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non-interpretability of a
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Ricardo Bianconi *

IMEUSP-Caixa Postal 20570-CEP 01452-990-São Paulo-SP-Brazil

e-mail: bianconi@ime.usp.br

Abstract

We prove the following theorem:

Theorem Let M be an o-minimal structure whose underlying order is dense and whose definable (n -ary) equivalence relations E are given by $x E y$ if, and only if, $f_E(x) = f_E(y)$, where $f_E : M^n \rightarrow M^n$ is a definable function. Then $\text{Th}(M)$ does not interpret the theory of a discrete ordered structure.

We also discuss some future work related to the lattice of chapters of mathematics.

1 Introduction

In [7], Świerczkowski proves that $\text{Th}((\omega, <))$ is not interpretable (with parameters) in RCF (the theory of real closed fields) by showing that a pre-ordering with successors is not definable in \mathbf{R} .

(We recall that a *pre-ordering with successors* is a reflexive and transitive binary relation \ll , satisfying $\forall x \forall y (x \ll y \vee y \ll x)$ and $\text{Succ}(x, y)$, where

$$\text{Succ}(x, y) \iff x \ll y \wedge x \not\ll y \wedge \forall z (x \ll z \ll y \rightarrow z \approx x \wedge z \approx y),$$

where $x \approx y$ means $x \ll y \wedge y \ll x$.)

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The proof uses certain properties of RCF, such as o-minimality and definable Skolem functions. Analysing the proofs of the lemmas and the main theorem of [7], which work in the reals (as a model of RCF), we can see that those results are still true in the more general setting of o-minimal densely ordered structures having definable Skolem functions.

The purpose of this note is to show this generality, that is:

Theorem 1 *Let M be an o-minimal structure whose underlying order is dense and whose definable (n -ary) equivalence relations E are given by $x E y$ if, and only if, $f_E(x) = f_E(y)$, where $f_E : M^n \rightarrow M^n$ is a definable function. Then $\text{Th}(M)$ does not interpret the theory of a discrete ordered structure.*

We follow very closely the proofs of [7], but depart from it when specific properties of the reals (such as elimination of quantifiers and the Baire Category Theorem) are used.

We assume familiarity with the notions of interpretability, definability and o-minimality. The reader should consult [3], [5], [1] and [6].

2 Proof of the main result

2.1 Suppose that M is an o-minimal structure whose underlying order is dense.

Lemma 2.1 *Let $L_0 \subset L_1 \subset \dots$ be a strictly increasing sequence of subsets of M such that the boundary ∂L_n of each L_n have at most K elements, for a fixed positive integer K . Then the difference $L_{i+1} \setminus L_i$ is infinite for infinitely many i .*

Proof: It is exactly the same as the proof of Lemma 3.1 of [7]. □

Lemma 2.2 *There does not exist a definable pre-ordering of M for which there is an infinite sequence x_0, x_1, \dots such that every x_{i+1} is an immediate successor of x_i .*

Proof: The proof is quite close to the proof of Lemma 3.2 in [7], where the author uses quantifier elimination of RCF. Here we replace its use by the Theorem 0.3(a) of [1] to obtain the required bound on the number of points of the boundaries of the sets L_n (see below).

Suppose that \ll is a definable pre-ordering of M , and $(x_i)_{i < \omega}$ is a sequence such that $M \models \text{Succ}(x_i, x_{i+1})$, for each i . Let $L_i = \{x \in M : x \ll x_i\}$, $i < \omega$. These sets are definable by a formula $\Lambda(x, x_i)$, whose parameters include the x_i 's. Let \approx denote the equivalence relation associated to \ll . By the o-minimality of M (using Theorem 0.3 (a) of [1]), there is a $K < \omega$ such that $|\partial L_n| \leq K$, for all $n < \omega$. Therefore, lemma 2.1 applies. This means that there are infinitely many infinite \approx -classes in M , a contradiction to o-minimality. \square

2.2 To finish the proof of Theorem 1.1 of [7], the author uses Baire Category Theorem to obtain a projection of a pre-ordering from \mathbb{R}^d to \mathbb{R} , with successors. Here we depart from this paper, substituting this argument by a more combinatorial one. But we still keep the idea of projecting the pre-ordering.

Now assume that M is a densely (totally) ordered structure, whose definable (n -ary) equivalence relations E are given by $x E y$ if, and only if, $f_E(x) = f_E(y)$, where $f_E : M^n \rightarrow M^n$ is a definable function.

Lemma 2.3 *If there is a definable pre-ordering of M^d , $d \geq 1$, for which there is an infinite sequence a_0, a_1, \dots such that every a_{i+1} is an immediate successor of a_i . Then there is a definable pre-ordering of M for which there is an infinite sequence b_0, b_1, \dots such that every b_{i+1} is an immediate successor of b_i .*

Proof: We do this by induction in $d \geq 1$, the case of $d = 1$ being trivial.

Now, suppose that the result is true for $d - 1 \geq 1$.

Let \ll be a definable pre-ordering in M^d and let a_0, a_1, \dots be a sequence in M^d such that a_{i+1} is an immediate successor of a_i , under \ll . Let f be the definable function associated to the equivalence relation $x \approx y$ if $x \ll y \wedge y \ll x$.

For simplicity we may assume that such f selects one representative of each \approx -class. Put $X = f(M^d)$. Observe that \ll induces a total order on X , by $f(x) \leq_X f(y)$ if $x \ll y$. Let $\Psi(\vec{x})$ be the formula $\exists \vec{y} (\text{Succ}(x, y) \vee \text{Succ}(y, x))$. This formula selects the (classes of) elements which have an immediate successor or predecessor under \ll . By choosing equivalent elements, we may suppose that $f(a_i) = a_i$, for each $i \geq 0$.

We define another auxiliary pre-order \ll' in M^d by

$$x \ll' y \iff ("x, y \in X" \wedge x \leq_X y \wedge \Psi(x) \wedge \Psi(y) \wedge a_0 \ll x) \vee$$

$$("x \in X" \wedge \neg \Psi(x)) \vee ("x \notin X") \vee \neg(a_0 \ll x).$$

This relation coincides with \ll on the elements of X having immediate successor or predecessor and which are \ll -greater than a_0 . These constitute the singleton classes under the associate equivalence relation, (which will be denoted by \approx'). Apart from these one-element classes there is possibly only another class, containing the remaining elements of M^d . This class is pre-ordered below all the others.

Observe that under this new pre-ordering, the element a_0 is put in the first or the second place, then there follow the other a_i and on the top of the sequence (a_i) there appear the remaining \approx' -classes.

We will use \ll' to define a pre-ordering in M .

To do this, let f' be the definable function selecting representatives of the \approx' -classes, and let $\pi : M^d \rightarrow M$ be the projection onto the first coordinate. For each $a \in M$, let $S_a = f'(M^d) \cap \pi^{-1}(a)$. It is easy to write a formula $\Sigma(x, t)$ such that $\Sigma(x, a)$ defines S_a in M^d , for each $a \in M$.

If there is $a \in M$ such that S_a contains a subsequence (a_{i_k}) of the sequence $(a_i)_{i \geq 0}$, then the restriction of \ll' to $\pi^{-1}(a)$ gives a definable pre-ordering \ll'' in M^{d-1} such that the corresponding elements a_{i_k} satisfy $a_{i_{k+1}}$ is an immediate successor of a_{i_k} . By the induction hypothesis, from this pre-ordering we can define a pre-ordering with successors in M , and we are done.

Otherwise, all S_a have finitely many (or none) elements of the sequence (a_i) . Let $L(x, t)$ be the formula $\forall y(\Sigma(y, t) \rightarrow x \ll' y)$. For each $a \in M$, if S_a contains an element of the sequence (a_i) then $L(x, a)$ selects the \ll' -least x in S_a . Such element exists because below any a_i there are only finitely many \approx' -classes under \ll' , and all but possibly one come from an element of this sequence. Let $(a'_i)_{i \geq 0}$ denote the subsequence of (a_i) selected in this way. Let $b_i \in M$ be such that $a'_i \in S_{b_i}$.

We define the relation \ll_1 in M by

$$a \ll_1 b \iff \exists x, y(L(x, a) \wedge L(y, b) \wedge x \ll' y) \vee \forall x(\neg L(x, a)).$$

This is more or less the projection on M of \ll' . It says that either a is the projection of the \ll' -least x in S_a , if there is such a x , and for such a 's we copy the pre-ordering \ll' ; or there is not such an x in S_a and this a is put in the minimal class of \ll_1 .

Observe that under this new pre-ordering, the element b_0 is put in the first or the second place (depending on having or not having a minimal class below it), then there follow the other b_i and on the top of the sequence (b_i) there appear the remaining classes.

Observe also that the sequence $(b_i)_{i \geq 0}$ satisfies b_{i+1} is the immediate successor of b_i under \ll_1 .

This proves the lemma. □

2.3 Proof of Main Theorem

(Compare with the proof of Theorem 1.1 of [7].) It is enough to prove that if α is a sentence which proves the existence of a pre-ordering with successors is not interpretable in $\text{Th}(M)$.

Putting Lemmas 2.2 and 2.3 together we have a contradiction which proves the theorem. □

3 Discussion

Here we want to discuss some problems which arise from this work.

3.1 An immediate consequence of Theorem 1 is that if T is an o-minimal theory of a dense order which can be interpreted in an o-minimal theory of a dense order and with definable Skolem functions, then T does not interpret any discrete order. This suggests the following questions.

Open problem 1 Can an o-minimal theory be expanded to one with definable Skolem functions?

This question seems likely to have a negative answer since $\text{Th}(\langle \omega < \rangle)$ has no o-minimal expansion by a non-definable function or relation (see [4]) although it has definable Skolem functions. The following question seems to be more likely to be true.

Open problem 2 Can an o-minimal theory be interpreted in an o-minimal theory with definable Skolem functions?

3.2 The lattice of chapters

The relation $T_1 \leq T_2$ defined by T_1 is interpretable in T_2 defines an equivalence relation $T_1 \approx T_2$ in the class of first-order theories. Each equivalent class is called *chapter of mathematics* (see [3]). We denote by $|T|$ the equivalence class of the theory T .

In [2], it is analysed the meet and the join of the classes (of local interpretability) of theories of order. In our context, we can pose the following problem.

Open problem 3 Describe the meet and the join of the classes of two o-minimal theories.

We believe that the techniques of [5], [1] and [6] will be useful to deal with this question.

A partial result in this direction is the following. Let M be an o-minimal L -structure, for some first order language L . Let $c(M) = \{x \in M : \exists a, b \in M (a < x < b \text{ and the open interval } (a, b) \text{ in } M \text{ is a dense linear order without endpoints})\}$ and let $d(M) = M \setminus c(M)$. These sets are L -definable in M . Let $\gamma(x)$ define $c(M)$ and $\delta(y)$ define $d(M)$. Let $T = \text{Th}(M)$, $T_c = \text{Th}(c(M))$ and $T_d = \text{Th}(d(M))$.

Lemma 3.1 *The join of $|T_c|$ and $|T_d|$ is $|T|$, i.e. $T_c \leq T$, $T_d \leq T$, and for all T' such that $T_c \leq T'$, $T_d \leq T'$, we have $T \leq T'$.*

Proof: Clearly we can interpret T_c and T_d in T . Suppose that the L' -theory T' can interpret each of T_c and T_d . We want to show that T' also interprets T . By introducing dummy variables if necessary, we can suppose that both T_c and T_d have an n -dimensional interpretation (I_c and I_d , respectively) in T' . We will make the “union” of these two interpretations.

Suppose that the equality symbol is interpreted by the equivalence relations E_c and E_d (under I_c and I_d respectively) on n -tuples. We define the new interpretation of the equality symbol by the following equivalence relation on $(n+1)$ -tuples,

$$E((\vec{x}, t), (\vec{y}, s)) \iff (t = s = a \wedge E_c(\vec{x}, \vec{y})) \vee (t = s \neq a \wedge E_d(\vec{x}, \vec{y})),$$

where a is a parameter (to be chosen in the $M \models T'$ intended to interpret those theories). This parameter is used to tag the n -tuples intended to interpret the continuous (or dense) part of T . The other symbols of L are interpreted analogously.

If $\phi(\vec{x}, \vec{y})$ is an L -formula then, by [6] Proposition 2.3, there is a Boolean combination $\phi_1(\vec{x}, \vec{y})$ of L -formulas $\psi(\vec{x})$ and $\chi(\vec{y})$ such that T proves the equivalence

$$\phi(\vec{x}, \vec{y}) \wedge \gamma(\vec{x}) \wedge \delta(\vec{y}) \leftrightarrow \phi_1(\vec{x}, \vec{y}) \wedge \gamma(\vec{x}) \wedge \delta(\vec{y}),$$

where γ and δ are the formulas defining $c(M)$ and $d(M)$ as above, and $\gamma(\vec{x})$ represents the conjunction of $\gamma(x_i)$ for all coordinates x_i of \vec{x} , and similarly for $\delta(\vec{y})$.

Since T also proves

$$\phi(\vec{w}) \leftrightarrow \bigvee_{\substack{\vec{w} \in c \\ \vec{w} = (\vec{x}, \vec{y})}} \phi(\vec{x}, \vec{y}) \wedge \gamma(\vec{x}) \wedge \delta(\vec{y}),$$

we can interpret T in T' by mapping separately the dense and the discrete parts of T . \square

To study the meet of these theories, we need to have more detailed information about them.

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