An Application of Singer's Theorem to Homogeneous Polynomials

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With the help of Singer's theorem we characterize the dual of $\mathcal{P}_c(^mE, F)$, the compact type m-homogeneous polynomials. Using knowns results on nuclear and integral polynomials we establish that the reflexivity of $\mathcal{P}(^mE, F)$ is equivalent to the coincidence of $\mathcal{P}(^mE, F)$ with its subspace $\mathcal{P}_c(^mE, F)$, when E and F are reflexive Banach spaces.

§0 – Introduction

In early paper [1] we established a similar condition for the reflexivity of $\mathcal{P}(^mE)$, the space of m-homogeneous polynomials scalar valued, where we use a characterization of the dual $\mathcal{P}_c^*(^mE)$ due to S. Dineen [6].

For vector valued polynomials, we can follow the scalar case by introducing the Bartle integral, where for the same vector measure we consider two kind of integrands, scalar valued functions and vector valued functions which permit to apply the Riesz representation theorem and the Singer theorem to characterize the dual of $\mathcal{P}_c(^mE, F)$ establishing a preliminary result. The main result follows from the fact that the spaces of nuclear and integral polynomials coincide when the dual space E^* has the Radon Nikodym property.

In the last paragraph we give some examples illustrating the results.

§1 – Notation and Definitions

E and F will represent Banach spaces, E^* and F^* their duals, B_E the closed unit ball of E.

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$L(^mE,F)$ the space of continuous m-linear mappings from $E\times\cdots\times E$ (m-fold) to F. $\mathcal{P}(^mE,F)$ the space of continuous m-homogeneous polynomials from E to F with the sup norm on the unit ball B_E. We recall that if $P\in\mathcal{P}(^mE,F)$ there is an unique symmetric m-linear mapping $A\in L(^m(E,F)$ such that $A(x,x,\ldots,x)=P(x)$ and in this case we denote $\hat{A}=P$. $\mathcal{P}_f(^mE,F)$ is the subspace of $\mathcal{P}(^mE,F)$ of finite type polynomials, i.e., if $P\in\mathcal{P}_f(^mE,F)$ there are finite sequences $\varphi_j\in E^*$, $y_j\in F$, $j=1,\ldots,r$ such that

$$P(x)=\sum_{j=1}^{r}\varphi_j^m(x)y_j \quad \text{for all} \quad x\in E.$$

$\mathcal{P}_c(^mE,F)$ denotes the closure of $\mathcal{P}_f(^mE,F)$ in $\mathcal{P}(^mE,F)$, called the compact type polynomials.

$\mathcal{P}_N(^mE,F)$ is the space of all $P\in\mathcal{P}(^mE,F)$, that can be represented by

$$P(x)=\sum_{j=1}^{\infty}\varphi_j^m(x)y_j \quad \text{where} \quad \varphi_j\in E^* \ , \ y_j\in F$$

with $\sum_{j=1}^{\infty}\|\varphi_j\|^m\|y_j\|<\infty$. $\mathcal{P}_N(^mE,F)$ endowed with the

norm $\|P\|_N=\inf\left\{\sum_{j=1}^{\infty}\|\varphi_j\|^m\|y_j\| : \text{over all representations}\right\}$,

is called the space of nuclear polynomials.

$\mathcal{P}_I(^mE,F)$ is the space of all $P\in\mathcal{P}(^mE,F)$ that can be represented by

$$P(x)=\int_{B_{E^*}}\varphi^m(x)d\mu(\varphi)$$

where μ is an F-valued regular σ-additive Borel measure of bounded variation, on the Borel sets of $(B_{E^*},\sigma(E^*,E))$. $\mathcal{P}_I(^mE,F)$ endowed with the norm,

$$\|P\|_I=\inf\{|\mu|(B_{E^*}) : \text{ over all representation } \mu \text{ of } P\},$$

is called the space of integral polynomials.

As usual, if K is a compact Hausdorff space, $C(K)$ and $C(K,F)$ denote the Banach space of continuous scalar-valued and F-valued functions respectively, endowed with the sup norm.

§2 – Integration with respect to a vector measure

We shall limit ourselves to some basic facts on the Bartle integral in two special situations that fulfill our needs.

Let Ω be a set and $\sum$ a σ-algebra of subsets of Ω and F a Banach space.

A vector measure $\mu : \sum \to F$ is a countably additive set function, i.e.: for all sequences $(A_n)_{n=1}^{\infty}$ of pairwise disjoint member of $\sum$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

where the convergence is in the norm of F.

Definition.

(a) the *variation* of μ is the extended nonnegative function $|\mu|$ given by:

$$A \in \sum \ , \ \ |\mu|(A) = \sup \sum_{i=1}^{n} ||\mu(A_i)||,$$

where the supremum is taken over all partition $\{A_i\}_{i=1}^{n}$, of A into finite disjoint members of $\sum$.

If $|\mu|(\Omega) < \infty$, then μ will be called a vector measure of bounded variation.

(b) the *semivariation* of μ is the extended nonnegative function $||\mu||$ given by:

$$A \in \sum \ , \ \ ||\mu||(A) = \sup\{|x^*\mu|(A) : x^* \in F^*, \ \ ||x^*|| \leq 1\},$$

where $|x^*\mu|$ is the variation of the scalar measure $x^*\mu$.

If $||\mu||(\Omega) < \infty$, then μ will be called a vector measure of bounded semivariation.

We notice that the semivariation is a subadditive set function and in general $||\mu||(\Omega) \leq |\mu|(\Omega)$.

We recall that a vector measure μ defined on the Borel σ-algebra of subsets of a compact Hausdorff space is regular if for each Borel set A and $\epsilon > 0$ there exists a compact set K and an open set O such that $K \subset A \subset O$ and $||\mu||(O \backslash K) < \epsilon$.

In the following we give the notion of Bartle integral $\int f \, d\mu$ in two special cases (see for example [4]):

- f is a scalar function and μ is an F-valued measure.
- f is a F-valued function and μ is an F^*-valued measure.

Let μ be a vector measure from $\sum$ into F (or F^*) of bounded semivariation. Let f be a scalar (or F-valued) simple function defined on Ω represented by

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where α_i are nonzero scalars (or F-elements) and $A_1, \ldots, A_n$ are pairwise disjoint members of $\sum$, then we define:

$$\int_\Omega f\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) \quad \left(\text{or} \quad \int_\Omega \langle f, d\mu \rangle = \sum_{i=1}^{n} \langle \alpha_i, \mu(A_i) \rangle \right)$$

It can be checked that to above integral is well defined and induces a continuous linear operator T_μ on the space of simple functions normed by the sup norm, given by

$$T_\mu(f) = \int_\Omega f\, d\mu \ . \ \left(\text{or} \quad \int_\Omega \langle f, d\mu \rangle \right)$$

If $\mathcal{B}(\Omega)$ is the Borel σ-algebra, of subsets of a compact Hausdorff space Ω, then every continuous scalar- valued (or F-valued) function is an uniform limit of a sequence of simple functions. Therefore we can extend the above definitions for f a continuous, scalar valued (or F-valued) function, and we have also a continuous operator

$$T_\mu(f) = \int_\Omega f\, d\mu, \quad f \in C(\Omega) \quad \left(\text{or} \quad \int_\Omega \langle f, d\mu \rangle, \quad f \in C(\Omega, F)\right)$$

Some basic properties are the following.

1 – $\|\int_\Omega f\, d\mu\| \leq \|f\| \|\mu\|(\Omega)$

2 – μ, F^*-valued, $f \in C(\Omega)$

$$\text{for every} \quad x^{**} \in F^{**}, \quad x^{**}\left(\int_\Omega f\, d\mu\right) = \int_\Omega f d(x^{**}\mu)$$

3 – μ, F^*-valued, $f(x) = \sum_{i=1}^{n} g_i(x) y_i$, $g_i \in C(\Omega)$, $y_i \in F$

$$\int_\Omega \langle f, d\mu \rangle = \sum_{i=1}^{n} \int_\Omega \langle g_i(x) y_i, d\mu \rangle$$

Now we recall two representation theorems.

The Singer Theorem.

"The dual $C^*(K, F)$ is isomorphic and isometric to the space of all F^*-valued regular, σ-additive Borel measures μ of bounded variation defined on the Borel sets of K, endowed with the variation norm."

The isomorphism is given by $T \in C^*(K, F) \longleftrightarrow \mu$, where $T(f) = \int_K \langle f(x), d\mu \rangle$.

The Riesz Representation Theorem.

"Let T be a bounded linear operator from $C(K)$ into F. Then there is a weak* σ-additive measure μ defined on the Borel sets of K with values in F^{**} such that:

(i) $\mu(\cdot)x^*$ is a regular σ-additive Borel measure for each $x^* \in F^*$.

(ii) the mapping $x^* \in F^* \longrightarrow \mu(\cdot)x^*$, from F^* into $C^*(K)$ is weak*-to-weak* continuous.

(iii) $x^*T(f) = \int_K f d(x^*\mu)$, for each $f \in C(K)$ and each $x^* \in F^*$.

(iv) $||T|| = ||\mu||(K)$

Conversely, if μ is an F^{**}-valued measure defined on the Borel sets of K for which (i) and (ii) hold, then (iii) defines a bounded linear operator from $C(K)$ to F which satisfies (iv)".

Definition: Let T be a bounded linear operator from $C(K)$ into F, then the measure μ satisfying (i) to (iv) will be called the representing measure of T.

The next theorem strengthens the utility of the Riesz Representation Theorem if we know more about T or its representing measure.

The Bartle–Dunford–Schwartz Theorem

"Let $T : C(K) \longrightarrow F$ be a bounded linear operator with representing measure μ. Any one of the following statements implies all the others.

(a) the operator T is weakly compact.

(b) the measure μ takes all its values in F.

(c) the measure μ is σ-additive.

For the above results on vector measures, the Riesz Represéntation theorem and the Bartle–Dunford–Schwartz theorem we refer to the book of Diestel and Uhl ([5]). For the Singer Theorem we refer to the Singer's paper ([9]) and also to the book of Jean Schmets ([8]).

§3 – Priliminary Result (Application of Singer's Theorem)

We start with the following remark: if μ is an F-valued regular σ-additive Borel measure of bounded variation, defined on the Borel sets of a compact Hausdorff space K, then it defines a bounded linear operator T from $C(K)$ into F given by

$$T(f) = \int_K f d\mu$$

where μ is just the representing measure of T. This is straight forward if we look at the Riesz Representation theorem and the Bartle- Dunford–Schwartz theorem.

Lemma: Let E and F be reflexive Banach spaces. Then $\mathcal{P}_I({}^mE^*, F^*)$ and $\mathcal{P}_c^*({}^mE, F)$ are isometric.

Proof: first of all we notice that if $P \in \mathcal{P}_f({}^mE, F)$, then P is $\sigma(E, E^*)$-continuous. As a consequence we have that the inclusion $\mathcal{P}_c({}^mE, F) \hookrightarrow C(B_E, F)$ is continuous, where B_E is endowed with the weak-topology. Therefore, if $T \in \mathcal{P}_c^*({}^mE, F)$, T can be extended via Hahn-Banach to some $\tilde{T}$ on $C(B_E, F)$, preserving the norm.

Now, by Singer's theorem we can find an F^*-valued measure μ defined on the Borel sets of B_E such that

$$\tilde{T}(F) = \int_{B_E} \langle f, d\mu \rangle \quad \text{for all} \quad f \in C(B_E, F) \quad \text{and} \quad \|\tilde{T}\| = |\mu|(B_E).$$

Now, by the previous remark, μ is the representing measure of the operator $\tau : C(B_E) \longrightarrow F^*$, given by

$$\tau(g) = \int_{B_E} g(x) d\mu(x), \quad \text{with} \quad \|\tau\| = \|\mu\|(B_E).$$

Suppose $\varphi \in E^*$, then $\varphi^m(x) = (\varphi(x))^m$ defines a continuous function on B_E, therefore

$$Q_T(\varphi) = \int_{B_E} \varphi^m(x) d\mu(x)$$

defines a m-homogeneous polynomial and it's clear that $Q_T \in \mathcal{P}_I({}^mE^*, F^*)$. Furthermore, we have,

$$\|Q_T\|_I \leq |\mu|(B_E) = \|\tilde{T}\| = \|T\|.$$

Conversely: Let Q be in $\mathcal{P}_I({}^mE^*, F^*)$, therefore Q can be represented by

$$Q(\varphi) = \int_{B_E} \varphi^m(x) d\mu(x).$$

Since $Q \in \mathcal{P}({}^mE^*, F^*)$ there is an unique symmetric $B \in L({}^mE^*, F^*)$ such that $\hat{B} = Q$.

Now, using the isomorphism isometric between $L({}^mE^*, F^*)$ and $L({}^mE^* \times F)$ (the $m+1$-linear scalar valued mappings), given $B \in L({}^mE^*, F^*)$ we can find $\tilde{B} \in L({}^mE^* \times F)$ such that

$$\tilde{B}(\varphi_1, \ldots, \varphi_m, y) = \langle y, B(\varphi_1, \ldots, \varphi_m) \rangle.$$

By the universal property of tensor product there is an unique linear form S on $E^* \otimes \cdots E^* \otimes F$ such that

$$\tilde{B}(\varphi_1, \ldots, \varphi_n, y) = S(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes y).$$

Now, let $P \in \mathcal{P}_f({}^mE, F)$ with the representation

$$P(x) = \sum_{j=1}^{r} \varphi_j^m(x) y_j, \quad \varphi_j \in E^* \quad \text{and} \quad y_j \in F, \quad j = 1, \ldots, r,$$

there is an unique symmetric $A \in L_f({}^mE,F)$ such that $\widehat{A} = P$. It is clear that A can be represented by

$$A(x_1,\ldots,x_m) = \sum_{j=1}^{r} \varphi_j(x_1)\cdots\varphi_j(x_m)y_j.$$

On the other hand A can be viewed as an element of $E^* \otimes \cdots \otimes E^* \otimes F$, represented by

$$A = \sum_{j=1}^{r} \varphi_j \otimes \cdots \otimes \varphi_j \otimes y_j.$$

We define a linear mapping scalar valued T on $\mathcal{P}_f({}^mE,F)$ by

$$\begin{aligned} T(P) &= S(A) = \sum_{j=1}^{r} S(\varphi_j \otimes \cdots \otimes \varphi_j \otimes y_j) = \sum_{j=1}^{r} \tilde{B}(\varphi_j,\ldots,\varphi_j,y_j) \\ &= \sum_{j=1}^{r} \langle y_j, B(\varphi_j,\ldots,\varphi_j)\rangle = \sum_{j=1}^{r} \langle y_j, Q(\varphi_j)\rangle \\ &= \sum_{j=1}^{r} \langle y_j, \int_{B_E} \varphi_j^m(x)d\mu(x)\rangle = \int_{B_E} \langle \sum_{j=1}^{r} \varphi_j^m(x)y_j, d\mu(x)\rangle \\ &= \int_{B_E} \langle P(x), d\mu(x)\rangle. \end{aligned}$$

This shows that T is well defined and we have

$$||T(P)|| \leq ||P|| |\mu|(B_E),$$

for every μ that represents Q, therefore $T \in \mathcal{P}_f^*({}^mE,F)$ and $||T|| \leq ||Q||_I$.
Now, T can be extended in an unique way to $\mathcal{P}_c({}^mE,F)$ and the lemma follows.

§4 – The Reflexivity of $\mathcal{P}({}^mE,F)$.

Now we recall some specific results on nuclear and integral polynomials.

The coincidence of Integral and Nuclear polynomials.

Theorem 1. [1]. If E^* has the Radon Nikodym property then $\mathcal{P}_N({}^mE,F)$ and $\mathcal{P}_I({}^mE,F)$ are isomorphic for $m = 1,2,\cdots$ and all F.

The characterization of the dual $\mathcal{P}_N^*({}^mE^*,F^*)$.

Theorem 2. [7]. Suppose E and F with the approximation property, then the spaces $\mathcal{P}_N^*({}^mE^*,F)$ and $\mathcal{P}({}^mE,F^*)$ are isomorphic and isometric.

Now we prove the result:

Theorem 3. Let E and F be reflexive Banach spaces E and F^* with the approximation property. Then $\mathcal{P}({}^mE,F)$ is reflexive if and only if $\mathcal{P}({}^mE,F)$ coincide with its subspace $\mathcal{P}_c({}^mE,F)$.

Proof: first of all we notice that E has the Radon Nikodym property since E is reflexive therefore by Theorem 1 we have that $\mathcal{P}_N({}^mE^*,F^*)=\mathcal{P}_I({}^mE^*,F^*)$ isomorphically. Now, combining this with Theorem 2 and the lemma we get the identities:

$$\mathcal{P}({}^mE,F)=\mathcal{P}_N^*({}^mE^*,F^*)\underset{\bullet}{=}\mathcal{P}_I^*({}^mE^*,F^*)=\mathcal{P}_c^{**}({}^mE,F).$$

To conclude it's enough to observe that $\mathcal{P}_c({}^mE,F)$ is a closed subspace of $\mathcal{P}({}^mE,F)$.

§5 – Examples.

1 – In early papers [2] and [3] we showed that if T^* is the space discovered by B. S. Tsirelson then the spaces $\mathcal{P}({}^nT^*)$ and $\mathcal{P}({}^nT^*,\ell_p)$ are reflexive for all $n=1,2,\dots$ and all p, $1<p<\infty$. Therefore by Theorem 3 we have that the elements of those spaces are in fact polynomials of compact type.

2 – Now we apply the theorem 3, in negative form, to some concrete space of polynomials. Now we consider only real ℓ_p spaces.

(a) Scalar valued polynomials

For p an even integer we have that $\mathcal{P}({}^p\ell_p)$ is not a reflexive space:
Let $x=(x_i)\in\ell_p$ and define a p–homogeneous polynomial P by,

$$P(x)=\sum_{i=1}^{\infty}x_i^p.$$

It is clear that $|P(x)|=||x||_p^p$ and $P\in\mathcal{P}({}^p\ell_p)$. Now if $Q\in\mathcal{P}_f({}^p\ell_p)$, Q has the form

$$Q(x)=\sum_{j=1}^{k}a_j\varphi_j^p(x)\quad\text{where}\quad\varphi_j\in\ell_p^*\quad\text{and}\quad a_j\in\mathbb{R},\quad j=1,\dots,k.$$

We can choose $y=(y_i)\in\bigcap_{j=1}^k\varphi_j^{-1}(0)$ such that $||y||_p=1$, therefore $Q(y)=0$ and $P(y)=||y||_p^p=1$ hence $||P-Q||\geq 1$ for all $Q\in\mathcal{P}_f({}^p\ell_p)$, then it follows that $\mathcal{P}_c({}^p\ell_p)\neq\mathcal{P}({}^p\ell_p)$ and by theorem 3 $\mathcal{P}({}^p\ell_p)$ is not reflexive.

(b) Vector valued polynomials

For n a positive integer and p a positive real number such that $p/n\geq 1$ we

have that $\mathcal{P}({}^n\ell_p, \ell_{p/n})$ is not a reflexive space.
Let $x = (x_i) \in \ell_p$ and define a n-homogeneous polynomial P by

$$P(x) = (x_i^n)$$

It is clear that $P(x) \in \ell_{p/n}$ and $P \in \mathcal{P}({}^n\ell_p, \ell_{p/n})$.
As in the scalar case, if $Q \in \mathcal{P}_f({}^n\ell_p, \ell_{p/n})$, Q has the form $Q(x) = \sum_{j=1}^k \varphi_j^n(x) a_j$, where $\varphi_j \in \ell_p^*$ and $a_j \in \ell_{p/n}$, $j = 1, \ldots, k$.
Therefore we can find $y = (y_i) \in \ell_p$ such that $||y||_p = 1$ and $Q(y) = 0$. From the definition of P we have that

$$||P(y)||_{p/n} = ||(y_i^n)||_{p/n} = ||y||_p^n = 1.$$

Hence $||P - Q|| \geq 1$ and $\mathcal{P}_c({}^n\ell_p, \ell_{p/n}) \neq \mathcal{P}({}^n\ell_p, \ell_{p/n})$ and by Theorem 3 $\mathcal{P}({}^n\ell_p, \ell_{p/n})$ is not reflexive.

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