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CONSTANT CURVATURE MODELS IN SUB-RIEMANNIAN GEOMETRY

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1. Introduction

This is an announcement of results which will appear in complete form in [FVV].

A sub-riemannian manifold is a differential manifold together with a smooth distribution of planes which carries a metric. We define a canonical connection on a sub-riemannian manifold analogous to the Levi-Civita connection for riemannian manifolds. We state a classification theorem of sub-riemannian manifolds of constant sectional curvature and vanishing torsion in dimension 3. The higher dimensional classification is completed in [FVV].

2. Adapted Connections and Curvature

Definition 2.1. A *Sub-Riemann manifold* is a triple (M, D, g) where M is a manifold, D is a smooth distribution on M and g is a smoothly varying quadratic form defined on D. We will say in this case that M is *a sub-riemannian manifold of codimension k* if D is of codimension k.

We will concentrate in this work in the case of sub-riemannian manifolds of codimension 1. Let M be of dimension m+1.

The G-structure associated to (M, D, g) is given by the set of 1-forms

$$\begin{cases} \theta' &= \lambda\theta \quad \text{with } \lambda \neq 0 \text{ real} \\ \theta^{i\prime} &= a^i_j\theta^j + v^i\theta \quad \text{where } (a^i_j) \in O(m) \end{cases} \tag{1}$$

Geometrically θ, θ^i is a basis of coframes satisfying $\theta(X_i) = 0$, $\theta^j(X_i) = \delta_i^j$ with $1 \leq i, j \leq m$ where X_i is an orthonormal basis of D.

Observe that, in general, there exists an antissymetric matrix (h_{ij}) such that

$$d\theta = h_{ij}\theta^i \wedge \theta^j + h_i\theta^i \wedge \theta$$

Although we could carry on with the theory without restrictive hypothesis, we will further restrict to the simplest case.

Definition 2.2. (M, D, g) is said to be *non-degenerate* if $det(h_{ij}) \neq 0$.

As (h_{ij}) is antissymetric, we have that in the non-degenerate case m = 2n is even. Furthermore, to show that the definition does not depend on the section of the G-structure, we choose another one as in (1). Then we see that

$$\begin{array}{rcl} h_{kl} & = & \frac{1}{\lambda}h_{ij}{}'a^i_k a^j_l \\ h_k & = & \frac{1}{\lambda}(2h_{ij}{}'v^j + \lambda h_i{}')a^i_k - \frac{\lambda_k}{\lambda} \end{array} \tag{2}$$

where $\sum \lambda_i\theta^i = d\lambda$.

It is clear now that the condition $det(h_{ij}) \neq 0$ is invariant. In fact it is equivalent to the condition that $\theta \wedge (d\theta)^n \neq 0$ for a section θ. In this case the G-structure can be reduced to a remarkably simple one.

Proposition 2.1. *The G-structure associated to a Sub-Riemannian manifold can be reduced to a* $\mathbf{Z}_2 \times O(2n)$*-structure in the non-degenerate case.*

Proof: We will impose the condition $det(h_{ij}) = 1$. Using the transformations (2) we see that this fixes the section θ modulo a sign. With this particular choice of the section, the second equation in (2) becomes $h_k = (2h'_{ij}v^j + h'_i)a^i_k$. Again, in the non-degenerate case we can choose v^j uniquely such that $h_k = 0$, reducing the G-structure (1) to, ignoring the $\mathbf{Z_2}$ term

$$\left\{ \begin{array}{rcl} \theta' & = & \theta \\ \theta^{i\prime} & = & a^i_j\theta^j \quad \text{where } (a^i_j) \in O(2n) \\ d\theta & = & h_{ij}\theta^i \wedge \theta^j \quad \text{with} \quad (h_{ij} = -h_{ji} \text{ and } det(h_{ij}) = 1) \end{array} \right. \tag{3}$$

In particular, if the distribution is orientable then it has an O(2n)-structure.

Let the Sub-Riemannian manifold (M, D, g) be given, and consider the associated O(2n)-structure (3). We will construct connection forms and torsion forms. We begin by considering the intrinsically defined tautological forms over the bundle (3) which we denote by the same letters θ, θ^i.

Theorem 2.1. *There exists unique forms ω^i_j and τ^i satisfying the equation*

$$d\theta^i = \theta^j \wedge \omega^i_j + \theta \wedge \tau^i$$

with conditions i) $\omega^i_j = -\omega^j_i$ and ii) $\sum \tau^i \wedge \theta^i = 0$.

Proof: Let $\tilde{\omega}^i_j$ and $\tilde{\tau}^i$ be any forms satisfying the first equation. If ω^i_j and τ^i also satisfy the equation, then

$$\theta^j \wedge (\omega^i_j - \tilde{\omega}^i_j) + \theta \wedge (\tau^i - \tilde{\tau}^i) = 0$$

From Cartan's lemma we have

$$\omega^i_j - \tilde{\omega}^i_j = a^i_{jk}\theta^k + b^i_j\theta$$

$$\tau^i - \tilde{\tau}^i = b^i_k\theta^k$$

with $a^i_{jk} = a^i_{kj}$. We will choose a^i_{jk}, b^i_j such that the conditions in the theorem be satisfied for ω^i_j, τ^i. To verify conditions ii) we must have

$$0 = \sum \tau^i \wedge \theta^i = \sum \tilde{\tau}^i \wedge \theta^i + \sum\sum b^i_k\theta^k \wedge \theta^i$$

If we write $\tilde{\tau}^i = \tilde{\tau}^i_k\theta^k$, then

$$\sum\sum(\tilde{\tau}^i_k + b^i_k)\theta^k \wedge \theta^i = 0$$

and using Cartan's lemma again $\tilde{\tau}^i_k + b^i_k = a^i_k$ with $a^i_k = a^k_i$. On the other hand if i) is satisfied, and writing $\tilde{\omega}^i_j = \tilde{\omega}^i_{jk}\theta^k + \tilde{w}^i_j\theta$

$$(\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik})\theta^k + (\tilde{w}^i_j + \tilde{w}^j_i + b^i_j + b^j_i)\theta = 0$$

We get two equations

$$\tilde{w}^i_j + \tilde{w}^j_i + a^i_j + a^j_i - \tilde{\tau}^i_j - \tilde{\tau}^j_i = 0$$

$$\tilde{\omega}^i_{jk} + \tilde{\omega}^j_{ik} + a^i_{jk} + a^j_{ik} = 0$$

The first equation, recalling that a^i_j is symmetric, has solution $a^i_j = \frac{\tilde{\tau}^i_j + \tilde{\tau}^j_i}{2} - \frac{\tilde{\omega}^i_j + \tilde{\omega}^j_i}{2}$ therefore b^i_j is determined. The second equation can be solved using the permutation trick, as in Riemannian geometry. □

The curvature forms are defined by

$$\Pi^i_k = d\omega^i_k + \omega^i_j \wedge \omega^j_k$$

We can state now the following theorem, see [FVV]

Theorem 2.2. *The curvature forms are given by*

$$\Pi^i_k = \frac{1}{2} R^i_{krs} \theta^r \wedge \theta^s + W^i_{ks} \theta^s \wedge \theta + h_{kl} \theta^l \wedge \tau^i - h_{il} \theta^l \wedge \tau^k$$

with the conditions $R^i_{krs} = -R^k_{irs}$, $R^i_{krs} = -R^i_{ksr}$, $R^i_{ksr} + R^i_{rks} + R^i_{srk} = 0$, $W^i_{ks} = -W^k_{is}$ *and* $W^i_{ks} + W^s_{ik} + W^k_{si} = 0$.

An important observation from this theorem is that the vanishing of torsion implies the vanishing of the curvature tensor W^i_{ks}.

In computations, we use a section of the the G-structure, that is, a moving frame which we write by the same letters as the tautological forms $\Theta^T = (\theta^1, \ldots, \theta^{2n})$ and $\tau^T = (\tau^1, \ldots, \tau^{2n})$. The structure equation is written also as $d\Theta = -\omega \wedge \Theta - \tau \wedge \theta$. If $\Theta' = g\Theta$ is a new moving frame, then

$$\begin{aligned} \omega' &= gdg^{-1} + g\omega g^{-1} \\ \tau' &= g\tau \end{aligned}$$

3. Constant Curvature in dimension 3

Let M be a manifold of dimension 3 with a metric distribution. The adapted bundle is

$$\begin{cases} \theta' &= \theta \\ \theta^{i'} &= a^i_j \theta^j \quad \text{where } (a^i_j) \in O(2) \\ d\theta &= \theta^1 \wedge \theta^2 - \theta^2 \wedge \theta^1 \end{cases}$$

Theorem 3.1 gives a connection form $\omega_2^1 = \omega$ and torsion forms τ^1, τ^2 such that

$$\begin{cases} d\theta^1 &= \quad\quad\quad \theta^2 \wedge \omega \quad + \theta \wedge \tau^1 \\ d\theta^2 &= -\theta^1 \wedge \omega \quad\quad\quad + \theta \wedge \tau^2 \end{cases}$$

with $\tau^1 \wedge \theta^1 + \tau^2 \wedge \theta^2 = 0$. The curvature forms are

$$\Pi = \Omega = \Omega_2^1 = d\omega$$

$$\Omega^1 = d\tau^1 - \tau^2 \wedge \omega$$

$$\Omega^2 = d\tau^2 - \tau^1 \wedge \omega$$

we write then

$$\Pi = \Omega = R\theta^1 \wedge \theta^2 + W_1\theta^1 \wedge \theta + W_2\theta^2 \wedge \theta$$

The metric defined by $\theta^1, \theta^2, \theta$ is called an adapted metric to the contact form θ on M. We will now make the computations for three examples of constant curvature. See [CH].

1) Consider the sphere $S^3 = \{(x_1, y_1, x_2, y_2) \in \mathbf{R}^4 \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}$. Define then

$$\begin{aligned} \theta^1 &= x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 \\ \theta^2 &= x_1dx_2 - x_2dx_1 + y_1dy_2 - y_2dy_1 \\ \theta &= x_1dy_2 - y_2dx_1 + y_1dx_2 - x_2dy_1 \end{aligned}$$

Let the distribution D be defined by the form θ. The metric induced in this distribution by the standard metric in $\mathbf{R}^4$ defines a metric distribution. We consider θ^1, θ^2 as an orthonormal coframe and verify that

$$d\theta = 2\theta^1 \wedge \theta^2$$

and cyclic permutations. We conclude that $\omega = 2\theta$ and $\tau^1 = \tau^2 = 0$. Also $\Pi = 4\theta^1 \wedge \theta^2$, therefore $R = 4$ and $W_1 = W_2 = 0$.

2) Let S be a compact orientable surface of genus different from 1. We suppose, furthermore, that a metric is given in S of curvature K. Let M denote the three dimensinal manifold of orthonormal coframes over S. Let θ^1, θ^2

$$\begin{aligned} d\theta^1 &= -\theta^2 \wedge \theta \\ d\theta^2 &= \theta^1 \wedge \theta \end{aligned}$$

Also $d\theta = -K\theta^1 \wedge \theta^2$. Suppose now that K = -1. Let D be the distribution on M defined by θ. The metric defined on D by the basis of forms has connection $\omega = -\theta$ and $\tau^1 = \tau^2 = 0$. We have $\Pi = K\theta^1 \wedge \theta^2$, therefore $R = -1$ and $W^1 = W^2 = 0$.

3) Let $H = \{(x,y,u,v) \in \mathbf{R}^4 \mid x^2 + y^2 - v = 0\}$ be the Heisenberg group. We define the metric distribution imposing that the distribution is generated by the orthonormal vector fields X_1, X_2.

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v} \\ X_2 &= \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v} \\ X &= 2\frac{\partial}{\partial u} \end{aligned}$$

The dual coframe satisfies

$$\begin{aligned} \theta^1 &= dx \\ \theta^2 &= dy \\ \theta &= du/2 - ydx + xdy \end{aligned}$$

Therefore $d\theta = 2\theta^1 \wedge \theta^2$. We compute then $\omega = \tau^1 = \tau^2 = 0$.

We can state now the following theorem of local classification for three dimensional metric distribution. A classification of higher dimensional metric distributions of null torsion and constant curvature is more involveld because it depends on the form h_{ij} and it is given in [FVV].

Theorem 3.1. *Suppose M and M' are two three dimensional metric distributions with null torsion and the same constant sectional curvature, then they are locally equivalent.*

Observe that this proves that the three dimensional examples presented above are, locally, the only ones of constant curvature and vanishing torsion. The simply connected forms are given by S^3, H and the universal cover of the unit bundle over the Poincaré disc with distribution and metric given as in the examples above.

Remark: As for 2-dimensional surfaces, a metric distribution in a manifold of dimension 3, defines a CR-structure. Conversely, a CR-structure defines a class of conformally related metrics on a distribution.

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