PERMANENCE OF NONUNIFORM NONAUTONOMOUS HYPERBOLICITY FOR INFINITE-DIMENSIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study stability properties of nonuniform hyperbolicity for evolution processes associated with differential equations in Banach spaces. We prove a robustness result of nonuniform hyperbolicity for linear evolution processes, that is, we show that the property of admitting a nonuniform exponential dichotomy is stable under perturbation. Moreover, we provide conditions to obtain uniqueness and continuous dependence of projections associated with nonuniform exponential dichotomies. We also present an example of evolution process in a Banach space that admits nonuniform exponential dichotomy and study the permanence of the nonuniform hyperbolicity under perturbation. Finally, we prove persistence of nonuniform hyperbolic solutions for nonlinear evolution processes under perturbations.

1. Introduction

In the framework of dynamical systems, hyperbolicity plays a fundamental role (see, e.g. [1, 2, 3] and the references therein). It is the key property for most of the results on permanence under perturbations. The permanence, on the other hand, is an essential property for dynamical systems that model real life phenomena. That importance is related to the fact that modelling always comes with approximations (due do the empiric nature that it carries) and/or with simplifications (introduced to make models treatable or simply because the complete set of variables that are related to the phenomenon is not known). Therefore, in order that the mathematical model reflects, in some way, the phenomenon modelled, it is essential that its dynamical structures are robust under perturbation. It starts with the robustness under perturbation of hyperbolicity itself. Here, we are concerned with the robustness of nonautonomous nonuniform hyperbolicity.

In the discrete case, *hyperbolic dynamical systems* $x_{n+1} = Bx_n$ appear when the spectrum of the bounded linear operator B does not intercept the unit circle in the complex plane. This implies the existence of a *hyperbolic decomposition* of the space, which means that exist two main directions: one where the evolution of the dynamical system decays exponentially and another where it grows exponentially.

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This property can be interpreted as a complete understanding of local or global dynamics. The set of operators that has such decomposition is an open set in the spaces of bounded linear operators and the operators in this set are called *hyperbolic operators*. In other words, if B is hyperbolic there is a neighborhood of B such that every operator in this neighborhood is hyperbolic. For autonomous differential equations, when A is a bounded linear operator, $\dot{x} = Ax$, by the spectral mapping theorem [4], hyperbolicity is associated with linear operators such that the spectrum does not intersect the imaginary line.

Generally, in nonautonomous differential equations, the notion of hyperbolicity is referred to as *exponential dichotomy*. More precisely, consider the following differential equation in a Banach space X,

$$\dot{x} = A(t)x, \quad x(s) = x_s \in X. \tag{1.1}$$

Under appropriate conditions, the solutions $x(t, s; x_s)$, $t \geq s$, of this initial value problem define an evolution process $\mathcal{S} := \{S(t, s)\,;\, t \geq s\}$, where $S(t, s)x_s = x(t, s; x_s)$. We say that the evolution process $\mathcal{S}$ admits an *(uniform) exponential dichotomy* if there exists a family of projections, $\{Q(t)\,;\, t \in \mathbb{R}\}$ such that for each $t \geq s$ we have that $S(t, s)Q(s) = Q(t)S(t, s)$, $S(t, s)$ is an isomorphism from $R(Q(s))$ onto $R(Q(t))$, and

$$\|S(t, s)(Id_X - Q(s))\|_{\mathcal{L}(X)} \leq Ke^{-\alpha(t-s)}, \quad t \geq s; \tag{1.2}$$

$$\|S(t, s)Q(s)\|_{\mathcal{L}(X)} \leq Ke^{\alpha(t-s)}, \quad t < s, \tag{1.3}$$

for some constants $K \geq 1$ and $\alpha > 0$. Note that, since the vector field is changing in time, it is natural to think that for each initial time we have a hyperbolic decomposition that resembles the properties in the autonomous case. There is a long list of works through these last decades about existence of exponential dichotomies and their stability properties, for instance: [5, 6, 7, 8, 9, 10, 11, 12, 13]. In Henry [11] a robustness result is proved for exponential dichotomies, in the discrete case, by characterizing dichotomy via admissibility for the associated difference equation and, in the continuous case, by a discretization method (see Chicone and Latushkin [7] for a comprehensive study of dichotomy and its robustness). The smooth dependence, with respect to parameters, of the projections is considered in Pötzsche [14], using two different approaches.

If we replace the constant K in the above definition by a continuous function $K(s)$ in (1.2) and (1.3), we say that (1.1) admits a *nonuniform exponential dichotomy* (for an introduction see [15]). Usually, the nonuniform bound is given by $K(s) \leq De^{\nu|s|}$ for some $\nu > 0$. As in the uniform case, there are many works concerning issues of existence and robustness for nonuniform exponential dichotomies [16, 17, 18, 19, 15, 20, 21, 22, 23].

The robustness of nonuniform exponential dichotomy for equation (1.1) can be interpreted as follows: suppose that the associated solution operator (evolution process) admits a nonuniform exponential dichotomy. The problem is to know for which family of bounded linear operators $\{B(t) : t \in \mathbb{R}\}$, the perturbed problem

$$\dot{x} = A(t)x + B(t)x, \tag{1.4}$$

admits a nonuniform exponential dichotomy.

Barreira and Valls [20] studied under which conditions the nonuniform exponential dichotomy is robust in the case of invertible evolution processes. Later, Zhou *et al.* [23], proved a similar result for random difference equations for linear operators without the invertibility requirement. More recently, Barreira and Valls [22], proved that nonuniform exponential dichotomy is robust for continuous evolution processes, also without invertibility. They consider an evolution process that admits a nonuniform exponential dichotomy with a general growth rate $\rho(\cdot)$. They proved that if $\alpha > 2\nu$ and $B : \mathbb{R} \to \mathcal{L}(X)$ is continuous satisfying $\|B(t)\|_{\mathcal{L}(X)} \leq \delta e^{-3\nu|\rho(t)|}\rho'(t)$, for all $t \in \mathbb{R}$, then the perturbed problem (1.4) admits a ρ-nonuniform exponential dichotomy.

We provide a interpretation of the robustness result as *open property.* In fact, if an evolution process $\mathcal{S}$ admits a nonuniform exponential dichotomy, there is an open neighborhood $N(\mathcal{S})$ of $\mathcal{S}$ such that every evolution process in $N(\mathcal{S})$ also admits a nonuniform exponential dichotomy. Our proof of the robustness result is inspired by the ideas of Henry [11]. We prove that if a continuous evolution process admits a nonuniform exponential dichotomy, then each discretization also admits it. Then we use the *roughness* of the nonuniform exponential dichotomy for discrete evolution processes, obtained by Zhou *et al.* [23], to obtain that each discretization of the perturbed evolution process also admits a nonuniform exponential dichotomy. Thus, to obtain our robustness result, we have to guarantee that if each discretization of a continuous evolution process $\mathcal{S}$ admits a nonuniform exponential dichotomy, then $\mathcal{S}$ also admits it.

With this method, we obtain uniqueness and continuous dependence of projections, and explicit expressions for the bound and exponent of the perturbed evolution process. Besides, since we preserve the condition $\alpha > \nu$ of Zhou *et al.* [23], we obtain an improvement of Barreira and Valls [22, Theorem 1] (see [18] for a situation where $\alpha > \nu$ is not required). We consider the case $\rho(t) = t$, the other cases follow by a change of scaling in time. Moreover, we do not assume that the evolution processes are invertible, then it is possible to apply our result on evolutionary differential equations in Banach spaces, as the ones that appears in [6, 24, 9, 11].

An important consequence of the robustness result regarding nonlinear evolution processes is the persistence under perturbation of *hyperbolic solutions.* More precisely, consider a semilinear differential equation

$$\dot{x} = A(t)x + f(t, x), \quad x(s) = x_s \in X, \tag{1.5}$$

and suppose that there for each $s \in \mathbb{R}$ and $x_s \in X$ there exist a solution $x(\cdot, s; x_s) : [s, +\infty) \to X$, then there exist a nonlinear evolution process $\mathcal{S}_f = \{S_f(t, s) : t \geq s\}$ defined by $S_f(t, s)x = x(t, s; x_s)$. A map $\xi : \mathbb{R} \to X$ is called a *global solution* for $\mathcal{S}_f$ if $S_f(t, s)\xi(s) = \xi(t)$ for every $t \geq s$ and we say that ξ is a *nonuniform hyperbolic solution* if the linearized evolution process over ξ admits a nonuniform exponential dichotomy. This notion also appears in Barreira and Valls [15] as *nonuniformly hyperbolic solutions.* In the uniform case, in Carvalho and Langa [6], it was studied the existence of hyperbolic solutions from nonautonomous perturbation of hyperbolic equilibria for a autonomous semilinear differential equation. The persistence of uniform hyperbolic solutions can be shown using the implicit function theorem, see Pötzsche [25].

Inspired by Carvalho *et al.* [24] we prove a result on the persistence of nonuniform hyperbolic solutions under perturbations. In fact, if ξ is a nonuniform hyperbolic solution for $\mathcal{S}_f$ and g is a map "close" to f, then there exists a nonuniform hyperbolic solution for $\mathcal{S}_g$ "close" to ξ. Additionally, we also prove that bounded nonuniform hyperbolic solutions are *isolated* in the space of bounded continuous functions $C_b(\mathbb{R}, X)$, i.e., if ξ is a nonuniform hyperbolic solution, then there exists a neighborhood of ξ in $C_b(\mathbb{R})$ such that ξ is the only bounded solution for $\mathcal{S}_f$ is this neighborhood.

Therefore, the aim of this work is to establish: uniqueness and continuity for family of projections associated with nonuniform exponential dichotomy; a robustness result with the condition $\alpha > \nu$; and persistence of nonuniform hyperbolic solutions. We restrict our attention to the study of robustness for nonuniform exponential dichotomies of linear evolution processes in the entire axis, $\mathbb{Z}$ and $\mathbb{R}$, to tackle the permanence of nonuniform hyperbolic solutions of semilinear differential equations.

The paper is organized as follows, in Section 2, we summarize some important facts for discrete evolution processes with nonuniform exponential dichotomy. We prove uniqueness and continuity of projections and briefly recall the robustness result of Zhou *et al.* [23] in our framework. Then, in Section 3, we prove uniqueness and continuous dependence for family of projections, and a robustness result of nonuniform exponential dichotomy for continuous evolution processes. In Section 4, we present a class of examples of evolutions processes in a Banach space that admit nonuniform exponential dichotomy. Finally, in Section 5, we consider *nonuniform hyperbolic solutions* for evolution processes associated with semilinear differential equations. We prove that these solutions are isolated in $C_b(\mathbb{R}, X)$ and that they persist under perturbations.

2. Nonuniform exponential dichotomy: discrete case

In this section, we present some basic facts of nonuniform exponential dichotomy for discrete evolution processes. We briefly recall some results of Zhou *et al.* [23] (without considering a random parameter) that we will use to prove our results for the continuous case. Furthermore, we establish uniqueness and continuous dependence of projections associated with the nonuniform exponential dichotomy.

We start with the definition of a *discrete evolution process* in a Banach space $(X, \|\cdot\|_X)$ in a particular case where the family of operators are linear bounded operators in X.

Definition 2.1. *Let $\mathcal{S} = \{S_{n,m} : n \geq m \text{ with } n, m \in \mathbb{Z}\}$ be a family of bounded linear operators in a Banach space X. We say that $\mathcal{S}$ is a **discrete evolution process** if*

(1) $S_{n,n} = Id_X$, for all $n \in \mathbb{Z}$;
(2) $S_{n,m}S_{m,k} = S_{n,k}$, for all $n \geq m \geq k$.

To simplify the notation, we write $\mathcal{S} = \{S_{n,m} : n \geq m\}$ as an evolution process, whenever is clear are dealing with discrete ones.

Remark 2.2. *Given a discrete evolution process $\mathcal{S} = \{S_{n,m} : n \geq m\}$ it is always possible to associate $\mathcal{S}$ with the family $\{S_n : n \in \mathbb{Z}\}$, where $S_n := S_{n+1,n}$ for all $n \in \mathbb{Z}$. Conversely, for any family of bounded linear operators $\{S_n : n \in \mathbb{Z}\} \subset \mathcal{L}(X)$ define $S_{n,m} := S_{n-1} \cdots S_m$ for $n > m$ and $S_{n,n} := Id_X$ so that $\mathcal{S} = \{S_{n,m} : n \geq m\}$ is discrete evolution process. Therefore, we often refer, indistinctly, to $\{S_n : n \in \mathbb{Z}\}$ or $\{S_{n,m} : n \geq m\}$ as the discrete evolution process.*

Thus it is possible to associate the evolution process $\{S_{n,m} : n \geq m\}$ with following difference equation

$$x_{n+1} = S_n x_n, \quad x_n \in X, \quad n \in \mathbb{Z}, \tag{2.1}$$

where $S_n = S_{n+1,n}$*,* $n \in \mathbb{Z}$*.*

Now, we present the definition of *nonuniform exponential dichotomy.*

Definition 2.3. *Let* $\mathcal{S} = \{S_{n,m} : n \geq m\} \subset \mathcal{L}(X)$ *be an evolution process in a Banach space* X*. We say that* S *admits a* ***nonuniform exponential dichotomy*** *if there is a family of continuous projections* $\{Q_n; n \in \mathbb{Z}\}$ *in* $\mathcal{L}(X)$ *such that*

(1) $Q_n S_{n,m} = S_{n,m} Q_m$*, for* $n \geq m$*;*
(2) $S_{n,m} : R(Q_m) \to R(Q_n)$ *is an isomorphism, for* $n \geq m$*, and we define* $S_{m,n}$ *as its inverse;*
(3) There exists a function $K : \mathbb{Z} \to [1, +\infty)$ *with* $K(n) \leq De^{\nu|n|}$*, for some* $D \geq 1$ *and* $\nu > 0$*, and* $\alpha > 0$ *such that*

$$\|S_{n,m}(Id_X - Q_m)\|_{\mathcal{L}(X)} \leq K(m)e^{-\alpha(n-m)}, \quad \forall n \geq m;$$

and

$$\|S_{n,m}Q_m\|_{\mathcal{L}(X)} \leq K(m)e^{\alpha(n-m)}, \quad \forall n \leq m.$$

In this theory, K and α are usually called the **bound** and the **exponent** of the exponential dichotomy, respectively.

We now recall the definition of a *Green's function.*

Definition 2.4. *Let* $\mathcal{S} = \{S_n : n \in \mathbb{Z}\}$ *be a discrete evolution process which admits a nonuniform exponential dichotomy with family of projections* $\{Q_n\}_{n\in\mathbb{Z}}$*. The* ***Green's function*** *associated to the evolution process* $\mathcal{S}$ *is given by*

$$G_{n,m} = \begin{cases} S_{n,m}(Id_X - Q_m), & \text{if } n \geq m, \\ -S_{n,m}Q_m, & \text{if } n < m. \end{cases}$$

A space that appears naturally when dealing with nonuniform exponential dichotomies is

$$l^{\infty}_{1/K}(\mathbb{Z}) := \big\{ f : \mathbb{Z} \to X : \sup_{n\in\mathbb{Z}} \big\{ \|f_n\|_X K(n+1) \big\} = M_f < +\infty \big\},$$

where $K : \mathbb{Z} \to \mathbb{R}$ is such that $K(n) \geq 1$ for all $n \in \mathbb{Z}$.

As in the uniform case, the next result shows that it is possible to obtain the solution for

$$x_{n+1} = S_n x_n + f_n. \tag{2.2}$$

using the *Green's function.*

Theorem 2.5. *Assume that the evolution process* $\mathcal{S} = \{S_n : n \in \mathbb{Z}\}$ *admits a nonuniform exponential dichotomy with bound* $K(n) \leq De^{\nu|n|}$ *and exponent* $\alpha > \nu$*. If* $f \in l^{\infty}_{1/K}(\mathbb{Z})$*, then* (2.2) *possesses a unique bounded solution given by*

$$x_n = \sum_{-\infty}^{+\infty} G_{n,k+1} f_k, \quad \forall\ n \in \mathbb{Z}.$$

For the proof of Theorem (2.5) see Zhou *et al.* [23]. Note that, in [23] they consider tempered exponential dichotomies, but their proof holds true with the condition $\alpha > \nu$.

As a consequence of Theorem 2.5, we obtain uniqueness of the family of projections associated with the nonuniform exponential dichotomy.

Corollary 2.6 (Uniqueness of projections)**.** *If $\mathcal{S} = \{S_n : n \in \mathbb{Z}\}$ admits a nonuniform exponential dichotomy with bound $K(n) \leq De^{\nu|n|}$ and exponent $\alpha > \nu$, then the family of projections is uniquely determined.*

Proof. Let $\{Q_n^{(i)}\,;\, n \in \mathbb{Z}\}$, for $i = 1, 2$, projections associated with the evolution process $\mathcal{S}$. Given $x \in X$ and $m \in \mathbb{Z}$ fixed, define $f_n = 0$, for all $n \neq m-1$, and $f_{m-1} = K(m)^{-1}x$. Thus, $f \in l^{\infty}_{1/K}(\mathbb{Z})$ and from Theorem 2.5 there exists a unique solution $\{x_n\}_{n\in\mathbb{Z}}$ for

$$x_{n+1} = S_n x_n + f_n, \quad n \in \mathbb{Z}.$$

Hence, $x_m = \sum_{-\infty}^{+\infty} G^{(i)}_{m,k+1} f_k = G^{(i)}_{m,m} f_{m-1} = K(m)^{-1}(Id_X - Q_m^{(i)})x$, for $i = 1, 2$. Therefore, $Q_m^{(1)} = Q_m^{(2)}$. □

Next, we establish a result on continuous dependence of projections.

Theorem 2.7 (Continuous dependence of projections)**.** *Suppose that $\{T_n\}_{n\in\mathbb{Z}}$ and $\{S_n\}_{n\in\mathbb{Z}}$ admit a nonuniform exponential dichotomy with projections $\{Q_n^{\mathcal{T}}\}_{n\in\mathbb{Z}}$ and $\{Q_n^{\mathcal{S}}\}_{n\in\mathbb{Z}}$, exponents $\alpha_{\mathcal{T}}$ and $\alpha_{\mathcal{S}}$, respectively, and the same bound $K(n) \leq De^{\nu|n|}$. If $\nu < \min\{\alpha_{\mathcal{T}}, \alpha_{\mathcal{S}}\}$ and*

$$\sup_{n\in\mathbb{Z}} \left\{ K(n+1) \|T_n - S_n\|_{\mathcal{L}(X)} \right\} \leq \epsilon,$$

then

$$\sup_{n\in\mathbb{Z}} \left\{ K(n)^{-1} \, \|Q_n^{\mathcal{T}} - Q_n^{\mathcal{S}}\|_{\mathcal{L}(X)} \right\} \leq \frac{e^{-\alpha_{\mathcal{S}}} + e^{-\alpha_{\mathcal{T}}}}{1 - e^{-(\alpha_{\mathcal{S}} + \alpha_{\mathcal{T}})}}\, \epsilon.$$

Proof. Let $z \in X$ and $m \in \mathbb{Z}$ be fixed and consider

$$f_n = \begin{cases} 0, & \text{if } n \neq m-1, \\ K(m)^{-1} z, & \text{if } n = m-1. \end{cases}$$

Thus, by Theorem 2.5, there exist bounded solutions $x^k = \{x_n^k\}_{n\in\mathbb{Z}}$ given by $x_n^k := G^k_{n,m} z K(m)^{-1}$ for $k = \mathcal{T}, \mathcal{S}$. Note that, for $n \in \mathbb{Z}$,

$$x^{\mathcal{T}}_{n+1} - S_n x_n^{\mathcal{T}} = T_n x_n^{\mathcal{T}} - S_n x_n^{\mathcal{T}} + f_n$$

and $x^{\mathcal{S}}_{n+1} - S_n x^{\mathcal{S}}_n = f_n$. Then, if $z_n := x_n^{\mathcal{T}} - x_n^{\mathcal{S}}$ we obtain that $z_{n+1} = S_n z_n + y_n$, where $y_n = (T_n - S_n)x_n^{\mathcal{T}}$ for all $n \in \mathbb{Z}$. Thanks to the boundedness of the sequence $\{x_n^{\mathcal{T}}\}_{n\in\mathbb{Z}}$ and by the hypothesis on $T_n - S_n$ we have that $\{y_n K(n+1)\}_{n\in\mathbb{Z}}$ is bounded, and by Theorem 2.5 we have that

$$z_n = \sum_{k=-\infty}^{\infty} G^{\mathcal{S}}_{n,k+1}(T_k - S_k) G^{\mathcal{T}}_{k,m} z K(m)^{-1},$$

and therefore, by the hypothesis on $\mathcal{T} - \mathcal{S}$, we deduce

$$\begin{aligned} \|z_m\|_X &\leq \sum_{k=-\infty}^{\infty} K(k+1) e^{-\alpha_{\mathcal{S}}|m-k-1|} \|T_k - S_k\|_{\mathcal{L}(X)} e^{-\alpha_{\mathcal{T}}|k-m|} \|z\|_X \\ &\leq \frac{e^{-\alpha_{\mathcal{S}}} + e^{-\alpha_{\mathcal{T}}}}{1 - e^{-(\alpha_{\mathcal{S}} + \alpha_{\mathcal{T}})}}\, \epsilon \, \|z\|_X. \end{aligned}$$

The definition of z in m yields

$$z_m = x_m^{\mathcal{T}} - x_m^{\mathcal{S}} = (G^{\mathcal{T}}_{m,m} - G^{\mathcal{S}}_{m,m}) K(m)^{-1} z = (Q_m^{\mathcal{S}} - Q_m^{\mathcal{T}}) K(m)^{-1} z.$$

Consequently,

$$\|(Q_m^{\mathcal{S}} - Q_m^{\mathcal{T}})K(m)^{-1}z\|_X \leq \frac{e^{-\alpha_{\mathcal{S}}} + e^{-\alpha_{\mathcal{T}}}}{1 - e^{-(\alpha_{\mathcal{S}}+\alpha_{\mathcal{T}})}}\, \epsilon\, \|z\|_X,$$

which concludes the proof of the theorem. □

Finally, we state a robustness result for discrete evolution processes with nonuniform exponential dichotomies.

Theorem 2.8 (Robustness for discrete evolution processes)**.** *Let $\mathcal{S} = \{S_n : n \in \mathbb{Z}\}$, $\mathcal{B} = \{B_n : n \in \mathbb{Z}\} \subset \mathcal{L}(X)$ be discrete evolution processes. Assume that $\mathcal{S}$ admits a nonuniform exponential dichotomy with bound $K(n) \leq De^{\nu|n|}$ and exponent $\alpha > \nu$, and that $\mathcal{B}$ satisfies*

$$\|B_k\|_{\mathcal{L}(X)} \leq \delta K(k+1)^{-1}, \ \forall k \in \mathbb{Z},$$

where $\delta > 0$ is such that $\delta < (1 - e^{-\alpha})(1 + e^{-\alpha})^{-1}$. Then the perturbed evolution process $\mathcal{T} = \mathcal{S} + \mathcal{B}$ admits a nonuniform exponential dichotomy with exponent

$$\tilde{\alpha} = -\ln(\cosh\alpha - [\cosh^2\alpha - 1 - 2\delta\sinh\alpha]^{1/2}),$$

and bound

$$\tilde{K}(n) = K(n)\left[1 + \frac{\delta}{(1-\rho)(1-e^{-\alpha})}\right]\max[D_1, D_2],$$

where $\rho := \delta(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1}$, $D_1 := [1 - \delta e^{-\alpha}/(1 - e^{-\alpha-\tilde{\alpha}})]^{-1}$, $D_2 := [1 - \delta e^{-\tilde{\beta}}/(1 - e^{-\alpha-\tilde{\beta}})]^{-1}$ and $\tilde{\beta} := \tilde{\alpha} + \ln(1 + 2\delta\sinh\alpha)$.

The proof of Theorem 2.8 follows, step by step, the proof [23, Theorem 1] with minimal changes. It is important to notice that all the arguments of their proof still hold with the assumption $\alpha > \nu$. We reinforce, that one of our goals is to prove a robustness result of nonuniform exponential dichotomy for *continuous evolution processes* with this same condition on the exponents ($\alpha > \nu$).

3. Nonuniform exponential dichotomy: continuous case

In this section, we consider evolution processes with parameters in $\mathbb{R}$. Inspired by the ideas of Henry [11], we prove theorems that allow us to obtain the continuous versions of the results presented in Section 2. The main theorem of this section is our robustness result for nonuniform exponential dichotomies, namely Theorem 3.11, and we also provide a version of it to be applied in differential equations, Theorem 3.14. In addition, we establish results on the uniqueness and continuous dependence of projections associated with nonuniform exponential dichotomy, Corollary 3.8 and Theorem 3.9, respectively.

We define a *continuous evolution process* in X as follows.

Definition 3.1. *Let $\mathcal{S} := \{S(t,s) : X \to X\,;\, t \geq s,\ t,s \in \mathbb{R}\}$ be a family of continuous operators in a Banach space X. We say that $\mathcal{S}$ is a* ***continuous evolution process*** *in X if*

(1) $S(t,t) = Id_X$, for all $t \in \mathbb{R}$;
(2) $S(t,s)S(s,\tau) = S(t,\tau)$, for $t \geq s \geq \tau$;
(3) $\{(t,s) \in \mathbb{R}^2; t \geq s\} \times X \ni (t,s,x) \mapsto S(t,s)x$ is continuous.

To simplify we usually say that $\mathcal{S} = \{S(t,s) : t \geq s\}$ is an ***evolution process****, whenever is implicit that $\mathcal{S}$ is a continuous evolution process.*

Remark 3.2. *Note that the operators $S(t,s): X \to X$, in the definition above, do not need to be linear. In fact, in Section 5, we study permanence of the nonuniform hyperbolic behavior for nonlinear evolution processes.*

We also recall the notion of a *global solution* for an evolution process.

Definition 3.3. *Let $\mathcal{S} = \{S(t,s) : t \geq s\}$ be an evolution process. We say that $\xi : \mathbb{R} \to X$ is a **global solution** for $\mathcal{S}$ if $S(t,s)\xi(s) = \xi(t)$ for every $t \geq s$.*

*We say that a global solution ξ is **backwards bounded** if there exists $t_0 \in \mathbb{R}$ such that $\xi(-\infty, t_0] = \{\xi(t) : t \leq t_0\}$ is bounded.*

Now, we present the definition of *nonuniform exponential dichotomy* for linear evolution processes:

Definition 3.4. *Let $\mathcal{S} = \{S(t,s)\,;\, t \geq s\} \subset \mathcal{L}(X)$ be a linear evolution process. We say that $\mathcal{S}$ admits a **nonuniform exponential dichotomy** if there exists a family of continuous projections $\{Q(t) : t \in \mathbb{R}\}$ such that*

(1) $Q(t)S(t,s) = S(t,s)Q(s)$, for all $t \geq s$;
(2) $S(t,s) : R(Q(s)) \to R(Q(t))$ is an isomorphism, for $t \geq s$, and we define $S(s,t)$ as its inverse;
(3) There exists a continuous function $K : \mathbb{R} \to [1, +\infty)$ and some constants $\alpha > 0$, $D \geq 1$ and $\nu \geq 0$ such that $K(s) \leq De^{\nu|s|}$ and

$$\begin{aligned} \|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} &\leq K(s)e^{-\alpha(t-s)}, \quad t \geq s; \\ \|S(t,s)Q(s)\|_{\mathcal{L}(X)} &\leq K(s)e^{\alpha(t-s)}, \quad t < s. \end{aligned}$$

Remark 3.5. *This definition also includes uniform exponential dichotomies, when $t \mapsto K(t)$ is bounded, and tempered exponential dichotomies, when $t \mapsto K(t)$ has a sub-exponential growth, see* [17, 23]. *From now on we assume that $K(t) = De^{\nu|t|}$, $t \in \mathbb{R}$.*

In the following result we study each "discretization at instant t" of an evolution process that admits a nonuniform exponential dichotomy.

Theorem 3.6. *Let $\mathcal{S}$ be a continuous evolution process that admits a nonuniform exponential dichotomy with bound $K(t) = De^{\nu|t|}$ and exponent $\alpha > 0$. Then for each $t \in \mathbb{R}$ and $l > 0$ the discrete evolution process*

$$\{S_{m,n}(t) \,:\, m,n \in \mathbb{Z} \text{ with } m \geq n\} := \{S(t+ml, t+nl) \,:\, m,n \in \mathbb{Z} \text{ with } m \geq n\}$$

admits a nonuniform exponential dichotomy with bound $\tilde{K}_t(m) := K(t+ml)$ and exponent $\tilde{\alpha} = \alpha l$.

Proof. Define, for each $t \in \mathbb{R}$, the family of projections $\{Q_m(t) = Q(t+ml) : m \in \mathbb{N}\}$, then

$$\begin{aligned} Q_m(t)S_{m,n}(t) &= Q(t+ml)S(t+ml, t+nl) \\ &= S(t+ml, t+nl)Q(t+nl) \\ &= S_{m,n}(t)Q_n(t), \end{aligned}$$

and the first property is proved. Note that, for $m \geq n$,

$$S_{m,n}(t)|_{R(Q_n(t))} = S(t+ml, t+nl)|_{R(Q(t+nl))}$$

and the right hand side of the equation is an isomorphism, so we define the inverse $S_{n,m}(t) : R(Q(t+ml)) \to R(Q(t+nl))$.

Finally, for $n \geq m$,

$$\begin{aligned} \|S_{n,m}(t)(Id_X - Q_m(t))\|_{\mathcal{L}(X)} &= \|S(t+ml, t+nl)(Id_X - Q(t+nl))\|_{\mathcal{L}(X)} \\ &\leq K(t+ml)e^{-\alpha l(n-m)}, \end{aligned}$$

and, for $n < m$,

$$\begin{aligned} \|S_{n,m}(t)Q_m(t)\|_{\mathcal{L}(X)} &= \|S_{n,m}(t)Q(t+ml)\|_{\mathcal{L}(X)} \\ &\leq K(t+ml)e^{\alpha l(n-m)}. \end{aligned}$$

Therefore, $\{S_{n,m}(t) : n \geq m\}$ admits a discrete nonuniform exponential dichotomy with exponent $\tilde{\alpha} = \alpha l$ and bound $\tilde{K}_t(m) = K(t+ml) \leq De^{\nu|t|}e^{\nu l|m|}$, which concludes the proof. □

Remark 3.7. *In Theorem 3.6, for a fixed $t \in \mathbb{R}$, the discretized evolution process $\{S_n(t) : n \in \mathbb{Z}\}$ possesses with a bound K_t dependent of the time t and the exponent $\tilde{\alpha}$ is independent of t. This is an expected difference with the the case of uniform exponential dichotomy, where both, the bound and the exponent of the discretization are independent of t, see Henry* [11].

Now, as a consequence of Theorem 3.6 and Corollary 2.6, we obtain the uniqueness of the family of projections.

Corollary 3.8 (Uniqueness of the family of projections)**.** *Let $\mathcal{S}$ be an evolution process which admits a nonuniform exponential dichotomy with bound $K(t) = De^{\nu|t|}$, $t \in \mathbb{R}$, and exponent $\alpha > \nu$. Then the family of projections is unique.*

As another application of Theorem 3.6, we prove a result on the continuous dependence of projections.

Theorem 3.9 (Continuous dependence of projections)**.** *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are linear evolution processes with nonuniform exponential dichotomy with projections $\{Q^{\mathcal{S}}(t) : t \in \mathbb{R}\}$ and $\{Q^{\mathcal{T}}(t) : t \in \mathbb{R}\}$ and exponents $\alpha_{\mathcal{T}}, \alpha_{\mathcal{S}}$ and with the same bound $K(t) = De^{\nu|t|}$, for $t \in \mathbb{R}$. If $\nu < \min\{\alpha_{\mathcal{T}}, \alpha_{\mathcal{S}}\}$ and*

$$\sup_{0 \leq t-s \leq 1} \left\{ K(t)\|T(t,s) - S(t,s)\|_{\mathcal{L}(X)} \right\} \leq \epsilon, \tag{3.1}$$

then

$$\sup_{t \in \mathbb{R}} \left\{ K(t)^{-1}\|Q^{\mathcal{T}}(t) - Q^{\mathcal{S}}(t)\|_{\mathcal{L}(X)} \right\} \leq \frac{e^{-\alpha_{\mathcal{S}}} + e^{-\alpha_{\mathcal{T}}}}{1 - e^{-(\alpha_{\mathcal{S}} + \alpha_{\mathcal{T}})}}\epsilon.$$

Proof. Let $t \in \mathbb{R}$, from Theorem 3.6 for $l = 1$, $\{T_n(t) = T(t+n+1, t+n) : n \in \mathbb{Z}\}$ and $\{S_n(t) = S(t+n+1, t+n) : n \in \mathbb{Z}\}$ admit a nonuniform exponential dichotomy with exponents $\alpha_{\mathcal{T}}$ and $\alpha_{\mathcal{S}}$ and the same bound $K_t(n) := K(t+n)$. Now, from Theorem 2.7 we conclude that

$$K(t+n)^{-1}\|Q^{\mathcal{T}}(t+n) - Q^{\mathcal{S}}(t+n)\|_{\mathcal{L}(X)} \leq \frac{e^{-\alpha_{\mathcal{S}}} + e^{-\alpha_{\mathcal{T}}}}{1 - e^{-(\alpha_{\mathcal{S}} + \alpha_{\mathcal{T}})}}\epsilon.$$

Since t is arbitrary and the right-hand side does not depend on t the proof is complete taking $n = 0$. □

Uniqueness and continuous dependence of projections were a simple consequence of Theorem 3.6, and of course the results in the discrete case. However, to prove our robustness result, we will need a sort of a reciprocal result of Theorem 3.6.

Theorem 3.10. *Let $\mathcal{S} = \{S(t,s) : t \geq s\} \subset \mathcal{L}(X)$ be a continuous evolution process. Suppose that*

(1) there exist $l > 0$ and $\nu \geq 0$ such that

$$L(\nu, l) := \sup_{0 \leq t-s \leq l} \left\{ \|S(t,s)\|_{\mathcal{L}(X)} \, e^{-\nu|t|} \right\} < +\infty,$$

(2) for each $t \in \mathbb{R}$ the discretized process,

$$\{T_{n,m}(t),\, n \geq m\} = \{S(t+nl, t+ml),\, n \geq m\}$$

possesses a nonuniform exponential dichotomy with bound $K_t(\cdot) : \mathbb{Z} \to [1, +\infty)$, with $K_t(m) \leq De^{\nu|t+m|}$ and exponent $\alpha > 0$ independent of t.

If $\nu\, l < \alpha$, the evolution process $\mathcal{S}$ admits a nonuniform exponential dichotomy with exponent $\hat{\alpha} = (\alpha - \nu l)/l$ and bound

$$\hat{K}(s) = D^2 e^{2\alpha} \max\{L(\nu, l), L(\nu, l)^2\}\, e^{2\nu|s|}.$$

Proof. First, we fix $t \in \mathbb{R}$ and define the linear operator $T_n(t) := T_{n+1,n}(t)$, for each $n \in \mathbb{Z}$. Then for each discrete evolution process $\{T_n(t) : n \in \mathbb{Z}\}$, there exists a family of projections $\{Q_n(t) : n \in \mathbb{Z}\}$ such that the nonuniform exponential dichotomy conditions are satisfied.

For each fixed $k \in \mathbb{Z}$ we have

$$T_{n+k}(t) = T_n(t + kl), \quad \forall n \in \mathbb{Z}.$$

Then this linear operator generates the same evolution process with associated projections $\{Q_{n+k}(t)\}_{n\in\mathbb{Z}}$ and $\{Q_n(t+kl)\}_{n\in\mathbb{Z}}$. Thus by uniqueness of the projections for the discrete case, namely Corollary 2.6, we obtain that for all $n, k \in \mathbb{Z}$,

$$Q_{n+k}(t) = Q_n(t + kl).$$

Now, for all $t \in \mathbb{R}$ we define $Q(t) := Q_0(t)$. These projections are the candidates to obtain the nonuniform exponential dichotomy.

Let us now prove the boundedness in the case $t \geq s$.

Claim 1: If $t \geq s$, then

$$\|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} \leq \hat{K}(s) e^{-\hat{\alpha}(t-s)},$$

where $\hat{K}$ is defined in the statement of the theorem.

In fact, choose $n \in \mathbb{N}$, such that $nl + s \leq t < (n+1)l + s$, then we write

$$S(t,s)(Id_X - Q(s)) = S(t, s+nl) S(s+nl, s)(Id_X - Q_0(s)).$$

Thus, by hypothesis,

$$\|S(s+nl, s)(Id_X - Q_0(s))\|_{\mathcal{L}(X)} = \|T_{n,0}(s)(Id_X - Q_0(s))\|_{\mathcal{L}(X)} \leq K_s(0) e^{-\alpha n},$$

which implies that

$$\begin{aligned} \|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} &\leq \|S(t, s+nl)\|_{\mathcal{L}(X)} K_s(0) e^{-\alpha n} \\ &= K(s) e^{\alpha(t-nl-s)/l} \|S(t, s+nl)\|_{\mathcal{L}(X)} e^{-\alpha(t-s)/l} \\ &\leq De^{\nu|s|}\, e^{\alpha}\, e^{\nu|t|}\, L(\nu, l)\, e^{-\alpha(t-s)/l}, \end{aligned}$$

where was used the fact that $0 \leq t - s - nl < l$.

Now, note that, if $t \geq s \geq 0$ we have

$$\nu|t| - \alpha(t-s)/l = -(\alpha - \nu l)(t-s)/l + \nu|s|,$$

and, for $s \leq t \leq 0$,

$$\nu|t| - \alpha(t-s)/l = -(\alpha + \nu l)(t-s)/l + \nu|s|,$$

then choose $\hat{\alpha} = (\alpha - \nu l)/l$. Thus, we obtain for $t \geq s \geq 0$ and $s \leq t \leq 0$ that

$$\begin{aligned}\|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} &\leq De^{\nu|s|}\, e^{\alpha}\, e^{\nu|t|}\, L(\nu,l)\, e^{-\alpha(t-s)/l} \\ &\leq DL(\nu,l)\, e^{\alpha}\, e^{2\nu|s|} e^{-\hat{\alpha}(t-s)}.\end{aligned}$$

Finally, for $t \geq 0 \geq s$ we have

$$\begin{aligned}\|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} &= \|S(t,s)(Id_X - Q(s)^2)\|_{\mathcal{L}(X)} \\ &\leq \|S(t,0)(Id_X - Q(0)\|_{\mathcal{L}(X)}\, \|S(0,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} \\ &\leq D^2 L(\nu,l)^2\, e^{2\alpha}\, e^{2\nu|s|} e^{-\hat{\alpha}(t-s)}.\end{aligned}$$

Therefore, for $t \geq s$,

$$\|S(t,s)(Id_X - Q(s))\|_{\mathcal{L}(X)} \leq D^2 e^{2\alpha} \max\{L(\nu,l), L(\nu,l)^2\} e^{2\nu|s|} e^{-\hat{\alpha}(t-s)}$$

and the first claim is proved.

Now, to prove the other inequality, for $t < s$, we take $n \leq 0$ such that $s + nl \leq t < s + (n+1)l$, and define for $z \in R(Q(s))$ the linear operator

$$S(t,s)z := S(t, s+nl) \circ [T_{0,n}(s)|_{R(Q_n(s))}]^{-1} z.$$

In other words,

$$S(t,s)z = S(t, s+nl) \circ T_{n,0}(s) z.$$

Claim 2: If $t < s$, we have

$$\|S(t,s)Q(s)\|_{\mathcal{L}(X)} \leq \hat{K}(s) e^{\hat{\alpha}(t-s)}.$$

Indeed, for $x \in X$ and $s + nl \leq t < s + (n+1)l$, for $n \leq 0$, by hypothesis,

$$\|T_{n,0}(s) Q_0(s) x\|_X \leq K_s(0) e^{\alpha n} \|x\|_X.$$

Hence, by a similar argument to that in the proof of Claim 1 we obtain that

$$\|S(t,s)Q(s)x\|_X \leq \|S(t, s+nl)\|_{\mathcal{L}(X)} De^{\nu|s|} e^{\alpha n} \|x\|_X \leq \hat{K}(s) e^{\hat{\alpha}(t-s)} \|x\|_X.$$

Now, to conclude the assertion we take the supremum for $\|x\|_X = 1$.

Claim 3: For all $t_0 \in \mathbb{R}$ we characterize the kernel of $Q(t_0)$, $N(Q(t_0)) = \{z \in X : Q(t_0)z = 0\}$, as

$$N(Q(t_0)) = \{z \in X : [t_0, +\infty) \ni t \mapsto S(t,t_0)z \text{ is bounded}\}.$$

Let $z \in N(Q(t_0))$, so by definition $Q(t_0)z = 0$ and for $t \geq t_0$ we can use Claim 1 to obtain

$$\|S(t,t_0)z\|_X = \|S(t,t_0)(Id_X - Q(t_0))z\|_X \leq \hat{K}(t_0) e^{-\hat{\alpha}(t-t_0)} \|z\|_X.$$

Therefore, $[t_0, +\infty) \ni t \mapsto S(t,t_0)z$ is bounded.

On the other hand, if $z \notin N(Q((t_0)))$ and $n > 0$,

$$\begin{aligned}\|Q(t_0)z\|_X &\leq \|T_{0,n}(t_0) Q_n(t_0)\|_{\mathcal{L}(X)} \|T_{n,0}(t_0)z\|_X \\ &\leq De^{\nu|t_0|} e^{\nu|n|} e^{-\alpha n} \|S(t_0 + nl, t_0)z\|_X.\end{aligned}$$

Thus, we obtain

$$\|Q(t_0)z\|_X D^{-1} e^{-\nu|t_0|} e^{n(\alpha-\nu)} \leq \|S(t_0 + nl, t_0)z\|_X.$$

Consequently, as $\nu < \alpha$ we have that $[t_0, +\infty) \ni t \mapsto S(t,t_0)z$ is not bounded.

Note that the last assertion implies that

$$S(t,t_0)N(Q(t_0)) \subset N(Q(t)).$$

Claim 4: The linear operator

$$S(t,t_0) : R(Q(t_0)) \to X$$

is injective for all $t \geq t_0$.

Indeed, let $z \in R(Q(t_0))$ with $S(t,t_0)z = 0$. Choose $n \in \mathbb{N}$ so that $t \leq nl + t_0$, then

$$0 = S(t_0 + nl, t)0 = S(t_0 + nl, t)S(t, t_0)z = T_{n,0}(t_0)z,$$

this implies that $z \in N(T_{n,0}(t_0)|_{R(Q_0(t_0))}) = \{0\}$.

Claim 5: For all $t_0 \in \mathbb{R}$ the range of $Q(t_0)$ is

$$R(Q(t_0)) = \{z \in X : \text{ there exists a backwards bounded solution } \xi \text{ with } \xi(t_0) = z\}.$$

Let $z \in R(Q(t_0))$ and $t < t_0$, then take $n \in \mathbb{Z}$ such that $t \in [t_0 + nl, t_0 + (n+1)l]$ and define

$$\xi(t) := S(t, t_0 + nl)T_{n,0}(t_0)z = S(t, t_0)z.$$

Now, choose $x \in X$ so that $z = Q(t_0)x$, thus by Claim 2

$$\|\xi(t)\|_X \leq \hat{K}(t_0)e^{\hat{\alpha}(t-t_0)}\|x\|_X.$$

Thus, ξ is a backward bounded solution such that $\xi(t_0) = z$. Suppose that $z \notin R(Q(t_0))$ and that there exists $\xi : \mathbb{R} \to X$ a global solution such that $\xi(t_0) = z$. For $n \leq 0$ we can write $z = S(t_0, t_0 + nl)\xi(t_0 + nl)$, thus

$$\begin{aligned} \|(Id_X - Q(t_0))z\|_X &\leq \|S(t_0, t_0 + nl)(Id_X - Q(t_0 + nl))\|_{\mathcal{L}(X)} \|\xi(t_0 + nl)\|_X \\ &\leq De^{\nu|t_0|}e^{\nu|n|}e^{\alpha n}\|\xi(t_0 + nl)\|_X. \end{aligned}$$

Therefore,

$$\|(Id_X - Q(t_0))z\|_X D^{-1}e^{-\nu|t_0|}e^{n(\nu-\alpha)} \leq \|\xi(t_0 + nl)\|_X.$$

Since $\nu < \alpha$, it follows that ξ is not backwards bounded, and the proof of Claim 5 is complete.

Claim 6: $S(t,t_0)R(Q(t_0)) = R(Q(t))$.

Indeed, if $z \in R(Q(t_0))$, then there exists a backwards bounded solution ξ through z in $t = t_0$. Thus, ξ is also a solution through $S(t,t_0)z$ in time t and we see that $S(t,t_0)z \in R(Q(t))$. On the other hand, if $z \in R(Q(t))$, there is a backwards bounded solution ξ with $\xi(t) = z$. Therefore, if $n \in \mathbb{Z}$ such that $nl + t \leq t_0 \leq t$, define

$$x = S(t_0, nl + t)S(nl + t, t)z \in R(Q(t_0)).$$

Therefore, $S(t,t_0)x = z$ and we conclude that $S(t,t_0)|_{R(Q(t_0))}$ is an isomorphism.

Finally, we prove that the family of projections commutes with the evolution process.

Claim 7: $Q(t)S(t,s) = S(t,s)Q(s)$. For $z \in X$, we have that

$$S(t,t_0)z = S(t,t_0)(Id_X - Q(t_0))z + S(t,t_0)Q(t_0)z.$$

Now, as $(Id_X - Q(t_0))z \in N(Q(t_0))$ and $S(t,t_0)Q(t_0)z \in R(Q(t))$, applying $Q(t)$ we obtain

$$Q(t)S(t,t_0)z = S(t,t_0)Q(t_0)z.$$

The proof is complete. □

We are ready to present the main result of this section.

Theorem 3.11 (Robustness for continuous evolution processes)**.** *Let $\mathcal{S} = \{S(t,s) : t \geq s\} \subset \mathcal{L}(X)$ be an evolution process that admits a nonuniform exponential dichotomy with bound $K(s) = De^{\nu|s|}$ and exponent $\alpha > \nu$. Assume that*

$$L_{\mathcal{S}}(\nu) := \sup_{0 \leq t-s \leq 1} \left\{ e^{-\nu|t|} \|S(t,s)\|_{\mathcal{L}(X)} \right\} < +\infty. \tag{3.2}$$

Then there exists $\epsilon > 0$ such that if $\mathcal{T} = \{T(t,s) : t \geq s\}$ is an evolution process such that

$$\sup_{0 \leq t-s \leq 1} \left\{ K(t) \|S(t,s) - T(t,s)\|_{\mathcal{L}(X)} \right\} < \epsilon, \tag{3.3}$$

then $\mathcal{T}$ admits a nonuniform exponential dichotomy with exponent $\hat{\alpha} := \tilde{\alpha} - \nu$ and bound

$$\hat{K}(s) = \tilde{D}^2 e^{2\tilde{\alpha}} \max\{L_{\mathcal{T}}(\nu), L_{\mathcal{T}}(\nu)^2\}\, e^{2\nu|s|}, \tag{3.4}$$

where $\tilde{D} := D(1 + \epsilon/(1-\rho)(1-e^{-\alpha})) \max\{D_1, D_2\}$, and $\rho, \tilde{\alpha}, D_1$ and D_2 are the same as in Theorem 2.8.

Proof. Let $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$, then, by Theorem 3.6, the discrete evolution process $\{S_n(t_0) := S(t_0+n+1, t_0+n) : n \in \mathbb{Z}\}$ admits a nonuniform exponential dichotomy with bound $K_t(n) \leq De^{\nu(|t+n|)}$ and exponent $\alpha > 0$. Let $\epsilon > 0$ be such that $\epsilon < (1-e^{-\alpha})/(1+e^{-\alpha})$ and $\mathcal{T} = \{T(t,s) : t \geq s\}$ an evolution process that satisfies (3.3). Let $\{T_n(t_0) : n \in \mathbb{Z}\}$ be the discretization of $\mathcal{T}$ at t_0 and define, for each $n \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$, the linear bounded operator

$$B_n(t_0) := T_n(t_0) - S_n(t_0).$$

Hence, from (3.3), we have that

$$\|B_n(t_0)\|_{\mathcal{L}(X)} < \epsilon K_{t_0}(n+1)^{-1}.$$

Therefore, by Theorem 2.8, the discrete evolution process $T_n(t_0) = S_n(t_0) + B_n(t_0)$ admits a nonuniform exponential dichotomy with exponent

$$\tilde{\alpha} := -\ln(\cosh\alpha - [\cosh^2\alpha - 1 - 2\epsilon\sinh\alpha]^{1/2}),$$

and bound

$$\tilde{K}_{t_0}(n) := K_{t_0}(n) \left[1 + \frac{\epsilon}{(1-\rho)(1-e^{-\alpha})} \right] \max[D_1, D_2],$$

where D_1, D_2, ρ are constants that can be found in Theorem 2.8.

Since each discretization at time t has the same exponent $\alpha > 0$ we see that ϵ can be choose independent of t. Thus for each $t \in \mathbb{R}$, the discrete evolution process $\{T_n(t) : n \in \mathbb{Z}\}$ admits nonuniform exponential dichotomy with bound $\tilde{K}_t(n)$ and exponent $\tilde{\alpha}$ defined above. Then condition (2) of Theorem 3.10 holds true for $\mathcal{T}$.

Moreover, from (3.3), $\mathcal{T}$ satisfies

$$\begin{aligned} \|T(t,s)\|_{\mathcal{L}(X)} &\leq \epsilon K(t)^{-1} + \|S(t,s)\|_{\mathcal{L}(X)} \\ &\leq \epsilon + \|S(t,s)\|_{\mathcal{L}(X)}, \text{ for } 0 \leq t-s \leq 1 \end{aligned}$$

then $\sup_{0 \leq t-s \leq 1}\{e^{-\nu|t|}\|T(t,s)\|_{\mathcal{L}(X)}\}$ is finite. Finally, note that it is possible to choose $\epsilon > 0$ small such that $\tilde{\alpha} > \nu$. Therefore, Theorem 3.10 implies that $\mathcal{T}$ admits nonuniform exponential dichotomy with bound $\hat{K}$ defined in (3.4) and exponent $\hat{\alpha} = \tilde{\alpha} - \nu > 0$. □

Remark 3.12. *Assumption* (3.2) *on the growth of* $\mathcal{S}$ *(analogous to that of* [11, Theorem 7.6.10] *with* $\nu = 0$*, that is, the uniform case) is expected for evolution processes that admit nonuniform exponential dichotomies, see Barreira and Valls* [15] *or Example 4.1 in Section 4.*

Remark 3.13. *Theorem 3.11 allows us to see the robustness as an open property. In fact, let* $\mathfrak{S}_\nu$ *be the space of all evolutionary processes that satisfy* (3.2) *and define a distance in* $\mathfrak{S}_\nu$ *as*

$$d_\nu(\mathcal{S},\mathcal{T}) := \sup_{0\leq t-s\leq 1} \left\{ e^{\nu|t|} \|S(t,s)-T(t,s)\|_{\mathcal{L}(X)} \right\}.$$

Then, from Theorem 3.11 we see that if $\mathcal{S}\in\mathfrak{S}_\nu$ *admits a nonuniform exponential dichotomy with bound* $K(t) = De^{\nu|t|}$ *and exponent* $\alpha>\nu$*, then there exists* $\epsilon>0$ *such that every evolution process* $\mathcal{T}$ *in a* ϵ*-neighborhood of* $\mathcal{S}$ *admits a nonuniform exponential dichotomy with bound and exponent given in Theorem 3.11.*

Now, we present another formulation of Theorem 3.11 that allows us to apply the result for differential equations in Banach spaces.

Theorem 3.14. *Let* $\mathcal{S} = \{S(t,s) : t\geq s\}\subset \mathcal{L}(X)$ *be an evolution process that admits a nonuniform exponential dichotomy with bound* $K(t) = De^{\nu|t|}$*,* $t\in\mathbb{R}$*, and exponent* $\alpha>\nu$*. Assume that*

$$L_{\mathcal{S}}(\nu) := \sup_{0\leq t-s\leq 1} \left\{ e^{-\nu|t|} \|S(t,s)\|_{\mathcal{L}(X)} \right\} < +\infty.$$

Let $\{B(t) : t\in\mathbb{R}\}\subset\mathcal{L}(X)$ *so that* $\mathbb{R}\ni t\mapsto B(t)x$ *is continuous for all* $x\in X$ *and*

$$\|B(t)\|_{\mathcal{L}(X)} < \delta e^{-3\nu|t|}, \quad \forall\, t\in\mathbb{R}.$$

Then any evolution process that satisfies the integral equation

$$T(t,s) = S(t,s) + \int_s^t S(t,\tau)B(\tau)T(\tau,s)d\tau \in \mathcal{L}(X), \quad t\geq s, \tag{3.5}$$

admits a nonuniform exponential dichotomy for suitably small $\delta>0$*, with bound and exponent given in Theorem 3.11.*

Proof. Let $\mathcal{T} = \{T(t,s) : t\geq s\}$ be a evolution process satisfying (3.5). Then

$$\|T(t,s)\|_{\mathcal{L}(X)} \leq \|S(t,s)\|_{\mathcal{L}(X)} + \int_s^t \|S(t,\tau)\|_{\mathcal{L}(X)}\|B(\tau)\|_{\mathcal{L}(X)}\, \|T(\tau,s)\|_{\mathcal{L}(X)} d\tau.$$

Thus, fix s and define the function $\phi(t) = e^{-\nu|t|}\|T(t,s)\|_{\mathcal{L}(X)}$, for $t\leq s+1$,

$$\phi(t) \leq L_{\mathcal{S}}(\nu) + L_{\mathcal{S}}(\nu)\int_s^t \|B(\tau)\|_{\mathcal{L}(X)} e^{\nu|\tau|}\phi(\tau)d\tau$$

By Grönwall's inequality, we obtain that

$$\phi(t) \leq L_{\mathcal{S}}(\nu) e^{L_{\mathcal{S}}(\nu)\int_s^t \|B(\tau)\|_{\mathcal{L}(X)} e^{\nu|\tau|}d\tau}, \text{ for } t\leq s+1.$$

Therefore,

$$L_{\mathcal{T}}(\nu) := \sup_{0\leq t-s\leq 1} \left\{ e^{-\nu|t|} \|T(t,s)\|_{\mathcal{L}(X)} \right\} < +\infty.$$

Now, for $0 \leq t-s \leq 1$,

$$\begin{aligned}\|S(t,s)-T(t,s)\|_{\mathcal{L}(X)} &\leq \int_s^t e^{\nu(|t|+|\tau|)} L_{\mathcal{S}}(\nu)\|B(\tau)\|_{\mathcal{L}(X)} L_{\mathcal{T}}(\nu) d\tau \\ &= L_{\mathcal{T}}(\nu) L_{\mathcal{S}}(\nu)\, e^{\nu|t|} \int_s^t e^{\nu|\tau|}\|B(\tau)\|_{\mathcal{L}(X)} d\tau.\end{aligned}$$

Then

$$K(t)\|S(t,s)-T(t,s)\|_{\mathcal{L}(X)} \leq L_{\mathcal{T}}(\nu) L_{\mathcal{S}}(\nu) D\,\delta,$$

and choose $\delta > 0$ suitably small in order to use Theorem 3.11 and conclude the proof. □

Theorem 3.14 is very useful when dealing with differential equations. In fact, let $\{A(t) : t \in \mathbb{R}\}$ be a family of linear operators, possibly unbounded, and consider

$$\dot{x} = A(t)x, \quad x(s) = x_s \in X. \tag{3.6}$$

Suppose that for each $s \in \mathbb{R}$ and $x_s \in X$ there exists a unique solution $x(\cdot, s, x_s) : [s, +\infty) \to X$. Thus there exists an evolution process $\mathcal{S} = \{S(t,s) : t \geq s\}$ defined by $S(t,s)x_s := x(t,s,x_s)$ for each $t \geq s$.

To study robustness of nonuniform exponential dichotomy of problem (3.6), we suppose that $\mathcal{S}$ admits a nonuniform exponential dichotomy and we want to know for which class of $\{B(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$ the perturbed problem

$$\dot{x} = A(t)x + B(t)x, \quad x(s) = x_s \in X, \tag{3.7}$$

admits a nonuniform exponential dichotomy with bound $K(t) = De^{\nu|t|}$ and exponent $\alpha > 0$. In this way, Theorem 3.14 ensures that the nonuniform hyperbolicity is preserved for exponentially small perturbations. In other words, if the norm of the perturbation of B does not grow more than $e^{-3\nu|t|}$ for $\nu < \alpha$, then the perturbed problem (3.7) admits a nonuniform exponential dichotomy.

Remark 3.15. *In Barreira and Valls* [22] *a version of Theorem 3.14 is proved under different assumptions. They considered a general growth rate $\rho(t)$ for the nonuniform exponential dichotomy and proved that if $\alpha > 2\nu$ and $B : \mathbb{R} \to \mathcal{L}(X)$ is continuous satisfying $\|B(t)\|_{\mathcal{L}(X)} \leq \delta e^{-3\nu|\rho(t)|}\rho'(t)$, for all $t \in \mathbb{R}$, then the perturbed problem* (3.7) *admits ρ-nonuniform exponential dichotomy. We note that our method does not work for general growth rates $\rho(t)$. We treat the case $\rho(t) = t$, the other cases can be achieved by a change of scaling in time, and since our condition on the exponents is only $\alpha > \nu$ we obtain a improvement of their robustness result.*

For $A(t)$ bounded and B satisfying $\|B(t)\|_{\mathcal{L}(X)} \leq \delta e^{-2\nu|t|}$, for all t, Barreira and Valls proved in [20] *a result similar to Theorem 3.14. Their result also holds when the evolution process $\mathcal{S}$ is invertible (which means that $S(t,s)$ is invertible for every $t \geq s$), even if $A(t)$ is not bounded.In their proof, thanks to invertibility, they can write explicit expressions of the projections for the perturbed evolution process. In applications to partial differential equations, in general, $A(t)$ is not bounded, see Section 4.*

4. An application in infinite-dimensional differential equations

In this section, we show an application of the robustness result in order to obtain examples of evolution processes that admits nonuniform exponential dichotomies. Inspired in an example of [15], we provide an evolution process defined on a Banach space that admits a nonuniform exponential dichotomy. Then, apply Theorem 3.14 to study for which class of perturbations the nonuniform hyperbolicity will be preserved.

Let X and Y be two Banach spaces. Suppose that A is a generator of a C_0-semigroup $\{e^{At} : t \geq 0\}$ in X and $B \in \mathcal{L}(Y)$ with $\sigma(A) \subset (-\infty, -\omega)$ and $\sigma(B) \subset (\omega, +\infty)$, for some $\omega > 0$, and there exists $M \geq 1$ such that

$$\begin{aligned} \|e^{A(t-s)}\|_{\mathcal{L}(X)} &\leq Me^{-\omega(t-s)},\ t \geq s; \\ \|e^{B(t-s)}\|_{\mathcal{L}(Y)} &\leq Me^{\omega(t-s)},\ t < s. \end{aligned}$$

Remark 4.1. *Let $\mathcal{C}$ be a generator of a hyperbolic C_0-semigroup $\{e^{\mathcal{C}t} : t \geq 0\}$, i.e., the associated evolution processes $\{e^{\mathcal{C}(t-s)} : t \geq s\}$ admits an uniform exponential dichotomy with a single projection $Q(t) = Q \in \mathcal{L}(X)$ for every $t \in \mathbb{R}$. Then, there is a decomposition $X = X^u \oplus X^s$ such that $A := C|_{X^s}$ and $B = \mathcal{C}|_{X^u}$ satisfy the conditions above over X^s and X^u, respectively, see* [24, 8, 11].

Let $\omega > a > 0$ and define the linear operator in $Z = X \times Y$

$$\mathcal{A}(t) := \begin{bmatrix} A - at\sin(t)Id_X & 0 \\ 0 & B + at\sin(t)Id_Y \end{bmatrix}.$$

Consider the differential equation

$$\dot{z} = \mathcal{A}(t)z, \quad z(s) = z_s \in Z. \tag{4.1}$$

Then, the evolution process associated with problem (4.1) is defined by

$$T(t,s) = (U(t,s), V(t,s))$$

where

$$\begin{aligned} U(t,s) &= e^{A(t-s)} \exp\Big\{ -\int_s^t a\tau\sin(\tau)d\tau \Big\} \text{ and} \\ V(t,s) &= e^{B(t-s)} \exp\Big\{ \int_s^t a\tau\sin(\tau)d\tau \Big\} \end{aligned}$$

are evolution processes in X and Y, respectively.

The proof of the next result is inspired by Proposition 2.3 of [15].

Proposition 4.2. *Let $\mathcal{T} = \{T(t,s) : t \geq s\}$ be the evolution process defined above. Then $\mathcal{T}$ admits a nonuniform exponential dichotomy with bound $K(t) = Me^{2a(1+|t|)}$ and exponent $\alpha = \omega - a > 0$.*

Proof. Define the linear operators $P(t) = P_X$ and $Q(t) = P_Y$ for all $t \in \mathbb{R}$ where P_X and P_Y are the canonical projections onto X and Y, respectively. Then $T(t,s)P(s) = U(t,s)$ and $T(t,s)Q(s) = V(t,s)$ for all $t \geq s$.

In this way we have that P_X commutes with $T(t,s)$, for all $t \geq s$ and since $B \in \mathcal{L}$(Y) generates a group in Y we have that $V(t,s)$ is an isomorphism over Y. Note that

$$\begin{aligned} \|U(t,s)\|_{\mathcal{L}(X)} &= \exp\Big\{ -\int_s^t a\tau\sin(\tau)d\tau \Big\} \|e^{A(t-s)}\|_{\mathcal{L}(X)} \\ &\leq Me^{-\omega(t-s)+at\cos(t)-as\cos(s)-a\sin(t)+a\sin(s)} \end{aligned}$$

Now, proceed as in Proposition 2.3 of [15] to obtain

$$U(t,s) \leq Me^{(-\omega+a)(t-s)+2a|s|+2a}, \text{ for } t \geq s. \tag{4.2}$$

Similarly, we obtain that

$$\|V(t,s)\|_{\mathcal{L}(Y)} \leq Me^{2a+2a|s|}e^{(\omega+a)(t-s)} \text{ for } t < s. \tag{4.3}$$

Therefore, $\mathcal{T}$ admits a nonuniform exponential dichotomy with bound $K(t) = Me^{2a(1+|t|)}$ and exponent $\alpha = \omega - a > 0$. □

Now, apply Theorem 3.14 to Example 4.1.

Theorem 4.3. *Consider for each $\epsilon > 0$ the operator $B_\epsilon(t) \in \mathcal{L}(Z)$ such that $\|B_\epsilon(t)\| \leq \epsilon e^{-6a|t|}$, and define the operator*

$$\mathcal{A}_\epsilon(t) := \mathcal{A}(t) + B_\epsilon(t), \quad \forall\, t \in \mathbb{R}.$$

If $\omega > 3a$, there exists $\epsilon > 0$ such that the evolution process associated with the problem

$$\dot{x} = \mathcal{A}_\epsilon(t)x, \quad x(s) = x_s \in Z. \tag{4.4}$$

admits a nonuniform exponential dichotomy.

Proof. Let us prove first that the evolution problem associated with (4.1) satisfies

$$\sup_{0\leq t-\tau\leq 1} \left\{ e^{-\nu|t|}\|T(t,\tau)\|_{\mathcal{L}(Z)} \right\} < +\infty. \tag{4.5}$$

In fact, we have for $t \geq s$ that

$$\|T(t,s)\|_{\mathcal{L}(Z)} \leq \|U(t,s)\|_{\mathcal{L}(X)} + \|V(t,s)\|_{\mathcal{L}(Y)},$$

where U and V are the evolution processes defined in the proof of Proposition 4.2. Then it is enough to prove that each evolution process satisfies (4.5) in the corresponding space. From (4.2) we have that

$$e^{-2a|t|}\|U(t,s)\|_{\mathcal{L}(X)} \leq Me^{2a+2a(|s|-|t|)}e^{-(\omega-a)(t-s)} \leq Me^{2a}e^{-(\omega-3a)(t-s)}.$$

Therefore

$$\sup_{0\leq t-s\leq 1} \{e^{-2a|t|}\|U(t,s)\|_{\mathcal{L}(X)}\} < +\infty, \text{ for all } t \geq s.$$

Now, since $\|e^{B(t-s)}\|_{\mathcal{L}(Y)} \leq \tilde{M}e^{\beta(t-s)}$ for some $\tilde{M} \geq 1$ and $\beta > 0$, for every $t \geq s$, we have that

$$\|V(t,s)\|_{\mathcal{L}(Y)} = \exp\left\{ \int_s^t a\tau \sin(\tau)d\tau \right\}\|e^{B(t-s)}\|_{\mathcal{L}(Y)} \leq \tilde{M}e^{4a+2a|t|}e^{(\beta+a)(t-s)},$$

which implies that

$$\sup_{0\leq t-s\leq 1} \{e^{-2a|t|}\|V(t,s)\|_{\mathcal{L}(Y)}\} < +\infty.$$

Now, from Proposition 4.2, $\mathcal{T}$ admits a nonuniform exponential dichotomy where the bound is $K(s) = Me^{2a+2a|s|}$ and exponent $\alpha = \omega - a > 0$. Since $\nu := 2a$ is such that $\alpha > \nu$, we apply Theorem 3.14 to conclude that the evolution process generated by (4.4) admits a nonuniform exponential dichotomy. □

Remark 4.4. *Note that, in Theorem 3.14 the assumption $\alpha > \nu$ of Theorem 4.3 is expressed by $\omega > 3a$. On the other hand, to apply Theorem 1 of* [22] *the hypothesis must be $\omega > 5a$, because their condition is $\alpha > 2\nu$.*

5. Persistence of nonuniform hyperbolic solutions

In this section, we study nonlinear evolution processes associated with a semilinear differential equation. Inspired by [24], we study persistence of *nonuniform hyperbolic solutions* under perturbation for evolutions processes in Banach spaces. More precisely, we use *Green's function* to characterize bounded global solutions for semilinear differential equations and conclude that nonuniform hyperbolic solutions are *isolated* in the set of bounded continuous functions, see Theorem 5.3. Finally, in Theorem 5.4, we provide conditions to prove that nonuniform hyperbolic solutions persist under perturbations.

Consider a semi-linear differential equation

$$\dot{y} = A(t)y + f(t,y), \quad y(s) = y_s. \tag{5.1}$$

Assume that f is continuous in the first variable and locally Lipschitz in the second and that $\{A(t) : t \in \mathbb{R}\}$ is a family of linear (possibly unbounded) operators associated with a linear bounded evolution process $\mathcal{T} = \{T(t,s) : t \geq s\}$, i.e., for each $s \in \mathbb{R}$ and $x_0 \in X$ the mapping $[s, +\infty) \ni t \to T(t,s)x_0$ is the solution of

$$\dot{x} = A(t)x, \;\; x(s) = x_0.$$

Then we have a *local mild solution* for problem (5.1), that is, for each $(s, y_s) \in \mathbb{R} \times X$ there exist $\sigma = \sigma(s, y_s) > 0$ and a solution y of the integral equation

$$y(t,s;y_s) = T(t,s)y_s + \int_s^t T(t,\tau) f(\tau, y(\tau,s;y_s)) d\tau, \tag{5.2}$$

for all $t \in [s, s+\sigma)$.

If for each $(s, y_s) \in \mathbb{R} \times X$, $\sigma(s, y_s) = +\infty$, we can consider the evolution process $S_f(t,s)y_s = y(t,s;y_s)$. We refer to $\mathcal{S}_f = \{S_f(t,s) : t \geq s\}$ as the evolution process obtained by a non-linear perturbation f of $\mathcal{T}$.

Suppose additionally that $f : \mathbb{R} \times X \to X$ is differentiable with continuous derivatives. Let ξ be a global solution of $\mathcal{S}_f$ (see Definition 3.3), and $\mathcal{L}_f = \{L_f(t,s) : t \geq s\}$ is the linearized evolution process of $\mathcal{S}_f$ on ξ. Thus $\mathcal{L}_f$ satisfies

$$L_f(t,s) = T(t,s) + \int_s^t T(t,\tau) D_x f(\tau, \xi(\tau)) L_f(\tau,s) d\tau.$$

Definition 5.1. *If $\mathcal{L}_f$ admits a nonuniform exponential dichotomy we say that ξ is a* ***nonuniform hyperbolic solution*** *for $\mathcal{S}_f$.*

In Barreira and Valls [15] this notion is called *nonuniformly hyperbolic trajectories.*

Remark 5.2. *Let φ be a global solution for $\mathcal{S}_f$. Then*

$$\varphi(t) = L_f(t,s)\varphi(s) + \int_s^t L_f(t,\tau)[f(\tau,\varphi(\tau)) - D_x f(\tau,\xi(\tau))\varphi(\tau)] d\tau, \quad t \geq s. \tag{5.3}$$

In particular, the global bounded solution ξ satisfies the integral equation (5.3)*.*

The next result allows us to characterize bounded nonuniform hyperbolic solutions.

Theorem 5.3. *Assume that there is a global nonuniform hyperbolic solution ξ for $\mathcal{S}_f$ and that $\mathcal{L}_f$ admits a nonuniform exponential dichotomy with bound $K(s) =$*

$De^{\nu|s|}$, $s \in \mathbb{R}$, exponent $\alpha > \nu$, and family of projections $\{Q(t) : t \in \mathbb{R}\}$. If φ is a bounded global solution for $\mathcal{S}_f$, then φ satisfies

$$\varphi(t) = \int_{-\infty}^{+\infty} G_f(t,\tau)[f(\tau,\varphi(\tau)) - D_x f(\tau,\xi(\tau))\varphi(\tau)]d\tau,$$

where G_f is the Green's function associated with the evolution process $\mathcal{L}_f$,

$$G_f(t,s) = \begin{cases} L_f(t,s)(Id_X - Q(s)), & \text{if } t \geq s, \\ -L_f(t,s)Q(s) & \text{if } t < s. \end{cases}$$

Moreover, if ξ is a bounded nonuniform hyperbolic solution of $\mathcal{S}_f$ and

$$\rho(\epsilon) = \sup_{\|x\| \leq \epsilon} \sup_{t \in \mathbb{R}} \frac{e^{\nu|t|} \|f(t,\xi(t)+x) - f(t,\xi(t)) - D_x f(t,\xi(t))x\|}{\|x\|} \to 0, \text{ as } \epsilon \to 0, \tag{5.4}$$

*then ξ is **isolated** in the set of bounded and continuous functions $C_b(\mathbb{R}, X)$, i.e., there is a ϵ-neighborhood of ξ, namely*

$$N_\epsilon = \{\varphi \in C_b(\mathbb{R}, X) : \sup_{t \in \mathbb{R}} \|\varphi(t) - \xi(t)\|_X < \epsilon\}, \quad \epsilon > 0,$$

such that ξ is the only bounded global solution of $\mathcal{S}_f$ in N_ϵ.

Proof. If $\tau > t$ we have that

$$\varphi(\tau) = L_f(\tau,t)\varphi(t) + \int_t^\tau L_f(\tau,s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{5.5}$$

Thus, applying $Q(\tau)$ in the previous equation we obtain

$$Q(\tau)\varphi(\tau) = L_f(\tau,t)Q(t)\varphi(t) + \int_t^\tau L_f(\tau,s)Q(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{5.6}$$

Now, use that $L_f(\tau,t)|_{R(Q(t))}$ is invertible with inverse $L_f(t,\tau)$ so we obtain

$$L_f(t,\tau)Q(\tau)\varphi(\tau) = Q(t)\varphi(t) + \int_t^\tau L_f(t,s)Q(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{5.7}$$

Since $\mathcal{L}_f$ admits a exponential dichotomy with exponent $\alpha > \nu$ and φ is bounded, we obtain that

$$\|L_f(t,\tau)Q(\tau)\varphi(\tau)\| \leq De^{\nu|\tau|}e^{\alpha(t-\tau)} \sup_{s \in \mathbb{R}} \|\varphi(s)\| \to 0, \text{ as } \tau \to +\infty.$$

Then

$$Q(t)\varphi(t) = -\int_t^{+\infty} L_f(t,s)Q(s)[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{5.8}$$

Similarly, for $t > \tau$, as

$$\|L_f(t,\tau)(Id_X - Q(\tau))\varphi(\tau)\| \leq De^{\nu|\tau|}e^{-\alpha(t-\tau)} \sup_{s \in \mathbb{R}} \|\varphi(s)\| \to 0, \text{ as } \tau \to -\infty,$$

thus

$$(Id_X - Q(t))\varphi(t) = \int_{-\infty}^t L_f(t,s)(Id_X - Q(s))[f(s,\varphi(s)) - D_x f(s,\xi(s))\varphi(s)]ds. \tag{5.9}$$

Therefore, the result follows by writing $\varphi(t) = (Id_X - Q(t))\varphi(t) + Q(t)\varphi(t)$ and using the previous expressions.

Finally, we prove that ξ is isolated. Let $\varphi \in C_b(\mathbb{R}, X)$ be a bounded global solution of $\mathcal{S}_f$ with $\sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\| \leq \epsilon$. Since ξ and φ are bounded we apply the first part of the proof for each and obtain

$$\varphi(t) - \xi(t) = \int_{-\infty}^{+\infty} G_f(t,\tau)[f(\tau,\varphi(\tau)) - f(\tau,\xi(\tau)) - D_x f(\tau,\xi(\tau))(\varphi(\tau)) - \xi(\tau)]d\tau.$$

Note that, the Green's function $\mathcal{G}_f$ satisfies

$$\|G_f(t,\tau)\|_{\mathcal{L}(X)} \leq De^{\nu|\tau|}e^{-\alpha|t-\tau|}, \text{ for all } t,\tau \in \mathbb{R},$$

this together with condition (5.4) we obtain

$$\sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\| \leq 2D\rho(\epsilon)\alpha^{-1} \sup_{t\in\mathbb{R}} \|\varphi(t) - \xi(t)\|.$$

For $\epsilon > 0$ such that $2D\rho(\epsilon)\alpha^{-1} < 1$ we see that $\varphi(t) = \xi(t)$ for all $t \in \mathbb{R}$. Therefore, ξ is isolated and the proof is complete. □

Now, as an application of Theorem 3.11 we prove a result on the *persistence of nonuniform hyperbolic solutions.*

Theorem 5.4 (Persistence of nonuniform hyperbolic solutions)**.** *Let $f : \mathbb{R}\times X \to X$ be a continuous map with continuous first derivative with respect to the second variable, $\mathcal{T}$ be a linear evolution processes and $\mathcal{S}_f$ be the evolution process generated by f and $\mathcal{T}$. Assume that*

(1) $\mathcal{T}$ satisfies

$$\sup_{0\leq t-s\leq 1} \{e^{-\nu|t|}\|T(t,s)\|_{\mathcal{L}(X)}\} < +\infty, \tag{5.10}$$

(2) there is a global nonuniform hyperbolic solution ξ for $\mathcal{S}_f$, i.e., $\mathcal{L}_f$ admits a nonuniform exponential dichotomy with bound $K(s) = De^{\nu|s|}$, for all $s \in \mathbb{R}$, and exponent $\alpha > \nu$.

(3) ξ is bounded with $\sup_{t\in\mathbb{R}} \|\xi(t)\| < M$;

(4) f satisfies Condition (5.4);

(5) the derivative of f satisfies

$$\sup_{\|x\|\leq M} \sup_{t\in\mathbb{R}} \{e^{\nu|t|}\|D_x f(t,x)\|_{\mathcal{L}(X)}\} < +\infty;$$

(6) $g : \mathbb{R}\times X \to X$ is differentiable with continuous first derivative with respect to the second variable and satisfies

$$\sup_{\|x\|_X\leq\epsilon} \sup_{t\in\mathbb{R}} e^{3\nu|t|}\|D_x g(t,\xi(t)+x) - D_x g(t,\xi(t))\|_{\mathcal{L}(X)} < \frac{\delta}{2}, \text{ and} \tag{5.11}$$

$$\sup_{\substack{t\in\mathbb{R} \\ \|x\|_X\leq M}} e^{3\nu|t|}\big\{\|f(t,x)-g(t,x)\|_X+\|D_x f(t,x)-D_x g(t,x)\|_{\mathcal{L}(X)}\big\} < \frac{\epsilon\alpha}{4D}, \tag{5.12}$$

for $0 < \epsilon < \epsilon_0 := \min\{M - \sup_{t\in\mathbb{R}} \|\xi(t)\|_X, 2\delta D\alpha^{-1}\}$ suitable small, where $\delta > 0$ is the same as in Theorem 3.14 applied for $\mathcal{L}_f$.

Then there exists a unique nonuniform hyperbolic solution ψ for $\mathcal{S}_g$ such that

$$\sup_{t\in\mathbb{R}} \|\xi(t) - \psi(t)\| < \epsilon.$$

Proof. If y is a global bounded solution for $\mathcal{S}_g$, then, as in Remark 5.2, we have that

$$\begin{aligned} y(t) &= L_f(t,s)y(s) + \int_s^t L_f(t,\tau)[g(\tau,y(\tau)) - D_x f(\tau,\xi(\tau))y(\tau)]d\tau, \\ \xi(t) &= L_f(t,s)\xi(s) + \int_s^t L_f(t,\tau)[f(\tau,\xi(\tau)) - D_x f(\tau,\xi(\tau))\xi(\tau)]d\tau. \end{aligned} \tag{5.13}$$

Thus $\phi(t) = y(t) - \xi(t)$ satisfies the following integral equation

$$\phi(t) = L_f(t,s)\phi(s) + \int_s^t L_f(t,\tau)h(\tau,\phi(\tau))d\tau, \tag{5.14}$$

where $h(t,\phi(t)) = g(t,\phi(t)+\xi(t)) - f(t,\xi(t)) - D_x f(t,\xi(t))\phi(t)$.

Then , by Theorem 5.3, there exists a bounded solution of (5.14) in

$$B_\epsilon := \{\phi : \mathbb{R} \to X \ : \phi \text{ is continuous and } \sup_{t\in\mathbb{R}} \|\phi(t)\| < \epsilon\},$$

if and only if, the operator

$$(\mathcal{F}\varphi)(t) = \int_{-\infty}^{+\infty} G_f(t,s)h(s,\varphi(s))ds,$$

has a fixed point in the space B_ϵ.

Now, we use the fact that $\mathcal{L}_f$ admits a nonuniform exponential dichotomy to show that $\mathcal{F}$ has a unique fixed point in B_ϵ, for suitable small $\epsilon > 0$. In order to use the Banach fixed point Theorem, we have to prove that $\mathcal{F}$ is a contraction and that $\mathcal{F}B_\epsilon \subset B_\epsilon$.

For $0 < \epsilon < \epsilon_0$ and $\phi \in B_\epsilon$, we have

$$\begin{aligned} \|(\mathcal{F}\phi)(t)\|_X &\leq D\int_{-\infty}^{+\infty} e^{\nu|s|}e^{-\alpha|t-s|}\|h(s,\phi(s))\|_X ds \\ &\leq 2D\alpha^{-1}\sup_{t\in\mathbb{R}} e^{\nu|t|}\,\|g(t,\xi(t)+\phi(t)) - f(t,\xi(t)+\phi(t))\|_X \\ &+ 2D\alpha^{-1}\epsilon \sup_{\|x\|\leq\epsilon}\sup_{t\in\mathbb{R}} \frac{e^{\nu|t|}\,\|f(t,\xi(t)+x) - f(t,\xi(t)) - D_x f(t,\xi(t))x\|_X}{\|x\|_X} \\ &\leq \epsilon/2 + 2\alpha^{-1}D\rho(\epsilon)\epsilon. \end{aligned}$$

Thus, choosing $\epsilon \in (0,\epsilon_0)$ such that $4\alpha^{-1}D\rho(\epsilon) < 1$, we see that $\mathcal{F}\phi \in B_\epsilon$. Now, we show that $\mathcal{F}$ is a contraction. In fact, with similar computations we are able to prove for $\phi_1,\phi_2 \in B_\epsilon$ that

$$\|(\mathcal{F}\phi_1)(t) - (\mathcal{F}\phi_2)(t)\|_X \leq \frac{1}{2}\sup_{t\in\mathbb{R}}\|\phi_1(t) - \phi_2(t)\|_X.$$

Therefore, there is a unique fixed point ϕ in B_ϵ and we obtain $\psi = \phi + \xi$ a global solution of $\mathcal{S}_g$.

Finally, we prove that ψ is a nonuniform hyperbolic solution, that means, the linear evolution process $\mathcal{L}_g := \{L_g(t,s) : t \geq s\}$ that satisfies

$$L_g(t,\tau) = T(t,\tau) + \int_\tau^t T(t,s)D_x g(s,\psi(s))L_g(s,\tau)ds$$

admits a nonuniform exponential dichotomy.

To that end, we show that $\mathcal{L}_f$ satisfies conditions of Theorem 3.14 and we see $\mathcal{L}_g$ as a small perturbation of $\mathcal{L}_f$. Indeed, since $\mathcal{T}$ satisfies (5.10) and

$$L_f(t,s) = T(t,s) + \int_s^t T(t,\tau) D_x f(\tau, \xi(\tau)) L_f(\tau, s) d\tau,$$

from a Grönwall's inequality and assumption (5) we see that

$$\sup_{0 \leq t-\tau \leq 1} \{e^{-\nu|t|} \, \|L_f(t,\tau)\|_{\mathcal{L}(X)}\} < +\infty.$$

Finally, note that

$$L_g(t,\tau) = L_f(t,\tau) + \int_\tau^t L_f(t,s)[D_x g(s,\psi(s)) - D_x f(s,\xi(s))] L_g(s,\tau) ds.$$

Now, define $B(s) := D_x g(s,\psi(s)) - D_x f(s,\xi(s))$ for all $s \in \mathbb{R}$. Since

$$\begin{aligned} \|B(s)\|_{\mathcal{L}(X)} \quad \leq \quad & \|D_x g(s,\psi(s)) - D_x g(s,\xi(s))\|_{\mathcal{L}(X)} \\ & + \|D_x g(s,\xi(s)) - D_x f(s,\xi(s))\|_{\mathcal{L}(X)}, \end{aligned}$$

hypotheses (5.11) and (5.12) imply that $\|B(t)\| \leq \delta e^{-3\nu|t|}$, $t \in \mathbb{R}$. Therefore $B : \mathbb{R} \to \mathcal{L}(X)$ satisfies conditions of Theorem 3.14 and we conclude ψ is a nonuniform hyperbolic solution of $\mathcal{S}_g$. □

Remark 5.5. *Note that, in Theorem 5.4, f and g have to be C^1-close with a exponential weight. In fact, the functions one has to consider are of the form $h : \mathbb{R} \times X \to X$ such that $h(t,x) = e^{-3\nu|t|} h_0(t,x)$ for some h_0 that satisfy the conditions for the uniform case (with $\nu = 0$ see* [24, Lemma 8.3]*), and this exponential weight has to be considered on the C^1-proximity of the functions as in* (5.12).

Remark 5.6. *We point out that the existence of a nonuniform hyperbolic solution can be achieved by an application of Theorem 5.4 in the special case where $f = 0$, so $\xi \equiv 0$ is a nonuniform hyperbolic solution of $\dot{y} = A(t)y$.*

6. Conclusions

The method of discretization, results 3.6 and 3.10, allowed us to compare continuous and discrete dynamical systems that exhibit nonuniform hyperbolicity. This approach was already known in the case of uniform exponential dichotomies, see for example [8, 11], and in this work we established it for the nonuniform case. Through this procedure we obtain:

(1) Uniqueness of the family of projections: Corollary 3.8.
(2) Continuous dependence of projections: Theorem 3.9.
(3) Robustness of nonuniform exponential dichotomies: theorems 3.11 and 3.14.

A disadvantage of the discretization method is that it is not possible to consider directly nonlinear growth rates $\rho(t)$, as in Barreira and Valls [22]. Note that, this case can be achieved by a change of scaling in time. Thus, it was possible to prove the robustness result with the assumption $\alpha > \nu$, which is the sharpest condition we can get with this technique. We also note that condition (3.2) is not required in [22], and its needed when applying this discretization method. However it is not a restrictive condition when dealing with evolution processes with nonuniform exponential dichotomies, see for instance Barreira and Valls [15] or Example 4.1. Therefore, we obtain an improvement on the exponents of the robustness result of [22] at the price of having to assume (3.2).

The continuous dependence of projections and the persistence of hyperbolic solutions play an important role in the study of continuity of local unstable and stable manifolds for an associated nonlinear evolution process. In [6] they use these permanence results to conclude continuity of pullback attractors under perturbation. On the other hand, it is not clear yet how to apply the results of stability of nonuniform hyperbolicity in the theory of attractors. However the persistence of nonuniform hyperbolic solutions and continuous dependence of projections should be important to study continuity of invariant manifolds associated to the nonuniform hyperbolic solutions. This, in turn, will be crucial in a possible application in the theory of attractors.

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