ARTICLE IN PRESS
Nonlinear Analysis xxx (xxxx) xxx

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Fractional problems in thin domains

Marcone C. Pereira [a], Julio D. Rossi [b,*], Nicolas Saintier [b]

[a] Dpto. de Matemática Aplicada, IME, Universidade de São Paulo, Rua do Matão 1010, São Paulo - SP, Brazil
[b] Dpto. de Matemáticas, FCEyN, Universidad de Buenos Aires, 1428, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 16 November 2018
Accepted 13 February 2019
Communicated by Enzo Mitidieri

MSC:
45A05
45C05
45M05

Keywords:
Thin domains
Nonlocal fractional equations
Neumann problem
Dirichlet problem

ABSTRACT

In this paper we consider nonlocal fractional problems in thin domains. Given open bounded subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, we show that the solution u_ε to

$$\Delta_x^s u_\varepsilon(x,y) + \Delta_y^t u_\varepsilon(x,y) = f(x, \varepsilon^{-1} y) \qquad \text{in } U \times \varepsilon V$$

with $u_\varepsilon(x,y) = 0$ if $x \notin U$ and $y \in \varepsilon V$, verifies that $\tilde{u}_\varepsilon(x,y) := u_\varepsilon(x, \varepsilon y) \to u_0$ strongly in the natural fractional Sobolev space associated to this problem. We also identify the limit problem that is satisfied by u_0 and estimate the rate of convergence in the uniform norm. Here $\Delta_x^s u$ and $\Delta_y^t u$ are the fractional Laplacian in the 1st variable x (with a Dirichlet condition, $u(x) = 0$ if $x \notin U$) and in the 2nd variable y (with a Neumann condition, integrating only inside V), respectively, that is,

$$\Delta_x^s u(x,y) = \int_{\mathbb{R}^n} \frac{u(x,y) - u(w,y)}{|x-w|^{n+2s}} \, dw$$

and

$$\Delta_y^t u(x,y) = \int_V \frac{u(x,y) - u(x,z)}{|y-z|^{m+2t}} \, dz.$$

1. Introduction

In this paper our main goal is to show that there is a limit problem for fractional type elliptic problems in thin domains, that is, when the thickness of the domain in one direction goes to zero.

Given two smooth open bounded domains $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, n and $m \geq 1$, real numbers $s, t \in (0,1)$ and $f \in L^2(U \times V)$, we consider the problem

$$\Delta_x^s u(x,y) + \Delta_y^t u(x,y) = f(x,y) \qquad \text{in } U \times V \tag{1.1}$$

* Corresponding author.
E-mail addresses: marcone@ime.usp.br (M.C. Pereira), jrossi@dm.uba.ar (J.D. Rossi), nsaintie@dm.uba.ar (N. Saintier).
URLs: http://www.ime.usp.br/~marcone (M.C. Pereira), http://mate.dm.uba.ar/~jrossi/ (J.D. Rossi), http://mate.dm.uba.ar/~nsaintie/ (N. Saintier).

https://doi.org/10.1016/j.na.2019.02.024

with a Dirichlet condition in $(\mathbb{R}^n \setminus U) \times V$, $u(x,y) = 0$ for $(x,y) \in (\mathbb{R}^n \setminus U) \times V$. Here $\Delta_x^s u$ and $\Delta_y^t u$ are the fractional Laplacian w.r.t. in the 1st variable x and the 2nd variable y, respectively, namely

$$\Delta_x^s u(x,y) = \int_{\mathbb{R}^n} \frac{u(x,y) - u(w,y)}{|x-w|^{n+2s}}\, dw$$

and

$$\Delta_y^t u(x,y) = \int_V \frac{u(x,y) - u(x,z)}{|y-z|^{m+2t}}\, dz.$$

In order to simplify the notation we have dropped the usual normalization constant that is usually in front of the integrals. Also, we note that we have a Neumann boundary condition on $U \times (\mathbb{R}^m \setminus V)$ since we are integrating only in V.

The purpose of this note is first to prove the existence and uniqueness of a weak solution to (1.1). Then we want to perturb the problem by replacing V by εV, $\varepsilon > 0$, and study the asymptotic behaviour of the corresponding solution u_ε as $\varepsilon \to 0$.

We will work in the space $H_0^{s,t}(\mathbb{R}^n \times V)$ of the functions $u \in L^2(\mathbb{R}^n \times V)$ such that $u(x,y) = 0$ if $x \notin U$ and $y \in V$, and such that

$$\begin{aligned} \|u\|^2 := &\int_{\mathbb{R}^n \times \mathbb{R}^n \times V} \frac{|u(x,y) - u(w,y)|^2}{|x-w|^{n+2s}} dxdydw \\ &+ \int_{\mathbb{R}^n \times V \times V} \frac{|u(x,y) - u(x,z)|^2}{|y-z|^{m+2t}} dxdydz \end{aligned} \tag{1.2}$$

is finite. Indeed, equipped with the norm $\|\cdot\|$, we have that $H_0^{s,t} := H_0^{s,t}(\mathbb{R}^n \times V)$ is a Hilbert space.

Now we are ready to state the main result of this note:

Theorem 1.1. *For any $f \in L^2(U \times V)$, there exists a unique weak solution $u \in H_0^{s,t}$ to (1.1). This weak solution is characterized as being the unique minimizer of the functional*

$$v \in H_0^{s,t}(\mathbb{R}^n \times V) \ \to \ \frac{1}{4}\|v\|^2 - (f,v),$$

where $(\cdot,\cdot)$ denotes the duality pairing.

Moreover, if $f \in L^a(U \times V)$ with $a > \frac{n+m}{2r}$ and $r = \min\{s,t\}$, then there exists a constant $K > 0$ (depending only on n, m, r, and a) such that the solution u satisfies

$$\|u\|_{L^\infty(\mathbb{R}^n \times V)} \le K \|f\|_{L^a(\mathbb{R}^n \times V)}.$$

For $\epsilon > 0$ denote by u_ε the solution to the thin domain problem

$$\Delta_x^s u_\varepsilon(x,y) + \Delta_y^t u_\varepsilon(x,y) = \tilde{f}^\varepsilon(x,y) \qquad \text{in } U \times \varepsilon V \tag{1.3}$$

with $u_\varepsilon(x,y) = 0$ if $x \notin U$ and $y \in \varepsilon V$, where

$$\tilde{f}^\varepsilon(x,y) = f(x, \varepsilon^{-1} y)$$

for some fixed $f \in L^2(U \times V)$. Then, the rescaled function

$$\tilde{u}_\varepsilon(x,y) := u_\varepsilon(x, \varepsilon y) \in H_0^{s,t}(\mathbb{R}^n \times V)$$

verifies

$$\tilde{u}_\varepsilon \to u_0, \qquad \text{strongly in } H_0^{s,t}(\mathbb{R}^n \times V), \tag{1.4}$$

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

where $u_0(x,y)$ depends only on the first variable x, that is, $u_0(x,y) = u_0(x)$ for all $(x,y) \in \mathbb{R}^n \times V$. Furthermore, u_0 is the solution to the limit problem

$$\Delta_x^s u_0(x) = \frac{1}{|V|}\int_V f(x,y)dy \qquad \text{in } U, \tag{1.5}$$

with $u_0 = 0$ in $\mathbb{R}^n \setminus U$.

In addition, if

$$\sup_{x\in\mathbb{R}^n} \|f(x,.)\|_{L^a(V)} + \sup_{x\in\mathbb{R}^n} \|\Delta_x^s f(x,.)\|_{L^a(V)} < \infty$$

for some $a > \max\{\frac{m}{2t}, 1\}$, then we have a uniform convergence result with an upper bound of order $2t$, that is,

$$\|\tilde{u}_\varepsilon(x,y) - u_0(x)\|_{L^\infty(U\times V)} \le C\varepsilon^{2t}. \tag{1.6}$$

We end this introduction with a brief description of related references. It is not difficult to see that thin structures occur naturally in many applications. For example, in oceanic models, one is dealing with fluid regions which are thin compared to the horizontal length scales. Other examples can include lubrication, nanotechnology, blood circulation, material engineering, meteorology, etc. In fact, many techniques and methods have been developed in order to understand the effect of the geometry and thickness of the domain on the solutions of such singular problems. From pioneering works to recent ones we mention [1–8] concerned with elliptic and parabolic equations, as well as [9–15] where the authors considered Stokes and Navier–Stokes equations from fluid mechanics. Concerning nonlocal equations in thin domains we mention the recent paper [16] where equations with smooth and compactly supported kernels are considered. For general references on fractional problems we refer to [17]. For other nonlocal models, see [18–22].

Finally, we want to mention that, when we look at the usual fractional Laplacian

$$\Delta^s u(x,y) = \int_{\mathbb{R}^n\times\mathbb{R}^m} \frac{u(x,y)-u(z,w)}{|(x,y)-(z,w)|^{n+m+2s}}\, dz\, dw$$

and we localize it in εV (to deal with a thin domain) taking

$$\Delta^s u(x,y) = \int_{\mathbb{R}^n\times\varepsilon V} \frac{u(x,y)-u(z,w)}{|(x,y)-(z,w)|^{n+m+2s}}\, dz\, dw$$

our results cannot be extended to this model. In fact, when one changes variables and considers the resulting kernel, one finds that it goes to zero as $\varepsilon \to 0$ (this is due to the fact that we take V bounded and therefore the effect of the tails of the fractional Laplacian in y is suppressed when considering the previous operator). We will comment more on this fact in the final section. Also remark that the usual local Laplacian $\Delta u(x,y)$ has the property that

$$\Delta u(x,y) = \Delta_x u(x,y) + \Delta_y u(x,y)$$

even if we consider it in $U \times \epsilon V$. In our problem (1.1) this property also holds, but it does not hold when we deal with the usual fractional Laplacian that we previously described.

The paper is organized as follows: in Section 2 we prove existence and uniqueness of weak solutions to our nonlocal problem; in Section 3 we deal with the problem in thin domains and compute the limit as $\epsilon \to 0$; in Section 4 we show that when f is smooth we have a corrector and hence we can show uniform convergence and obtain a bound for the rate of order $2t$; finally, in Section 5 we collect some possible extensions of our results.

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

2. Existence and uniqueness

Notice that a reasonable weak formulation of (1.1) is the following one: $u \in H_0^{s,t}$ is a weak solution if for every $\phi \in H_0^{s,t}$ it holds

$$\begin{aligned}
&\int_{\mathbb{R}^n \times V} f(x,y)\phi(x,y)\, dxdy \\
&= \int_{\mathbb{R}^n \times V} \phi(x,y)\Big(\int_{\mathbb{R}^n} \frac{u(x,y)-u(w,y)}{|x-w|^{n+2s}}\, dw\Big) dxdy \\
&\quad + \int_{\mathbb{R}^n \times V} \phi(x,y)\Big(\int_{V} \frac{u(x,y)-u(x,z)}{|y-z|^{m+2t}}\, dz\Big) dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n \times V} \frac{[u(x,y)-u(w,y)][\phi(x,y)-\phi(w,y)]}{|x-w|^{n+2s}}\, dxdwdy \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n \times V \times V} \frac{[u(x,y)-u(x,z)][\phi(x,y)-\phi(x,z)]}{|y-z|^{m+2t}}\, dxdydz.
\end{aligned} \tag{2.1}$$

In view of this expression, it is natural to introduce, as we did in the introduction, the space $H_0^{s,t}(\mathbb{R}^n \times V)$ of the functions $u \in L^2(\mathbb{R}^n \times V)$ such that $u(x,y) = 0$ if $x \notin U$ and $y \in V$, and such that

$$\begin{aligned}
\|u\|^2 :&= \int_{\mathbb{R}^n \times \mathbb{R}^n \times V} \frac{|u(x,y)-u(w,y)|^2}{|x-w|^{n+2s}} dxdydw \\
&\quad + \int_{\mathbb{R}^n \times V \times V} \frac{|u(x,y)-u(x,z)|^2}{|y-z|^{m+2t}} dxdydz \\
&= \int_V [u(\cdot,y)]^2_{H^s(\mathbb{R}^n)}\, dy + \int_{\mathbb{R}^n} [u(x,\cdot)]^2_{H^t(V)}\, dx < +\infty.
\end{aligned} \tag{2.2}$$

Recall that, given any open subset $\Omega \subset \mathbb{R}^N$, the expression

$$[w]^2_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|w(\xi)-w(\eta)|^2}{|\xi-\eta|^{N+2s}} d\xi d\eta$$

is the so-called Gagliardo semi-norm of w. Also, equipped with the norm $\|\cdot\|$, it is clear that the space $H_0^{s,t}(\mathbb{R}^n \times V)$ sets a Hilbert space.

It follows as an immediate consequence of Lax–Milgram's theorem that when f belongs to the dual space of $H_0^{s,t}$, Eq. (1.1) has a unique weak solution.

Proposition 2.1. *For any $f \in (H_0^{s,t})'$ there exists a unique weak solution $u \in H_0^{s,t}$ to (1.1). This weak solution is the unique minimizer of the functional*

$$v \in H_0^{s,t}(\mathbb{R}^n \times V) \to \frac{1}{4}\|v\|^2 - (f,v).$$

We now verify that we can take for instance $f \in L^2(U \times V)$.

Proposition 2.2. *Letting $r = \min\{s,t\}$, there holds*

$$H_0^{s,t} \hookrightarrow H_0^{r,r} \hookrightarrow H_0^r, \tag{2.3}$$

where H_0^r is the subspace of the usual fractional space $H^r(\mathbb{R}^n \times V)$ composed of the functions $u \in H^r(\mathbb{R}^n \times V)$ such that $u(x,y) = 0$ for $y \in V$ and $x \notin U$.

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

Proof. Recall that H_0^r is equipped with the norm

$$\begin{aligned}\|u\|^2_{H_0^r} &= \|u\|_2^2 + [u]^2_{H^r} \\ &= \|u\|_2^2 + \int_{\mathbb{R}^n\times V}\int_{\mathbb{R}^n\times V} \frac{|u(x,y)-u(w,z)|^2}{|(x,y)-(w,z)|^{n+m+2r}}\, dx\, dy\, dw\, dz.\end{aligned}$$

The first embedding follows from the two embeddings $H^s \hookrightarrow H^r$, $H^t \hookrightarrow H^r$ and the last line in (2.2).

Now, we prove the second embedding, $H_0^{r,r} \hookrightarrow H_0^r$. To this end we first write

$$\begin{aligned}|u(x,y)-u(w,z)|^2 &= |u(x,y)-u(x,z)+u(x,z)-u(w,z)|^2 \\ &\le 2|u(x,y)-u(x,z)|^2 + 2|u(x,z)-u(w,z)|^2\end{aligned}$$

and then we obtain

$$\begin{aligned}[u]^2_{H^r} &= \int_{\mathbb{R}^n\times V}\int_{\mathbb{R}^n\times V} \frac{|u(x,y)-u(w,z)|^2}{|(x,y)-(w,z)|^{n+m+2r}}\, dx\, dy\, dw\, dz \\ &\le 2\int_{\mathbb{R}^n\times V}\int_{\mathbb{R}^n\times V} \frac{|u(x,y)-u(x,z)|^2}{(|x-w|^2+|y-z|^2)^{\frac{n+m+2r}{2}}}\, dx\, dw\, dy\, dz \\ &\quad + 2\int_{\mathbb{R}^n\times V}\int_{\mathbb{R}^n\times V} \frac{|u(x,z)-u(w,z)|^2}{(|x-w|^2+|y-z|^2)^{\frac{n+m+2r}{2}}}\, dx\, dy\, dw\, dz \\ &=: 2I_1 + 2I_2.\end{aligned}$$

Letting $a = |y-z|$, we bound I_1 writing first that

$$\begin{aligned}I_1 &= \int_{V\times V}\left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(ax,y)-u(ax,z)|^2}{[1+|x-w|^2]^{\frac{n+m+2r}{2}}}\, dx\, dw\right)\frac{dy\, dz}{|y-z|^{m-n+2r}} \\ &= \int_{V\times V}\left(\int_{\mathbb{R}^n} |u(ax,y)-u(ax,z)|^2 \right.\\ &\qquad\qquad \left.\times\left(\int_{\mathbb{R}^n}\frac{dw}{[1+|x-w|^2]^{\frac{n+m+2r}{2}}}\right)dx\right)\frac{dy\, dz}{|y-z|^{m-n+2r}}.\end{aligned}$$

The integral in w is bounded,

$$\int_{\mathbb{R}^n}\frac{dw}{[1+|w|^2]^{\frac{n+m+2r}{2}}} \le C.$$

Thus changing variables in x we obtain

$$\begin{aligned}I_1 &\le C\int_{V\times V}\left(\int_{\mathbb{R}^n}\left(|u(x,y)-u(x,z)|^2\right)dx\right)\frac{dy\, dz}{|y-z|^{m+2r}} \\ &= C\int_{\mathbb{R}^n}[u(x,\cdot)]^2_{H^r(V)}\, dx.\end{aligned}$$

We tackle I_2 in a similar way. □

We need a result like the following one.

Proposition 2.3. *The embedding*

$$H_0^r(\mathbb{R}^n\times V) \hookrightarrow L^2(\mathbb{R}^n\times V)$$

is continuous and compact.

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

Proof. Since U and V are smooth bounded domains, the result follows from [17, Theorem 6.7 and Corollary 7.2]. □

As a consequence of this result and Proposition 2.2 we get

Corollary 2.1. *The embedding*

$$H_0^{s,t}(\mathbb{R}^n \times V) \hookrightarrow L^2(\mathbb{R}^n \times V)$$

is continuous and compact.

We thus obtain from Proposition 2.1 the existence and uniqueness part of Theorem 1.1:

Proposition 2.4. *For any $f \in L^2(U \times V)$ there exists a unique weak solution $u \in H_0^{s,t}$ to (1.1). This weak solution is the unique minimizer of the functional*

$$v \in H_0^{s,t}(\mathbb{R}^n \times V) \to \frac{1}{4}\|v\|^2 - (f, v).$$

Now let us introduce a condition on f in order to guarantee that the solutions to (1.1) belong to L^∞. We will need this result in Section 4.

Proposition 2.5. *Let $u \in H_0^{s,t}(\mathbb{R}^n \times V)$ be a weak solution of (1.1) for some $f \in L^a(U \times V)$ with $a > \frac{n+m}{2r}$ where $r = \min\{s,t\}$.*

Then $u \in L^\infty(\mathbb{R}^n \times V)$ with

$$\|u\|_{L^\infty(\mathbb{R}^n \times V)} \leq K \|f\|_{L^a(U \times V)} \tag{2.4}$$

where $K > 0$ depends only on n, m, r, and a.

Proof. Let $A_k = \{(x,y) \in \mathbb{R}^n \times V : u(x,y) > k\}$ for $k \in \mathbb{N}$. According to Proposition 2.2, we have $H_0^{s,t}(A_k) \hookrightarrow H_0^r(A_k)$ where $r = \min\{s,t\}$. By Sobolev embeddings (see for instance [17, Theorem 6.7 and Remark 6.8]), we deduce that

$$H_0^{s,t}(A_k) \hookrightarrow L^{\frac{2(n+m)}{n+m-2r}}(A_k).$$

Then

$$\begin{aligned}
\int_{A_k} (u(x,y) - k)\, dxdy &= \int_{A_k} (u(x,y) - k)_+ \, dxdy \\
&\leq \|1\|_{L^{\frac{2(n+m)}{n+m+2r}}(A_k)} \|(u-k)_+\|_{L^{\frac{2(n+m)}{n+m-2r}}(A_k)} \\
&\leq C|A_k|^{\frac{1}{2}+\frac{r}{n+m}} \|(u-k)_+\|.
\end{aligned} \tag{2.5}$$

Here $\phi_\pm$ denote the positive and negative part of a function ϕ defined as $\phi_+ = \max\{\phi, 0\}$ and $\phi_- = \max\{-\phi, 0\}$. Notice that $\phi = \phi_+ - \phi_-$.

To estimate $\|(u-k)_+\|$, we take $\varphi = (u-k)_+$ in the weak formulation (2.1). We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \times V} f(x,y)(u(x,y) - k)_+ \, dx\, dy = \\
&\qquad \int_{\mathbb{R}^n \times V} \int_{\mathbb{R}^n} \frac{(u(x,y)-k)_+^2 - (u(x,y)-k)_+(u(w,y)-k)}{|x-w|^{n+2s}} dw\, dx\, dy \\
&\qquad + \int_{\mathbb{R}^n \times V} \int_V \frac{(u(x,y)-k)_+^2 - (u(x,y)-k)_+(u(x,z)-k)}{|y-z|^{m+2t}} dz\, dx\, dy.
\end{aligned}$$

Notice that

$$\begin{aligned}&(u(x,y)-k)_+(u(w,y)-k)\\&\quad=(u(x,y)-k)_+(u(w,y)-k)_+-(u(x,y)-k)_+(u(w,y)-k)_-\\&\quad\le(u(x,y)-k)_+(u(w,y)-k)_+,\end{aligned}$$

and, analogously that

$$(u(x,y)-k)_+(u(x,z)-k)\le(u(x,y)-k)_+(u(x,z)-k)_+.$$

We thus get

$$\begin{aligned}&\int_{\mathbb{R}^n\times V}f(x,y)(u(x,y)-k)_+\,dx\,dy\\&\ge\int_{\mathbb{R}^n\times V}\int_{\mathbb{R}^n}\frac{(u(x,y)-k)_+^2-(u(x,y)-k)_+(u(w,y)-k)_+}{|x-w|^{n+2s}}dw\,dx\,dy\\&\quad+\int_{\mathbb{R}^n\times V}\int_V\frac{(u(x,y)-k)_+^2-(u(x,y)-k)_+(u(x,z)-k)_+}{|y-z|^{m+2t}}dz\,dx\,dy\\&=\frac{1}{2}\int_{\mathbb{R}^n\times V\times\mathbb{R}^n}\frac{\left[(u(x,y)-k)_+-(u(w,y)-k)_+\right]^2}{|x-w|^{n+2s}}dw\,dx\,dy\\&\quad+\frac{1}{2}\int_{\mathbb{R}^n\times V\times V}\frac{\left[(u(x,y)-k)_+-(u(x,z)-k)_+\right]^2}{|y-z|^{m+2t}}dz\,dx\,dy\\&=\frac{1}{2}\|(u-k)_+\|^2.\end{aligned}$$

Hence

$$\|(u-k)_+\|^2\le 2\int_{\mathbb{R}^n\times V}f(x,y)(u(x,y)-k)_+\,dx\,dy.$$

We now take $a,b\in(1,\infty)$ such that

$$\frac{1}{a}+\frac{1}{b}+\frac{n+m-2r}{2(n+m)}=1, \tag{2.6}$$

and apply Hölder inequality to get

$$\begin{aligned}\|(u-k)_+\|^2&\le 2\|f\|_{L^a(\mathbb{R}^n\times V)}\|1\|_{L^b(A_k)}\|(u-k)_+\|_{L^{\frac{2(n+m)}{n+m-2r}}(\mathbb{R}^n\times V)}\\&\le C|A_k|^{1/b}\|f\|_{L^a(\mathbb{R}^n\times V)}\|(u-k)_+\|.\end{aligned}$$

We thus conclude that

$$\|(u-k)_+\|\le C|A_k|^{1/b}\|f\|_{L^a(\mathbb{R}^n\times V)}.$$

Plugging this estimate in (2.5) yields

$$\int_{A_k}(u(x,y)-k)\,dxdy\le C|A_k|^{\frac{1}{2}+\frac{1}{b}+\frac{r}{n+m}}\|f\|_{L^a(\mathbb{R}^n\times V)}. \tag{2.7}$$

We now choose $a,b\in(1,+\infty)$ in such a way that

$$\frac{1}{2}+\frac{1}{b}+\frac{r}{n+m}>1.$$

Since a,b must verify (2.6), this is possible if we assume $a>\frac{n+m}{2r}$. It then follows from (2.7) and [23, Chap. 2, Lemma 5.1] that $u\in L^\infty(\mathbb{R}^n\times V)$ with

$$\|u\|_{L^\infty(\mathbb{R}^n\times V)}\le\frac{1+\varepsilon}{\varepsilon}\gamma^{\frac{1}{1+\varepsilon}}\|u\|_{L^1(\mathbb{R}^n\times V)} \tag{2.8}$$

where

$$\gamma=C\|f\|_{L^a(U\times V)},\qquad 1+\varepsilon=\frac{1}{2}+\frac{1}{b}+\frac{r}{n+m}.$$

Moreover notice that since u is a weak solution of (1.1),

$$\begin{aligned}\|u\|^2 &\leq 2\|f\|_{L^a(U\times V)}\|u\|_{L^{a'}(\mathbb{R}^n\times V)}\\ &\leq C\|f\|_{L^a(U\times V)}\|u\|.\end{aligned}$$

In the last equality we used the embeddings

$$H_0^{s,t}(\mathbb{R}^n\times V)\hookrightarrow H_0^r(\mathbb{R}^n\times V)\hookrightarrow L^{a'}(U\times V)$$

which hold since $a'\leq\frac{2(n+m)}{n+m-2r}$. We deduce that

$$\|u\|_{L^1(\mathbb{R}^n\times V)}\leq\|u\|\leq C\|f\|_{L^a(U\times V)}.$$

Combining with (2.8), we obtain

$$\|u\|_{L^\infty(\mathbb{R}^n\times V)}\leq C\|f\|_{L^a(U\times V)}^{1+\frac{1}{1+\varepsilon}},$$

where the constant C depends only on b, r, n and m. We deduce in particular that the linear operator $L: f\to u$ is continuous from $L^a(U\times V)$ into $L^\infty(\mathbb{R}^n\times V)$. This proves (2.4). □

Remark 2.1. Notice that Proposition 2.5 also gives us boundedness results to the solutions for the usual fractional Laplacian operator: the Dirichlet problem

$$\begin{gathered}\int_{\mathbb{R}^n}\frac{u(x)-u(w)}{|x-w|^{n+2s}}dw=f(x),\quad x\in U\\ u(x)=0\quad x\in\mathbb{R}^n\setminus U,\end{gathered}$$

and the regional fractional Laplacian (Neumann conditions)

$$\begin{gathered}\int_V\frac{u(y)-u(z)}{|y-z|^{m+2t}}dz=f(y),\quad y\in V\\ \int_V u(y)\,dy=0.\end{gathered}$$

Indeed, such solutions are unique and satisfy the nonlocal Eq. (1.1) in a trivial way.

3. Thin domains

We now perturb V by replacing it by εV, $\varepsilon>0$. Given some $f\in L^2(U\times V)$, there exists, according to the previous Proposition 2.4, a unique function $u_\varepsilon\in H_0^{s,t}(\mathbb{R}^n\times\varepsilon V)$ solution of

$$\Delta_x^s u_\varepsilon(x,y)+\Delta_y^t u_\varepsilon(x,y)=\tilde{f}^\varepsilon(x,y)\qquad\text{in } U\times\varepsilon V \tag{3.1}$$

with $u_\varepsilon(x,y)=0$ if $x\notin U$ and $y\in\varepsilon V$, and where

$$\tilde{f}^\varepsilon(x,y):=f(x,\varepsilon^{-1}y).$$

Let us study the asymptotic behaviour of u_ε as $\varepsilon\to 0$ proving the second part of Theorem 1.1. We first rescale u_ε considering the function $\tilde{u}_\varepsilon\in H_0^{s,t}(\mathbb{R}^n\times V)$ defined as

$$\tilde{u}_\varepsilon(x,y):=u_\varepsilon(x,\varepsilon y). \tag{3.2}$$

Note that the whole family $\tilde{u}_\varepsilon$ belongs to the same space $H_0^{s,t}(\mathbb{R}^n\times V)$ (that is independent of $\varepsilon>0$). Using this approach we obtain the limit problem for (3.1).

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

Theorem 3.1. *There holds*

$$\tilde{u}_\varepsilon \to u_0 \qquad \text{strongly in } H_0^{s,t}(\mathbb{R}^n \times V), \tag{3.3}$$

where u_0 depends only on variable x, belongs to the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ and is the solution to

$$\begin{aligned} \Delta_x^s u_0(x) &= \frac{1}{|V|}\int_V f(x,y)dy \qquad \text{in } U, \\ u_0 &= 0 \qquad \text{in } \mathbb{R}^n \backslash U. \end{aligned} \tag{3.4}$$

Proof. A change of variable in the weak formulation of (3.1) shows that $\tilde{u}_\varepsilon$ defined in (3.2) satisfies

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n\times V} \frac{[\tilde{u}_\varepsilon(x,y)-\tilde{u}_\varepsilon(w,y)][\phi(x,y)-\phi(w,y)]}{|x-w|^{n+2s}}\, dx\, dw\, dy \\ &\quad + \frac{1}{2\varepsilon^{2t}}\int_{\mathbb{R}^n\times V\times V} \frac{[\tilde{u}_\varepsilon(x,y)-\tilde{u}_\varepsilon(x,z)][\phi(x,y)-\phi(x,z)]}{|y-z|^{m+2t}}\, dx\, dy\, dz \\ &= \int_{\mathbb{R}^n\times V} f(x,y)\phi(x,y)\, dxdy \end{aligned} \tag{3.5}$$

for any $\phi \in H_0^{s,t}(\mathbb{R}^n \times V)$. Thus $\tilde{u}_\varepsilon \in H_0^{s,t}(\mathbb{R}^n \times V)$ is the weak solution to

$$\Delta_x^s \tilde{u}_\varepsilon(x,y) + \varepsilon^{-2t}\Delta_y^t \tilde{u}_\varepsilon(x,y) = f(x,y) \qquad \text{in } U \times V.$$

It follows in particular that $\tilde{u}_\varepsilon$ is the unique minimum point of the convex functional

$$J_\varepsilon(u) = \frac{1}{4}\|u\|_\varepsilon^2 - (f,u)$$

defined for any $u \in H_0^{s,t}(\mathbb{R}^n \times V)$ with

$$\|u\|_\varepsilon^2 = \int_V [u(\cdot,y)]_{H^s}^2\, dy + \frac{1}{\varepsilon^{2t}}\int_{\mathbb{R}^n} [u(x,\cdot)]_{H^t(V)}^2\, dx.$$

Note that $\|\cdot\| \le \|\cdot\|_\varepsilon$ for all $\varepsilon \in (0,1)$.

Taking $\phi = \tilde{u}_\varepsilon$ in (3.5), we obtain

$$\|\tilde{u}_\varepsilon\|_\varepsilon^2 \le 2\|f\|_{L^2(\mathbb{R}^n\times V)}\|\tilde{u}_\varepsilon\|_{L^2(U\times V)}.$$

Hence, using Corollary 2.1, we get that

$$\|\tilde{u}_\varepsilon\|_\varepsilon^2 \le C\|\tilde{u}_\varepsilon\|_{L^2(U\times V)} \le C\|\tilde{u}_\varepsilon\|.$$

It follows that for any $\varepsilon \in (0,1)$,

$$\|\tilde{u}_\varepsilon\| \le \|\tilde{u}_\varepsilon\|_\varepsilon \le C$$

for some $C > 0$ independent of ε. As a consequence there exists $\tilde{u} \in H_0^{s,t}(\mathbb{R}^n \times V)$ such that, up to a subsequence, $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ weakly in $H_0^{s,t}(\mathbb{R}^2 \times V)$ and also strongly in $L^2(\mathbb{R}^n \times V)$ in view of Proposition 2.3, that is,

$$\tilde{u}_\varepsilon \to u_0 \qquad \text{strongly in } L^2(\mathbb{R}^n \times V) \text{ and weakly in } H_0^{s,t}(\mathbb{R}^n \times V). \tag{3.6}$$

We obtain in particular that

$$\|u_0\| \le \liminf_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|. \tag{3.7}$$

Let us check that u_0 does not depend on y. Notice that

$$H_0^{s,t}(\mathbb{R}^n \times V) = L^2(\mathbb{R}^n, H^t(V)) \cap L^2(V, H_0^s(\mathbb{R}^n)).$$

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

In particular

$$(L^2(\mathbb{R}^n, H^t(V)))' \subset (H_0^{s,t}(\mathbb{R}^n \times V))'.$$

Hence, it follows that

$$\tilde{u}_\varepsilon \to u_0 \qquad \text{weakly in } L^2(\mathbb{R}^n, H^t(V)). \tag{3.8}$$

In the same way,

$$\tilde{u}_\varepsilon \to u_0 \qquad \text{weakly in } L^2(V, H_0^s(\mathbb{R}^n)). \tag{3.9}$$

So,

$$\|u_0\|_{L^2(\mathbb{R}^n, H^t(V))} \le \liminf_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|_{L^2(\mathbb{R}^n, H^t(V))}.$$

Since $\tilde{u}_\varepsilon \to u_0$ in $L^2(\mathbb{R}^n \times V)$, we deduce that

$$\int_{\mathbb{R}^n} [u_0(x,\cdot)]^2_{H^t(V)}\, dx \le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^n} [\tilde{u}_\varepsilon(x,\cdot)]^2_{H^t(V)}\, dx.$$

On the other hand, it follows from $\|\tilde{u}_\varepsilon\|_\varepsilon \le C$, $0 < \varepsilon < 1$, that

$$\int_{\mathbb{R}^n} [\tilde{u}_\varepsilon(x,\cdot)]^2_{H^t(V)}\, dx \le C\varepsilon^{2t},$$

and then we get

$$\int_{\mathbb{R}^n} [u_0(x,\cdot)]^2_{H^t(V)}\, dx = 0.$$

Therefore, the limit function u_0 does not depend on y.

Now, we take a test function $\phi(x)$ independent of y in (3.5). We get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n\times V} \frac{[\tilde{u}_\varepsilon(x,y) - \tilde{u}_\varepsilon(w,y)][\phi(x)-\phi(w)]}{|x-w|^{n+2s}}\, dx\, dw\, dy \\ &\qquad = \int_{\mathbb{R}^n\times V} f(x,y)\phi(x)\, dx\, dy. \end{aligned} \tag{3.10}$$

Thus, passing to the limit in (3.10) as $\epsilon \to 0$ using (3.9) and recalling that u_0 does not depend on y, we arrive at

$$\begin{aligned} &\frac{|V|}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{[u_0(x) - u_0(w)][\phi(x)-\phi(w)]}{|x-w|^{n+2s}}\, dx\, dw \\ &\qquad = \int_{\mathbb{R}^n\times V} f(x,y)\phi(x)\, dx\, dy. \end{aligned} \tag{3.11}$$

Hence $u_0 \in H_0^s(\mathbb{R}^n)$ is a weak solution to

$$\Delta_x^s u_0(x) = \frac{1}{|V|}\int_V f(x,y)dy \qquad \text{in } U,$$

with

$$u_0 = 0 \qquad \text{in } \mathbb{R}^n \setminus U,$$

as we wanted to show.

Finally, let us prove the strong convergence in $H_0^{s,t}(\mathbb{R}^n \times V)$ of $\tilde{u}_\varepsilon$ to u_0. Since we already have the weak convergence, it suffices to prove the convergence of the norms. Notice that $\|u_0\|^2 = |V|[u_0]^2_{H^s}$ because u_0 does not depend on y. In view of (3.7), we have

$$|V|[u_0]^2_{H^s} = \|u_0\|^2 \le \liminf_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|^2 \le \limsup_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|^2 \le \limsup_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|^2_\varepsilon.$$

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

Now let us take $\phi = \tilde{u}_\varepsilon$ in the weak formulation (3.5) and we obtain

$$\limsup_{\varepsilon\to 0} \|\tilde{u}_\varepsilon\|_\varepsilon^2 = \lim_{\epsilon\to 0} 2\int_{\mathbb{R}^n\times V} f(x,y)\tilde{u}_\varepsilon(x,y)\,dx\,dy$$
$$= 2\int_{\mathbb{R}^n\times V} f(x,y)u_0(x)\,dx\,dy,$$

where we used the strong convergence of $\tilde{u}_\varepsilon \to u_0$ in $L^2(\mathbb{R}^n\times V)$. Eventually taking $\phi = u_0$ in the weak formulation (3.11) shows that the r.h.s. is equal to $|V|[u_0]^2_{H^s}$. We thus deduce that

$$|V|[u_0]^2_{H^s} = \|u_0\|^2 \le \liminf_{\varepsilon\to 0}\|\tilde{u}_\varepsilon\|^2 \le \limsup_{\varepsilon\to 0}\|\tilde{u}_\varepsilon\|^2 = |V|[u_0]^2_{H^s}.$$

We conclude that all the inequalities are in fact equalities. The claim follows. □

4. Correctors

In this section we need that fractional Laplacians in different variables commute, that is,

$$\Delta_y^t(\Delta_x^s b)(x,y) = \Delta_x^s(\Delta_y^t b)(x,y)$$
$$= \iint \frac{b(x,y)-b(x,y')-b(x',y)+b(x',y')}{|x-x'|^{n+2s}|y-y'|^{m+2t}}\,dx'dy'.$$

Note that here, to simplify the notation, we will neglect the integration domain.

Proposition 4.1. *If $\Delta_y^t b(x,y) = g(x,y)$ weakly, then $\Delta_y^t(\Delta_x^s b)(x,y) = \Delta_x^s g(x,y)$ weakly.*

Proof. Using a density argument we can assume that all the involved functions are smooth. For any ϕ smooth, we have for any x that

$$\iint \frac{(\Delta_x^s b(x,y)-\Delta_x^s b(x,y'))(\phi(y)-\phi(y'))}{|y-y'|^{m+2t}}\,dydy'$$
$$= \iint \frac{\phi(y)-\phi(y')}{|y-y'|^{m+2t}}\left\{\int \frac{b(x,y)-b(x',y)}{|x-x'|^{n+2s}}dx' - \int \frac{b(x,y')-b(x',y')}{|x-x'|^{n+2s}}dx'\right\}dydy'$$
$$= \int\left(\iint \frac{(b(x,y)-b(x,y'))(\phi(y)-\phi(y'))}{|y-y'|^{m+2t}}\,dy\,dy' - \iint \frac{(b(x',y)-b(x',y'))(\phi(y)-\phi(y'))}{|y-y'|^{m+2t}}\,dy\,dy'\right)\frac{dx'}{|x-x'|^{n+2s}}$$
$$= 2\int\left(\int g(x,y)\phi(y)\,dy - \int g(x',y)\phi(y)\,dy\right)\frac{dx'}{|x-x'|^{n+2s}}$$
$$= 2\int \phi(y)\int \frac{g(x,y)-g(x',y)}{|x-x'|^{n+2s}}\,dx'\,dy$$
$$= 2\int \phi(y)\Delta_x^s g(x,y)\,dy.$$

Then, we get

$$\frac{1}{2}\iint \frac{(\Delta_x^s b(x,y)-\Delta_x^s b(x,y'))(\phi(y)-\phi(y'))}{|y-y'|^{m+2t}}\,dydy' = \int \phi(y)\Delta_x^s g(x,y)\,dy$$

for any point x and smooth function ϕ, concluding the proof. □

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

We will also use a maximum principle.

Lemma 4.1. *Let $u \in H_0^{s,t}(\mathbb{R}^n \times V)$, $u \le 0$ in $(\mathbb{R}^n \backslash U) \times V$, be a weak solution to*

$$\Delta_x^s u + \Delta_y^t u \le 0 \qquad \text{in } \mathbb{R}^n \times V.$$

Then $u \le 0$ in $\mathbb{R}^n \times V$.

Proof. The proof is the same as in [24, Lemma 4.6]. We include a brief sketch for completeness.

Since $u_+ \in H_0^{s,t}(\mathbb{R}^n \times V)$, $u_+ \ge 0$, we can use it as a test-function:

$$\begin{aligned} 0 \ge & \int_{\mathbb{R}^n \times \mathbb{R}^n \times V} \frac{(u(x,y) - u(x',y))(u_+(x,y) - u_+(x',y))}{|x-x'|^{n+2s}} \, dx \, dx' \, dy \\ & + \int_{V \times V \times \mathbb{R}^n} \frac{(u(x,y) - u(x,y'))(u_+(x,y) - u_+(x,y'))}{|y-y'|^{m+2t}} \, dx \, dy \, dy'. \end{aligned}$$

Both integrals are non-negative. Indeed, let us show that for the first term. Writing $u = u_+ - u_-$ and using that $u_+(x,y)u_-(x,y) = 0$ for any (x,y), the numerator of the integrand of the first integral verifies

$$\begin{aligned} & (u(x,y) - u(x',y))(u_+(x,y) - u_+(x',y)) \\ & \quad = (u_+(x,y) - u_+(x',y))^2 + u_-(x,y)u_+(x',y) + u_-(x',y)u(x',y) \\ & \quad \ge (u_+(x,y) - u_+(x',y))^2. \end{aligned}$$

We thus obtain

$$\begin{aligned} 0 \ge & \int_{\mathbb{R}^n \times \mathbb{R}^n \times V} \frac{(u_+(x,y) - u_+(x',y))^2}{|x-x'|^{n+2s}} \, dx \, dx' \, dy \\ & + \int_{V \times V \times \mathbb{R}^n} \frac{(u_+(x,y) - u_+(x,y'))^2}{|y-y'|^{m+2t}} \, dx \, dy \, dy' \ge 0. \end{aligned}$$

It thus follows from the first integral that for any $y \in V$, the function $u_+(\cdot, y)$ is constant. Since $u_+ = 0$ in $(\mathbb{R}^n \backslash U) \times V$, we obtain that $u_+ = 0$. $\square$

Now we are ready to prove the last part of Theorem 1.1. Recall that for each $\epsilon > 0$ we consider the problem

$$\begin{cases} \Delta_x^s u(x,y) + \Delta_y^t u(x,y) = f(x, \varepsilon^{-1} y) & \text{in } U \times \varepsilon V, \\ u = 0 & \text{in } (\mathbb{R}^n \setminus U) \times \varepsilon V. \end{cases} \tag{4.1}$$

If we change variables as we did in the previous section in (4.1) we get the following problem

$$\begin{cases} \Delta_x^s u(x,y) + \dfrac{1}{\varepsilon^{2t}} \Delta_y^t u(x,y) = f(x,y) & \text{in } U \times V, \\ u = 0 & \text{in } (\mathbb{R}^n \setminus U) \times V. \end{cases} \tag{4.2}$$

For any $x \in U$ consider

$$\theta(x) = \frac{1}{|V|} \int_V f(x,y) \, dy.$$

Since $f \in L^2(U \times V)$, we have that $\theta \in L^2(U)$ and so, there exists $u_0 \in H_0^s(\mathbb{R}^n)$ solution to

$$\begin{cases} \Delta_x^s u_0(x) = \theta(x) & \text{in } U, \\ u_0 = 0 & \text{in } \mathbb{R}^n \setminus U. \end{cases}$$

Moreover, we have the following result.

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.

Proposition 4.2. *For any $x \in U$, there exists a unique weak solution $b(x,\cdot) \in H^t(V)$ to*

$$\Delta_y^t b(x,y) = f(x,y) - \theta(x) \quad \text{in } V,$$
$$\frac{1}{|V|}\int_V b(x,y)\,dy = 0.$$

Moreover, if f satisfies

$$\sup_{x\in U} \|f(x,\cdot)\|_{L^a(V)} < \infty \tag{4.3}$$

with $a > \max\{\frac{m}{2t}, 1\}$, then $b \in L^\infty(U \times V)$ with

$$\|b\|_{L^\infty(U\times V)} \le C$$

where $C > 0$ depends only on V, r, t.

In addition, if

$$\sup_{x\in\mathbb{R}^n} \|\Delta_x^s f(x,\cdot)\|_{L^a(V)} < \infty \qquad \text{for some } a > \max\left\{\frac{m}{2t}, 1\right\}, \tag{4.4}$$

then $\Delta_x^s b \in L^\infty(\mathbb{R}^n \times V)$.

Proof. Note that such a function b is well defined since we have

$$\frac{1}{|V|}\int_V [f(x,y) - \theta(x)]\,dy = 0.$$

Moreover the existence of $b(x,\cdot)$ follows by direct minimization of the associated functional using the Poincaré inequality $[u]_{H^t(V)} \ge C\|u\|_{L^2(V)}$ which holds for any $u \in H^t(V)$ such that $\int_V u = 0$.

In view of Proposition 2.5 and Remark 2.1, we have for any $x \in U$ that

$$\|b(x,\cdot)\|_{L^\infty(V)} \le K$$

with $a > \frac{m}{2t}$ and $K > 0$ depends only on V, a, t and $f(x,\cdot)$. Assumption (4.3) then shows that the r.h.s. is bounded uniformly for $x \in U$.

Now notice that $\Delta_x^s b$ is the solution of

$$\Delta_y^t \Delta_x^s b(x,y) = \Delta_x^s f(x,y) - \Delta_x^s \theta(x).$$

Here we are using that Δ_y^t and Δ_x^s commute (this fact can be easily obtained from a density argument since it holds for smooth functions using Fubini's theorem, see Proposition 4.1). We then have from Proposition 2.5 and Remark 2.1 that for any $x \in \mathbb{R}^n$, $\Delta_x^s b(x,\cdot) \in L^\infty(V)$ with

$$\|\Delta_x^s b(x,\cdot)\|_{L^\infty(V)} \le C.$$

Notice that $\Delta_x^s \theta(x) = \int_V \Delta_x^s f(x,y)\,\frac{dy}{|V|}$ so that $|\Delta_x^s \theta(x)| \le C\|\Delta_x^s f(x,\cdot)\|_{L^a(V)}$. The result follows. □

Assuming (4.3) and (4.4), we know from the previous result that there exist constants $k, K > 0$ such that for any $(x,y) \in \mathbb{R}^n \times V$,

$$|b(x,y)| \le k \qquad \text{and} \qquad |\Delta_x^s b(x,y)| \le K. \tag{4.5}$$

We also need $h \in H_0^s(\mathbb{R}^n)$ the solution to

$$\Delta_x^s h(x) = 1 \quad \text{in } U,$$
$$h = 0 \quad \text{in } \mathbb{R}^n \setminus U.$$

Notice that $h \in L^\infty(U)$ in view of Remark 2.1. We then consider the functions $\underline{v}$ and $\overline{v}$ defined in $\mathbb{R}^n \times V$ by

$$\overline{v}(x,y) = u_0(x) + \varepsilon^{2t}(b(x,y) + K\ h(x) + k),$$

and

$$\underline{v}(x,y) = u_0(x) + \varepsilon^{2t}(b(x,y) - K\ h(x) - k).$$

We claim that $\overline{v}$ and $\underline{v}$ are super-solution and sub-solution respectively of (4.2). Let us see, for instance, that $\overline{v}$ is a supersolution to (4.2). In fact, in $\mathbb{R}^n \times V$,

$$\begin{aligned} \Delta_x^s \overline{v}(x,y) + \frac{1}{\varepsilon^{2t}} \Delta_y^t \overline{v}(x,y) &= \Delta_x^s u_0(x) + \varepsilon^{2t}(\Delta_x^s b(x,y) + K) + \Delta_y^t b(x,y) \\ &= f(x,y) + \varepsilon^{2t}(\Delta_x^s b(x,y) + K) \\ &\geq f(x,t), \end{aligned}$$

by the definition of K. Moreover in $(\mathbb{R}^n \backslash U) \times V$,

$$\overline{v}(x,y) = \varepsilon^{2t}(b(x,y) + k) \geq 0$$

by the definition of k (we used here that $u_0 = h = 0$ in $\mathbb{R}^n \backslash U$).

It then follows from the weak maximum principle Lemma 4.1 that

$$\underline{v} \leq \tilde{u}_\varepsilon \leq \overline{v} \qquad \text{in } \mathbb{R}^n \times V$$

which implies exactly (1.6). This ends the proof of Theorem 1.1.

Remark 4.1. Note that we needed that $\Delta_x^s b$ is bounded. This fact is obtained from smoothness assumptions on f, in fact we assumed that $\Delta_x^s f \in L^a(\mathbb{R}^n \times V)$ with $a > \frac{n+m}{2s}$. This assumption is quite restrictive and is not needed when one looks at this problem using a variational approach like we did in the previous section using only that $f \in L^2(U \times V)$.

5. Possible extensions of our results

In this final section we comment briefly on possible extensions of our results. We can consider different problems.

1. The Neumann problem I. We consider the equation

$$f = \Delta_x^s u_\epsilon + \epsilon^{-2t} \Delta_y^t u_\epsilon,$$

given by,

$$f(x,y) = \int_U \frac{u_\epsilon(z,y) - u_\epsilon(x,y)}{|z-x|^{n+2s}} dz + \int_V \frac{u_\epsilon(x,w) - u_\epsilon(x,y)}{\epsilon^{2t}|w-y|^{m+2t}} dw. \tag{5.1}$$

In this case we are taking Neumann boundary conditions both in x and y variables and we need to impose that

$$\int_U \int_V u_\epsilon(x,y)\ dxdy = 0$$

in order to obtain uniqueness of a solution.

The computations that we made in the previous sections can be used to pass to the limit as $\epsilon \to 0$ in this problem.

2. The Neumann problem II. We can also deal with the following version of the previous Neumann problem

$$f(x,y) = \int_{U\times V} \frac{u_\epsilon(z,y) - u_\epsilon(x,y)}{|z-x|^{n+2s}} dz\, dy + \int_{U\times V} \frac{u_\epsilon(x,w) - u_\epsilon(x,y)}{\epsilon^{2t}|w-y|^{m+2t}} dw\, dy, \tag{5.2}$$

assuming again that

$$\int_U \int_V u_\epsilon(x,y)\, dx\, dy = 0.$$

3. The regional fractional Laplacian. However, when we look at the fractional Laplacian

$$\Delta^s u(x,y) = \int_{\mathbb{R}^n\times\mathbb{R}^m} \frac{u(x,y) - u(z,w)}{|(x,y)-(z,w)|^{n+m+2s}}\, dz\, dw$$

and we localize it in ϵV (to deal with a thin domain) taking

$$\Delta^s u(x,y) = \int_{\mathbb{R}^n\times\epsilon V} \frac{u(x,y) - u(z,w)}{|(x,y)-(z,w)|^{n+m+2s}}\, dz\, dw$$

it seems that our results cannot be extended to this model.

In fact, if we assume that $u(x,y)$ does not depend on y (this is the expected limit situation in which the limit is independent of the y variable) we get

$$\Delta^s u(x,y) = \Delta^s u(x) = \int_{\mathbb{R}^n} (u(x)-u(z)) \left(\int_{\varepsilon V} \frac{dw}{|(x,y)-(z,w)|^{n+m+2s}} \right) dz.$$

Then, we have to look at the limit of the kernel

$$\int_{\varepsilon V} \frac{1}{(|x-z|^2 + |y-w|^2)^{\frac{1}{2}(n+m+2s)}}\, dw.$$

Changing variables we obtain

$$\varepsilon^m \int_V \frac{1}{(|x-z|^2 + \varepsilon^2|y-w|^2)^{\frac{1}{2}(n+m+2s)}}\, dw.$$

Now if we use that V is bounded and we take polar coordinates we get that the last integral is bounded above by

$$C\varepsilon^m \int_0^R \frac{r^{m-1}}{(|x-z|^2 + \varepsilon^2 r^2)^{\frac{1}{2}(n+m+2s)}}\, dr.$$

Now, we change variables again, taking $\frac{\varepsilon}{|x-z|} r = s$, and we arrive to

$$\frac{C}{|x-z|^{n+2s}} \int_0^{\varepsilon\frac{R}{|x-z|}} \frac{s^{m-1}}{(1+s^2)^{\frac{1}{2}(n+m+2s)}}\, ds$$

that goes to zero as $\varepsilon \to 0$. Consequently, if the function u does not depend on the y variable, we have $\Delta^s u \to 0$ as $\varepsilon \to 0$.

Therefore our ideas are not applicable to handle this situation.

Acknowledgements

The first author (MCP) is partially supported by CNPq (Brazil) 302960/2014-7 and FAPESP 2015/17702-3 (Brazil) and the second author (JDR) by MINCYT (Argentina) grant MTM2016-68210 (Spain).

References

[1] J. Shuichi, Y. Morita, Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels, Comm. Part. Diff. Eq. 17 (3) (1991) 189–226.
[2] J.K. Hale, G. Raugel, Reaction–diffusion equations on thin domains, J. Math. Pures et Appl. 9 (71) (1992) 33–95.
[3] M. Prizzi, K.P. Rybakowski, Recent results on thin domain problems ii, Top. Meth. Nonlinear Anal. 19 (2002) 199–219.
[4] R. Ferreira, M.L. Mascarenhas, A. Piatnitski, Spectral analysis in thin tubes with axial heterogeneities, Portugal. Math. 72 (2015) 247–266.
[5] M.C. Pereira, R.P. Silva, Remarks on the p-Laplacian on thin domains, Progr. Nonlinear Differential Equations Appl. (2015) 389–403.
[6] S.R.M. Barros, M.C. Pereira, Semilinear elliptic equations in thin domains with reaction terms concentrating on boundary, J. Math. Anal. Appl. 441 (1) (2016) 375–392.
[7] J.M. Arrieta, M. Villanueva-Pesqueira, Thin domains with non-smooth periodic oscillatory boundaries, J. Math. Anal. Appl. 446 (2017) 130–164.
[8] N. Saintier, Asymptotics of best Sobolev constants on thin manifolds, J. Differential Equations 246 (2009) 2876–2890.
[9] R. Aris, On the dispersion of a solute in a fluid flowing through a tube, Proc. Roy. Soc. London Sect. A 235 (1956) 67–77.
[10] D. Iftimie, The 3D Navier–Stokes equations seen as a perturbation of the 2D Navier- Stokes equations, Bull. Soc. Math. France 127 (1999) 473–518.
[11] L.T. Hong, G.R. Sell, Navier–stokes equations with navier boundary conditions for an oceanic model, J. Dynam. Differential Equations 22 (3) (2010) 563–616.
[12] P. Bella, E. Feireisl, A. Novotny, Dimension reduction for compressible viscous fluids, Acta Appl. Math. 134 (2014) 111–121.
[13] J. Fabricius, Y.O. Koroleva, A. Tsandzana, P. Wall, Asymptotic behavior of Stokes flow in a thin domain with a moving rough boundary, Proc. R. Soc. A 470 (2014).
[14] M. Benes, I. Pazanin, F.J. Suárez-Grau, Heat flow through a thin cooled pipe filled with a micropolar fluid, J. Theoret. Appl. Mech. 53 (3) (2015) 569–579.
[15] X. Liao, On the strong solutions of the inhomogeneous incompressible Navier–Stokes equations in a thin domain, Differential Integral Equations 29 (2016) 167–182.
[16] M.C. Pereira, J.D. Rossi, Nonlocal problems in thin domains. Preprint.
[17] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[18] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J. Toledo, Nonlocal Diffusion Problems, in: Mathematical Surveys and Monographs, Vol. 165, AMS, 2010.
[19] C. Cortazar, M. Elgueta, J.D. Rossi, N. Wolanski, How to approximate the heat equation with neumann boundary conditions by nonlocal diffusion problems, Arch. Ration. Mech. Anal. 187 (2007) 137–156.
[20] C. Cortazar, M. Elgueta, J.D. Rossi, Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions, Israel J. Math. 170 (1) (2009) 53–60.
[21] M.C. Pereira, R.P. Silva, Correctors for the Neumann problem in thin domains with locally periodic oscillatory structure. Quart, Appl. Math. 73 (2015) 537–552.
[22] G. Raugel, Dynamics of Partial Differential Equations on Thin Domains, in: Lecture Notes in Mathematics, vol. 1609, Springer-Verlag, 1995.
[23] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and quasilinear elliptic equations. Vol. 46, Academic Press, 1968.
[24] G. Molica Bisci, V. Radulescu, R. Servadei, Variational methods for nonlocal fractional problems. With a foreword by Jean Mawhin, in: Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, ISBN: 978-1-107-11194-3, 2016, p. xvi+383.

Please cite this article as: M.C. Pereira, J.D. Rossi and N. Saintier, Fractional problems in thin domains, Nonlinear Analysis (2019), https://doi.org/10.1016/j.na.2019.02.024.