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A NOTE ON THE RELIABILITY PROPERTIES OF USED COMPONENTS FROM A COHERENT SYSTEM

by

Vanderlei da Costa Bueno

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A note on the reliability properties of used components from a coherent system

Vanderlei da Costa Bueno,
Institute of Mathematics and Statistics,
São Paulo University,
Cx Postal 66281 - 05311-970, São Paulo, Brazil.

Summary In this paper we analyse the reliability properties of the residual lifetimes of live components in a failed coherent system. We study coherent system reliability assembled from these used components under dependence conditions.

Keywords: Stochastic order, hazard rate order, cumulative hazard rate order, association, conditional increasing failure rate distribution, .

1. Introduction In recent researchers authors analyse stochastic properties of residual lifetimes of live components of a coherent system conditioned to either a system survival time t or even failed at time t. See Nama, M.K. et al. (2013a), Nama, M.K. and Asadi (2013b), Eryilmaz, S.(2012), (2013). In general, theses studies consider an independent, identically, and continuous distribution function assumption of the component lifetimes which allows the signature theory approach from Samaniego (1985), and its extension for absolutely continuous exchangeable distribution function as in Navarro et al.(2005).

In this paper we intend to analyse the residual lifetimes of live components of a failed coherent system under a continuous but dependent approach where simultaneous failures are ruled out. We are going to use a point process martingale approach to reliability theory.

2. Preliminaries.

In order to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, always hold with probability one.

Consider a coherent system of n components with lifetime T_i, $1 \leq i \leq n$, being positive random variables defined on a complete probability space $(\Omega, \Im, P)$ with $P(T_i = T_j) = 0, \quad 1 \leq i, j \leq n$, that is, the components can be dependent but simultaneous failures are null-sets. As in Barlow and Proschan (1981), a coherent system lifetime T has a series-parallel decomposition

$$T = \Phi(\mathbf{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i,$$

where $\mathbf{T} = (T_1, .., T_n)$ and where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system failure.

The system is monitored through a family of sub σ-algebras of $\Im$, denoted $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma(1_{\{T_i>s\}}, 1 \leq i \leq n,\ s \leq t)$$

satisfies the Dellacherie conditions of right continuity and completeness. Let $\Im_0 = \{\Omega, \emptyset\}$.

The counting process $(N_i(t))_{t\geq 0}$, $N_i(t) = 1_{\{T_i \leq t\}}$, is an $\Im_t$-submartingale and from the Doob-Meyer decomposition, there exists a unique right continuous, $\Im_t$-predictable, nondecreasing and integrable process $(A_i(t))_{t\geq 0}$, with $A_i(0) = 0$ and such that $(N_i(t) - A_i(t))_{t\geq 0}$ is a zero mean $\Im_t$-martingale called the $\Im_t$-compensator of $N_i(t)$. We denote $A_i(t) = A_i(t|\Im_{t-})$ to understand that it is a conditioned process to the strictly past.

In the sequel we consider absolutely continuous compensator, in which case $A_i(t) = \int_0^t 1_{\{T_i>s\}}\lambda_i(s)ds$. where $\lambda_i(s) = \lambda_i(s|\Im_{s-})$ is called the intensity process of $N_i(t)$.

We analyse the remain lifetimes of live components after system failure, that is, the lifetimes $S_i = E[(T_i - T)^+|\Im_T]$ where

$$\Im_T = \{A \in \Im_\infty : A \bigcap \{T \leq s\} \in \Im_s, s \geq 0\}$$

The counting process of this remain lifetime is

$$M_i(t) = 1_{\{S_i \leq t\}} = E[1_{\{T<T_i\leq T+t\}}|\Im_T] = E[N_i(T+t) - N_i(T)|\Im_T]$$

and the corresponding $\Im_{T+t}$-compensator of $M_i(t)$ is $B_i(t) = E[A_i(T+t) - A_i(T)|\Im_T]$ with respect to the observations after T, $(\Im_{T+t})_{t\geq 0}$, where

$$\Im_{T+t} = \{A \in \Im_\infty : A \bigcap \{T+t \leq s\} \in \Im_s, s \geq 0\}$$

For a mathematical basis on stochastic processes applied to reliability theory see the book by Aven and Jensen (1999).

3. Preservation properties.

3.1. Stochastic order.

In order to study stochastic comparison of random lifetimes we introduce the following concept:

Definition 3.1.1 A Borel set $U \subset \Re^n$ is called an upper set if for any $\mathbf{x}, \mathbf{y} \in \Re^n$, $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \in U$ implies that $\mathbf{y} \in U$. In the univariate case U is equal to either (u, ∞) or $[u, \infty)$, $u \in \Re$.

If $\mathbf{T}$ and $\mathbf{T}^*$ are two $\Re^n$-valued random vectors, defined respectively on $(\Omega, \Im, P)$ and $(\Omega^*, \Im^*, P^*)$, then we say that $\mathbf{T}$ is stochastically smaller than $\mathbf{T}^*$, denoted by $\mathbf{T} \leq^{st} \mathbf{T}^*$, if for all upper set $U \in \Re^n$,

$$P\{\mathbf{T} \in U\} \leq P^*\{\mathbf{T}^* \in U\}.$$

Equivalently, it can be prove that, $\mathbf{T} \leq^{st} \mathbf{T}^*$, if for all bounded and increasing Borel measurable functions $f : \Re^n \to \Re$ we have

$$E[f(\mathbf{T})] \leq E[f(\mathbf{T}^*)].$$

The following result proves that the multivariate stochastic inequality between component's lifetime vectors is preserved to the corresponding lifetimes vectors of live components.

Theorem 3.1.2 If $\mathbf{T}$ and $\mathbf{T}^*$ are lifetimes defined in $(\Omega, \Im, P)$ such that $\mathbf{T} \leq^{st} \mathbf{T}^*$, then $\mathbf{S} \leq^{st} \mathbf{S}^*$ where $S_i = E[(T_i - T)^+ | \Im_T]$ and $S_i^* = E[(T_i^* - T)^+ | \Im_T]$ are the the live component lifetimes of $\mathbf{S}$ and $\mathbf{S}^*$.

Proof

First we note that, if U is an upper set in $\Re^n$, for any realization of T we have that $U+T = \{\mathbf{u}+T : \mathbf{u} \in U\}$ is also an upper set in $\Re^n$, because if $\mathbf{x} \in U+T$ and $\mathbf{z} \geq \mathbf{x} = \mathbf{u}+T$, for some $\mathbf{u} \in U$, we have that $\mathbf{z} - T \geq \mathbf{u}$, that is, $\mathbf{z} - T = \mathbf{u}^* \in U$, by hypothesis, and $\mathbf{z} = \mathbf{u}^* + T \in U + T$.

Therefore, by hypothesis

$$P(\mathbf{T}^* - T \in U) = P(\mathbf{T}^* \in U + T) \geq P(\mathbf{T} \in U + T) = P(\mathbf{T}^* - T \in U)$$

and, for all $\Lambda \in \Im_T$ we have

$$\int_\Lambda P(\mathbf{T}^* - T \in U | \Im_T) dP = \int_\Lambda P(\mathbf{T}^* \in U + T) dP \geq \int_\Lambda P(\mathbf{T} \in U + T) dP =$$

$$\int_\Lambda P(\mathbf{T} - T \in U | \Im_T) dP$$

implying

$$\int_\Lambda [P(\mathbf{T}^* - T \in U | \Im_T) - P(\mathbf{T} - T \in U | \Im_T)] dP \geq 0$$

for all $\Lambda \in \Im_T$. In particular if

$$\Lambda = \{P(\mathbf{T}^* - T \in U | \Im_T) - P(\mathbf{T} - T \in U | \Im_T)] < 0\} \in \Im_T$$

we conclude that

$$P(\mathbf{S}^* \in \mathrm{U}) = \mathrm{P}(\mathbf{T}^* - \mathrm{T} \in \mathrm{U} | \Im_\mathrm{T}) \geq \mathrm{P}(\mathbf{T} - \mathrm{T} \in \mathrm{U} | \Im_\mathrm{T}) = \mathrm{P}(\mathbf{S} \in \mathrm{U}),$$

that is, $\mathbf{S} \leq^{st} \mathbf{S}^*$.

Let

$$\Phi(\mathbf{S}) = \min_{1 \leq j \leq k} \max_{i \in K_j} S_i$$

be a coherent system assembled with the used components S. As ϕ is increasing in each coordinate, from Theorem 3.1.2, we have

Corollary 3.1.3 If $\mathbf{T}$ and $\mathbf{T}^*$ are lifetimes defined in $(\Omega, \Im, P)$ such that $\mathbf{T} \leq^{st} \mathbf{T}^*$, then the reliability of the system S at time t satisfies

$$P(\phi(\mathbf{S}) > t) \leq P(\phi(\mathbf{S}^*) > t), \forall t \in \Re.$$

The reliability of the system S satisfies

$$E[\phi(\mathbf{S}] \leq E[\phi(\mathbf{S}^*].$$

3.2. Hazard rate order.

An other important stochastic inequality in reliability theory is the multivariate hazard rate order.

Definition 3.2.1 If $\mathbf{T} = (T_1, .., T_n)$ is a non negative random vector with an absolutely continuous distribution function and if T_i alive t its multivariate conditional hazard rate is defined as

$$\lambda_i(t) = \lambda_i(t|\Im_{t-}) = \lim_{dt \downarrow 0} \frac{1}{dt} P(t < T_i \leq t + dt|\Im_{t-}).$$

The multivariate conditional hazard rate is denoted by $\lambda(t) = (\lambda_1(t), ..., \lambda_n(t))$.
It is important to note that the sets in $\Im_{-t}$ is of the form

$$\{\mathbf{T_I} = \mathbf{t_I}, \mathbf{T_I^c} > te\}, \quad oe \leq \mathbf{t_I} \leq te, \quad I \subseteq \{1, ..., n\}$$

where e is the unitary vector of appropriated dimension.

Clearly, I is the set of failed components at time t and I^c is the set of the components surviving t. Follows that $\lambda_i(t)$ is defined by parts, each one is deterministic in each interval of the form $[T_{(j)}, T_{(j+1)})$ and random at $T_{(j)}$, where $\{T_{(1)}, ..., T_{(n)}\}$ are the ordered $\{T_1, ..., T_n\}$.

If $\mathbf{T}$ and $\mathbf{T}^*$ are two $\Re^n$-valued random vectors, defined on $(\Omega, \Im, P)$, with multivariate conditional hazard rate $\lambda(t)$ and $\eta(t)$, respectively and

$$\eta_i(t) = \eta_i(t|\mathbf{t_I^*}) \leq \lambda_i(t|\mathbf{t_J}) = \lambda_i(t), \quad 1 \leq i \leq n$$

whenever, $I \subseteq J$, $\mathbf{t_I} \leq \mathbf{t_I^*} \leq te$ and $\mathbf{t_{J-I}} \leq te$. we say that $\mathbf{T}$ is smaller than $\mathbf{T}^*$ in multivariate conditional hazard rate ordering, denoted by $\mathbf{T} \leq^{hr} \mathbf{T}^*$.

These comparisons follows from its detailed form as in Shaked and Shanthikumar (1994).

We are interested in the following Theorem

Theorem 3.2.2 If $\mathbf{T}$ and $\mathbf{T}^*$ are lifetimes defined in $(\Omega, \Im, P)$ with multivariate conditional hazard rate $\lambda_i(t)$ and $\eta_i(t)$, respectively such that $\mathbf{T} \leq^{hr} \mathbf{T}^*$, then $\mathbf{S} \leq^{hr} \mathbf{S}^*$ where

$S_i = E[(T_i - T)^+|\Im_T]$ and $S_i^* = E[(T_i^* - T)^+|\Im_T]$ are the the live component lifetimes of $\mathbf{S}$ and $\mathbf{S}^*$.

Proof First we prove that the multivariate conditional hazard rate of S_i is

$$E[1_{\{S_i>s\}}\lambda_i(s+T)|\Im_T].$$

This result follows from the assumption relation between the hazard rate and the particular form of the continuous compensator process $A_i(t) = \int_0^t 1_{\{T_i>s\}}\lambda_i(s)ds$.

The compensator function of $M_i(t)$ is

$$B_i(t) = E[A_i(T+t) - A_i(T)|\Im_T] = E[\int_T^{T+t} 1_{\{T_i>s\}}\lambda_i(s)ds|\Im_T] =$$

$$E[\int_0^t 1_{\{T_i>u+T\}}\lambda_i(u+T)du|\Im_T] = \int_0^t E[1_{\{S_i>u\}}\lambda_i(u+T)|\Im_T]du.$$

Therefore, from the conditional expectation definition, for all $\Lambda \in \Im_T$, we have

$$\int_\Lambda E[1_{\{S_i>s\}}\lambda_i(s+T)|\Im_T]dP = \int_\Lambda 1_{\{S_i>s\}}\lambda_i(s+T)dP \geq$$

$$\int_\Lambda 1_{\{S_i>u\}}\eta_i(s+T)dP = \int_\Lambda E[1_{\{S_i>s\}}\eta_i(s+T)|\Im_T]dP$$

and

$$\int_\Lambda [E[1_{\{S_i>s\}}\lambda_i(s+T)|\Im_T] - E[1_{\{S_i>s\}}\eta_i(s+T)|\Im_T]]dP \geq 0.$$

In particular, we can take

$$\Lambda = \{E[1_{\{S_i>s\}}\lambda_i(s+T)|\Im_T] - E[1_{\{S_i>s\}}\eta_i(s+T)|\Im_T]] < 0\}$$

and conclude that

$$\{E[1_{\{S_i>s\}}\lambda_i(s+T)|\Im_T] \geq E[1_{\{S_i>s\}}\eta_i(s+T)|\Im_T]],$$

that is, $\mathbf{S} \leq^{hr} \mathbf{S}^*$.

3.2. Cumulative hazard rate order.

As long as the component is alive it accumulates hazard at the rate $\lambda_i(t)|\Im_{-t})$ at time t. The cumulative hazard of component $i \in I^c$, at time t is

$$A_i(t) = A_i(t|\Im_{t-}) = \int_0^{T_{(1)}} \lambda_i(s)ds + \sum_{j=2}^{k} \int_{T_{(j)}}^{T_{(j+1)}} \lambda_i(s|\Im_{s-})ds + \int_{T_{(j+1)}}^{t} \lambda_i(s|\Im_{s-})ds.$$

Let $\mathbf{T}^* = (T_1^*, ..., T_n^*)$ be another lifetime vector with absolutely continuous distribution function and component cumulative hazard $A_i^*(t) = A_i^*(t|\Im_{t^*-})$, as above and where

$$\{\mathbf{T}_\mathbf{J}^* = \mathbf{t}_\mathbf{J}^*, \mathbf{T}_{\mathbf{J}^c}^* > t\mathbf{e}\}, \quad o\mathbf{e} \leq \mathbf{t}_{\mathbf{J}^*} \leq t\mathbf{e}, \quad J \subseteq \{1, ..., n\}.$$

Select two integers j and l in $\{1, ..., n\}$, $j \leq l$. Let

$$t_{(1)}, t_{(2)}, ..., t_{(j)}, ..., t_{(l)}$$

and

$$t_{(1)}^*, t_{(2)}^*, ..., t_{(j)}^*$$

be ordered failures times such that $0 \leq t_{(i)} \leq t_{(i)}^*, \quad 1 \leq i \leq j$ and $t_{(i)} > 0, \quad j+1 \leq i \leq l$.
If for any $i > l, \quad 1 \leq i \leq n$

$$A_i^*(t|t_{(1)}^*, t_{(2)}^*, ..., t_{(j)}^*) \leq A_i(t|t_{(1)}, t_{(2)}, ..., t_{(l)}),$$

whenever $t \geq \max\{t_j^*, t_{(1)}, t_{(2)}, ..., t_{(l)}\}$, and if $1, 2, ..., j$ can be replaced by $\pi_1, \pi_2, ..., \pi_j$ for every permutation π of $\{1, ..., n\}$, then $\mathbf{T}^*$ is said to be greater than $\mathbf{T}$ in the cumulative hazard order, denoted by $\mathbf{T} \leq^{ch} \mathbf{T}^*$.

We observe that if $\mathbf{T}$ and $\mathbf{T}^*$ are two $\Re^n$-valued random vectors, with multivariate conditional hazard rate $\lambda(t)$ and $\eta(t)$, respectively with

$$\eta_i(t) = \eta_i(t|\mathbf{t}_\mathbf{I}^*) \leq \lambda_i(t|\mathbf{t}_\mathbf{J}) = \lambda_i(t), \quad 1 \leq i \leq n$$

whenever, $I \subseteq J$, $\mathbf{t}_\mathbf{I} \leq \mathbf{t}_\mathbf{I}^* \leq t\mathbf{e}$ and $\mathbf{t}_{\mathbf{J}-\mathbf{I}} \leq t\mathbf{e}$, that is $\mathbf{T} \leq^{hr} \mathbf{T}^*$ we have

$$A_i(t) = \int_0^t \lambda_i(s)ds \geq \int_0^t \eta_i(s)ds = A_i^*(t).$$

We resume that $A_i(t|\Im_{t-})$ is the $\Im_t$ compensator of $N_i(t) = 1_{\{T_i \leq t\}}$, in the sense that $N_i(t) - A_i(t)$ is a zero mean martingale. Follows that, if $A_i(t) \geq A_i^*(t)$ we have

$$P(T_i \leq t) = E[N_i(t)] = E[A_i(t)] \geq E[A_i^*(t)] = E[N_i^*(t)] = P(T_i^* \leq t),$$

, that is $T_i \leq^{st} T_i^*, \quad 1 \leq i \leq n$ and $\mathbf{T} \leq^{ch} \mathbf{T}^*$.

It is remarkable, see Norros (1986) that the continuous compensator processes at its final points, $A_i(T_i), \quad 1 \leq i \leq n$ are independent and identically distributed standard exponential random variables. This holds no matter how dependent the actual lifetimes are and what the history, as long as simultaneous failures are ruled out. Furthermore, there exists a bijective relationship between the lifetime vector $\mathbf{T}$ and the random vector of the compensator processes at its final points, $(A_1(T_1), ..., A_n(T_n))$.
Follows that , if $A_i(t) \leq A_i^*(t), \quad 1 \leq i \leq n$ we have

$$E[f(\mathrm{T})] = E[f(A_1(T_1), ..., A_n(T_n))] \leq E[f(A_1^*(T_1^*), ..., A_n^*(T_n^*))] = E[f(\mathrm{T}^*)],$$

for all all bounded and increasing Borel measurable functions $f : \Re^n \to \Re$, that is $\mathrm{T} \leq^{st} \mathrm{T}^*$.

We can apply the above results and Theorem 3.2.2 to get, again, Corollary 3.1.3.

3.3. Association.

We introduce the association definition.

Definition 3.3.1 The random variables $T_1, ..., T_n$ (or the corresponding random vector T) are associated, if for all upper sets U_1 and U_2 of $\Re^n$

$$P\{\mathrm{T} \in U_1 \cap U_2\} \geq P\{\mathrm{T} \in U_1\} P\{\mathrm{T} \in U_2\}.$$

It can be prove that

$$E[< M^1, M^2 >_\infty] = P\{\mathrm{T} \in U_1 \cap U_2\} - P\{\mathrm{T} \in U_1\} P\{\mathrm{T} \in U_2\}$$

where $< M^1, M^2 >_t$ is the covariance processe of the uniformly integrable $\Im_t$-martingales, $M_t^1 = E[1_{U_1}(\mathrm{T})|\Im_t]$ and $M_t^2 := E[1_{U_2}(\mathrm{T})|\Im_t]$. The covariance process $< M^1, M^2 >_t$ is an unique $\Im_t$-predictable process, increasing, right continuous, with $< M^1, M^2 >_0 = 0$ such that $M_t = M_t^1 M_t^2 - < M^1, M^2 >_t$ is an $\Im_t$-martingale. Therefore we can define

Definition 3.3.2 The random variables $T_1, ..., T_n$ (or the corresponding random vector T) are associated if, and only if

$$E[< M^1, M^2 >_\infty] \geq 0.$$

However, since M_t is uniformly integrable, we can apply the Optimal Sampling Theorem to conclude that

$$M_{T+t} = M_{T+t}^1 M_{T+t}^2 - < M^1, M^2 >_{T+t}$$

is an $\Im_{T+t}$-martingale.

Now, note that, if U is an upper set in $\Re^n$, for any realization of T , $U + T = \{\mathrm{u} + T : \mathrm{u} \in \Re^n\}$ is also an upper set in $\Re^n$, because if $\mathrm{x} \in U + T$ and $\mathrm{z} \geq \mathrm{x} = \mathrm{u} + T$, for some $\mathrm{u} \in U$, we have that $\mathrm{z} - T \geq \mathrm{u}$, that is, $\mathrm{z} - T = \mathrm{u}^* \in U$ and $\mathrm{z} = \mathrm{u}^* + T \in U + T$.

It follows that

$$E\{E[1_{U_1}(\mathrm{S})|\Im_{T+t}]\} = E\{E[1_{U_1+T}(\mathrm{T})|\Im_{T+t}]\}$$

and

$$E[< M^1, M^2 >_{T+\infty}] = E[< M^1, M^2 >_\infty] \geq 0.$$

Therefore, we can enunciate the preservation property:

Theorem 3.3.3 If **T** is associated, then also S.

Let **T** be the associated component lifetimes vector of a coherent system with lifetime T and let

$$\Phi(\mathrm{S}) = \min_{1\leq j\leq k} \max_{i\in K_j} S_i$$

be a coherent system assembled with the used components S of a coherent system with lifetime T.

As the performance of a coherent system is always better than the performance of a series system, Barlow and Proschan (1981) derived the natural series system lower bound: If **T** is associated, then

$$P(T > t) \geq P(\min_{1\leq i\leq n} T_i > t) \geq \pi_{i=1}^{n} P(T_i > t), \forall t \in \Re.$$

However, if **T** is associated, then, by Theorem 3.3.3, also S and we have

$$P(S > t) \geq P(\min_{1\leq i\leq n} S_i > t) \geq \pi_{i=1}^{n} P(S_i > t), \forall t \in \Re.$$

3.4. Conditional increasing failure rate distribution.

In order to generalize classes of non-parametric distributions based on conditional stochastic order concepts, of Arjas (1981), we introduce some related terminology.

Let θ_t be a shift in time and define for $1 \leq i \leq n$

$$\theta_t T_i = (T_i - t)^+ = \max\{T_i - t, 0\}.$$

We may think of $\theta_t T_i$ as the residual lifetime of T_i at time t. Let $\theta_t \mathrm{T} = (\theta_t T_1, .., \theta_t T_n)$.

If $U \subset \Re^n$ is an open upper set, we consider the process $M_t = P\{\theta_t \mathrm{T} \in U | \Im_t\}$.
Note that

$$E[M_t|\Im_s] = E[P\{\theta_t \mathrm{T} \in U|\Im_t\}|\Im_s] = P\{\theta_t \mathrm{T} \in U|\Im_s\} \geq P\{\theta_s \mathrm{T} \in U|\Im_t\} = M_s$$

is an uniformly $\Im_t$-super martingale.

Using a stochastic process approach to multivariate reliability systems, Arjas (1981) introduced the definition

Definition 3.4.1 We say that **T** is multivariate increasing failure rate relative to $(\Im_t)_{t\geq 0}$, denoted by MIFR$|\Im_t$, if for all $0 \leq s \leq t$ and all open upper sets $U \in \Re^n$

$$P\{\theta_t \mathrm{T} \in U|\Im_t\} \leq P\{\theta_s \mathrm{T} \in U|\Im_s\}.$$

As M_t is uniformly integrable we can use the Random Sample Theorem to define

Definition 3.4.2 Let T be an $\Im_t$-stopping time. We say that $\mathbf{T}$ is multivariate IFR relative to $(\Im_{T+t})_{t\geq 0}$, denoted by MIFR$|\Im_{T+t}$, if for all $0 \leq s \leq t$ and all open upper sets $U \in \Re^n$,

$$P\{\theta_t(\mathrm{T}-T)^+ \in U|\Im_{T+t}\} \leq P\{\theta_s(\mathrm{T}-T)^+ \in U|\Im_{T+s}\}.$$

This extend the definition of Arjas (1981):

Note that if $P(T=0)=1$, then this definition coincides with the Arjas definition . In the case where the σ-algebra is generated by the lifetime S, that is, $\Im_t = \sigma\{1\{S>s\}, s \leq t\}$, the former definition coincides with the classical definition.

Remark It turns out that this new definition retains several desirable properties of any extension of the conventional MIFR class:

a) Let T be an $\Im_t$-stopping time and suppose that T is MIFR$|\Im_{T+t}$. Then any subvector $\mathrm{T}_0 = (T_i)_{i\in I_0}, I_0 \subset \{1,...,n\}$, is also MIFR$|\Im_{T+t}$. In particular, each T_i is IFR$|\Im_{T+t}$.

b) Let T be an $\Im_t$-stopping time and suppose that T_1 and T_2 are each MIFR$|\Im_{T+t}$ and that $\theta_t(\mathrm{T}_1-T)^+$ and $\theta_t(\mathrm{T}_2-T)^+$ are independent given $\Im_{T+t}$, then $(\mathrm{T}_1,\mathrm{T}_2)$ is MIFR$|\Im_{T+t}$.

Also, from the Random Sample Theorem, the preservation property of the MIFR$|\Im_t$ class is straightforward:

Theorem 3.4.3 Let T be an $\Im_t$-stopping time. If $\mathbf{T}$ is multivariate IFR relative to $(\Im_t)_{t\geq 0}$, then S is multivariate IFR relative to $(\Im_{T+t})_{t\geq 0}$.

Remark In our context, from Theorem 3.4.3, the MIFR$|\Im_t$-property of $\mathbf{T}$ implies the MIFR$|\Im_{T+t}$-property of S and this is carried over the coherent system lifetime S of used lifetimes S.

Corollary 3.4.4 If T is MIFR$|\Im_t$, then the coherent system $S=\phi(\mathrm{S}) = \min_{1\leq j\leq k}\max_{i\in K_j} S_i$ is IFR$|\Im_{T+t}$.

Proof Note that

$$\theta_t\phi(\mathrm{S}) = \theta_t(\min_{1\leq j\leq k}\max_{i\in K_j} S_i) = [\min_{1\leq j\leq k}\max_{i\in K_j} S_i - t]^+ = [\min_{1\leq j\leq k}\max_{i\in K_j}(S_i-t)]^+ = \phi(\theta_t\mathrm{S}).$$

Since the function $\mathrm{S}\to\phi(\mathrm{S})$ is increasing for all bounded and increasing Borel measurable functions $f:\Re^m\to\Re$, we have

$$f(\theta_t\phi(\mathrm{S})) = f(\phi(\theta_t\mathrm{S}) = (fo\phi)(\theta_t\mathrm{S}).$$

Moreover, since the composite function $fo\phi : \Re^m \to \Re$ is increasing, we may conclude that, by the assumed MIFR$|\Im_t$ of T and Theorem 3.1.3,

$$E[(fo\phi)(\theta_t \mathrm{S})|\Im_{T+t})] \leq E[(fo\phi)(\theta_s \mathrm{S})|\Im_{T+s})],$$

for all $s \leq t$. Hence, S is IFR$|\Im_{T+t}$.

Remark Corollary 3.4.4 contradicts with the following well-known property: Monotone systems with independent IFR component lifetimes need not to be IFR.

From Arjas (1981) we known that if **T** is MIFR$|\Im_t$, then the sample paths of the $\Im_t$-compensator process $(A_i(t))_{t\geq 0}$ are convex for $t \in (0, T_i]$.

It follows that the $\Im_{T+t}$-compensator of $1_{\{S_i \leq t\}}$, $B_i(t) = E[A_i(t+T) - A_i(T)|\Im_T]$, is such that, for any $0 \leq \alpha \leq 1$ and any positive real numbers s and t,

$$B_i(\alpha t + (1-\alpha)s) = E[A_i((\alpha t + (1-\alpha)s) + T) - A_i(T)|\Im_T] =$$

$$E[A_i((\alpha(t+T) + (1-\alpha)(s+T)) - A_i(T)|\Im_T]$$

$$\leq \alpha E[A_i(t+T) - A_i(T)|\Im_T] + (1-\alpha)E[A_i(s+T) - A_i(T)|\Im_T] = \alpha B_i(t) + (1-\alpha)B_i(s).$$

Therefore, we proved the following

Corollary 3.4.5 Let **T** be the random vector representing the component lifetimes of a coherent system. If **T** is MIFR$|\Im_t$, then the sample paths of the $\Im_{T+t}$-compensator process $(B_i(t))_{t\geq 0}$ are $P-a.s.$ convex for $t \in (T, T_i - T]$.

From Theorem 3.4.3 and Theorem 3.4.5 we obtain bounds for the survival function.

Theorem 3.4.6 If T_i is $IFR|\Im_t^i$ for all $1 \leq i \leq n$ and $P(T_i \neq T_j) = 1$ for all $1 \leq i, j \leq n$, then the survival function $\overline{F}(u)$ satisfies

$$\overline{F}(u) \geq \begin{cases} \exp\{-u \sum_{i=1}^n \frac{1}{E[(T_i - T)^+|\Im_T]}\}, & \text{if } u < m, \\ 0, & \text{otherwise,} \end{cases}$$

where $m := \min_{1\leq i \leq n} \{E[(T_i - T)^+|\Im_T]\}$, $\Im_t^i := \sigma(1_{\{T_j > s\}}, 1 \leq j \leq n, j \neq i,\ s \leq t)$.

Proof $B_i((T_i - T)^+)$, $1 \leq i \leq n$ are random variables independent and identically distributed with unit exponential distribution. Follows from Theorem 3.4.6 that the compensator $B_i(u)$ are Q-a.s. convex on $(0, (T_i - T)^+]$. We can use Jensen's Inequality to show that

$$1 = E\{B_i((T_i - T)^+)\} \geq B_i\{E((T_i - T)^+|\Im_T)\}$$

Now as the system $\Im_{T+u}$ compensator is $B_\Phi(u) = \sum_{i=1}^n B_i(u)$

$$\frac{B_\Phi(u)}{u} = \frac{\sum_{i=1}^n B_i(u)}{u} \leq \sum_{i=1}^n \frac{B_i(E((T_i - T)^+|\Im_{Y_i}))}{E((T_i - T)^+|\Im_T)}$$

$$\leq \sum_{i=1}^{n} \frac{1}{E((T_i - Y_i)^+ | \Im_T)}.$$

As

$$\overline{F}(u|\Im_u) = P(T > u|\Im_u) = \exp\{-B_\Phi(u)\},$$

the result follows.

3.5. The new better than used distribution.

If we take $s = 0$ in definition 3.4.2, we clearly have that the class of distributions MIFR$|\Im_{T+t}$ is a subclass of distributions MNBU$|\Im_{T+t}$ defined by:

Definition 3.5.1 Let T be an $\Im_t$-stopping time. We say that **T** is multivariate NBU relative to $(\Im_{T+t})_{t\geq 0}$, denoted by MNBU$|\Im_{T+t}$, if

$$P\{\theta_t \mathbf{T} \in U | \Im_{T+t}\} \leq P\{\mathbf{T} \in U | \Im_T\}.$$

If we have $P(T = 0) = 1$ we get the Arjas (1981) definition of MNBU$|\Im_t$:

Definition 3.5.2 (Arjas (1981)) We say that **T** is multivariate NBU relative to $(\Im_t)_{t\geq 0}$, denoted by MNBU$|\Im_t$, if

$$P\{\theta_t \mathbf{T} \in U | \Im_t\} \leq P\{\mathbf{T} \in U | \Im_0\}.$$

Clearly, from the $\Im_t$-super martingale property of $M_t = P\{\theta_t \mathbf{T} \in U | \Im_t\}$ we have preserved the MNBU$|\Im_t$ property of the vector lifetime **T** to the MNBU$|\Im_{T+t}$. property of S.

We always have $\forall i$,

$$P\{\min_{1\leq i\leq n}[(T_i - T)^+ > t|\Im_T]\} \leq P\{(T_i - T)^+ > t|\Im_T\},$$

which constitutes an upper bound for a series system reliability. Also,

$$P\{\min_{1\leq i\leq n}[(T_i - T)^+ > t|\Im_T]\} \leq \min_{\{1\leq i\leq n\}} P\{(T_i - T)^+ > t|\Im_T\}.$$

If **T** is MNBU$|\Im_t$, then we have

$$P\{(T_i - T)^+ > s|\Im_T\} \leq P\{T_i > s\}.$$

Therefore, under the MNBU$|\Im_t$ assumption, our upper bound is sharper than the minimax upper bound obtained by Barlow and Proschan (1981).

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Departamento de Estatística
IME-USP
Caixa Postal 66.281
05314-970 - São Paulo, Brasil