPT) approach [5] the expansion of Eq. (3) in powers of $\alpha_s(-s)$ is employed so that a non-trivial integration over the strong coupling's renormalization scale is carried out in the contour integral. So the series for $\delta_w^{(0)}(s_0)$ is not a power series. When determining the strong coupling from τ hadronic spectral function moments the FOPT and CIPT prescriptions lead to differences that are larger than the uncertainty estimates of each individual series. In general, CIPT leads to smaller values for $\delta_W^{(0)}(s_0)$, which means that $\alpha_s(m_\tau^2)$ extractions based on CIPT expansion are systematically larger. This CIPT-FOPT discrepancy has been discussed controversially [6, 7], and had constituted one of the dominant uncertainties in the determination of α_s from hadronic τ decays.

2 Inconsistency of CIPT with the OPE and RF Gluon Condensate Scheme

Recently it was shown by Hoang and Regner [8, 9] that the CIPT-FOPT discrepancy can be caused by a difference concerning the infra-red (IR) sensitivity that emerges from the (factorially divergent) IR renormalon contributions in the $c_{n,1}$ perturbative coefficients of the underlying Adler function. Each term in the OPE series of Eq. (5) formally compensates for a particular IR renormalon contribution, where the lower dimensional terms are associated to a stronger factorially divergent contribution. It was demonstrated in Ref. [8, 9] that using CIPT for $\delta_w^{(0)}(s_0)$ upsets this one-to-one correspondence between IR renormalons and the terms in the OPE. In particular, for weight functions where a particular OPE term from Eq. (5) becomes suppressed, the corresponding IR renormalon in the Adler function still survives the contour integration. In contrast, for FOPT this problem does not arise. The effect, which can be quantified analytically (called the "asymptotic separation"), is particularly sizeable for GC-suppressed moments and can nicealy explain the CIPT-FOPT discrepancy problem. Since all strong coupling determinations from τ hadronic spectral function moments are based on GC-suppressed moments, the work of Hoang and Regner implies that all previous analyses based on CIPT are incorrect, if one accepts the natural view that the known coefficients $c_{n,1}$ ($n = 3, 4$) already have a sizeable contribution from the GC renormalon. The analogue issue for the $d \leq 6$ OPE corrections is much smaller numerically and can be ignored in practice — at least for the τ hadronic spectral function moments. So even though it is not possible to fully remedy the CIPT method, it may be cured practically focusing on the GC renormalon only. Still, the CIPT expansion resums logarithmic phase corrections associated to the contour path (essentially powers of $\ln(\pm pi)$) which may have some valuable use in phenomenological applications, so that it is worth to consider such a cure and to not simply abandon the CIPT method. Furthermore some valuable additional insights may be gained on the way, particularly if the cause behind this problematic behavior is understood more thoroughly.

The results of Refs. [8, 9] imply that, if the GC renormalon is eliminated from Adler function, also the CIPT-FOPT discrepancy should be eliminated (at least for phenomenological

applications). In Ref. [10] we have achieved this aim through a redefinition of the GC matrix element of the form

$$\langle \bar{G}^2 \rangle \equiv \langle G^2 \rangle(R^2) - R^4 \sum_{\ell=1} N_g \, r_\ell^{(4,0)} \bar{a}^\ell(R^2) \,, \tag{6}$$

where $\langle \bar{G}^2 \rangle$ is the $\overline{\rm MS}$ scheme GC and the coefficients $r_\ell^{(4,0)}$ of the subtraction series contain the divergent asymptotic series related to a pure $O(\Lambda^4_{\rm QCD})$ renormalon. In the C-scheme for the strong coupling [11] (here we always use $C = 0$ and indicate the C-scheme coupling by a bar)[2], the coefficients can be given in closed form and read $r_\ell^{(4,0)} = (\frac{1}{2})^{\ell+4\hat{b}_1} \, \frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)}$. The in principle arbitrary scale R^2 acts as an infrared cutoff scale and, eventually, has to be set of the order of the physical dynamical scale of the observable (which is s_0 for $\delta_w^{(0)}(s_0)$). The term N_g is the GC renormalon norm and needs to be provided through a separate analysis. The subtraction series is combined with the series for $\delta_w^{(0)}(s_0)$. For either CIPT or FOPT, it is mandatory to consistently expand the combined series coherently in powers of $a^\ell(-s)$ or $a^\ell(s_0)$, i.e. using the strong coupling at a common renormalization scale. The latter ensures the systematic removal of the GC renormalon from the Adler function, and any scheme for α_s can be used for this expansion. This defines the renormalon-free GC matrix element $\langle G^2 \rangle(R^2)$.

It is now advantageous to define in a second step a scale-invariant GC matrix element $\langle G^2 \rangle^{\rm RF}$ through the relation

$$\langle G^2 \rangle(R^2) \equiv \langle G^2 \rangle^{\rm RF} + N_g \, \bar{c}_0(R^2) \,, \tag{7}$$

where $\bar{c}_0(R^2)$ satisfies the same R-evolution equation as the subtraction series on the RHS in Eq. (6) and thus of $\langle G^2 \rangle(R^2)$. Because the subtraction series contains a pure $O(\Lambda^4_{\rm QCD})$ renormalon this R-evolution equation is a convergent series [12] and can even be given in closed form [10]. A suitable choice is ($\hat{b}_1 = \beta_1/2\beta_0^2, \beta_0 = 11 - 2/3 n_f, \beta_1 = 102 - 38\, n_f/3$)

$$\bar{c}_0(R^2) \equiv -\frac{R^4 \, e^{-\frac{2}{\bar{a}(R^2)}}}{(\bar{a}(R^2))^{4\hat{b}_1}} \, {\rm Re}\left[e^{4\pi\hat{b}_1 i} \, \Gamma\Big(-4\hat{b}_1, -\frac{2}{\bar{a}(R^2)}\Big)\right] , \tag{8}$$

which is the Borel sum of the subtraction series itself in the C-scheme using the common PV prescription. The $\bar{c}_0(R^2)$ term must be treated strictly as a tree-level term (i.e. not being reexpanded again at any later point and numerically evaluated in the C-scheme for the strong coupling). This defines the renormalon-free and scale-invariant GC matrix element $\langle G^2 \rangle^{\rm RF}$, called RF GC scheme. When employing the RF GC scheme in practical applications, apart from the cancellation of the GC renormalon contributions, the R-dependence formally vanishes at large orders in analogy to the renormalization scale dependence. The RF GC scheme furthermore automatically takes care of the resummation of large logarithms when the GC is extracted from quantities with widely separated dynamical scale, since the term $\bar{c}_0(R^2)$ provides a solution of the R-evolution equation for the GC renormalon series [12] (see also Sec. 3.1 in [10]). In addition, the definition ensures that uncertainties in the knowledge of N_g are naturally suppressed. As we see in Sec. 3, the uncertainty in determinations of N_g is (still) rather large, so that the latter feature is particularly useful for high precision applications of the RF GC scheme.

It was shown in Ref. [10] for concrete renormalon models of the Adler function (where a concrete form for the Borel transform of the Adler function is employed and N_g has a definite value) that using the RF GC scheme for the GC-suppressed moments indeed leads to a

[2]Numerically, for $C = 0$, the C-scheme for the strong coupling is very close to the $\overline{\rm MS}$ scheme, see the appendix of Ref. [10] for details.

substantial modification of the CIPT series so that the CIPT-FOPT discrepancy is eliminated. Furthermore, it was demonstrated that using the RF GC scheme for GC-enhanced moments (for which $w(x)$ contains a linear term x) leads to a considerably improved and mutually consistent perturbative behavior for CIPT and FOPT since the factorically diverging contributions from the GC renormalon are eliminated. Thus GC-enhanced moments may now be employed for high-precision phenomenological analyses.

3 Gluon Condensate Renormalon Norm

In practice one cannot rely on Borel function models, and the GC renormalon norm N_g can only be determined approximately based on the assumption that known perturbative corrections (up to 5 loops) contains sizeable GC renormalon contributions. The consistency of the N_g determinations using different approaches serves as an essential reconfirmation of that assumption. In Ref. [13] we have used three different methods to determine N_g.

The first method, called *Borel model approach*, is based on the construction of models for the Adler function's Borel function $B[\hat{D}](u)$ using the known coefficients $c_{n,1}$ ($n = 1, 2, 3, 4$) and estimates for $c_{5,1}$ as an input. Using the natural assumption [14] that the (IR and UV) renormalons close to the origin of the Borel plane are most relevant a typical Borel model function ansatz consists of sum of non-analytic term for the renormalons accounted for and a polynomial. The free parameters are the renormalons norms and the polynomial coefficients which are fixed by the values the input $c_{n,1}$'s. Variations of the concrete ansatz and the range of input values for $c_{5,1}$ allow to determine N_g with an uncertainty. We obtained $N_g = 0.64$ with a relative uncertainty of $\pm 40\%$ that is dominated by the variation $c_{5,1}$. This method allows to determine the ball park of N_g values one should expect, but it relies strongly on $c_{5,1}$ which is still unknown. It can therefore only serve as a starting point.

The second method, called *conformal mapping approach*, uses that the non-analytic structure of $B[\hat{D}](u)$ related to the IR renormalons and the location of the UV renormalons (i.e. the structure of the renormalon cuts in the u-plane) is known from the dimensional dependence of the OPE corrections and the UV power corrections (i.e. all effects that are integrated out in the low-energy $n_f = 3$ QCD effective theory used to calculate $\delta_w^{(0)}(s_0)$). Using a conformal mapping in the u-plane that leaves the origin invariant, one can move the GC renormalon branch cut located at $u = 2$ and the other renormalon branch cuts such that that the GC branch cut is the one closest to the origin. On can then extract the norm N_g from a (convergent!) perturbation series using the coefficients $c_{n,1}$ ($n = 1, \ldots, 5$) as input. This method, first applied in this context by Lee [15], is substantially less model-independent than the first approach, but it relies on a good choice of the mapping so that truncating the N_g series at order 5 is already close to the final answer. We found for the models designed above that Lee's mapping function $z = u/(1 + u)$ yields a particularly slowly converging series for N_g, which yields at order 5 results substantially lower than the model value. We found that the mapping functions $w(u, p) = (\sqrt{1 + u} - \sqrt{1 - u/p})/(\sqrt{1 + u} + \sqrt{1 - u/p})$, with integer values $p \geq 5$ [16], yield much better convergence and N_g results within 10% of the true model value at order 5. We found $0.45 < N_g < 0.96$. The conformal mapping approach is more reliable than the Borel model approach, but still has a substantial dependence on the input value of $c_{5,1}$.

The third method, called *optimal subtraction approach*, encodes the two major improvements the RF GC scheme achieves over the previously used $\overline{\text{MS}}$ GC scheme for the $\delta_{w,V/A}(s_0)$: (1) the reduction of the CIPT-FOPT discrepancy for GC-suppressed moments and (2) the improvement of the badly behaved perturbation series for GC-enhanced moments. The basis of the optimal subtraction approach is therefore completely orthogonal from the previous two approaches. For the proper choice of N_g these two types of improvements are realized simultaneously for any possible choice of GC-suppressed (GCS) or GC-enhanced (GCE) moments.

We devised a function $\chi^2_m(N_g) = \chi^2_{m,\mathrm{GCS}}(N_g) + \chi^2_{m,\mathrm{GCE}}(N_g)$ which encodes conditions (1) and (2) for five GCS and five GCE moments for linearly-independent order 7 polynomials with $w(1) = 0$, which covers the polynomial orders used in phenomenological analyses. The exact form of these polynomials does not matter for the analysis, since they essentially form a complete basis for pinched weight functions. The optimal subtraction approach shows a minor dependence on the choice of renormalization scale and moreover very little dependence on the value of $c_{5,1}$ so that the known 5-loop coefficients already provide reliable results. Furthermore, we have tested that the central value for N_g is very close to the Borel function models used for the Borel model approach above. The final result reads

$$N_g = 0.57 \pm 0.23\,, \tag{9}$$

where the uncertainties come from renormalization scale variation and using only the known 5-loop coefficients as input. We consider this result as more reliable than the results from the first and second method, but the mutual agreement of the outcome of the three methods, all of which are based on completely different aspects and starting points, is essential for our confidence in this result for N_g. For the following analysis we therefore adopt Eq. (9) as the currently best estimate for the GC renormalon norm N_g.

4 Strong coupling predictions in the RF GC scheme

To demonstrate the improvements that can be achieved using the RF scheme for the GC instead of the original $\overline{\mathrm{MS}}$ scheme in realistic α_s analyses of τ hadronic spectral function moments, we exemplarily carry out in Ref. [13] two full-fledged phenomenological determinations of $\alpha_s(m_\tau^2)$ closely following the analysis setups employed in the recent references by Pich and Rodríguez-Sanchez [17] and by Boito, Golterman, Maltman, Peris, Rodrigues and Schaaf [18]. These two analyses are representatives of the two major approaches concerning the treatment of non-perturbative corrections currently used in the literature, the former employing a truncated OPE approach (referred to as tOPE), and the latter relying besides the OPE corrections on DV contributions (referred to as DVM). The two approaches are discussed controversially among the respective groups. Both determinations exclusively rely on GC-suppressed spectral function moments. We refer to Ref. [13] for all details and references.

For the tOPE approach a number of linearly independent integrated spectral function moments calculated at $s_0 = m_\tau^2$ are used, all based on (at least) doubly-pinched weight functions intended to suppress DV effects. This entails that the polynomial weight functions $w(x)$ contain x^n powers so that the number of OPE terms that would be needed to consistently account for all non-perturbative corrections it too large to carry out an α_s-fit (since they have to be fitted as well). In the tOPE approach one therefore truncates the tower of OPE condensates in order to have a manageable number of parameters in the fit procedure. This is based on the argument that the contributions from the neglected higher power OPE corrections are hierarchical and small. Concretely we have followed Sec. 5.3 of Ref. [17] using ALEPH V+A data. (The five double-pinched GCS moments used in this analysis were also employed for the term $\chi^2_{m,\mathrm{GCS}}(N_g)$ used for the optimal subtraction approach discussed in Sec. 3.)

The DVM approach avoids the truncation of the OPE by adopting weight functions with less (or even no) pinching. This allows to employ low-degree polynomial weight functions $w(x)$ which strongly suppress higher-dimension (or even all) OPE corrections. But it also enhances the DV contributions $\delta^{(\mathrm{DV})}_{w,V/A}(s_0)$, which therefore are included included in the fit analysis. This is done with a parametrization ansatz based on general QCD arguments. The fits can be carried out using moments for the same weight function using different values for

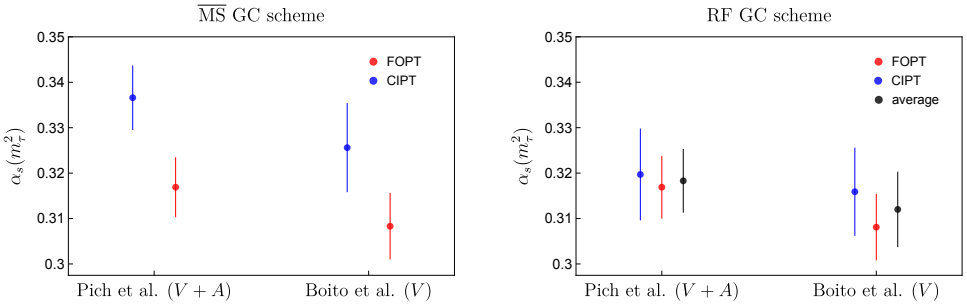

Figure 1. Left panel: Results for $\alpha_s(m_\tau^2)$ in FOPT (red) and CIPT (blue) in the $\overline{\rm MS}$ GC scheme, following the strategies of Ref. [17] (Pich et al.), based on the tOPE strategy applied to the $V + A$ ALEPH data [19], and Ref. [18] (Boito et al.), based on the DV-model strategy applied to the new vector spectral function of Ref. [18]. Right panel: Results for $\alpha_s(m_\tau^2)$ in FOPT (red) and CIPT (blue) and their average (black) in the RF GC scheme for the same analysis set-ups. The substantial reduction in the CIPT-FOPT discrepancy is clearly evident.

$s_0 \leq m_\tau^2$ under the assumption that the ansatz function is adequate. Concretely, we followed analysis of Ref. [18] using the weight function $w(x) = 1$ and an updated V spectral function that includes information from recent $e^+e^- \to$ hadrons cross-section data related by isospin symmetry.

The result of our analysis is displayed in Fig. 1. The left panel shows the results in the original $\overline{\rm MS}$ GC scheme. We can clearly see the discrepancy between the CIPT (blue) and FOPT (red) results for both analyses, which largely exceeds the size of the individual theoretical uncertainties based on variations of the renormalization scale and $c_{5,1}$. For the combination of CIPT and FOPT results it has been argued in Ref. [17] that a conservative way to account for the additional uncertainty related to the CIPT-FOPT discrepancy is to quadratically add half the CIPT-FOPT discrepancy to the smallest of the individual uncertainties. Given that the CIPT results are inconsistent, however, any averaging with the FOPT results is meaningless and inconsistent as well irrespective of the way how the CIPT-FOPT discrepancy is accounted for. The α_s result from CIPT in the original $\overline{\rm MS}$ GC scheme is simply incorrect.

The right panel shows the outcome for the analogous determinations in the RF GC scheme. While the FOPT results are virtually unchanged (because in FOPT the major effects of the GC scheme change are eliminated through contour integration), the CIPT results have shifted downward significantly. For CIPT the effects of the scheme change are not eliminated through the contour integration, but remain significant and instead eliminate the inconsistency of CIPT in the $\overline{\rm MS}$ OPE scheme (at least for the dominant and most relevant GC renormalon effects). The CIPT error bars slightly increase due to the additional variations of the IR subtraction scale R and the error in the GC renormalon norm N_g in Eq. (9). The effects of these additional two sources of uncertainty are, however, very mild. Only at this point the CIPT and FOPT results can be consistently combined. Using the prescription of from Ref. [17] already mentioned in the previous paragraph, the outcome is shown in black.

Comparing the results obtained in the $\overline{\rm MS}$ GC scheme with those in the RF GC scheme, we see that that the outcome of a CIPT-FOPT combination in the $\overline{\rm MS}$ GC scheme leads to a result for $\alpha_s(m_\tau^2)$ that is too large. Furthermore, the error of the combination (based on quadratically adding half of the CIPT-FOPT discrepancy) may not cover the individual cen-

tral values. So if the $\overline{\text{MS}}$ GC scheme is used, only FOPT should be used in phenomenological analyses. In contrast, a combination of the α_s determinations from CIPT and FOPT is perfectly meaningfull in the RF GC scheme.

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