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Finite conjugacy in group rings

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FINITE CONJUGACY IN GROUP RINGS

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1. INTRODUCTION

Let KG be the group ring of a group G over a field K and let U(KG) denote the group of units of this ring. The set $S = S_K(G)$ of elements in G having a finite number of conjugates in U(KG) is called the K-supercenter of G. Clearly, if $S = G$ then U(KG) is itself an FC group.

This concept was introduced by S.K. Sehgal and H. J. Zassenhaus in the context of integral group rings [11]. C. Polcino Milies and S.K. Sehgal [8] studied the supercenter of a group G over an infinite field K in two cases: when G is a torsion group and when $char(K) = p > 0$ and G contains a normal p-subgroup.

In this paper we are concerned with supercenters of FC groups, but our techniques allow us to include also the case where K is infinite and G arbitrary. In this way, our results include those in [8]; however, we do not need to impose the restriction of the existence of a normal p-subgroup and we are able to describe S also in the case where $char(K) = 0$.

Group rings whose unit groups form an FC group have been studied in a series of papers: [10], [7] and [1]. Considering the

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case where $S = G$ we are able to fully describe these rings. Our results, given in section 4, are equivalent to those in these papers but we are able to describe conditions for the existence of FC unit groups solely in terms of the given group G and the field K.

2. SOME LEMMAS

Throughout this section we shall always assume either that K is an infinite field or that G is an infinite FC group.

(2.1) Lemma. Let $x \in KG$ be an element such that $x^2 = \beta x$ for some $\beta \in K$. Then $[x,KS] = \{xy-yx \mid y \in KS\} = 0$.

Proof. We wish to show that x commutes with every element $y \in S$. So, set $c = yxy^{-1}$. We claim that there exist infinitely many elements in $K^\circ \cup G$ which commute simultaneously with x, y and c. In fact, if K is infinite then the assertion is obvious. If not, then G is an infinite FC group and thus, all centralizers of elements in G are infinite.

We ennumerate explicitely $\mathrm{supp}(x) \cup \mathrm{supp}(c) = \{g_1, g_2, \ldots, g_n\}$. Denote $C_o = C_G(y)$. Since g_1 has finitely many conjugates by elements of C_o then $C_1 = C_{C_o}(g_1)$ is also infinite. Inductively, we see that $C_m = C_{C_{m-1}}(g_m)$ is infinite, and the elements in C_n verify our claim.

Set $\alpha \in K^\circ \cup G$. If $\beta \neq 0$ we consider the unit $u_\alpha = 1 - \beta^{-1}x + \beta^{-1}\alpha x$ whose inverse is $u_\alpha^{-1} = 1 - \beta^{-1}x + \beta^{-1}\alpha^{-1}x$ and compute:

$$w_\alpha = u_\alpha y u_\alpha^{-1} = (1-\beta^{-1}x+\beta^{-1}\alpha x)y(1-\beta^{-1}x+\beta^{-1}\alpha^{-1}x) =$$

$$= y-\beta^{-1}xy-\beta^{-1}yx+\beta^{-1}\alpha xy+\beta^{-1}\alpha^{-1}yx+(2\beta^{-2}-\beta^{-2}\alpha-\beta^{-2}\alpha^{-1})xyx.$$

Since $yx = cy$ we get:

$$w_\alpha = (1+\beta^{-1}\alpha x-\beta^{-1}x-\beta^{-1}c+\beta^{-1}\alpha^{-1}c+2\beta^{-2}xc-\beta^{-2}\alpha xc-\beta^{-2}\alpha^{-1}xc)y.$$

Hence:

$$xw_\alpha = x(\alpha+\beta^{-1}c-\beta^{-1}\alpha c)y = \alpha(x-\beta^{-1}xc) + \beta^{-1}xc.$$

Thus, if $x-\beta^{-1}xc \neq 0$ we would have infinitely many possible values for xw_α, a contradiction. So, $x = \beta^{-1}xc$ i.e. $\beta x = xc$. Back in the expression of w_α we obtain:

$$w_\alpha = (1-\beta^{-1}c+\beta^{-1}\alpha^{-1}(c-x))y.$$

Once again, we obtain infinitely many values for w_α unless $x = c$, as we wished to show.

If $\beta = 0$ we consider the unit $u_\alpha = 1+\alpha x$ whose inverse is $u_\alpha^{-1} = 1-\alpha x$ and a similar argument will show again that $x = c$. ▲

(2.2) <u>Corollary</u>. Every idempotent of KS is central in KG.

<u>Proof</u>. Let $e \in KS$ be an idempotent element and set $x \in G$. The elements $\alpha = ex(1-e)$ and $\beta = (1-e)xe$ are such that $\alpha^2 = \beta^2 = 0$.

From lemma (2.1) we have that:

$$\alpha = e\alpha = \alpha e = ex(1-e)e = 0.$$

In a similar way we see that $\beta = 0$. Hence $\alpha = \beta$ and thus $ex = xe$. ▲

(2.3) Lemma. Let t be a torsion element in G. Then, for every element $s \in S$ we have that $t^s \in \langle t \rangle$.

Proof. Set $x = \sum_{i=1}^{o(t)} t^i$. Then $x^2 = \beta x$ where $\beta = o(t)$. The lemma above shows that $x^s = x$ and the result follows immediately. Δ

We shall denote by T(S) the set of all torsion elements in S. Lemma (2.3) shows that T(S) is a subgroup of G. Moreover, it shows that each of its subgroups is normal in T(S) hence it is either an abelian group or a group of the form $K_8 \times A \times B$ where K_8 is the quaternion group of eight elements, A is an elementary abelian 2-group and B an abelian group all of whose elements are of odd order.

(2.4) Lemma. Assume that $char(K) = p > 0$. Then every p-element of T(S) is central in G.

Proof. Let $t \in T(S)$ be a p-element and let x be an arbitrary element in G. As in lemma (2.1) we can find infinitely many elements α in $K^o \cup G$ which commute with both t and x.

It follows from the description of T(S) above that t is contained in a finite p-group H which is normal in G. Hence, the element $\alpha x(t-1) \in \Delta(G:H)$ is nilpotent and thus $u_\alpha = 1-\alpha x(t-1)$ is a unit. We compute:

$$w_\alpha = u_\alpha t u_\alpha^{-1} = (1-\alpha x(t-1))t(1-\alpha x(t-1))^{-1}$$

$$= (1-\alpha x(t-1))t(1+\alpha x(t-1)+\alpha^2(x(t-1))^2+\ldots+\alpha^n(x(t-1))^n,$$

where we assume that $[x(t-1)]^{n+1} = 0$.

Hence:

$$w_\alpha = t + (tx-xt)(t-1) \sum_{i=1}^{n} \alpha^i (x(t-1))^{i-1}.$$

Notice that the set of all p-elements in T(S) forms a subgroup; hence, $\gamma = t^{-1}x^{-1}tx$ is also a p-element of T(S). We have that:

$$\theta = (tx-xt)(t-1) = xt(\gamma-1)(t-1) ,$$

$$w_\alpha = t + \theta \sum_{i=1}^{n} \alpha^i (x\ t-1))^{i-1}.$$

Assume, by contradiction, that $xt \neq tx$ i.e. that $\theta \neq 0$. If we can choose infinitely many values of α in K it is easy to see that $tx \neq xt$ implies that the coefficient of xt in w_α is of the form

$$\alpha + k_1\alpha^2 + k_2\alpha^3 + \ldots \quad , \quad k_i \in K \quad ,$$

a non zero polynomial which assumes infinitely many values as varies in K, a contradiction.

If, on the other hand, we can choose infinitely many values for α in G, but not in K, it is easy to see that we would have infinitely many values for w_α unless $\alpha^i \in$ <x,t> for some positive integer i. Let α^k be the smallest power of α that belongs to <x,t>. We can write:

$$w_\alpha = t + \alpha\theta(1+\alpha^k(x(t-1))^k + \alpha^{2k}(x(t-1))^{2k} + \ldots) +$$
$$+ \alpha^2\theta(x(t-1) + \alpha^k(x(t-1))^{k+1} + \ldots) +$$
$$+ \alpha^k\theta(x(t-1)^k + \alpha^k(x(t-1)^2)^{k-1} + \ldots) .$$

Since w_α must assume finitely many values for infinitely many values of α, if $k \neq 1$ we must have that all the coefficients of the powers of α in the expression above, which belongs to $K\langle x,t\rangle$ must be equal to 0. The coefficient of α is of the form $\theta(1+\delta)$ where $\delta \in \Delta(G:H)$; hence δ is nilpotent, $1+\delta$ is a unit and thus $\theta = 0$, a contradiction.

Finally, let us consider the case where $\alpha \in \langle x,t\rangle$. In order to have infinitely many such α, we must have that $o(x) = \infty$. As a power x^m must commute with t, we may assume that α is of the form $\alpha = x^{mj}$, for a positive integer j.

Notice that since $tx = xt\gamma$ we can write $(x(t-1))^{i-1} = x^{i-1}\gamma_{i-1}$ with $\gamma_{i-1} \in K\langle t,\gamma\rangle$ where $\langle t,\gamma\rangle \subset T(S)$ is finite and $\theta = x(t\gamma t - t\gamma - t^2 + t)$; hence:

$$w_\alpha = t + x^{mj+1}(t\gamma t - t\gamma - t^2 + t)(1 + x^{mj+1}\gamma_1 + x^{2mj+2}\gamma_2 + \dots) .$$

Once more, if $xt \neq tx$ i.e. if $\neq 1$, we would have infinitely many values for w , a contradiction. Δ

(2.5) <u>Lemma</u>. Let $char(K) = p > 0$ and denote by $P(K)$ the prime field of K. If $T(S)$ is not central in G, then the algebraic closure Ω of $P(K)$ in K is finite and for every $x \in G$ and every $t \in T(S)$ there exists an integer $r = r(x,t)$ such that $t^x = t^{p^r}$ where r is a multiple of $[\Omega:P(K)]$.

<u>Proof</u>. Let L be a finite subfield of K and let $t \in T(S)$ be a non central element. Then, by lemma 2.4, we may assume that t is a p-element, so $L\langle t\rangle = \bigoplus_i K_i$ where one summand, K_1 say, is of the form $K_1 = L(\xi)$ with ξ a root of unity such that $o(\xi) = o(t)$.

Clearly, t is mapped to ξ in the natural projection $L\langle t\rangle \to K_1$. Since idempotents of KS are central in KG, a routine argument shows that, for any element $x \in G$ we have $t^x = t^i$ for some i.

Corollary (2.2) also implies that $K_1^x = K_1$, hence conjugation by x defines an automorphism ϕ of K_1 and clearly $\phi(\xi) = \xi^i$.

Since K_1 is finite, ϕ is a power F^r of the Frobenius automorphism of K_1, $F: x \mapsto x^p$; thus $\xi^{p^r} = \phi(\xi) = \xi^i$ and so $o(\xi)|(p^r-i)$; consequently t^x is of the form $t^x = t^i = t^{p^r}$, as stated.

Also notice that $K_1 = L(\xi)$ contains a copy of L which is fixed by ϕ so, for every element $\alpha \in L$ we have that $\alpha^{p^r} = \alpha$, thus L is contained in a field with p^r elements i.e. $[L:P(K)] | r$. It follows easily that also $[\Omega:P(K)]|r$. Δ

We shall need the following elementary result on matrices.

(2.6) <u>Lemma</u>. Let D be a division ring. If $A \in M_n(D)$ commutes with all the idempotent matrices of $M_n(D)$ then $A = dI$ for some $d \in D$.

<u>Proof</u>. Let $A = (a_{ij}) \in M_n(D)$. Since $AE_{ii} = E_{ii}A$, it follows easily that $a_{ij} = 0$, if $i \neq j$.

Also let E be the matrix such that all entries in the first column are equal to 1 and all other entries are 0. Then $E^2 = E$ and thus $AE = EA$. This shows immediately that $a_{ii} = a_{11}$, for $i = 1,\dots,n$ completing the proof. Δ

(2.7) <u>Lemma</u>. Let $s \in T(S)$ and $t \in T(G)$. Then $(s,t)=1$.

<u>Proof</u>. Let us consider first the case where $car(K) = p > 0$. The group $<s,t>$ is finite so the group ring $\mathbb{Z}_p<s,t>$ is also finite and hence its Jacobson Radical, J, is nilpotent. We have that:

$$\frac{\mathbb{Z}_p<s,t>}{J} \cong \bigoplus_{i=1}^{r} M_{n_i}(K_i) \quad ,$$

where K_i is a finite extension of $\mathbb{Z}_p$, $1 \leq i \leq r$.

Let $(s_1,\ldots,s_r)$ denote the image of s in the left-hand member of the isomorphism above. Any idempotent $e_i \in M_{n_i}(D_i)$ can be lefted to an idempotent $e \in \mathbb{Z}_p<s,t>$. Therefore, because of lemma (2.2), we have that $es = se$ hence also $e_i s_i = s_i e_i$. After lemma (2.6) this shows that s_i is central im $M_{n_i}(K_i)$, $i=1,\ldots,r$. Hence $st = ts$ modulo J. We know, from lemma (2.3) that $sts^{-1}t^{-1} = t^j \in S$ for some positive integer j; hence $t^j-1 \in J$ is nilpotent. Consequently, $(t^j-1)^{p^n} = 0$ for some n i.e. $t^{jp^n} = 1$.

Hence, lemma (2.3) shows that t^j is central in G.

Since s can be written as the product of a p-element and a p'-element and p-elements of S are central in G it will <u>suf</u>fice to prove our statement assuming that s is a p'-element. We can write $ts^{-1}t^{-1} = s^{-1}t^j$. Since the left-hand side of this equation is a p'-element it follows immediately that $t^j = 1$ and thus $sts^{-1}t^{-1} = 1$ as we wished to prove.

Assume now that $char(K) = 0$. Then

$$K<s,t> \cong \bigoplus_{i=1}^{n} M_{n_i}(D_i)$$

where D_i is a division ring containing K, $i = 1,\dots,r$. Let $s \mapsto (s_1,\dots,s_r)$ in this isomorphism. Lemma (2.2) implies that s_i commutes with every idempotent in $M_{n_i}(D_i)$, so $s_i = d_i I$ for some $d_i \in D_i$ and, since s_i has a finite number of conjugates in $M_{n_i}(D_i)$ it follows from Herstein [5] that d_i is central in D_i, $i = 1,\dots,r$. Hence, our statement follows. Δ

(2.8) Lemma. Assume that S is non central and that T(S) is infinite such that, if $car(K) = p \neq 0$ then T(S) contains no p-elements. Then $T(S) = \mathbb{Z}(q^\infty) \times B$, for some prime $q \neq p$, where B is finite and central in G, $\mathbb{Z}(q^\infty)$ is central in S and $(G,S) \subset \mathbb{Z}(q^\infty)$. Furthermore, there exists a positive integer k such that K does not contain primitive roots of unity of order q^k.

Proof. We shall give the proof in several steps.

Claim 1. (G,S) is contained in every infinite subgroup of T(S).

Let $x \in G$, $s \in S$ and set $\gamma = s^{-1}x^{-1}sx \in S$, $\gamma \neq 1$. Assume by contradiction that there exists an infinite subgroup H in T(S) such that $\gamma \notin H$. As T(S) is abelian by Lemma (2.7) we can construct an infinite sequence $H_1 \underset{\neq}{\subset} H_2 \underset{\neq}{\subset} \dots \underset{\neq}{\subset} H_n \underset{\neq}{\subset} \dots$ of finite subgroups of H such that $\gamma \notin H_i$ and, consequently, if we set $e_i = |H|^{-1} \sum_{h \in H_i} h$ we obtain idempotents such that $(1-\gamma)e_i = (1-\gamma)e_j$ whenever $i \neq j$. By Corollary (2.2), these idempotents are central and hence $u_i = e_i x + (1-e_i)$ is a unit, for all i.

As in [1,p.166], we would obtain infinitely many conjugates s^{u_i} for s, a contradiction.

We know from Fuchs [4, Theorem 21.3] that $T(S)$ can be wirtten in the form $T(S) = D \times B$ where D is divisible and B a reduced group.

<u>Claim 2.</u> $D = \mathbb{Z}(q^{\infty})$ for some prime $q \neq p$, and B is finite and central in G.

If B were infinite a result of Kulikov [4, Corollary 27.2] shows that B contains a direct product of infinitely many cyclic groups and it would be easy to exibit two disjoint infinite groups, contradicting Claim 1.

The structure theorem for divisible groups [4, Theorem 23.1] shows that D is the direct product of indecomposable factors which are isomorphic either to $\mathbb{Z}(q^{\infty})$ for some prime q or to Q. Once again, Claim 1 shows that there can be only one factor and, since (G,S) must be contained in all infinite subgroups, it cannot be isomorphic to Q so $D = \mathbb{Z}(q^{\infty})$ for some prime q.

To show that B is central in G consider $b \in B$, $x \in G$ and set $\hat{b} = \sum_{i=1}^{o(b)} b^i$. Since $\hat{b}^x = \hat{b}$ by Lemma (2.1), it follows that $b^x = b^i$ for some i and thus $b^{-1}x^{-1}bx = b^{i-1} \in \mathbb{Z}(q^{\infty}) \cap B$ so $b^{i-1} = 1$ and then $bx = xb$.

<u>Claim 3.</u> $\mathbb{Z}(q^{\infty})$ is central in S.

Assume, by contradiction, that there exists $s \in S$ which does not centralize $\mathbb{Z}(q^{\infty})$. Since $(s,T(\mathbb{Z}(q^{\infty}))) \subset s\,Cl(s^{-1})$, where $Cl(s^{-1})$ denotes the conjugacy class of s^{-1}, it is finite. So, we can choose an element $t_1 = sts^{-1}t^{-1}$ of maximal order.

Set $\ell \in \mathbb{Z}(q^{\infty})$ such that $\ell^q = t$ and consider $t_2 = s\ell s^{-1}\ell^{-1}$. Note that both $s\ell s^{-1}$ and ℓ^{-1} belong to $T(S)$, which is abelian, thus

$$t_2^q = s\ell^q s^{-1}\ell^{-q} = sts^{-1}t^{-1} = t_1 .$$

consequently $o(t_2) > o(t_1)$, a contradiction.

To conclude, we prove

Claim 4. There exists a positive integer k such that K does not contain primitive roots of units of order q^k.

Once again, assume by contradiction that K contains all roots of unity of order a power of q.

Choose $x \in G$, $s \in S$ such that $1 \neq t = xsx^{-1}s^{-1} \in T(Z(q^\infty))$, with $o(t) = q^r$. Then, $xsx^{-1} = ts$ and $xs^{q^{r-1}}x^{-1} = t^{q^{r-1}}s^{q^{r-1}}$ so, if we set $y = s^{q^{r-1}}$ we see that $xyx^{-1}y^{-1} = xs^{q^{r-1}}x^{-q}s^{-q^{r-1}} = t^{q^{r-1}}$ is a commutator of order q.

Now, for each positive integer n we define an idempotent $e_n = q^{-n}\sum_{i=0}^{q^n-1}(\xi_n y_n)^i$ where $_n$ is a $(q^n)^{th}$-root of unity and y_n is an element in $Z(q^\infty)$ such that $y_n^{q^{n-1}} = t$. We can construct units $u_n = e_n x + (1-e_n)$ and compute:

$$y^{u_n} = [e_n x + (1-e_n)]y[e_n x + (1-e_n)]$$

$$= e_n xyx^{-1} + y(1-e_n) = (t-1)e_n y + y.$$

Since y has only finitely many conjugates, we would have $(t-1)e_n = (t-1)e_m$ for $n \neq m$, a contradiction. Δ

The final lemma of this section deals with the case where S contains p-elements.

(2.9) Lemma. Assume that $char(K) = p > 0$. If S is non-central and contains p-elements then $p = 2$, $T(S) = \langle t \rangle \times A$ where $o(t) = 2$, A is a finite subgroup of odd order, $(G,S) = \langle t \rangle$ and T(S) is central.

Proof. Let $t \in S$ be a p-element and let $s \in S$, $g \in G$ be such that $sg \neq gs$. Since lemma (2.4) shows that t is central, it follows that $g(t-1)$ is nilpotent. As before, we can find infinitely many elements $\alpha \in K^{o} \cup G$ which commute with t, s and g so we can consider units of the form $u_{\alpha} = 1 + \alpha g(1-t)$ and compute:

$$w_{\alpha} = s^{\alpha} = s+\alpha(sg-gs)+\alpha^{2}(sg-gs)g(t-1)^{2}+\ldots+\alpha^{n}(sg-gs)g^{n-1}(t-1)^{n}$$

where we are assuming that $(t-1)^{n+1} = 0$.

Writing $s^{-1}g^{-1}sg = \gamma$ we get:

$$w_{\alpha} = s + gs(\gamma-1)(t-1)[\alpha+\alpha^{2}g(t-1)^{2}+\ldots+\alpha^{n}g^{n-1}(t-1)^{n}].$$

Arguments similar to those in lemma (2.4) show that we will obtain a contradiction unless $(\gamma-1)(t-1) = 0$. This can only happen if $\mathrm{char}(K) = 2$, $\gamma t = 1$ and $\gamma = t$ which shows that $o(t) = 2$, that $(S,G) = \langle t\rangle$ and that t is the only element of order a power of 2 in $T(S)$.

If we write $T(S) = \langle t\rangle \times A$ it follows as in claim 2 of lemma (2.4) that A is central. If A were infinite, KA would contain an infinite family of idempotents $\{e_i\}$ such that $(t-1)e_i \neq (t-1)e_j$ whenever $i \neq j$. Setting $u_i = e_i g+(1-e_i)$ and computing s^{u_i}, we would obtain a contradiction similar to that in claim 4 of lemma (2.4). ▲

3. SUPERCENTERS

We are now ready to state our results. As before, we shall assume either that K is an infinite field or that G is an infinite FC-group.

Lemma (2.7) describes the supercenter in the case where G is torsion.

<u>Theorem A</u>. Let G be a torsion group. Then $S = Z(G)$, the center of G.

To describe the supercenter when G contains elements of infinite order we shall discuss two separate cases, according to the characteristic of the field K.

<u>Theorem B</u>. Assume that $char(K) = p > 0$ and that G is not a torsion group. Then one of the following holds.

(i) $S = Z(G)$.

(ii) $p = 2$, $T(S) = \langle t \rangle \times A$ where $o(t) = 2$, A is a finite subgroup of odd order, $(G,S) = \langle t \rangle$ and T(S) is central.

(iii) T(S) is an abelian p'-group such that for all $t \in T(S), x \in G$, we have that $t^x = t^{p^r}$ for some non negative integer $r = r(x,t)$. If T(S) is not central in G then the algebraic closure Ω of P(K) in K is finite and r is a multiple of $[\Omega : P(K)]$.

Furthermore, if T(S) is infinite, then it is of the form $T(S) = Z(q^\infty) \times B$ for some prime $q \neq p$, B is finite central in G, $Z(q^\infty)$ is central in S, $(G,S) \subset Z(q^\infty)$ and there exists an integer k such that K does not contain roots of unity of order q^k.

<u>Proof</u>. Assume that S is not central. If it contains a p-element then lemma (2.9) applies and gives (ii). Δ

On the other hand, if S contains no p-elements we can use both lemma (2.5) and lemma (2.8) to obtain (iii).

A few changes will allow us to deal with the case where char(K) = 0.

<u>Theorem C</u>. Assume that char(K) = 0 and that G is not a torsion group. Then one of the following holds:

(i) $S = Z(G)$

(ii) T(S) is an abelian group such that for all $t \in T(S)$, $x \in G$ we have that $t^x = t^i$ for some positive integer i and for each non central element $t \in T(S)$ the field K does not contain roots of unity of order o(t). Furthermore if T(S) is infinite then T(S) and K can be described as in part (iii) of Theorem B above.

<u>Proof</u>. Assume that S is non-central. Since idempotents of KS are central in KG, it follows easily that $t^x = t^i$ for some i.

Assume, by contradiction, that we can find a non-central element t such that K contains a root of unity ξ with $o(\xi) = o(t)$. Let $x \in G$ be such that $xt \neq tx$ and consider the idempotent

$$e = o(t)^{-1} \sum_{i=0}^{o(t)-1} (\xi t)^i \in KS \quad .$$

Since e is central, we have that $e^x = e$. Also, we have that $t^x = t^i$ so, computing the coefficient of t^i in this equation we obtain $\xi = \xi^i$. But $i < o(\xi)$ so we would have $i = 1$, a contradiction.

If T(S) is infinite it must be as in Theorem B above since Lemma (2.8) applies also in this case. ▲

4. FC UNIT GROUPS

In this section we consider the question as to when the group of units U(KG) is an FC-group, i.e. when $S_K(G) = G$. Theorem A of section §3 gives immediately our first result.

Theorem A'. Let G be a torsion group. Then U(KG) is an FC group if and only if either KG is finite or G is abelian.

In order to simplify the statement of the results, we shall divide the case where $char(K) = p > 0$ in two subcases, according to the existence of p-elements.

Theorem B'. Assume that $char(K) = p > 0$ and that G is a non-torsion group which contains p-elements. Then U(KG) is an FC group if and only if either G is abelian or G is a non abelian FC-group, $p = 2$, $T(G) = \langle t\rangle \times A$ where $o(t) = 2$, A is a finite group of odd order, $G' = \langle t\rangle$ and T(G) is central.

Proof. If U(KG) is FC then $S = G$ and theorem B part (i) and (ii) gives the "only if" part of our statement.

The converse is precisely the same as in [1, p.168]. ∆

Theorem B". Let $char(K) = p > 0$ and let G be a non torsion group with no p-elements. Then U(KG) is an FC group if and only if either G is abelian or G is a non abelian FC group, T(G) is abelian and one of the following conditions holds:

(i) KT(G) is finite and for all $t \in T(G)$, $x \in G$, we have that $t^x = t^{p^r}$ for some non negative integer $r = r(x,t)$ which is a multiple of [K:P(K)].

(ii) T(G) is finite, central.

(iii) T(G) is central, of the form $T(G) = Z(q^\infty)$ B for some prime $q \neq p$, $G' \subset Z(q^\infty)$ and there exists an integer k such that K does not contain roots of unity of order q^k.

Proof. If $S = G$ and G is non abelian, Theorem B gives directly that T(G) is abelian, that conjugacy is as stated and that if T(G) is infinite then it is as described in (iii) above.

So, it remains to prove that if T(G) is finite then either it is central or K is finite and $r = r(x,t)$ is a multiple [K:P(K)]. Assume then that T(G) is finite and non central. Lemma (2.3) of [7] shows that K must be finite, hence K coincides with Ω in theorem B and the conclusion follows.

To prove the converse we remark that condition (i) implies, after [2], that every idempotent of KT(G) is central in KG hence, by [1] theorem A, part (iii) it follows that U(KG) is FC.

If (ii) holds then either theorem A part (iii) or theorem C part (ii) of [1] shows that U(KG) is FC.

Finally, the fact that (iii) implies U(KG) FC is also due to theorem A and C of [1]. Δ

Theorem C'. Let $char(K) = 0$ and assume that G is a non-torsion group. Then U(KG) is an FC group if and only if G is either abelian or a non abelian FC group with T(G) central and, if T(G) is infinite then T(G) and K can be described as in part (iii) of Theorem B" above.

Proof. To prove the "only if" part of the statement, in view of theorem C, it suffices to show that T(G) is central. This, as well as the converse, is done in [10] or [9, pp.209-214]. ▵

5. FINAL EXAMPLES

Both in theorems B and C we have shown that T(S) is an abelian subgroup of G but not its centrality, as obtained in [8]. Our first example shows that T(S) can actually be non-central. It also illustrates the fact that, though theorem B shows that the algebraic closure of P(K) in K is finite, K itself needs not be finite.

Example 1. Let $G = \langle g,t \mid t^3 = 1,\ gtg^{-1} = t^2 \rangle$. It is easy to see that $T(G) = \langle t \rangle$. Set $K = \mathbb{Q}$ or $K = \mathbb{Z}_5(x)$, a field of rational functions.

Since $K\langle t \rangle \cong \dfrac{K[X]}{X^3-1} \cong \dfrac{K[X]}{(X-1)} \oplus \dfrac{K[X]}{(X^2+X+1)}$ and X^2+X+1 is irreducible in K[X] in both cases, we see that $K\langle t \rangle$ is of the form $K\langle t \rangle = K_1 \oplus K_2$ where $K_1 = K\langle t \rangle e$, $K_2 = K\langle t \rangle(1-e)$ are fields, with $e = \dfrac{1+t+t^2}{3}$.

Since the first part of [9, Lemma VI.3.22] holds also in the semisimple case, we have that a unit $u \in KG$ can be written in the form $u = u_1 g^{i_1} + u_2 g^{i_2}$ with inverse $u^{-1} = g^{-i_1} u_1^{-1} + g^{-i_2} u_2^{-1}$ with $u_i \in K_i$, $i = 1,2$, $i_1, i_2 \in \mathbb{Z}$. Hence:

$$t^u = u_1 g^{i_1} t g^{-i_1} u_1^{-1} + u_2 g^{i_2} t g^{-i_2} u_2^{-1} = t^{\delta_1} e_1 + t^{\delta_2} e_2$$

where $\delta_i = 1,2$, $i = 1,2$.

Consequently, $T(S) = \langle t\rangle$ which is not a central subgroup.

Again, both in theorems B and C we have shown that $Z(q^\infty)$ is central in S but not in G. Our next example shows a situation where this actually happens.

Example 2. Let $G = \langle Z(3^\infty), g \mid gtg^{-1} = t^2, \forall\, t \in Z(3^\infty)\rangle$. As before, it is easy to see that $T = T(G) = Z(3^\infty)$ and we set again $K = Q$ or $K = Z_5(x)$.

We wish to show that $T = T(S)$. To do so, consider $t \in Z(q^\infty)$ and let $e = o(t)^{-1} \sum_{i=1}^{o(t)} t^i$. Then $1-e \in \Delta(T)$ and since K is a field of the first kind, [6, lemma 14.4.3] shows that $1-e = \sum_{i=1}^{m} e_i$, a finite sum of orthogonal primitive idempotents of KT.

Given a unit $u \in KG$, it can be written as $u = \sum_r \gamma_r g^r$, $\gamma_r \in KT$, $r \in Z$. We consider the subgroup $H \subset G$ generated by t, the supports of the elements γ_r and the supports of the idempotents e_i. Clearly H is finite and we can write:

$KT = KTe \oplus KT(1-e) = KTe \oplus L_1 \oplus \dots \oplus L_m$, where $L_i = KTe_i$, $1 \le i \le m$

and

$KH = KHe \oplus KH(1-e) = KHe \oplus KHe_1 \oplus \dots \oplus KHe_m$, where $KHe_i \subset L_i$, $1 \le i \le m$

Writing $e = \sum_{j=1}^{n} f_j$ a sum of orthogonal primitive idempotents of KH we can write:

$$KH = KHf_1 \oplus \dots \oplus KHf_n \oplus KHe_1 \oplus \dots \oplus KHe_m \quad ,$$

a direct sum of fields.

Since the theorems in [2] and [3] show that all idempotents of KT are central in KG we can apply again [9, Lemma VI.3.22] and write u in the form:

$$u = \sum_{j=1}^{n} k_j g^{\alpha_j} + \sum_{i=1}^{m} \ell_i g^{\beta_i}, \quad k_j \in KHf_j, \quad \ell_i \in KHe_i, \quad \alpha_j, \beta_i \in \mathbb{Z}.$$

Now, writing $t = te + \sum_{i=1}^{m} te_i = e + \sum_{i=1}^{m} te$ we obtain:

$$t^u = e^u + \sum_{i=1}^{m} (te_i)^u = e + \sum_{i=1}^{m} (te_i)^{\ell_i g^{\beta_i}} = e + \sum_{i=1}^{m} t^{g^{\beta_i}} e_i.$$

Consequently, t has finitely many conjugates in KG, so $t \in S$ and thus $T(S) = T(G) = \mathbb{Z}(3^\infty)$ is not central in G.

REFERENCES

[1] G.H. Cliff and S.K. Sehgal: Group Rings whose Unit form an FC Group, Math. Z. 161(1978), 163-168.

[2] S.P. Coelho: A note on Central Idempotents in Group Rings, Proc.Edinburgh Math. Soc. 30(1987), 69-72.

[3] S.P. Coelho and C. Polcino Milies: A Note on Central Idempotents in Group Rings II, Proc. Edinburgh Math. Soc. 31(1988), 211-215.

[4] L. Fuchs: Infinite Abelian Groups, vol. I, Academic Press, New York, 1970.

[5] I.N. Herstein: Conjugates in Division Rings, Proc.Amer.Math. Soc. 7(1956), 1021-1022.

[6] D.S. Passman: The Algebraic Structure of Group Rings, John Wiley & Sons, New York, 1977.

[7] C. Polcino Milies: Group Rings whose Units form an FC group Archiv der Math. 30(1978), 380-384.

[8] C. Polcino Milies and S.K.Sehgal: FC elements in Group Rings, Comm. in Algebra 9(1981), 1285-1293.

[9] S.K. Sehgal: Topics in Group Rings, Marcel Dekker, New York, 1978.

[10] S.K. Sehgal and H.J. Zassenahus: Group Rings whose Units form an FC group, Math. Z. 153(1977), 29-35.

[11] S.K. Sehgal and H.J. Zassenhaus: On the Supercenter of a Group and its Ring-Theoretical Generalizations in Integral Representations and Applications, Lecture Notes in Math, nº 882, Springer-Verlag, New York, 1981, pp.117-144.

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