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The Čech number of $C_p(X)$ when X is an ordinal space

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Abstract. The Čech number of a space Z, $\check{C}(Z)$, is the pseudocharacter of Z in βZ. In this article we obtain, in ZFC and assuming SCH, some upper and lower bounds of the Čech number of spaces $C_p(X)$ of realvalued continuous functions defined on an ordinal space X with the pointwise convergence topology.

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1. Notations and Basic results

In this article, every space X is a Tychonoff space. The symbols ω (or $\mathbb{N}$), $\mathbb{R}$, I, $\mathbb{Q}$ and $\mathbb{P}$ stand for the set of natural numbers, the real numbers, the closed interval $[0,1]$, the rational numbers and the irrational numbers, respectively. Given two spaces X and Y, we denote by $C(X,Y)$ the set of all continuous functions from X to Y, and $C_p(X,Y)$ stands for $C(X,Y)$ equipped with the topology of pointwise convergence, that is, the topology in $C(X,Y)$ of subspace of the Tychonoff product Y^X. The space $C_p(X,\mathbb{R})$ is denoted by $C_p(X)$. The restriction of a function f with domain X to $A \subset X$ is denoted by $f \restriction A$. For a space X, βX is its Stone-Čech compactification.

Recall that for $X \subset Y$, the *pseudocharacter of* X *in* Y is defined as

$$\Psi(X,Y) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a family of open sets in } Y \text{ and } X = \bigcap \mathcal{U}\}.$$

Definition 1.1.

(1) *The Čech number of a space* Z *is* $\check{C}(Z) = \Psi(Z, \beta Z)$.
(2) *The k-covering number of a space* Z *is* $kcov(Z) = min\{|\mathcal{K}| : \mathcal{K}$ *is a compact cover of* $Z\}$.

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We have that (see Section 1 in [8]): $\check{C}(Z) = 1$ if and only if Z is locally compact; $\check{C}(Z) \leq \omega$ if and only if Z is Čech-complete; $\check{C}(Z) = kcov(\beta Z \setminus Z)$; if Y is a closed subset of Z, then $kcov(Y) \leq kcov(Z)$ and $\check{C}(Y) \leq \check{C}(Z)$; if $f : Z \to Y$ is an onto continuous function, then $kcov(Y) \leq kcov(Z)$; if $f : Z \to Y$ is perfect and onto, then $kcov(Y) = kcov(Z)$ and $\check{C}(Y) = \check{C}(Z)$; if bZ is a compactification of Z, then $\check{C}(Z) = \Psi(Z, bZ)$.

We know that $\check{C}(C_p(X)) \leq \aleph_0$ if and only if X is countable and discrete ([7]), and $\check{C}(C_p(X, I)) \leq \aleph_0$ if and only if X is discrete ([9]).

For a space X, $ec(X)$ (*the essential cardinality of* X) is the smallest cardinality of a closed and open subspace Y of X such that $X \setminus Y$ is discrete. Observe that, for such a subspace Y of X, $\check{C}(C_p(X, I)) = \check{C}(C_p(Y, I))$. In [8] it was pointed out that $ec(X) \leq \check{C}(C_p(X, I))$ and $\check{C}(C_p(X)) = |X| \cdot \check{C}(C_p(X, I))$ always hold. So, if X is discrete, $\check{C}(C_p(X)) = |X|$, and if $|X| = ec(X)$, $\check{C}(C_p(X)) = \check{C}(C_p(X, I))$.

Consider in the set of functions from ω to ω, ${}^{\omega}\omega$, the partial order $\leq^*$ defined by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A collection D of $({}^{\omega}\omega, \leq^*)$ is *dominating* if for every $h \in {}^{\omega}\omega$ there is $f \in D$ such that $h \leq^* f$. As usual, we denote by $\mathfrak{d}$ the cardinal number $\min\{|D| : D \text{ is a dominating subset of } {}^{\omega}\omega\}$. It is known that $\mathfrak{d} = kcov(\mathbb{P})$ (see [3]); so $\mathfrak{d} = \check{C}(\mathbb{Q})$. Moreover, $\omega_1 \leq \mathfrak{d} \leq \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

We will denote a cardinal number τ with the discrete topology simply as τ; so, the space τ^{κ} is the Tychonoff product of κ copies of the discrete space τ. The cardinal number τ with the order topology will be symbolized by $[0, \tau)$.

In this article we will obtain some upper and lower bounds of $\check{C}(C_p(X, I))$ when X is an ordinal space; so this article continues the efforts made in [1] and [8] in order to clarify the behavior of the number $\check{C}(C_p(X, I))$ for several classes of spaces X.

For notions and concepts not defined here the reader can consult [2] and [4].

2. The Čech number of $C_p(X)$ when X is an ordinal space

For an ordinal number α, let us denote by $[0, \alpha)$ and $[0, \alpha]$ the set of ordinals $< \alpha$ and the set of ordinals $\leq \alpha$, respectively, with its order topology. Observe that for every ordinal number $\alpha \leq \omega$, $[0, \alpha)$ is a discrete space, so, in this case, $\check{C}(C_p([0, \alpha), I)) = 1$. If $\omega < \alpha < \omega_1$, then $[0, \alpha)$ is a countable metrizable space, hence, by Theorem 7.4 in [1], $\check{C}(C_p([0, \alpha), I)) = \mathfrak{d}$. We will analyze the number $\check{C}(C_p([0, \alpha), I))$ for an arbitrary ordinal number α.

We are going to use the following symbols:

Notations 2.1. For each $n < \omega$, we will denote as $\mathcal{E}_n$ the collection of intervals

$$[0, 1/2^{n+1}), (1/2^{n+2}, 3/2^{n+2}), (1/2^{n+1}, 2/2^{n+1}), (3/2^{n+2}, 5/2^{n+2}), \ldots$$
$$\ldots, ((2^{n+2} - 2)/2^{n+2}, (2^{n+2} - 1)/2^{n+2}), ((2^{n+1} - 1)/2^{n+1}, 1].$$

Observe that $\mathcal{E}_n$ is an irreducible open cover of $[0, 1]$ and each element in $\mathcal{E}_n$ has diameter $= 1/2^{n+1}$. For a set S and a point $y \in S$, we will use the symbol $[yS]^{<\omega}$ in order to denote the collection of finite subsets of S containing y.

Moreover, if γ and α are ordinal numbers with $\gamma \leq \alpha$, $[\gamma, \alpha]$ is the set of ordinal numbers λ which satisfy $\gamma \leq \lambda \leq \alpha$. The expression $\alpha_0 < \alpha_1 < ... < \alpha_n < ... \nearrow \gamma$ will mean that the sequence $(\alpha_n)_{n<\omega}$ of ordinal numbers is strictly increasing and converges to γ.

Lemma 2.2. *Let γ be an ordinal number such that there is $\omega < \alpha_0 < \alpha_1 < ... < \alpha_n < ... \nearrow \gamma$. Then $\check{C}(C_p([0,\gamma], I) \leq \check{C}(C_p([0,\gamma), I) \cdot kcov(|\gamma|^\omega)$.*

Proof. For $m < \omega$, $F \in [\gamma[\alpha_m, \gamma]]^{<\omega} = \{M \subset [\alpha_m, \gamma] : |M| < \aleph_0 \text{ and } \gamma \in M\}$ and $n < \omega$, define

$$B(m, F, n) = \bigcup_{E \in \mathcal{E}_n} B(m, F, E)$$

where $B(m, F, E) = \prod_{x \in [0,\gamma]} J_x$ with $J_x = E$ if $x \in F$, and $J_x = I$ otherwise. (So, $B(m, F, n)$ is open in $I^{[0,\gamma]}$.) Define

$$B(m, n) = \bigcap \{B(m, F, n) : F \in [\gamma[\alpha_m, \gamma]]^{<\omega}\}.$$

Observe that $B(m, n)$ is the intersection of at most $|\gamma|$ open sets $B(m, F, n)$.

Define $G(n) = \bigcup_{m<\omega} B(m, n)$, and $G = \bigcap_{n<\omega} G(n)$.

Claim: G is the set of all functions $g : [0, \gamma] \to [0, 1]$ which are continuous at γ.

Proof of the claim: Let $g : [0, \gamma] \to [0, 1]$ be continuous at γ. Given $n < \omega$ there is $E \in \mathcal{E}_n$ such that $g(\gamma) \in E$. Since g is continuous at γ, there is $\beta < \gamma$ so that $g(t) \in E$ if $t \in [\beta, \gamma]$. Fix $m < \omega$ so that $\beta < \alpha_m$. For every $F \in [\gamma[\alpha_m, \gamma]]^{<\omega}$ we have that $g \in B(m, F, E) \subset B(m, F, n)$; hence, $g \in B(m, n) \subset G(n)$. We conclude that g belongs to G.

Now, let $h \in G$. We are going to prove that h is continuous at γ. Assume the contrary, that is, there exist $\epsilon > 0$ and a sequence $t_0 < t_1 < ... < t_n < ... \nearrow \gamma$ such that

$$|f(t_j) - f(\gamma)| \geq \epsilon, \tag{1}$$

for every $j < \omega$. Fix $n < \omega$ such that $1/2^{n+1} < \epsilon$.

Since $h \in G$, then $h \in G(n)$ and there is $m \geq 0$ such that $h \in B(m, n)$. Choose $t_{n_p} > \alpha_m$ and take $F = \{t_{n_p}, \gamma\}$. Thus $h \in B(m, F, n)$, but if $E \in \mathcal{E}_n$ and $h(\gamma) \in E$, then $h(t_{n_p}) \notin E$, which is a contradiction. So, the claim has been proved.

Now, we have $I^{[0,\gamma]} \setminus G = \bigcup_{n<\omega} (I^{[0,\gamma]} \setminus G(n))$, and

$$I^{[0,\gamma]} \setminus G(n) = \bigcap_{m<\omega} \bigcup_{F \in \gamma[\alpha_m, \gamma]^\omega} (I^{[0,\gamma]} \setminus B(m, F, n)).$$

So $I^{[0,\gamma]} \setminus G(n))$ is an $F_{|\gamma|\delta}$-set. By Corollary 3.4 in [8], $kcov(I^{[0,\gamma]} \setminus G(n)) \leq kcov(|\gamma|^\omega)$. Hence, $\check{C}(G) = kcov(I^{[0,\gamma]} \setminus G) \leq \aleph_0 \cdot kcov(|\gamma|^\omega)$. Thus, it follows that

$$\check{C}(C_p([0,\gamma], I) \leq \check{C}(C_p([0,\gamma), I) \cdot kcov(|\gamma|^\omega).$$

□

Lemma 2.3. *If $\gamma < \alpha$, then $\check{C}(C_p([0,\gamma), I)) \leq \check{C}(C_p([0,\alpha), I))$.*

Proof. First case: $\gamma = \beta + 1$.

In this case, $[0,\gamma) = [0,\beta]$ and the function $\phi : [0,\alpha) \to [0,\beta]$ defined by $\phi(x) = x$ if $x \leq \beta$ and $\phi(x) = \beta$ if $x > \beta$ is a quotient. So, $\phi^{\#} : C_p([0,\beta], I) \to C_p([0,\alpha), I)$ defined by $\phi^{\#}(f) = f \circ \phi$, is a homeomorphism between $C_p([0,\beta], I)$ and a closed subset of $C_p([0,\alpha), I)$ (see [2], pages 13,14). Then, in this case, $\check{C}(C_p([0,\gamma), I)) \leq \check{C}(C_p([0,\alpha), I))$.

Now, in order to finish the proof of this Lemma, it is enough to show that for every limit ordinal number α, $\check{C}(C_p([0,\alpha), I)) \leq \check{C}(C_p([0,\alpha], I))$.

Let $\kappa = cof(\alpha)$, and $\alpha_0 < \alpha_1 < \ldots < \alpha_\lambda < \ldots \nearrow \alpha$ with $\lambda < \kappa$. For each of these λ, we know, because of the proof of the first case, that $\kappa_\lambda = \check{C}(C_p([0,\alpha_\lambda], I)) \leq \check{C}(C_p([0,\alpha], I))$. Let, for each $\lambda < \kappa$, $\{V^\lambda_\xi : \xi < \kappa_\lambda\}$ be a collection of open subsets of $I^{[0,\alpha_\lambda]}$ such that $C_p([0,\alpha_\lambda], I) = \bigcap_{\xi<\kappa_\lambda} V^\lambda_\xi$. For each $\lambda < \kappa$ and each $\xi < \kappa_\lambda$, we take $W^\lambda_\xi = V^\lambda_\xi \times I^{(\alpha_\lambda,\alpha)}$. Each W^λ_ξ is open in $I^{[0,\alpha)}$ and $\bigcap_{\lambda<\kappa}\bigcap_{\xi<\kappa_\lambda} W^\lambda_\xi = C_p([0,\alpha), I))$. Therefore, $\check{C}(C_p([0,\alpha), I)) \leq \kappa \cdot sup\{\kappa_\lambda : \lambda < \kappa\} \leq \kappa \cdot \check{C}(C_p([0,\alpha], I))$. But $\kappa \leq |\alpha| = ec([0,\alpha]) \leq \check{C}(C_p([0,\alpha], I))$.

Then, $\check{C}(C_p([0,\alpha), I)) \leq \check{C}(C_p([0,\alpha], I))$. □

Lemma 2.4. *Let α be a limit ordinal number $> \omega$. Then*

$$\check{C}(C_p([0,\alpha), I)) = |\alpha| \cdot sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma), I)).$$

In particular, $\check{C}(C_p([0,\alpha), I)) = sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma), I))$ if $cof(\alpha) < \alpha$.

Proof. By Lemma 2.3, $sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma), I)) \leq \check{C}(C_p([0,\alpha), I))$, and, by Corollary 4.8 in [8], $|\alpha| \leq \check{C}(C_p([0,\alpha), I))$.

For each $\gamma < \alpha$, we write κ_γ instead of $\check{C}(C_p([0,\gamma), I))$. Let $\{V^\gamma_\lambda : \lambda < \kappa_\gamma\}$ be a collection of open sets in I^γ such that $C_p([0,\gamma), I) = \bigcap_{\lambda<\kappa_\gamma} V^\gamma_\lambda$. Now we put $W^\gamma_\lambda = V^\gamma_\lambda \times I^{[\gamma,\alpha)]}$. We have that W^γ_λ is open for every $\gamma < \alpha$ and every $\lambda < \gamma$, and $C_p([0,\alpha), I) = \bigcap_{\gamma<\alpha}\bigcap_{\lambda<\kappa_\gamma} W^\gamma_\lambda$. So, $\check{C}(C_p([0,\alpha), I)) = |\alpha| \cdot sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma), I))$. □

In order to prove the following result it is enough to mimic the prove of 5.12.(c) in [5].

Lemma 2.5. *If α is an ordinal number with $cof(\alpha) > \omega$ and $f \in C_p([0,\alpha), I))$, then there is $\gamma_0 < \alpha$ for which $f \restriction [\gamma_0, \alpha)$ is a constant function.*

Lemma 2.6. *If α is an ordinal number with cofinality $> \omega$, then $\check{C}(C_p([0,\alpha], I)) = \check{C}(C_p([0,\alpha), I))$.*

Proof. Let $\kappa = \check{C}(C_p([0,\alpha), I))$. There are open sets V_λ $(\lambda < \kappa)$ in $I^{[0,\alpha)}$ such that $C_p([0,\alpha), I) = \bigcap_{\lambda<\kappa} V_\lambda$. For each $\lambda < \kappa$, we take $W_\lambda = V_\lambda \times I^{\{\alpha\}}$. Each W_λ is open in $I^{[0,\alpha]}$ and $\bigcap_{\lambda<\kappa} W_\lambda = \{f : [0,\alpha] \to I \mid f \restriction [0,\alpha) \in C_p([0,\alpha), I)\}$.

For each $(\gamma, \xi, E) \in \alpha \times \alpha \times \mathcal{E}_n$, we take $B(\gamma, \xi, E) = \prod_{\lambda\leq\alpha} J_\lambda$ where $J_\lambda = E$ if $\lambda \in \{\xi + \gamma, \alpha\}$, and $J_\lambda = I$ otherwise. Let $B(\gamma, \xi, n) = \bigcup_{E\in\mathcal{E}_n} B(\gamma, \xi, E)$.

Finally, we define $B(\gamma) = \bigcup_{\xi<\alpha} B(\gamma,\xi,n)$, which is an open subset of $I^{[0,\alpha]}$. We denote by M the set $\bigcap_{\lambda<\kappa} W_\lambda \cap \bigcap_{\gamma<\alpha} B(\gamma)$. We are going to prove that $C_p([0,\alpha],I) = M$.

Let $f \in C_p([0,\alpha],I)$. We know that $f \in \bigcap_{\lambda<\kappa} W_\lambda$, so we only have to prove that $f \in \bigcap_{\gamma<\alpha} B(\gamma)$. For $n < \omega$, there is $E \in \mathcal{E}_n$ such that $f(\alpha) \in E$. Since $f \in C([0,\alpha],I)$, there are $\gamma_0 < \alpha$ and $r_0 \in I$ such that $f(\lambda) = r_0$ if $\gamma_0 \leq \lambda < \alpha$. Let $\chi < \alpha$ such that $\chi + \gamma \geq \gamma_0$. Thus, $f \in B(\gamma,\chi,n) \subset B(\gamma)$. Therefore, $C_p([0,\alpha],I) \subset M$.

Take an element f of M. Since $f \in \bigcap_{\lambda<\alpha} W_\lambda$, f is continuous at every $\gamma < \alpha$, thus $f \restriction [\gamma_0,\alpha) = r_0$ for a $\gamma_0 < \alpha$ and an $r_0 \in I$.

For each $n < \omega$, and each $\gamma \geq \gamma_0$, $f \in B(\gamma,\xi,n)$ for some $\xi < \alpha$. Then, $|r_0 - f(\alpha)| = |f(\gamma+\xi) - f(\alpha)| < 1/2^n$. But, these relations hold for every n. So, $f(\alpha)$ must be equal to r_0, and this means that f is continuous at every point.

Therefore, $\check{C}(C_p([0,\alpha],I)) \leq |\alpha| \cdot \check{C}(C_p([0,\alpha),I))$. Since $\check{C}(C_p([0,\alpha),I)) \geq ec([0,\alpha)) = |\alpha|$, $\check{C}(C_p([0,\alpha],I)) \leq \check{C}(C_p([0,\alpha),I))$. Finally, Lemma 2.3 gives us the inequality $\check{C}(C_p([0,\alpha),I)) \leq \check{C}(C_p([0,\alpha],I))$. □

Theorem 2.7. *For every ordinal number $\alpha > \omega$,*

$$|\alpha| \cdot \mathfrak{d} \leq \check{C}(C_p([0,\alpha),I)) \leq kcov(|\alpha|^\omega).$$

Proof. Because of Theorem 7.4 in [1], Corollary 4.8 in [8] and Lemma 2.3 above, $|\alpha| \cdot \mathfrak{d} \leq \check{C}(C_p([0,\alpha),I))$.

Now, if $\omega < \alpha < \omega_1$, we have that $\check{C}(C_p([0,\alpha),I)) \leq kcov(|\alpha|^\omega)$ because of Corollary 4.2 in [1].

We are going to finish the proof by induction. Assume that the inequality $\check{C}(C_p([0,\gamma),I)) \leq kcov(|\gamma|^\omega)$ holds for every $\omega < \gamma < \alpha$. By Lemma 2.4 and inductive hypothesis, if α is a limit ordinal, then

$$\check{C}(C_p([0,\alpha),I)) \leq |\alpha| \cdot sup_{\gamma<\alpha} kcov(|\gamma|^\omega) \leq kcov(|\alpha|^\omega).$$

If $\alpha = \gamma_0+2$, then $\check{C}(C_p([0,\alpha),I)) = \check{C}(C_p([0,\gamma_0+1),I)) \leq kcov(|\gamma_0+1|^\omega) = kcov(|\alpha|^\omega)$.

Now assume that $\alpha = \gamma_0 + 1$, γ_0 is a limit and $cof(\gamma_0) = \omega$. We know by Lemma 2.2 that $\check{C}(C_p([0,\gamma_0+1),I)) \leq \check{C}(C_p([0,\gamma_0),I) \cdot kcov(|\gamma_0|^\omega)$. So, by inductive hypothesis we obtain what is required.

The last possible case: $\alpha = \gamma_0 + 1$, γ_0 is limit and $cof(\gamma_0) > \omega$.

By Lemma 2.6, we have $\check{C}(C_p([0,\gamma_0+1),I)) = |\alpha| \cdot \check{C}(C_p([0,\gamma_0),I)$. By inductive hypothesis, $\check{C}(C_p([0,\gamma_0),I) \leq kcov(|\alpha|^\omega)$. Since $|\alpha| \leq kcov(|\alpha|^\omega)$, we conclude that $\check{C}(C_p([0,\alpha),I)) \leq kcov(|\alpha|^\omega)$. □

As a consequence of Proposition 3.6 in [8] (see Proposition 2.11, below) and the previous Theorem, we obtain:

Corollary 2.8. *For an ordinal number $\omega < \alpha < \omega_\omega$, $\check{C}(C_p([0,\alpha),I)) = |\alpha| \cdot \mathfrak{d}$.*

In particular, we have:

Corollary 2.9. $\check{C}(C_p([0,\omega_1),I)) = \check{C}(C_p([0,\omega_1],I)) = \mathfrak{d}$.

By using similar techniques to those used throughout this section we can also prove the following result.

Corollary 2.10. *For every ordinal number $\alpha > \omega$ and every $1 \leq n < \omega$,*

$$|\alpha| \cdot \mathfrak{d} \leq \check{C}(C_p([0,\alpha)^n, I)) \leq kcov(|\alpha|^\omega).$$

For a generalized linearly ordered topological space X, $\chi(X) \leq ec(X)$, so $\chi(X) \leq \check{C}(C_p(X,I))$, where $\chi(X)$ is the character of X. This is not the case for every topological space, even if X is a countable EG-space, as was pointed out by O. Okunev to the authors. Indeed, let X be a countable dense subset of $C_p(I)$. We have that $\chi(X) = \chi(C_p(I)) = \mathfrak{c}$ and $\check{C}(C_p(X,I)) = \mathfrak{d}$. So, it is consistent with ZFC that there is a countable EG-space X with $\chi(X) > \check{C}(C_p(X,I))$.

One is tempted to think that for every linearly ordered space X, the relation $\check{C}(C_p(X,I)) \leq kcov(\chi(X)^\omega)$ is plausible. But this illusion vanishes quickly; in fact, when $\mathfrak{d} < 2^\omega$ and X is the doble arrow, then X has countable character and $ec(X) = |X| = 2^\omega$. Hence, $\check{C}(C_p(X,I)) \geq 2^\omega > \mathfrak{d} = kcov(\chi(X)^\omega)$ (compare with Theorem 2.7, above, and Corollary 7.7 in [1]).

In [8] the following was remarked:

Proposition 2.11.

(1) *For every cardinal number $\omega \leq \tau < \omega_\omega$, $kcov(\tau^\omega) = \tau \cdot \mathfrak{d}$,*
(2) *for every cardinal $\tau \geq \lambda$, $kcov((\tau^+)^\lambda) = \tau^+ \cdot kcov(\tau^\lambda)$, and,*
(3) *if $cf(\tau) > \lambda$, then $kcov(\tau^\lambda) = \tau \cdot sup\{kcov(\mu^\lambda) : \mu < \tau\}$.*

Lemma 2.12. *For every cardinal number κ with $cof(\kappa) = \omega$, we have that $kcov(\kappa^\omega) > \kappa$.*

Proof. Let $\{K_\lambda : \lambda < \kappa\}$ be a collection of compact subsets of κ^ω. Let $\alpha_0 < \alpha_1 < ... < \alpha_n < ...$ be an strictly increasing sequence of cardinal numbers converging to κ. We are going to prove that $\bigcup_{\lambda<\kappa} K_\lambda$ is a proper subset of κ^ω. Denote by $\pi_n : \kappa^\omega \to \kappa$ the n-projection. Since π_n is continuous and K_λ is compact, $\pi_n(K_\lambda)$ is a compact subset of the discrete space κ, so, it is finite. Thus, we have that $|\bigcup_{\lambda<\alpha_n} \pi_n(K_\lambda)| \leq \alpha_n < \kappa$ for each $n < \omega$. Hence, for every $n < \omega$, we can take $\xi_n \in \kappa \setminus \bigcup_{\lambda<\alpha_n} \pi_n(K_\lambda)$. Consider the point $\xi = (\xi_n)_{n<\omega}$ of κ^ω. We claim that $\xi \notin \bigcup_{\lambda<\kappa} K_\lambda$. Indeed, assume that $\xi \in K_{\lambda_0}$. There is $n < \omega$ such that $\lambda_0 < \alpha_n$. So, $\xi_n \in \bigcup_{\lambda<\alpha_n} \pi_n(K_\lambda)$ which is not possible. □

Recall that *the Singular Cardinals Hypothesis (SCH)* is the assertion:

For every singular cardinal number κ, if $2^{cof(\kappa)} < \kappa$, then $\kappa^{cof(\kappa)} = \kappa^+$.

A proposition, apparently weaker than SCH, is: *"for every cardinal number κ with $cof(\kappa) = \omega$, if $2^\omega < \kappa$, then $\kappa^\omega = \kappa^+$."* But this last assertion is equivalent to SCH as was settled by Silver (see [6], Theorem 23).

Proposition 2.13. *If we assume SCH and $\mathfrak{c} \leq (\omega_\omega)^+$, and if τ is an infinite cardinal number, then*

$$(*) \qquad kcov(\tau^\omega) = \begin{cases} \tau \cdot \mathfrak{d} & \text{if } \omega \leq \tau < \omega_\omega \\ \tau & \text{if } \tau > \omega_\omega \text{ and } cof(\tau) > \omega \\ \tau^+ & \text{if } \tau > \omega \text{ and } cof(\tau) = \omega \end{cases}$$

Proof. Our proposition is true for every $\omega \leq \tau < \omega_\omega$ because of (1) in Proposition 2.11.

Assume now that $\kappa \geq \omega_\omega$ and that $(*)$ holds for every $\tau < \kappa$. We are going to prove the assertion for κ.

Case 1: $cof(\kappa) = \omega$. By Lemma 2.12, $kcov(\kappa^\omega) > \kappa$. On the other hand, $kcov(\kappa^\omega) \leq \kappa^\omega$.

First two subcases: Either $\mathfrak{c} < \omega_\omega$ or $\kappa > \omega_\omega$. In both subcases, we can apply SCH and conclude that $kcov(\kappa^\omega) = \kappa^+$.

Third subcase: $\mathfrak{c} = (\omega_\omega)^+$ and $\kappa = \omega_\omega$. In this case we have $kcov((\omega_\omega)^\omega) \leq (\omega_\omega)^\omega \leq \mathfrak{c}^\omega = \mathfrak{c} = (\omega_\omega)^+$. Moreover, by Lemma 2.12, $(\omega_\omega)^+ \leq kcov((\omega_\omega)^\omega)$. Therefore, $kcov((\omega_\omega)^\omega) = (\omega_\omega)^+$.

Case 2: $cof(\kappa) > \omega$. By Proposition 2.11 (3), $kcov(\kappa^\omega) = \kappa \cdot \sup\{kcov(\mu^\omega) : \omega \leq \mu < \kappa\}$. By inductive hypothesis we have that for each $\mu < \kappa$

$$(**) \qquad kcov(\mu^\omega) = \begin{cases} \mu \cdot \mathfrak{d} & \text{if } \omega \leq \mu < \omega_\omega \\ \mu & \text{if } \mu > \omega_\omega \text{ and } cof(\mu) > \omega \\ \mu^+ & \text{if } \mu > \omega \text{ and } cof(\mu) = \omega \end{cases}$$

First subcase: κ is a limit cardinal. For every $\mu < \kappa$, $kcov(\mu^\omega) < \kappa$ (because of $(**)$ and because we assumed that $\kappa > (\omega_\omega)^+ \geq \mathfrak{c} \geq \mathfrak{d}$); and so $\sup\{kcov(\mu^\omega) : \mu < \kappa\} = \kappa$. Thus, $kcov(\kappa^\omega) = \kappa$.

Second subcase: Assume now that $\kappa = \mu_0^+$. In this case, by Proposition 2.11, $kcov(\kappa^\omega) = \kappa \cdot kcov(\mu_0^\omega)$. Because of $(**)$ and because $\mu_0 \geq \omega_\omega$, $kcov(\mu_0)^\omega \leq \kappa$. We conclude that $kcov(\kappa^\omega) = \kappa$. □

Proposition 2.14. *Let κ be a cardinal number with $cof(\kappa) = \omega$. Then*

$$\check{C}(C_p([0,\kappa], I)) > \kappa.$$

Proof. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \ldots$ be a strictly increasing sequence of cardinal numbers converging to κ. Assume that $\{V_\lambda : \lambda < \kappa\}$ is a collection of open sets in $I^{[0,\kappa]}$ which satisfies $C_p([0,\kappa], I) \subset \bigcap_{\lambda<\kappa} V_\lambda$. We are going to prove that $\bigcap_{\lambda<\kappa} V_\lambda$ contains a function $h : [0,\kappa] \to I$ which is not continuous. In order to construct h, we are going to define, by induction, the following sequences:

(i) elements $t_0, \ldots, t_n, \ldots$ which belong to $[0, \kappa]$ such that

(1) $0 = t_0 < t_1 < \cdots < t_n < \ldots$,
(2) $t_i \geq \alpha_i$ for each $0 \leq i < \omega$,
(3) each t_i is an isolated ordinal, and
(4) $\kappa = \lim(t_n)$;

(ii) subsets $G_0, ..., G_n, ... \subset [0, \kappa]$ with $|G_i| \leq \alpha_i$ for every $i < \omega$, and such that each function which equals 0 in G_i and 1 in $\{t_0, ..., t_i\}$ belongs to $\bigcap_{\lambda < \alpha_i} V_\lambda$ for every $0 \leq i < \omega$ and $(\bigcup_n G_n) \cap \{t_0, ..., t_n, ...\} = \emptyset$;

(iii) functions $f_0, f_1, ..., f_n, ...$ such that $f_0 \equiv 0$, and f_i is the characteristic function defined by $\{t_0, ..., t_{i-1}\}$ for each $0 < i < \omega$.

Let f_0 be the constant function equal to 0. Assume that we have already defined $t_0, ..., t_{s-1}$, $G_0, ..., G_{s-1}$ and $f_0, ..., f_{s-1}$. We now choose an isolated point $t_s \in [\alpha_s, \kappa] \setminus G_0 \cup ... \cup G_{s-1}$ (this is possible because $|G_0 \cup ... \cup G_{s-1}| < \kappa$). Consider the characteristic function defined by $\{t_0, ..., t_{s-1}, t_s\}$, f_s. This function is continuous, so it belongs to $\bigcap_{\lambda < \alpha_s} V_\lambda$. For each $\lambda < \alpha_s$, there is a canonical open set A^s_λ of the form $[f_s; x^s_1, ..., x^s_{n^s(\lambda)}; 1/m^s(\lambda)] = \{f \in I^{[0,\kappa]} : |f_s(x^s_i) - f(x^s_i)| < 1/m^s(\lambda) \ \forall\ 1 \leq i \leq n^s(\lambda)\}$ satisfying $f_s \in A^s_\lambda \subset V_\lambda$. For each $\lambda < \alpha_s$ we take $F^s_\lambda = \{x^s_1, ..., x^s_{n^s(\lambda)}\}$. Put $G_s = \bigcup_{\lambda < \alpha_s} F^s_\lambda \setminus \{t_0, ..., t_s\}$. It happens that $\{f \in I^{[0,\kappa]} : f(x) = 0 \ \forall\ x \in G_s \text{ and } f(t_i) = 1 \ \forall\ 0 \leq i \leq s\}$ is a subset of $\bigcap_{\lambda < \alpha_s} V_\lambda$. This finishes the inductive construction of the required sequences.

Now, consider the function $h : [0, \kappa] \to [0, 1]$ defined by $h(x) = 0$ if $x \notin \{t_0, ..., t_n, ...\}$, and $h(t_n) = 1$ for every $n < \omega$. This function h is not continuous at κ because $h(\kappa) = 0$, $\kappa = \lim(t_n)$, and $h(t_n) = 1$ for all $n < \omega$.

Now, take $\lambda_0 \in \kappa$. There exists $l < \omega$ such that $\lambda_0 < \alpha_l$. Since h is equal to 0 in G_l and 1 in $\{t_0, ..., t_l\}$, then $h \in \bigcap_{\lambda < \alpha_l} V_\lambda$. Therefore, $h \in V_{\lambda_0}$. So, $C_p([0, \kappa], I)$ is not equal to $\bigcap_{\lambda < \kappa} V_\lambda$. This means that $\check{C}(C_p([0, \kappa], I)) > \kappa$. □

Theorem 2.15. *SCH + $\mathfrak{c} \leq (\omega_\omega)^+$ implies:*

$$\check{C}(C_p([0,\alpha), I)) = \begin{cases} 1 & \text{if } \alpha \leq \omega \\ |\alpha| \cdot \mathfrak{d} & \text{if } \alpha > \omega \text{ and } \omega \leq |\alpha| < \omega_\omega \\ |\alpha| & \text{if } |\alpha| > \omega_\omega \text{ and } cof(|\alpha|) > \omega \\ |\alpha| & \text{if } cof(|\alpha|) = \omega \text{ and } \alpha \text{ is a cardinal number} > \omega_\omega \\ |\alpha| & \text{if } |\alpha| = \omega_\omega \text{ and } \mathfrak{d} < (\omega_\omega)^+ \\ |\alpha|^+ & \text{if } cof(|\alpha|) = \omega,\ |\alpha| > \omega_\omega,\ \alpha \text{ is not a cardinal number} \\ |\alpha|^+ & \text{if } |\alpha| = \omega_\omega \text{ and } \mathfrak{d} = (\omega_\omega)^+ \end{cases}$$

Proof. If $\alpha \leq \omega$, $C_p([0, \alpha), I) = I^{[0,\alpha)}$, so $\check{C}(C_p([0, \alpha), I)) = 1$.

If $\alpha > \omega$ and $\omega \leq |\alpha| < \omega_\omega$, we obtain our result because of Theorem 2.7 and Proposition 2.13.

If $|\alpha| > \omega_\omega$ and $cof(|\alpha|) > \omega$, by Theorem 2.7 and Proposition 2.13,

$$|\alpha| \cdot \mathfrak{d} = |\alpha| \leq \check{C}(C_p([0, \alpha), I)) \leq kcov(|\alpha|^\omega) = |\alpha|.$$

If $cof(|\alpha|) = \omega$ and α is a cardinal number $> \omega_\omega$, by Lemma 2.4,

$$\check{C}(C_p([0,\alpha),I)) = |\alpha| \cdot \sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma),I)).$$

The number α is a limit ordinal and for every $\gamma < \alpha$,

$$\check{C}(C_p([0,\gamma),I)) \leq |\gamma|^+ \cdot \mathfrak{d}.$$

Since $\mathfrak{d} \leq (\omega_\omega)^+ < |\alpha|$, then $\check{C}(C_p([0,\alpha),I)) = |\alpha|$.

By Lemma 2.4 and Theorem 2.7, if $|\alpha| = \omega_\omega$, then

$$\omega_\omega \cdot \mathfrak{d} \leq \check{C}(C_p([0,\alpha),I)) = |\alpha| \cdot \sup_{\gamma<\alpha}\check{C}(C_p([0,\gamma),I)) \leq |\alpha| \cdot \sup_{\gamma<\alpha}(|\gamma|^+ \cdot \mathfrak{d}).$$

Thus, if $|\alpha| = \omega_\omega$ and $\mathfrak{d} < (\omega_\omega)^+$, $\check{C}(C_p([0,\alpha),I)) = |\alpha|$.

Assume now that $cof(|\alpha|) = \omega$, $|\alpha| > \omega_\omega$ and α is not a cardinal number. There exists a cardinal number κ such that $\kappa = |\alpha|$ and $[0,\alpha) = [0,\kappa] \oplus [\kappa+1,\alpha)$. So, $\check{C}(C_p([0,\alpha),I)) = \check{C}(C_p([0,\kappa],I)) \cdot \check{C}(C_p([\kappa+1,\alpha),I)) = \check{C}(C_p([0,\kappa],I))$ (see Proposition 1.10 in [8] and Lemma 2.3). By Theorem 2.7 and Proposition 2.14, $\kappa \cdot \mathfrak{d} \leq \check{C}(C_p([0,\kappa],I)) \leq \kappa^+$. Being κ a cardinal number $> \omega_\omega$ with cofinality ω, it must be $> (\omega_\omega)^+$; so $\kappa > \mathfrak{d}$ and, then, $\kappa \leq \check{C}(C_p([0,\kappa],I)) \leq \kappa^+$. Now we use Proposition 2.14, and conclude that $\check{C}(C_p([0,\alpha),I)) = \kappa^+ = |\alpha|^+$.

Finally, assume that $|\alpha| = \omega_\omega$ and $\mathfrak{d} = (\omega_\omega)^+$. By Theorems 2.7 and Proposition 2.13 we have

$$|\alpha| \cdot \mathfrak{d} \leq \check{C}(C_p([0,\alpha),I)) \leq kcov(|\alpha|^\omega) = (\omega_\omega)^+.$$

And we conclude: $\check{C}(C_p([0,\alpha),I)) = |\alpha|^+$. □

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