Nº 2

Wild categories of periodic modules

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WILD CATEGORIES OF PERIODIC MODULES

1.

Let K be a field of characteristic $p>0$ and $G = \langle x,y \rangle$ be an elementary abelian group of order p^2. It has long been known (Heller - Reiner [2]), that if $p>2$ the category of left KG-modules is wild. Basically this means that there is no possibility of classifying the indecomposable objects in the category. However, there seems to be a general misconception that suitable subcategorie such as the full subcategory of periodic modules, should be better behaved. The purpose of this note is to demonstrate that such is not the case. Indeed we show that the full subcategory of all KG-modules whose cohomology rings are annihilated by a fixed non-nilpotent element ζ of $H^2(G,K)$ is a wild category. We will not belabor the point by proving it in every possible case. For simplicity, only the case in which $p \geq 7$ and ζ is the Bockstein of a nonzero element of $H^1(G,F_p)$ will be considered. We hope that the reader will regard this example as sufficient.

Concerning our terminology and notation we refer to Carlson [1].

2.

For convenience let $X = x-1$ and $Y = y-1$. Then a KG-module may be considered as a $K[X,Y]$-module in which $X^pM = 0$ and $Y^pM = 0$.

Let $\zeta \in H^2(G,K)$ be the Bockstein of the element $\eta \in H^1(G,K)$ represented by $\eta : \Omega(K) \to K$, where $\eta(X) = 1$ and $\eta(Y) = 0$.

Lemma. Let M be any KG-module such that $M_{\langle x\rangle}$ is a free $K\langle x\rangle$-module and $Y^{(p-1)/2}M = 0$. Then ζ annihilates $\mathrm{Ext}^*_{KG}(M,M)$.

Proof. The basic trick is to note that ζ is represented by the sequence

$$E : 0 \to K \to K^G_{\langle y\rangle} \to K^G_{\langle y\rangle} \to K \to 0,$$

where the middle map is multiplication by X. If I is the identity of $\mathrm{Ext}^*_{KG}(M,M)$, then ζI is represented by $E \otimes M$. For $(\alpha, \beta) \in K^2$ and $u = 1 + \alpha X + \beta Y$, it is easy to see that $M_{\langle u\rangle}$ is free as a $K\langle u\rangle$-module provided $\alpha \neq 0$, hence $V_G(M) = \{(0, \beta); \beta \in K\}$. So M is periodic. Moreover, M must be periodic of order 2. Therefore we have the diagram

$$\begin{array}{ccccccccc}
0 \to & M & \to & P_2 & \to & P_1 & \to & M & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \| & \\
0 \to & M & \to & K^G_{\langle y\rangle} \otimes M & \to & K^G_{\langle y\rangle} \otimes M & \to & M & \to 0
\end{array}$$

where the upper sequence is part of a projective resolution of M, and μ is the chain map coming from the projectivity of P_1 and P_2.

Now $j(M) \subset Y^{(p+1)/2}P_2$, so

$$\mu_2 j(M) \subset Y^{(p+1)/2}(K^G_{\langle y\rangle} \otimes M) = 0,$$

since $Y^{(p+1)/2}M = 0$ and $YK^G_{\langle y\rangle} = 0$. Therefore $\mu = 0$.

Let $\mathcal{C}_\zeta$ be the full subcategory of KG-modules M such that $\zeta\,\mathrm{Ext}^*_{KG}(M,M) = 0$.

Let $\mathcal{M}$ be the full subcategory of $K[X,Y]$-modules consisting of all M that satisfy (1) to (7).

(1) $X^pM = 0$.

(2) $Y^3M = 0$.

(3) $\mathrm{Dim}_K X^{p-1}M = \mathrm{Dim}_K X/XM$.

(4) $YM \subseteq XM$.

(5) $Y^2M \subseteq X^{p-1}M$.

For any $Z \in K[X,Y]$ and any submodule $L \subseteq M$, let $Z^{-n}L = \{m \in M \; ; \; Z^n m \in L\}$, and let $\psi: M \longrightarrow M/\mathrm{Rad}\, M$ be the natural quotient.

(6) $\psi(X^{1-p}YX^{-1}YX^{-1}YY^{-2}(0)) \subset \psi(X^{-1}YX^{-2}(0))$.

(7) $\psi(X^{1-p}Y^2M) \subset \psi(X^{-1}YY^{-2}(0))$.

Note that condition (3) guarantees that M is free as a K<x>-module. Thus, by the lemma, for $p \geqslant 7$, conditions (1), (2) and (3) say that any module in $\mathcal{M}$ is an object in $\mathcal{C}_{\gamma}$.

<u>Theorem</u>. The category $\mathcal{M}$ is a wild category, therefore $\mathcal{C}_{\gamma}$ is wild if $p \geqslant 7$.

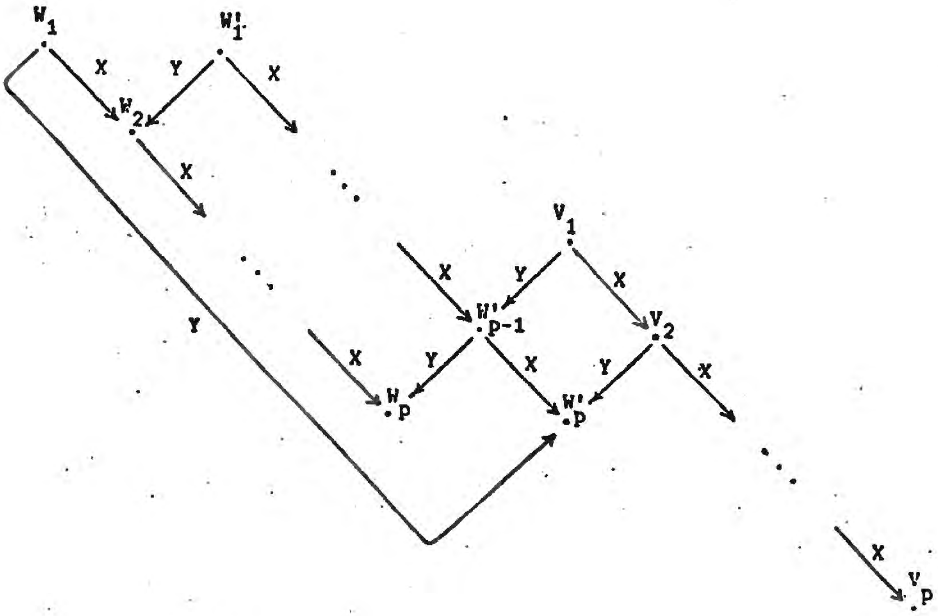
<u>Proof</u>. The proof follows Ringel [3].

Let Q be the quiver

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \circlearrowleft$$

A K-representation of Q consists of a triple $R = (V, W, \varphi)$, where W is the K-space corresponding to the vertex 2, V is the subspace of W corresponding to vertex 1, the inclusion $V \longrightarrow W$ corresponds to the arrow from 1 to 2, and the linear transformation $\varphi : W \longrightarrow W$ corresponds to the loop on 2. The category of K representations of Q, Rep(Q), is known to be a wild category.

To any $R = (V, W, \varphi)$ in Rep(Q) we associate a KG-module M_R in $\mathcal{M}$ represented by the following diagram.



That is, as a K-space,

$$M_R = W_1 \oplus \ldots \oplus W_p \oplus W_1' \oplus \ldots \oplus W_p' \oplus V_1 \oplus \ldots \oplus V_p ,$$

where

$$W_1 = \ldots = W_p = W_1' = \ldots = W_p' = W ,$$

and

$$V_1 = \ldots = V_p = V .$$

The action of X on M is given by the identity maps

$$I_W : W_i \longrightarrow W_{i+1} , \quad I_W : W_i' \longrightarrow W_{i+1}' , \quad I_V : V_i \longrightarrow V_{i+1} ,$$

for $i = 1, \ldots , p - 1$, and X is 0 on W_p, W_p' and V_p.
The action of Y on M is given by

$$\varphi : W_1 \longrightarrow W_p' , \quad I_W : W_i' \longrightarrow W_{i+1}$$

for $i = 1, \ldots , p-1$, and by the inclusions

$V_1 \longrightarrow W_{p-1}'$, $V_2 \longrightarrow W_p'$, and Y is 0 on $W_2, \ldots , W_p, W_p', V_3, \ldots , V_p$.

It is clear that $X^p M_R = Y^3 M_R = 0$, and $XY = YX$ on M_R . Also

$$M_R/\text{Rad } M_R \cong W_1 \oplus W_1' \oplus V_1 \cong W_p \oplus W_p' \oplus V_p' = X^{p-1} M_R .$$

It is easy to verity 4 and 5, and the proof of 6 and 7 is also straightforward.

Define then a functor $f : \text{Rep}(Q) \longrightarrow \mathfrak{M}$ by $f(R) = M_R$. A morphism

$$\Theta : R = (V, W, \varphi) \longrightarrow \hat{R} = (\hat{V}, \hat{W}, \hat{\varphi})$$

is such that $\Theta_V(V) \subset \Theta_W(W)$ and $\Theta_W \varphi = \hat{\varphi}\, \Theta_W$. So define a KG-module homomorphism $\mu : M_R \longrightarrow M_{\hat{R}}$ by $\mu = \Theta_W$ on W_i or W_i', and $\mu = \Theta_V$ on V_i, for $i = 1, \ldots, p$.

We now define a functor $g : \mathfrak{M} \longrightarrow \text{Rep}(Q)$. Given $M \in \mathfrak{M}$, let

$$W = \Psi(X^{-1}YY^{-2}(0), \quad V = \Psi(X^{1-p}Y^2M).$$

In order to define $\varphi : W \longrightarrow W$, given $w \in W$ choose a representative $\bar{w} \in X^{-1}YY^{-2}(0)$. Then $XY\bar{w} = 0$. Choose w_1 such that $Xw_1 = Y\bar{w}$. Then $Yw_1 \in YX^{-1}XX^{-1}YY^{-2}(0)$, and it is easily seen that Yw_1 is uniquely determined by w. Since $Yw_1 \in X^{p-1}M$, we have $Yw_1 = X^{p-1}w_2$, where $w_2 \in X^{1-p}YX^{-1}YX^{-1}YY^{-2}(0)$. Again, modulo Rad M, w_2 is uniquely determined by w. Let then $\varphi(w) = \Psi(w_2)$.

Define now $g : \mathfrak{M} \longrightarrow \text{Rep}(Q)$ by $g(M) = (V, W, \varphi)$, with V, W and φ as above.

Any KG-homomorphism, $\Theta : M \longrightarrow M_1'$, must preserve all of the subspaces used in the definition. So it must induce a morphism $\hat{\Theta} : g(M) \longrightarrow g(M')$.

Finally note that for all R in Rep (Q), $gf(R) \cong R$.

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