RT-MAT 95-11

Attractors for Parabolic Problems with Non linear Boundary Conditions in Fractional Power Spaces

Sergio Muniz Oliva
Antonio Luiz Pereira

Abril 1995

Attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces

Sergio Muniz Oliva*

Antonio Luiz Pereira†

Abstract

In this work we prove existence of global attractors for reaction-diffusion problems with nonlinear boundary conditions in fractional power spaces X^α which are embedded in $\mathbf{C}$, without assuming growth conditions on the reaction term. These hypotheses are natural and easy to verify in many applications. The tools employed are comparison principles and interpolation theory.

1 Introduction

Let Ω be a bounded smooth domain of $\mathbf{R}^n$. In this paper we consider reaction diffusion systems with dispersion of the form

$$\begin{cases} u_t = \mathrm{Div}(a\nabla u) - \sum_{j=1}^{n} B_j(x)\dfrac{\partial u}{\partial x_j} - \lambda u + f(u), & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n_a} = g(u), & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

*Departamento de Mat. Aplicada, Instituto de Matemática e Estatística, USP - Brazil
†Departamento de Matemática, Instituto de Matemática e Estatística, USP - Brazil

where $u = (u_1, \cdots, u_N)^\top$, $N \geq 1$, $a(x) = \text{diag}(a_1(x), \cdots, a_N(x))$, $a_i \in C^1(\overline{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq N$, $\frac{\partial u}{\partial n_a} = \langle a\nabla u, \vec{n}\rangle$, $\vec{n}$ is the outward normal, λ is a positive constant and $B_j = \text{diag}(b_j^1, \cdots, b_j^N)$ is continuous in $\bar{\Omega}$, $j = 1, \cdots, n$. Let $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$, $g = (g_1, \cdots, g_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ be smooth functions.

It has been shown by Pao [15] that if f is a source of heat and if $g = 0$ then we have blow up in finite time. Our aim is to control the increase of heat by means of a dissipative flux through the boundary. To accomplish this goal we need to introduce some kind of "competition" between f and g. In fact one of the basic questions is: If g dissipates heat through the boundary, can we find a relation between the dissipation g and the source of heat f in such a way that we can assure the existence of global attractors?

This problem is not new, Pao [15] introduced a relation between f and g and almost completely solved the problem for classical solutions working in the space of continuous functions. Later on, Alikakos [1], following the ideas of Friedman [8], imposing some growth conditions on f and g, showed global existence and some asymptotic behavior of the solutions. Working in $\mathbb{R}$, Henry [11], solved the problem completely, and assuming a nice relation between f and g (similar to Pao [15]), showed that the system is Morse-Smale.

Another major concern is to relax the growth conditions on f and g. In this direction Carvalho, Oliva, Pereira and Rodriguez-Bernal [5] proved, for the scalar case, N=1, and $\Omega \subset \mathbb{R}^n$, $n \leq 3$, that under some growth assumptions on the nonlinearity f, the problem (1), with $B_j \equiv 0$, $j = 1, \cdots, n$, has a global attractor in $H^1(\Omega)$. More specifically, for $n = 2$, f and g are required to satisfy:

$$\left.\begin{aligned} \liminf_{|s|\to\infty} \frac{f(s)}{s} \leq 0 \\ \liminf_{|s|\to\infty} \frac{g(s)}{s} \leq 0 \end{aligned}\right\} \text{ one of the inequalities being strict,} \tag{2}$$

(the dissipative conditions)

and

$$\lim_{|s|\to\infty} \frac{|f'(s)|}{e^{\eta|s|^2}} = \lim_{|s|\to\infty} \frac{|g''(s)|}{e^{\eta|s|^2}} = 0, \ \forall \eta > 0, \tag{3}$$

(the growth conditions).

These growth assumptions are used to obtain local existence of solutions for (1) and also play a role in obtaining energy estimates necessary to guarantee that the solution operator for (1) defines a global dynamical system which is bounded dissipative.

In the work above, for $n = 2$, it has also been proved the existence of global attractors assuming only dissipative properties on f and g. The key idea is to restrict the space of initial data in such a way that no growth assumptions are needed for local existence of solutions for (1).

The goal of this paper is to extend the later results to arbitrary dimensions. To accomplish this we work in L^p spaces for a suitable choice of $1 < p < \infty$, instead of $L^2(\Omega)$, and then follow the general approach developed by Amann [2], [3]. The main difference is that, in our approach, instead of working in the Sobolev spaces $W^{k,p}(\Omega)$, we work in the fractional power spaces associated to the operator defined by the linear part of (1) with homogeneous boundary conditions. These fractional power spaces turn out to be the so called "Lebesgue Spaces" $H_p^s(\Omega)$ (see Triebel [20], for a general discussion about these spaces and their relation with differential operators and interpolation theory). In this way we are able to use the well developed theory of sectorial operators as described, for example, in Henry [10]. This approach leads, in our opinion, to a considerable simplification in Amann's arguments.

To better describe our results, let us be somewhat more precise.

Let $A = diag(A_1, \cdots, A_N)$ be the operator in $L^p(\Omega; \mathbb{C}^N)$, defined by

$$D(A_i) = H^2_{p,\{B\}}(\Omega)$$

$$A_i u = -Div(a_i \nabla u) + \sum_{j=1}^{n} b_j^i(x) \frac{\partial u}{\partial x_j} + \lambda u$$

where $L^p(\Omega; \mathbb{C}^N)$ is the *complex* space of p-integrable functions in Ω and $H^2_{p,\{B\}}(\Omega)$ is the subspace of functions in $H^2_p(\Omega)$ satisfying the homogeneous boundary conditions $Bu \equiv \dfrac{\partial u}{\partial n_a} = 0$.

It is convenient to work first in complex spaces since their interpolation theory is simpler. Afterwards we need to return to real spaces in order to use comparison principles. This can be done simply by taking real parts, since our operator has real coefficients.

We then define the operator A_{-1} in the dual space of $\left(H^2_{p,\{\mathcal{B}\}}(\Omega;\mathbf{C}^N)\right)$, with domain $L^p(\Omega;\mathbf{C}^N)$, by duality. We prove that A_{-1} is a sectorial operator and thus so are its restrictions to the fractional power spaces $X^\alpha = D(A^\alpha_{-1})$. Actually we need to choose α in order to meet two competing requirements. Since we are going to use comparison principles, we want X^α imbedded in $C(\bar{\Omega})$. On the other hand, we do not want to incorporate the boundary conditions $\mathcal{B}u = 0$, since we want our solutions to satisfy the nonlinear boundary conditions of the original problem ($\mathcal{B}u = g(u)$).

With this choice of α, we will consider an abstract problem, whose solutions include the classical solutions of our original problem. Actually, we prove that our solutions $u(t,\cdot)$ are classical solutions for positive t.

Within this abstract framework we are then able to prove existence of solutions and, imposing suitable dissipation conditions on the pair (f,g) (which we call competition conditions) we finally establish existence of a global attractor.

The approach we follow here is to show that the solution operator associated to (1) is globally defined, that orbits of bounded subsets of X^α, under the flow defined by (1), are bounded subsets of X^α and that there is a bounded set that attracts points of X^α. Since the solution operator associated to (1) is compact, Theorem 3.4.6 in Hale [9] guarantees the existence of a global attractor.

The paper will proceed as follows: in Section 2 we introduce the notations to be used in this work, and also define the Lebesgue Spaces and the basic results of Interpolation Theory. We also define in this Section the negative fractional powers of our operator A. In Section 3 we state our general hypotheses, in Section 4 we define the spaces we work in and show local existence of solutions for (1) and prove some regularity results (Section 5). In Section 6 we use the notion of sub- and super-solutions to show that the semigroup is bounded and that the solutions are defined for all time. Finally in Section 7 we prove existence of global attractors and a bound for such attractors.

2 Definitions and Notation

In this section we define the spaces and fix our notation. We will also state some results of interpolation theory for the sake of completeness and refer all the proofs and further information to Triebel [20].

2.1 Lebesgue Spaces

Let $S = S(\mathbf{R}^n)$ be the set of all complex-valued rapidly decreasing infinitely differentiable functions defined on the n-dimensional real Euclidean space $\mathbf{R}^n$. As usual, $S' = S'(\mathbf{R}^n)$ denotes the space of tempered distributions, which is the dual of S. We denote by

$$(\mathcal{F}\phi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle}\phi(x)dx, \ \ \phi \in S,$$

$\langle x, \xi\rangle = \sum_{j=1}^{n} x_j\xi_j$, the Fourier transformation, and

$$(\mathcal{F}^{-1}\phi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle}\phi(x)dx, \ \ \phi \in S,$$

the inverse Fourier transformation.

With this we can define the *Lebesgue Spaces* in $\mathbf{R}^n$, as follows

Definition 2.1 *Let* $-\infty < s < \infty$ *and* $1 < p < \infty$. *Then*

$$H_p^s(\mathbf{R}^n) = \left\{f | f \in S'(\mathbf{R}^n), \|f\|_{H_p^s} = \|\mathcal{F}^{-1}(1+|x|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L^p} < \infty\right\}$$

The first result relates the negative exponents with the dual spaces.

Theorem 2.1 (Triebel [20], pag. 198) *Let* $-\infty < s < \infty$ *and* $1 < p < \infty$. *Then*

$$\left(H_p^s(\mathbf{R}^n)\right)' = H_{p'}^{-s}(\mathbf{R}^n), \ \ \frac{1}{p} + \frac{1}{p'} = 1,$$

where $\left(H_p^s(\mathbf{R}^n)\right)'$ *denotes the dual space of* $H_p^s(\mathbf{R}^n)$.

The next theorem shows the relation between the usual Sobolev Spaces ($W^{k,p}$) and the Lebesgue Spaces, and in particular will imply that, for $p = 2$, our approach coincides with the one of Amann [2], [3] or Carvalho, Oliva, Pereira and Bernal [5].

Theorem 2.2 **(Triebel [20], pag. 242)** *Let* $-\infty < s_0, s_1 < \infty$ *and* $1 < p_0, p_1 < \infty$. *Then*

$$W^{s_0,p_0}(\mathbf{R}^n) = H^{s_1}_{p_1}(\mathbf{R}^n)$$

if and only if, either

$$s_0 = s_1 \text{ is an integer, and } p_0 = p_1;$$

or

$$s_0 = s_1 \text{ and } p_0 = p_1 = 2.$$

Now let us define the Lebesgue Spaces in a domain $\Omega \subset \mathbf{R}^n$.

Definition 2.2 *Let* $\Omega \subset \mathbf{R}^n$ *be an arbitrary (bounded or unbounded) domain. Further, let* $-\infty < s < \infty$ *and* $1 < p < \infty$. *Then* $H^s_p(\Omega)$ *is the restriction of* $H^s_p(\mathbf{R}^n)$ *to* Ω,

$$\|f\|_{H^s_p(\Omega)} = \inf_{\substack{g|_\Omega = f \\ g \in H^s_p(\mathbf{R}^n)}} \|g\|_{H^s_p(\mathbf{R}^n)}.$$

Here $g|_\Omega$ *denotes the restriction of* g *to* Ω.

Remark 2.1 $H^s_p(\Omega)$ *is a Banach Space for any* s *(see Triebel [20]).*

There are other ways to restrict a function $f \in H^s_p(\mathbf{R}^n)$ to Ω. Here are some related spaces.

Definition 2.3 *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded* C^∞ *domain. Further, let* $-\infty < s < \infty$ *and* $1 < p < \infty$. *Then*

$$\tilde{H}^s_p(\Omega) = \{f | f \in H^s_p(\mathbf{R}^n), supp f \subset \bar{\Omega}\};$$
$$\overset{\circ}{H}{}^s_p(\Omega) = \{\text{completion of } C^\infty_0(\Omega) \text{ in } H^s_p(\mathbf{R}^n)\}.$$

The following theorem establishes the relations between the above spaces.

Theorem 2.3 **(Triebel [20], pag. 318)** *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded* C^∞ *domain.*

1. *If* $1 < p < \infty$ *and* $-\infty < s \leq \frac{1}{p}$, *then*

$$H^s_p(\Omega) = \overset{\circ}{H}{}^s_p(\Omega)$$

2. *If* $-\infty < s < \infty$, $1 < p < \infty$ *and* $1 \leq q < \infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $\tilde{H}_p^s(\Omega)$. *It holds that*

$$\tilde{H}_p^s(\Omega) \subset \overset{\circ}{H}{}_q^s(\Omega).$$

3. *If* $1 < p < \infty$ *and* $\frac{1}{p} - 1 < s < \infty$, *with* $s - \frac{1}{p} \neq$ *integer, then*

$$\tilde{H}_p^s(\Omega) = \overset{\circ}{H}{}_p^s(\Omega).$$

We have the following duality relations.

Theorem 2.4 (Triebel [20], pag. 332) *Let* $\Omega \subset \mathbf{R}^n$ *be an arbitrary (bounded or unbounded) domain. Further, let* $-\infty < s < \infty$ *and* $1 < p < \infty$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\left(\tilde{H}_p^s(\Omega)\right)' = H_{p'}^{-s}(\Omega).$$

Theorem 2.5 (Triebel [20], pag. 332) *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded* C^∞ *domain.*

1. *Let* $1 < p < \infty$ *and* $\frac{1}{p} < s < \infty$, *with* $s - \frac{1}{p} \neq$ *integer, and suppose that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\left(\overset{\circ}{H}{}_p^s(\Omega)\right)' = H_{p'}^{-s}(\Omega).$$

2. *Let* $1 < p < \infty$ *and* $\frac{1}{p} - 1 < s < \frac{1}{p}$, *and suppose that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\left(H_p^s(\Omega)\right)' = H_{p'}^{-s}(\Omega).$$

3. *Let* $1 < p < \infty$ *and* $-\infty < s < \frac{1}{p}$, *and suppose that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then*

$$\left(H_p^s(\Omega)\right)' = \tilde{H}_{p'}^{-s}(\Omega).$$

We will also need the embedding results stated below.

Definition 2.4

1. *If* $t \geq 0$ *is an integer then* $C^t(\mathbf{R}^n)$ *denotes the completion of* $S(\mathbf{R}^n)$ *in the norm*
$$\|f\|_{C^t} = \sum_{|\alpha|\leq t} \sup_{x\in\mathbf{R}^n} |D^\alpha f(x)|.$$

2. *If* $0 < t = [t] + \{t\}$, *where* $[t]$ *is an integer and* $0 < \{t\} < 1$, *then*
$$C^t(\mathbf{R}^n) = \{f | f \in C^{[t]}(\mathbf{R}^n), \|f\|_{C^t} < \infty\}, \text{ where}$$
$$\|f\|_{C^t} = \|f\|_{C^{[t]}} + \sum_{|\alpha|=[t]} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{t\}}}.$$

3. *Let* $\Omega \subset \mathbf{R}^n$ *be a* C^∞*-domain. If* $t \geq 0$, *then* $\bar{C}^t(\Omega)$ *denotes the restriction of* $C^t(\mathbf{R}^n)$ *to* Ω.

Theorem 2.6 (Triebel [20], pag. 328) *Let* $\Omega \subset \mathbf{R}^n$ *be an arbitrary domain,* $1 < p < \infty$, $t \geq 0$ *and* $s > t + \frac{n}{p}$. *Then*
$$H_p^s(\Omega) \subset \bar{C}^t(\Omega).$$

We also have the following trace theorem.

Theorem 2.7 (Triebel [20], pag. 330) *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded* C^∞*-domain* $1 < p < \infty$ *and* $\frac{1}{p} < s < 1 + \frac{1}{p}$. *Then the map* γ,
$$\gamma(f) = f_{|\partial\Omega},$$
is a continuous linear map from $H_p^s(\Omega)$ *onto* $W^{s-\frac{1}{p},p}(\partial\Omega) \subset L^p(\partial\Omega)$, *such that there exists a continuous linear map* $\tilde{\gamma}$, *with* $\gamma \circ \tilde{\gamma} = I$, *the identity operator from* $W^{s-\frac{1}{p},p}(\partial\Omega)$ *onto itself.*

Now, let us define our spaces taking into account the boundary condition $\mathcal{B}$. To do this, we first define the admissible boundary conditions, the so called "Normal System" of boundary conditions.

Definition 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain. Further let*

$$\mathcal{B}_j f(x) = \sum_{|\alpha|\le m_j} b_{j,\alpha}(x) D^\alpha f, \; b_{j,\alpha}(x) \in C^\infty(\partial\Omega),$$

$j = 1, \cdots, k$, be differential operators on $\partial\Omega$. Then $\{\mathcal{B}_j\}_{j=1}^k$ is said to be a normal system if

$$0 \le m_1 < m_2 < \cdots < m_k$$

and if for any normal vector ν_x with respect to $\partial\Omega$ and the point $x \in \partial\Omega$ it holds that

$$\sum_{|\alpha|=m_j} b_{j,\alpha}(x)\nu_x^\alpha \neq 0, \; j = 1, \cdots, k.$$

Definition 2.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain. Further, let $\{\mathcal{B}_j\}_{j=1}^k$ be a normal system. For $s \ge 0$ and $1 < p < \infty$, we have*

1. *If $s - \frac{1}{p} < m_1$, then*

$$H^s_{p,\{\mathcal{B}_j\}}(\Omega) = H^s_p(\Omega)$$

2. *If for some l, $m_l < s - \frac{1}{p} < m_{l+1}$, then*

$$H^s_{p,\{\mathcal{B}_j\}}(\Omega) = \{f \mid f \in H^s_p(\Omega), \mathcal{B}_j f|_{\partial\Omega} = 0, \; for \; j \le l\}$$

Remark 2.2 *In our case, that is in (1), we have that $k = 1$, $m_1 = 1$, $\mathcal{B}_1 u = \mathcal{B}u = \dfrac{\partial u}{\partial n_a}$. The definition takes into account the boundary condition $\mathcal{B}_j f = 0$, as long as it makes sense, that is, as long as the m_j derivatives have trace.*

2.2 Interpolation Theory and Domains

Now we summarize some results in interpolation theory applied to the Lebesgue Spaces defined above, and relate them to the domain of definition of fractional powers of positive operators.

Let $X_1 \subset X_0$ be a continuous dense injection of Banach Spaces, $\mathcal{S} = \{z \in \mathbb{C} : 0 \le Re(z) \le 1\}$, and $\mathcal{S}^0$ denote the interior of $\mathcal{S}$. Let $H(X_0, X_1)$ denote all continuous bounded functions $F : \mathcal{S} \to X_0$ which are holomorphic in $\mathcal{S}^0$, and such that the following norm is finite:

$$\|F\| = \max_j \sup_y \|F(j + iy)\|_j$$

where $j = 0$ or $1; -\infty < y < \infty$; and $\|\cdot\|_j$ is the norm in X_j. For $0 \le \theta \le 1$, the interpolation space X_θ is defined by

$$X_\theta = [X_0, X_1]_\theta = \{F(\theta) : F \text{ in } H(X_0, X_1)\},$$

with norm $\|f\|_\theta = inf\{\|F\| : F(\theta) = f\}$.

The following facts can be proved (see Triebel [20]):

1. X_1 is densely and continuously injected in X_θ, and X_θ in X_0.
2. (*Iteration Property*) If $\theta_1 \le \theta \le \theta_2$, then
$$[X_0, X_1]_\theta = [X_{\theta_1}, X_{\theta_2}]_s,$$
where $\theta = \theta_1 + s(\theta_2 - \theta_1)$.
3. (*Interpolation Property*) Let $Y_1 \subset Y_0$ be a continuous dense injection of Banach spaces, and let $A : X_0 \to Y_0$ be a linear map such that $\|Ax\|_0 \le C_0\|x\|_0$ and $\|Ax\|_1 \le C_1\|x\|_1$. Then $\|Ax\|_\theta \le C_\theta\|x\|_\theta$, $C_\theta = C_0^{1-\theta}C_1^\theta$.

Remark 2.3 *We can also define X_θ if $X_0 \subset X_1$, simply by setting*

$$X_\theta = [X_0, X_1]_\theta = [X_1, X_0]_{(1-\theta)}.$$

With this we have the following *Duality Property*.

Theorem 2.8 (Triebel [20], pag. 72) *If one of the two spaces X_0 or X_1 is reflexive, then*

$$[X_0, X_1]'_\theta = [X'_0, X'_1]_\theta, \ \ 0 < \theta < 1.$$

Now we state some interpolation formulas for the Lebesgue spaces.

Theorem 2.9 (Triebel [20], pag. 185) *Let* $-\infty < s_0, s_1 < \infty, 1 < p_0, p_1 < \infty$, *and* $0 < \theta < 1$. *Then*

$$[H^{s_0}_{p_0}(\mathbb{R}^n), H^{s_1}_{p_1}(\mathbb{R}^n)]_\theta = H^s_p(\mathbb{R}^n),$$

where

$$s = (1-\theta)s_0 + \theta s_1 \text{ and } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Theorem 2.10 (Triebel [20], pag. 317) *Let* $\Omega \subset \mathbb{R}^n$ *be a* C^∞ *bounded domain. Let* $0 \leq s_0, s_1 < \infty, 1 < p_0, p_1 < \infty$, *and* $0 < \theta < 1$. *Then*

$$[H^{s_0}_{p_0}(\Omega), H^{s_1}_{p_1}(\Omega)]_\theta = H^s_p(\Omega),$$

where

$$s = (1-\theta)s_0 + \theta s_1, \text{ and } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Theorem 2.11 (Triebel [20], pag. 321) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* C^∞*-domain. Further let* $\{B_j\}_{j=1}^k$ *be a normal system of boundary conditions. Let* m *be a natural number such that* $m > m_k$, $1 < p < \infty$, *and* $0 < \theta < 1$.

1. *If there does not exist a number* m_j, $j = 1, \cdots k$, *such that* $m\theta - \frac{1}{p} = m_j$, *then*
$$[L^p(\Omega), H^m_{p,\{B_j\}}(\Omega)]_\theta = H^{\theta m}_{p,\{B_j\}}(\Omega).$$

2. *Let* $m_l = m\theta - \frac{1}{p}$. *Extending the coefficients* $b_{l,\alpha}(x)$ *and their first derivatives continuously to* Ω, *then*
$$[L^p(\Omega), H^m_{p,\{B_j\}}(\Omega)]_\theta = \{f | f \in H^{\theta m}_{p,\{B_j\}}(\Omega), B_l f \in \tilde{H}^{\frac{1}{p}}_p(\Omega)\}.$$

Now we can state a result that connects these interpolation spaces with the domain of fractional powers of positive operators.

Theorem 2.12 (Triebel [20], pag. 103) *Let* Λ *be a positive operator. It is supposed that there exist two positive numbers* ϵ *and* C *such that* Λ^{it} *is a bounded operator for* $-\epsilon \leq t \leq \epsilon$ *and* $\|\Lambda^{it}\| \leq C$. *If* α *and* β *are two complex numbers,* $0 \leq Re\alpha < Re\beta < \infty$ *and* $0 < \theta < 1$, *then*

$$[D(\Lambda^\alpha), D(\Lambda^\beta)]_\theta = D(\Lambda^{\alpha(1-\theta)+\theta\beta}).$$

2.3 Negative Fractional Power Spaces

Now we want to define the fractional power spaces related to the operator defined in (1), including negative powers. Let us mention that if we work in L^2, these negative fractional powers can be easily defined using duality and Fourier transforms (see for example, Rodriguez-Bernal [18]), but in L^p things get a little more delicate.

Let us start by defining the operator. Consider $A = \mathrm{diag}(A_1, \cdots, A_N)$ in $L^p(\Omega; \mathbb{C}^N)$, the operator defined by

$$D(A_i) = H^2_{p,\{B\}}(\Omega)$$

$$A_i u = -Div(a_i \nabla u) + \sum_{j=1}^{n} b^i_j(x) \frac{\partial u}{\partial x_j} + \lambda u'$$

where B is the boundary operator $Bu = \frac{\partial u}{\partial n_a}$. Let $A' = diag(A'_1, \cdots, A'_N)$ (the dual operator) be the operator in $L^{p'}(\Omega; \mathbb{C}^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, defined by

$$D(A'_i) = H^2_{p',\{C\}}(\Omega)$$
$$A'_i v = -Div(a_i \nabla v) - div(vB_i) + \lambda v,$$

where C is the boundary operator $Cv = \frac{\partial v}{\partial n_a} + vB \cdot \vec{n}$.

We have that (see Triebel [20], pag. 401):

- A' is an isomorphism from $H^2_{p',\{C\}}(\Omega; \mathbb{C}^N)$ onto $L^{p'}(\Omega; \mathbb{C}^N)$;
- Denote by A'' the dual operator of A'. Then A'' is an isomorphism from $L^p(\Omega; \mathbb{C}^N)$ onto $(H^2_{p',\{C\}}(\Omega; \mathbb{C}^N))'$;
- $A'' \equiv A$, in $H^2_{p,\{B\}}(\Omega; \mathbb{C}^N)$.

With this, let us define the operator A_{-1} in $(H^2_{p',\{C\}}(\Omega; \mathbb{C}^N))'$ by

$$D(A_{-1}) = (L^p(\Omega; \mathbb{C}^N))$$

$$A_{-1} u = A'' u, \text{ for all } u \in L^p(\Omega; \mathbb{C}^N).$$

Therefore, we have the following diagram

$$\begin{array}{ccc} L^p(\Omega;\mathbf{C}^N) & \xrightarrow{A''} & \left(H^2_{p',\{C\}}(\Omega;\mathbf{C}^N)\right)' \\ A \big\downarrow & & \big\downarrow A_{-1} \\ L^p(\Omega;\mathbf{C}^N) & \xrightarrow{A''} & \left(H^2_{p',\{C\}}(\Omega;\mathbf{C}^N)\right)' \end{array}$$

Since $A_{-1} \equiv A'' \circ A \circ (A'')^{-1}$ in a dense subspace of $(H^2_{p',\{C\}}(\Omega;\mathbf{C}^N))'$ (for example C_0^∞), and they are closed operators, we have that $A_{-1} \equiv A'' \circ A \circ (A'')^{-1}$. From this the following result follows easily.

Proposition 2.1 *A_{-1} is a sectorial operator, with $\rho(A) = \rho(A_{-1})$. Moreover, given $\theta \geq 0$, if we define*

$$X^\theta_{-1} = D(A^\theta_{-1})$$

then A_{-1} is also a sectorial operator in X^θ_{-1}, which we denote by $A_{\theta-1}$.

Proof:

From the diagram above, and the fact that A'' is an isomorphism, we have the following identity

$$(A_{-1} - \mu I)^{-1} = (A''(A - \mu I)(A'')^{-1})^{-1} \tag{4}$$

thus it is clear that $\rho(A) = \rho(A_{-1}) = \rho(A_{\theta-1})$.

To finish the proof, we have just to observe that, if $u \in X^\theta_{-1}$, then

$$\begin{aligned} \|(A_{\theta-1} &- \mu I)^{-1}u\|_{X^\theta_{-1}} = \|A^\theta_{-1}(A_{\theta-1} - \mu I)^{-1}u\|_{X_{-1}} \\ &= \|(A_{\theta-1} - \mu I)^{-1}A^\theta_{-1}u\|_{X_{-1}} \\ &= \|(A_{-1} - \mu I)^{-1}A^\theta_{-1}u\|_{X_{-1}} = \|A''(A - \mu I)^{-1}(A'')^{-1}A^\theta_{-1}u\|_{X_{-1}} \\ &\leq \|A''(A - \mu I)^{-1}\|_{\mathcal{L}(L^p,X_{-1})}\|(A'')^{-1}A^\theta_{-1}u\|_{L^p} \\ &\leq \|A''\|_{\mathcal{L}(L^p,X)}\|(A - \mu I)^{-1}\|_{L^p}\|(A'')^{-1}\|_{\mathcal{L}(X,L^p)}\|A^\theta_{-1}u\|_{X_{-1}} \\ &\leq C\|\|(A - \mu I)^{-1}\|_{L^p}\|\|A^\theta_{-1}u\|_{X_{-1}} < \tfrac{M}{\mu - a}\|A^\theta_{-1}u\|_{X_{-1}} \\ &= \tfrac{M}{\mu - a}\|u\|_{X^\theta_{-1}}. \end{aligned}$$

∎

We want to use Theorem 2.12 to determine the space X^θ_{-1}, to this end we need the following lemma.

Lemma 2.1 *There exist two positive numbers ϵ and C such that*

$$\|A_{-1}^{it}\| \leq C, \text{ for all } -\epsilon \leq t \leq \epsilon. \tag{5}$$

Proof: Let $u \in X_{-1}$, then

$$\|A_{-1}^{it}u\|_{X_{-1}} \leq C\|(A'')^{-1}A_{-1}^{it}u\|_{L^p} = \|A^{it}(A'')^{-1}u\|_{L^p}$$

But from Seeley [19] we have that A satisfies the property (5). Thus

$$\|A_{-1}^{it}u\|_{X_{-1}} \leq C\|(A'')^{-1}u\|_{L^p} = C\|u\|_{X_{-1}}$$

□

Applying Proposition 2.1 and Theorems 2.12, 2.8 and 2.11, we have that, for $2\theta \neq 1 + \frac{1}{p}$,

$$\begin{aligned} X_{-1}^{\theta} = D(A_{-1}^{\theta}) &= [(H^2_{p',\{C\}}(\Omega;\mathbf{C}^N))', (L^{p'}(\Omega;\mathbf{C}^N))']_\theta \\ &= [H^2_{p',\{C\}}(\Omega;\mathbf{C}^N), L^{p'}(\Omega;\mathbf{C}^N)]'_\theta = (H^{2(1-\theta)}_{p',\{C\}}(\Omega;\mathbf{C}^N))'. \end{aligned} \tag{6}$$

If we define $X^\theta = D(A^\theta)$, then we have the following result.

Theorem 2.13 *If $0 \leq \theta \leq 1$, then*

$$X_{-1}^{\theta+1} = X^\theta = H^{2\theta}_{p,\{B\}}$$

Proof:

Let us first show that $X_{-1}^2 = X^1$.

$$\begin{aligned} X_{-1}^2 &= \{u \in X_{-1} : A_{-1}^2 u \in X_{-1}\} \\ &= \{u \in X_{-1} : A_{-1} \circ A_{-1}u \in X_{-1}\} \\ &= \{u \in X_{-1} : A_{-1}u \in L^p(\Omega;\mathbf{C}^N)\} \\ &= \{u \in L^p(\Omega;\mathbf{C}^N) : A_{-1}u \in L^p(\Omega;\mathbf{C}^N)\} \\ &= \{u \in L^p(\Omega;\mathbf{C}^N) : Au \in L^p(\Omega;\mathbf{C}^N)\} = D(A) = X^1. \end{aligned}$$

Thus, we can write

$$D(A^\theta) = [D(A^0), D(A^1)]_\theta = [D(A_{-1}^1), D(A_{-1}^2)]_\theta = D(A_{-1}^{1+\theta}).$$

To finish the proof, we have to observe that

$$D(A^\theta) = [L^p(\Omega;\mathbf{C}^N), H^2_{p,\{B\}}(\Omega;\mathbf{C}^N)]_\theta = H^{2\theta}_{p,\{B\}}(\Omega;\mathbf{C}^N).$$

□

Notation 2.1 *Having this result in mind we will define, for all* $0 \le s \le 1$,

$$X^{-s} = X_{-1}^{1-s}$$

3 Hypotheses

In this Section we fix the hypotheses to be used throughout this paper.

(H1) $f = (f_1, \cdots, f_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and* $g = (g_1, \cdots, g_N) \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ *are such that, there exists* $\xi^0 = (\xi_1^0, \cdots, \xi_N^0)$ *such that, for all* $\xi = (\xi_1, \cdots, \xi_N)$, *there are constants* $c_i^0 = c_i^0(\xi^0)$ *and* $d_i^0 = d_i^0(\xi^0)$ *with*

$$\begin{array}{l} \frac{f_i(s)}{s_i} \le c_i^0 \ \forall \ \xi_i^0 < |s_i| < \xi_i \\ \frac{g_i(s)}{s_i} \le d_i^0 \ \forall \ \xi_i^0 < |s_i| < \xi_i \end{array} \tag{7}$$

Moreover, if f, g satisfy (3), and given the eigenvalue problem

$$\left. \begin{array}{l} -\mathrm{Div}(a\nabla v_i) + \sum_{j=1}^{n} B_j(x) \frac{\partial v_i}{\partial x_j} + \lambda v_i - c_0 v_i = \mu_i v_i, \qquad \text{in} \quad \Omega, \\ \\ \frac{\partial v_i}{\partial n_a} = d_0 v_i \qquad \text{on} \quad \partial\Omega \end{array} \right\} \tag{8}$$

we will assume the following,

(H2) c_i^0 *and* d_i^0 *are such that the first eigenvalue* (μ_1) *of the problem (8) is positive.*

Remark 3.1 *This condition is satisfied if one of the following conditions hold*

1.

$$\begin{array}{l} \liminf_{|s| \to \infty} \frac{f_i(s)}{s_i} \le c_i^0 \\ \liminf_{|s| \to \infty} \frac{g_i(s)}{s_i} \le d_i^0 \end{array} \tag{9}$$

where c_i^0, d_i^0 *are such that the first eigenvalue of the problem (8) is positive.*

2.

$$s_i(f_i(s)) < 0 \qquad\qquad (10)$$
$$s_i g_i(s) < 0,$$

$s_i \notin [-\xi_i^0, \xi_i^0]$, $1 \le i \le N$.

Remark 3.2 *From the results of Protter & Weinberger [16], [17] and Krein & Rutman [12], we have that the first eigenvalue of (8) is always real. Here, we mean first, in the sense that all the others have greater real part.*

Remark 3.3 *To avoid notational complications we will treat only the case $N = 1$, but it will be clear from the proofs that the results remain true in higher dimensions and the same arguments apply if we assume (H1).*

Remark 3.4 *(H1) is the dissipation condition on the equation. Note that we allow either c_0 or d_0 to be positive. In other words, we allow either f or g to be a source of heat.*

Remark 3.5 *(H2) is a precise formulation of the "competition" between f and g that we mentioned in the Introduction. Notice that we cannot have both c_0 and d_0 positive. Moreover this condition states that our problem "behaves" as an intermediate case between the Dirichlet case ($d_0 = \infty$) and the Newmann case ($d_0 = 0$).*

4 Local Existence

It follows from Proposition 2.1 and the results of Henry [10] that $A_{-\beta}$ generates an analytic semigroup in $X^{-\beta}$ for $0 < \beta < 1$ which satisfies, for suitable λ and for $-\beta < \alpha < 1 - \beta$

$$\left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} \le M e^{-\epsilon t} \|u_0\|_{X^\alpha}, \quad t \ge 0$$
$$\left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} \le M e^{-\epsilon t} t^{-(\alpha+\beta)} \|u_0\|_{X^{-\beta}}, \quad t > 0. \qquad (11)$$

for some $\epsilon > 0, M > 0$. In particular, if $B_j \equiv 0$, $j = 1, \cdots, n$, λ can be any positive number.

We want to choose α, β and p in such a way that

1. $X^\alpha \subset \bar{C}(\Omega)$;

2. $X^{1-\beta} = H_p^{2(1-\beta)}(\Omega)$, in other words $X^{1-\beta}$ does not incorporate the boundary condition;

3. $\alpha + \beta < 1$.

So we will take p, α and β satisfying

$$\frac{n}{2p} < \alpha < 1 - \beta < 1 - \frac{1}{2p'} = \frac{1}{2} + \frac{1}{2p}. \tag{12}$$

Notice that

- 1. follows from Theorem 2.13 and Theorem 2.6;
- 2. follows from Definition 2.6, Proposition 2.1 and Theorem 2.5.

Remark 4.1 *It is easy to check that (12) can be realized if p is big enough (for instance $p = n$ is enough).*

One can easily check the following result.

Corollary 4.1 *If α, β and p satisfy (12), then*

$$X^\alpha = H_p^{2\alpha}(\Omega) \text{ and } X^{-\beta} = \left(H_{p'}^{2\beta}(\Omega)\right)'.$$

Since we are going to use the linear operator a with homogeneous boundary conditions to define the abstract problem, we need to introduce the nonlinear boundary conditions in the equation.

Notation 4.1 *Consider the map $g_\gamma : X^\alpha \to X^{-\beta}$ defined by*

$$\langle g_\gamma(u), \phi\rangle := \int_{\partial\Omega} \gamma(g(u))\gamma(\phi), \text{ for all } \phi \in H_{p'}^{2\beta}(\Omega),$$

where γ denotes the trace operator.

Similarly, we define $f_\Omega : X^\alpha \to X^{-\beta}$ by

$$\langle f_\Omega(u), \phi\rangle := \int_\Omega f(u)\phi, \text{ for all } \phi \in H_{p'}^{2\beta}(\Omega).$$

We will also denote by $h := f_\Omega + g_\gamma$.

It is easy to show that f_Ω and g_γ are well defined. The following lemma is necessary to establish local existence for the abstract equation.

Lemma 4.1 *If (H1) holds then h is Lipschitz continuous in bounded sets of X^α.*

Proof:

Let $u, v \in V \subset X^\alpha$, where V is bounded. Then we have that,

$$\|g_\gamma(u) - g_\gamma(v)\|_{X^{-\beta}} = \sup_{\substack{\phi \in H^{2\beta}_{p'}(\Omega) \\ \|\phi\|_{H^{2\beta}_{p'}(\Omega)} = 1}} |\langle g_\gamma(u) - g_\gamma(v), \phi\rangle|,$$

and

$$\begin{aligned} |\langle g_\gamma(u) - g_\gamma(v), \phi\rangle| &\leq \int_{\partial\Omega} |\gamma(g(u) - g(v))\gamma(\phi)| \\ &\leq \|\gamma(g(u) - g(v)\|_{L^p(\partial\Omega)} \|\gamma(\phi)\|_{L^{p'}(\partial\Omega)} \\ &\leq K\|\gamma(u - v)\|_{L^p(\Omega)} \|\gamma(\phi)\|_{L^{p'}(\partial\Omega)} \\ &\leq K'\|u - v\|_{X^\alpha} \|\phi\|_{H^{2\beta}_{p'}(\Omega)} \end{aligned}$$

Similarly we prove that f_Ω is Lipschitz.

$\square$

With this, we can state the following

Theorem 4.1 *Suppose that (H1) holds and that α, β and p satisfy (12). Then the abstract parabolic problem*

$$\begin{cases} \dfrac{du}{dt} + A_{-\beta}u = h(u) \\ \\ u(0) = u_0 \in X^\alpha \end{cases} \tag{13}$$

has an unique solution for any $u_0 \in X^\alpha$, which is given by the variation of constants formula

$$T(t)u_0 = e^{-A_{-\beta}t}u_0 + \int_0^t e^{-A_{-\beta}(t-s)}h(T(s)u_0)ds. \tag{14}$$

Moreover, if the maximal interval of existence of the solution $T(t)u_0$ is $[0, t_{max}[$ then either $t_{max} = +\infty$ or $\|T(t)u_0\|_{X^\alpha} \to \infty$ as $t \to t_{max}$.

Proof:

The result follows from Henry's result [10], since from Lemma 4.1 $h : X^\alpha \to X^{-\beta}$ is Lipschitz continuous in bounded sets of X^α.

$\square$

Remark 4.2 *Now that we have local existence for (13) and since all functions and coefficients in the equation are real, we can take the real part of the solution, and we still have a solution.*

5 Regularity Result

In this section, we want to show that the solution given by Theorem 4.1 is in $\bar{C}^{2+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$, for any $t > 0$.

Theorem 5.1 *Suppose that (H1) holds and that α, β and p satisfy (12). Let $u_0 \in X^\alpha$ and let u be the solution of (13). Then, there exists $\varepsilon > 0$ such that $u(t,\cdot) \in C^{2+\varepsilon}(\bar{\Omega})$, for all $t > 0$. Moreover, $u(t,\cdot)$ is a classical solution of (1), for any $t > 0$.*

Proof: Applying the results of Henry [10], we know that, for $t > 0$ $u, \dfrac{du}{dt} \in X^\gamma$, for all $\dfrac{n}{2p} < \gamma < \dfrac{1}{2} + \dfrac{1}{2p}$. Therefore, using the characterization of X^γ, we have that $u, \dfrac{du}{dt} \in H^s_p(\Omega)$, for all $\dfrac{n}{p} < s < 1 + \dfrac{1}{p}$. Thus using Theorem 2.6 we obtain that $\dfrac{du}{dt} \in \bar{C}^\delta(\Omega)$ for all $\delta < \frac{1}{p}$. Furthermore $u \in H^1_p(\Omega) = W^1_p(\Omega)$ and thus, using the regularity of f and g; $f(u) - \dfrac{du}{dt} \in L^p(\Omega), g(u) \in W^1_p(\Omega)$. So, by the trace theorem (2.7), $\gamma(g(u)) \in W^{1-\frac{1}{p},p}(\partial\Omega)$.

If we fix u and consider the elliptic problem,

$$\begin{cases} -\mathrm{Div}(a\nabla v) + \displaystyle\sum_{j=1}^{n} B_j(x)\frac{\partial v}{\partial x_j} + \lambda v = f(u) - \frac{du}{dt}, & \text{in} \quad \Omega, \\ \dfrac{\partial v}{\partial n_a} = g(u), & \text{on} \quad \partial\Omega. \end{cases} \tag{15}$$

we can apply elliptic regularity results (see Lions and Magenes [14]), to conclude that $v \in W^{2,p}(\Omega) = H_p^2(\Omega)$.

Now, we want to show that $v = u$. From the Green's Formula, if follows that for all $w \in H_p^2$ and $\phi \in H^2_{p',\mathcal{C}}(\Omega)$

$$\begin{aligned}\int_\Omega (Aw)(x)\phi(x)dx \quad &- \quad \int w(x)(A'\phi)(x)dx = -\int_{\partial\Omega} \frac{\partial w}{\partial n}(y)(\gamma(\phi))(y)dy \\ &+ \quad \int_{\partial\Omega}(\gamma(w))(y)\left[\frac{\partial v}{\partial n_a}(y) + B(y)\cdot\vec{n}(y)\right]dy \\ &= \quad \int_{\partial\Omega} -(\mathcal{B}w)(y)(\gamma(\phi))(y) + (\gamma(w))(y)(\mathcal{C}\phi)(y)dy \\ &= \quad \int_{\partial\Omega} -(\mathcal{B}w)(y)(\gamma(\phi))(y)dy. \end{aligned} \tag{16}$$

Applying (16) to v and having in mind that (from (15))

$$\begin{aligned}(Av)(x) &= f(u)(x) - \frac{du}{dt}, \\ (\mathcal{B}v)(y) &= (\gamma(g(u)))(y),\end{aligned}$$

for all $x \in \Omega$ and $y \in \partial\Omega$ it follows that v satisfies

$$\begin{aligned}\int_\Omega \left[f(u)(x) - \frac{du}{dt}\right]\phi(x)dx \quad &- \quad \int v(x)(A'\phi)(x)dx \\ &= \quad \int_{\partial\Omega} -(\gamma(g(u)))(y)(\gamma(\phi))(y)dy.\end{aligned} \tag{17}$$

Therefore, since $H^2_{p',\mathcal{C}}(\Omega)$ is dense in $H^{2\beta}_{p'}(\Omega)$,
it follows that v satisfies, in $X^{-\beta}$, the equation

$$A_{-\beta}v = -\frac{du}{dt} + f_\Omega(u) + g_\gamma(u).$$

But u is the unique solution of (13), so $u = v \in H_p^2(\Omega)$.

Applying Theorem 2.6 once more we get that $u \in \bar{C}^{1+\epsilon}(\Omega)$ (and thus $u \in C^{1+\epsilon}(\partial\Omega)$). Now applying regularity and existence theorems for (15) (see Ladyženskaja and Ural'ceva [13], pag.128) we conclude that $u \in C^{2+\epsilon}(\bar{\Omega})$.

Moreover, since $u = v$ satisfies (15), u is a classical solution of (1). ◻

6 Boundedness of the Semigroup

In this section we prove that solutions of (1) with initial data in X^α, are globally defined and orbits of bounded subsets of X^α, under the flow determined by (1), are also bounded in X^α.

To accomplish this goal, we use comparison results. We start by defining the concepts of sub- and super-solutions.

Definition 6.1 *Let $u_0 \in W^{2+\epsilon,2}(\Omega)$, $T > 0$ and $\bar{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ ($\underline{u}$ respectively) a function which is continuous in $[0,T] \times \bar{\Omega}$, continuously differentiable in t and twice continuously differentiable in x for $(t,x) \in (0,T] \times \Omega$. Then $\bar{u}$ ($\underline{u}$ respectively) is a super-solution (sub-solution) of the problem*

$$\begin{cases} u_t = Div(a\nabla u) - \sum_{j=1}^{n} B_j(x)\dfrac{\partial u}{\partial x_j} - \lambda u + f(u), & in \quad \Omega, \\ \dfrac{\partial u}{\partial n_a} = g(u), & on \quad \partial\Omega \\ u(0) = u_0. \end{cases} \tag{18}$$

if it satisfies

$$\begin{aligned} &\bar{u}_t \geq Div(a\nabla \bar{u}) - \textstyle\sum_{j=1}^{n} B_j(x)\frac{\partial \bar{u}}{\partial x_j} - \lambda \bar{u} + f(\bar{u}), && in \quad (0,T] \times \Omega, \\ &\frac{\partial \bar{u}}{\partial n_a} \geq g(\bar{u}), && on \quad (0,T] \times \partial\Omega \\ &\bar{u}(0) \geq u_0. \end{aligned} \tag{19}$$

(and respectively with the $\geq$ sign replaced by the $\leq$ sign).

A basic result for our arguments is the following

Theorem 6.1 *(Pao [15])*

If f is locally Lipschitz and $\bar{u}$ and $\underline{u}$ are respectively a super- and sub-solution of the problem (18), satisfying

$$\underline{u} \leq \bar{u}, \ \ in\ \Omega \times (0,T),$$

then, there exists a solution u of (18) such that

$$\underline{u} \leq u \leq \bar{u}, \ \ in\ \Omega \times (0,T).$$

Let φ be the first positive normalized eigenfunction of (8) and $m = \min_{x \in \bar{\Omega}} \varphi(x)$. We know that $m > 0$. For each $\theta \in \mathbf{R}_+$, define

$$\Sigma_\theta = \left\{ u \in X^\alpha : |u(x)| \leq \theta\varphi(x), \text{ for all } x \in \bar{\Omega} \right\}.$$

From the dissipative hypothesis (H1) on f and g, we know that there exists $\xi \in \mathbf{R}$, such that

$$\frac{f(s)}{s} \leq c_0 \text{ and } \frac{g(s)}{s} \leq d_0,$$

for all s with $|s| \geq \xi$.

Lemma 6.1 *If $\theta m \geq \xi$ then Σ_θ is a positively invariant set for the local solution of (1).*

Proof:

Let

$$\Sigma_\theta^1 = \{ u \in X^\alpha : u(x) \leq \theta\varphi(x), \text{ for all } x \in \bar{\Omega} \}$$

$$\Sigma_\theta^2 = \{ u \in X^\alpha : u(x) \geq -\theta\varphi(x), \text{ for all } x \in \bar{\Omega} \}$$

Since $\Sigma_\theta = \Sigma_\theta^1 \cap \Sigma_\theta^2$ it is enough to show that Σ_θ^1 and Σ_θ^2 are positively invariant.

Let $u_0 \in \Sigma_\theta^1$, and suppose, for contradiction, that there exists $t_0 \in [0, t_{max}[$ and $x_0 \in \bar{\Omega}$ such that

$$T(t_0)u_0(x_0) > \theta\varphi(x_0).$$

Consider $\bar{v}(t) = e^{\mu(t-t_0)}\theta\varphi$, where μ is the eigenvalue associated with φ. We have that

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} = & \quad \mathrm{Div}(a\nabla\bar{v}) - \textstyle\sum_{j=1}^{n} B_j(x)\frac{\partial \bar{v}}{\partial x_j} - \lambda\bar{v} + c_0\bar{v} \geq \\ & \quad \mathrm{Div}(a\nabla\bar{v}) - \textstyle\sum_{j=1}^{n} B_j(x)\frac{\partial \bar{v}}{\partial x_j} - \lambda\bar{v} + f(\bar{v}) \\ \frac{\partial \bar{v}}{\partial n_a} = & \quad d_0\bar{v} \geq g(\bar{v}), \end{aligned}$$

for all $t \in]0, t_0]$.

Thus $\bar{v}$ is a super-solution for the problem (1). It follows from Theorem 6.1 that

$$T(t)u_0 \leq \bar{v}(t), \text{ in } \bar{\Omega} \text{ for all } t \in [0, t_0[.$$

In particular, $T(t_0)u_0(x_0) \leq \theta\varphi(x_0)$ and we reach a contradiction.

To prove that Σ_θ^2 is positively invariant we proceed in a similar way, using now that $\underline{v} = -\bar{v}$ is a sub-solution for the problem (1).

∎

Lemma 6.2 *If V is a bounded subset of X^α then $\bigcup_{t\geq 0} T(t)V$ is also a bounded subset of X^α.*

Proof:

Since the inclusion map $i : X^\alpha \hookrightarrow C^0(\bar{\Omega})$ is continuous, there exists $\theta \in \mathbb{R}$ such that $V \subset \Sigma_\theta$. We can, of course assume that $\theta m \geq \xi$. Lemma 6.1 implies that $T(t)u_0 \in \Sigma_\theta$, for all $t \in [0, t_{max}[$ so

$$\|T(t)u_0\|_\infty \leq \theta\|\varphi\|_\infty.$$

Applying the variation of constants formula, we obtain

$$\|T(t)u_0\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^t (t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}\|h(T(s)u_0)\|_{X^{-\beta}}ds,$$

where $M, \epsilon > 0$ are constants depending only on the semigroup $e^{A_{-\beta}t}$ and α, β and p satisfy (12).

To compute the norm $\|h(T(s)u_0)\|_{X^{-\beta}}$, let $\phi \in H^{2\beta}_{p'}(\Omega)$. We have

$$\langle h(T(s)u_0), \phi\rangle = \int_\Omega f_\Omega(T(s)u_0)\phi(x)dx + \int_{\partial\Omega} \gamma(g(T(s)u_0))\gamma(\phi(x))dx$$

$$\leq \|f_\Omega(T(s)u_0)\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)} + \|\gamma(g(T(s)u_0))\|_{L^p(\partial\Omega)}\|\gamma(\phi)\|_{L^{p'}(\partial\Omega)}$$

$$\leq \left(\|f_\Omega(T(s)u_0)\|_{L^p(\Omega)} + K\|\gamma(g(T(s)u_0))\|_{L^p(\partial\Omega)}\right)\|\phi\|_{H^{2\beta}_{p'}},$$

where K is a bound for the continuous linear map $\gamma : H^{2\beta}_{p'}(\Omega) \to L^{p'}(\partial\Omega)$. Thus,

$$\|T(t)u_0\|_{X^\alpha} \leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^{t_{max}} \Big(K\|\gamma(g(T(s)u_0))\|_{L^p(\partial\Omega)}$$

$$+\|f_\Omega(T(s)u_0)\|_{L^p(\Omega)}\Big)(t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}ds$$

$$\leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\int_0^{t_{max}} \Big[K\|\gamma(g(T(s)u_0))\|_\infty|\partial\Omega|^{\frac{1}{p}}$$

$$+\|f_\Omega(T(s)u_0)\|_\infty|\Omega|^{\frac{1}{p}}\Big](t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}ds$$

$$\leq Me^{-\epsilon t}\|u_0\|_{X^\alpha} + M\Big[\sup_{|\tau|\leq\theta\|\varphi\|_\infty} |g(\tau)||\partial\Omega|^{\frac{1}{p}}$$

$$+\sup_{|\tau|\leq\theta\|\varphi\|_\infty} |f(\tau)||\Omega|^{\frac{1}{p}}\Big]\int_0^\infty (t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}ds,$$

for all $t \in [0, t_{max}[$. Therefore, $\|T(t)u_0\|_{X^\alpha}$ is bounded by a constant depending only on V. In particular, $t_{max} = \infty$.

∎

7 Existence of Global Attractors

The first step to show the existence of global attractors will be to show a "contraction property" of the sets Σ_θ, similar to the property for rectangles, considered by Carvalho [4]. It is interesting to notice that we cannot use rectangles here since they are not invariant (unless f and g are both negative). In fact if f is positive at some point $x_0 \in \Omega$ and a constant function u_0 is chosen as an initial condition at time t_0 then $\frac{\partial u}{\partial t}(u_0, t_0, x_0) > 0$, so $T(t)u_0$ grows at the point x_0 for some time. A similar argument can be used for a point on $\partial\Omega$ if $g > 0$. We show in Fig 1 the result of a simulation (for N=1) taking $c_0 > 0$ and $d_0 < 0$.

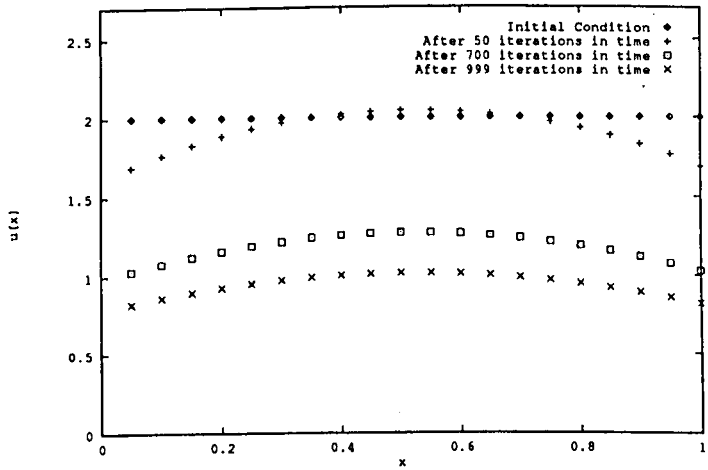

Figure 1: Simulation for $N = 1$, and with initial condition $u_0 \equiv 2$. We also used $c_0 > 0$ and $d_0 < 0$ (defined in (H1)).

Lemma 7.1 *Suppose $\bar{\theta} \in R$ satisfy $\bar{\theta} m > \xi$. Then, for any θ there exists a $\bar{t}$ such that*

$$T(t)\Sigma_\theta \subset \Sigma_{\bar{\theta}},$$

for all $t \geq \bar{t}$.

Proof:

Let $u \in \Sigma_\theta$. We can suppose without loss of generality that $\theta \geq \bar{\theta}$. Let $\bar{v} = e^{t\mu}\theta\varphi$, $\underline{v} = -\bar{v}$. As in Lemma 6.1, we can prove that $\bar{v}$ and $\underline{v}$ are superand sub-solutions respectively. Thus, using Theorem 6.1 and the uniqueness of solution, we have that

$$\underline{v} \leq T(t)u \leq \bar{v},$$

as long as $e^{\mu t}\theta \geq \bar{\theta}$.

So, $T(t)u$ enters $\Sigma_{\bar{\theta}}$ eventually. Since $\Sigma_{\bar{\theta}}$ is positively invariant, the result follows.

∎

Theorem 7.1 *The problem (1) has a global attractor $\mathcal{A}$ in X^α. Furthermore $\mathcal{A} \subset \Sigma_\theta$ if $\theta m \geq \xi$.*

Proof:

Since, by Lemma 6.2, $T(t)$ takes bounded sets of X^α into bounded sets of X^α for any $t \geq 0$, and the semigroup regularizes the solutions, only point dissipativeness remains to be proved (see Hale [9]).

Let $\bar{\theta} \in \mathbb{R}$ be such that $\bar{\theta} m \geq \xi$. If u is any element of X^α, it follows from the continuity of the imbedding $X^\alpha \hookrightarrow C^0(\bar{\Omega})$ that $u \in \Sigma_\theta$, for some θ and then, applying Lemma 7.1, we conclude that $T(t)u \in \Sigma_{\bar{\theta}}$, for t big enough. Let $v = T(t_0)u \in \Sigma_{\bar{\theta}}$.

Applying the variation of constants formula, as in Lemma 6.2, we obtain

$$\begin{aligned}\|T(t)v\|_{X^\alpha} \leq\ & Me^{-\epsilon t}\|v\|_{X^\alpha} + M\int_0^\infty \Big[K\|\gamma(g(T(s)v))\|_\infty |\partial\Omega|^{\frac{1}{p}} \\ & + \|f_\Omega(T(s)v)\|_\infty |\Omega|^{\frac{1}{p}}\Big](t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}ds,\end{aligned}$$

where M and K are independent of v.

Observing that $T(s)v \in \Sigma_{\bar{\theta}}$, for any $s \geq 0$, we conclude that, for t sufficiently large,

$$\|T(t_0+t)u\|_{X^\alpha} = \|T(t)v\|_{X^\alpha} \leq M +$$

$$M\left[\sup_{|\tau|\leq\bar{\theta}\|\varphi\|_\infty} |g(\tau)||\partial\Omega|^{\frac{1}{p}} + \sup_{|\tau|\leq\bar{\theta}\|\varphi\|_\infty} |f(\tau)||\Omega|^{\frac{1}{p}}\right]\int_0^\infty (t-s)^{-(\alpha+\beta)}e^{-\epsilon(t-s)}ds,$$

for t large.

Thus, the set in X^α bounded by the right-hand side above, attracts points. This proves point dissipativeness. Furthermore, since $\mathcal{A} \subset \Sigma_\theta$, for some θ, it follows from Lemma 7.1, taking t large, that

$$\mathcal{A} = T(t)\mathcal{A} \subset \Sigma_{\tilde{\theta}}, \qquad \text{if } \tilde{\theta} > \xi$$

so

$$\mathcal{A} \subset \Sigma_{\bar{\theta}} = \bigcap_{\tilde{\theta}m > \xi} \Sigma_{\tilde{\theta}},$$

which proves the second part of the thesis.

□

References

[1] **Alikakos, N.D.**, *Regularity and Asymptotic Behavior for the Second Order Parabolic Equation with Nonlinear Boundary Conditions*, Journal of Diff. Eq. bf 39 (1981), 311-344.

[2] **Amann, H.**, *Semigroups and Nonlinear Evolution Equations*, Linear Algebra and its Appl. 84 (1986), 3-32.

[3] **Amann, H.**, *Parabolic Evolution Equations and Nonlinear Boundary Conditions*, Journal of Diff Eq. 72 (1988), 201-269.

[4] **Carvalho, A.N.**, *Contracting Sets and Dissipation*, ICMSC-USP Preprint Series # 93-11, (1993).

[5] **Carvalho, A.N., Oliva, S.M., Pereira, A.L. and Rodriguez-Bernal, A.**, *Parabolic Problems with Nonlinear Boundary Conditions*, to appear.

[6] **Carvalho, A.N. and Rodriguez-Bernal, A.**, *Global attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces*, ICSMS-USP Preprint Series (to appear) (1994).

[7] **Carvalho, A.N. and Ruas-Filho, J. G.**, *Global Attractors for Parabolic Problems in Fractional Power Spaces*, to appear in SIAM J. Math. Analysis, ICMSC-USP Preprint Series # 93-128 (1993).

[8] **Friedman, A.**, Partial Differential Equations, Krieger, 1964.

[9] **Hale, J.K.**, Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs, 25, AMS, 1988.

[10] **Henry, D.**, Geometric Theory of Semilinear Parabolic Equations, Lectures Notes in Mathematics, **840**, Springer-Verlag, 1981.

[11] **Henry, D.**,*Some Infinite Dimensional Morse-Smale Systems Defined by Parabolic Differential Equations*, journal of Differential Equations **59** (1985), 165-205.

[12] **Krein and Rutman**, "Linear Operators Leaving Invariant a Cone in a Banach Space", Amer. Math. Soc. Trans. (1) 10 [1962], 3-95.

[13] **Ladyženskaja, O.A. and Ural'ceva, N.N.**, Équations aux dérivées partielles de type elliptique, Monographies Universitaires de Mathématiques, Dunod Paris, 1968.

[14] **Lions J.L. and Magenes, E.**, *Problemi ai Limiti Non Omogenei V*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 1-44.

[15] **Pao, C.V.**, *Asymptotic Behavior and Nonexistence of Global Solutions for a Class of Nonlinear Boundary Value Problems of Parabolic Type*, Journal of Math. Anal. and Appl. **65** (1978), 616-637.

[16] **Protter and Weinberger**, Maximum Principles in Differential Equations, Prentice Hall, inc. [1967].

[17] **Protter and Weinberger**, "On the Spectrum of General Second Order Operators", Bulletin Amer. Math. Soc., **72**, 251-255, [1966].

[18] **Rodriguez-Bernal, A.**, *Some Remarks on PDEs with Nonlinear Newman Boundary Conditions*, CDSNS Preprint Series #93-145 (1993).

[19] **Seeley, R.**, *Norms and Domains of Complex Powers* A_B^z, Amer. J. Math. **93** (1971), 299-309.

[20] **Triebel, H.**, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdan, 1978.

Electronic Mail: smo@ime.usp.br, alpereir@ime.usp.br

TRABALHOS DO DEPARTAMENTO DE MATEMÁTICA

TÍTULOS PUBLICADOS

94-01 BIANCONI, R. A note on the construction of a certain class of Kleinian groups. 9p.

94-02 HENTZEL, I.R., JACOBS, D.P., PERESI, L.A. & SVERCHKOV, S.R. Solvability of the ideal of all weight zero elements in Bernstein Algebras. 11p.

94-03 CORREA, I. & PERESI, L.A. Bernstein-Jordan Algebras of dimension five, 6p.

94-04 ABADIE, B. The range of traces on the K_0-group of quantum Heisenberg manifolds. 13p.

94-05 BIANCONI, R., GUSEVSKII, N. and KLIMENKO, H. Schottky type groups and Kleinian groups acting on S^3 with the limit set a wild Cantor set. 30p.

94-06 GORODSKI, C. Minimal Hyperspheres in Rank Two Compact Symmetric Spaces. 20 p.

94-07 BARROS, L.G.X. de and WATSON, B. On Modular Loop Algebras of RA2 Loops. 19p.

94-08 BARROS, L.G.X. de Loop Rings of I.P. Loops. 12p.

94-09 BIANCONI, R. A note on the non-interpretability of a discrete order in an o-minimal (dense) structure. 9p.

94-10 GOODAIRE, E.G. and POLCINO MILIES, C. On the loop of units of an alternative loop ring, 10p.

94-11 GOODAIRE, E.G. and POLCINO MILIES, C. Finite subloops of units in an alternative loop ring. 8p.

94-12 JURIAANS, S.O. Trace Properties of Torsion Units in Group Rings. 7p.

94-13 FALBEL, E. and GORODSKI, C. On Contact Sub-Riemannian Symmetric Spaces. 22p.

94-14 GORODSKI, C. Mínimal Sphere Bundles in Euclidean Spheres. 32p.

94-15 COELHO, F.U., MARCOS, E.N., MERKLEN, H.A. and SKOWRONSKI, A. Module categories with infinite radical cube zero. 28p.

94-16 CHAVES, R.M.B. On two parameter envelopes of N-Spheres in a real space form. 17p.

94-17 BARROS, L.G.X and JURIAANS, S.O. Units in Integral Loop Rings. 13p.

94-18 BRASIL, JR. A. Hypersurfaces in $S^4(1)$ with constant mean curvature and constant Gauss-Kronecker curvature. 7p.

94-19 COSTA R., NEUBURG, M. and SUAZO, A. Some Results in the Theory of Bernstein Algebras. 19p.

94-20 FALBEL E., GORODSKI C. Sub-Riemannian Homogeneous Spaces in Dimensions 3 and 4. 32p.

94-21 BARROS, L. G. X. de and POLCINO MILIES, C. Loops Algebras of Code Loops. 13p.

94-22 CARRARA, V.L., RUAS, M.A.S. and SAEKI, O. A note on codimension two submanifolds with at most four critical points. 11p.

94-23 COELHO F.U., MARCOS E.N., MERKLEN H.A., and SKOWRONSKI, A. Domestic Semiregular Branch Enlargements of Tame Cancealed Algebras. 14p.

94-24 DOKUCHAEV M. A., and JURIAANS, O. Finite Subgroups in Integral Group Rings. 20p

94-25 FALBEL E., GORODSKI C. and VELOSO J.M. Conformal Sub-Riemannian Geometry in Dimension 3. 12p.

95-01 BARROS, L.G.X. de and Juriaans, S.O. Loops whose Loop Algebras are Flexible. 25p.

95-02 GUIDORIZZI, H.L. - Jordan canonical form: an elementary proof, 12 p.

95-03 CATALAN A., and COSTA R. E-Ideals in Baric Algebras. 14p.

95-04 MARTIN, P.A. On the generating function of the decimal expansion of an irracional real numbers. 7p.

95-05 COELHO F.U., MARCOS E.N., MERKLEN H.A. and PLATZECK.M.I. Modules of Infinite Projective Dimension over Algebras whose Idempotent Ideals are Projective. 13p.

95-06 GUIDORIZZI, H. L. The family of functions $S_{\alpha,k}$ and the Liénard Equation. 22p.

95-07 GUIDORIZZI, H. L. On the Existence of Periodic Solution for the Equation $\ddot{x}+\alpha x^{2n+1}\dot{x}+x^{4n+3}=0$. 5p.

95-08 CORTIZO, S.F. Extensões Virtuais. 27p.

95-09 CORTIZO, S.F. Cálculo Virtual. 31p.

95-10 GUIDORIZZI, H. L. On Periodic Solutions of Systems of the Type $\ddot{x}=H(y)$, $\dot{y}=-\sum_{i=1}^{n} f_i(x)H_i(y)-g(x)$. 16p.

95-11 OLIVA, S. M., PEREIRA, A. L. Attractors for Parabolic Problems with Non linear Boundary Conditions in Fractional Power Spaces. 28p.

NOTA: Os títulos publicados dos Relatórios Técnicos dos anos de 1980 a 1993 estão à disposição no Departamento de Matemática do IME-USP. Cidade Universitária "Armando de Salles Oliveira" Rua do Matão, 1010 - Butantã
Caixa Postal - 66281 (Ag. Cidade de São Paulo)
CEP: 05389-970 - São Paulo - Brasil