Bull Braz Math Soc, New Series (2025) 56:5
https://doi.org/10.1007/s00574-024-00429-0

Dynamics of the Korteweg–de Vries Equation on a Balanced Metric Graph

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Received: 8 February 2024 / Accepted: 4 November 2024 / Published online: 7 December 2024

Abstract

In this work, we establish local well-posedness for the Korteweg-de Vries model on a balanced star graph with a structure represented by semi-infinite edges, by considering a boundary condition of δ-type at the unique graph-vertex. Additionally, we extend the linear instability result in Angulo and Cavalcante (Nonlinearity 34:3373–3410, 2021) to one of nonlinear instability. For the proof of local well posedness theory, the principal new ingredient is the utilization of the special solutions by Faminskii in the context of half-lines. As far as we are aware, this approach is being used for the first time in the context of star graphs and can potentially be applied to other boundary classes. In the case of the nonlinear instability result, the principal ingredients are the linearized instability known result and the fact that data-to-solution map determined by the local theory is at least of class C^2.

Keywords Korteweg–de Vries model · Star graph · Bumps · δ-type · Nonlinear instability

Mathematics Subject Classification Primary 35Q51 · 35Q53 · 35J61; Secondary 47E05

In memoriam: Rafael José Iório Junior (1947–2023).

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1 Introduction

The focus of our study to follow will be the well-known Korteweg-de Vries equation (KdV henceforth)

$$\partial_t u + \partial_x^3 u + \partial_x u + 2u\partial_x u = 0 \tag{1.1}$$

on a metric graph of balanced type (see Fig. 2 below). We recall that an evolution model on a metric graph is equivalent to a system of PDEs defined on the edges (intervals) with a coupling given exclusively through the boundary conditions at the vertices (known as the "topology of the graph") and which will determine the dynamic on the network or graph. Moreover, the freedom in setting the topology in a graph allows us to create different dynamics much closer to the real world applications.

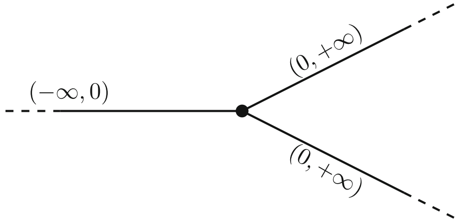
We recall that the KdV was first derived by Korteweg and de Vries (1895) as a model for long waves propagating on a shallow water surface as well as provided an explanation of the existence of *the Great Wave of Translation* a phenomena first discovered by Russell (1844). Actually, this type of solutions are known as solitary wave solutions or soliton profiles (Boussinesq 1877; Zabusky and Kruskal 1965). The KdV equation also arise naturally in the modeling of various types of wave phenomena in other physical context, such as, the nonlinear mass-spring system (FPU recurrence phenomenon), ion-acoustic waves in a collisionless plasma and magnetosonic waves in a magnetized plasma (see Ablowitz and Clarkson 1991 and the references therein). In particular, in 1987, Sigeo (1987) took the lead in combining nonlinear science with hemodynamics and derived the KdV equation for the velocity of blood flow. Also, in Crépeau and Sorine (2007) the KdV equation has been used as a model to study blood pressure waves in large arteries.

Because of the range of its potential application, dynamics of the KdV equation from a mathematical context have been well studied in the last decades and a larger quantity of manuscripts have been published. These studies have generally concentrated on aspects of the pure initial-value problem, namely, $u = u(x, t)$, $x \in \mathbb{R}$ and $t \geqq 0$, satisfying (1.1) with the initial condition $u(x, 0) = f(x)$. Thus, the Cauchy problem for the KdV posed on the real axis, on the torus, on half-lines and on a finite interval has been studied comprehensively (see Bona et al. 2008; Colliander and Kenig 2002; Faminskii 2004; Holmer 2006; Jia et al. 2004; Kenig et al. 1993; Kishimoto 2009; Killip and Visan 2019 and the references therein). Also, closely related to these studies is the large-time asymptotic behavior of solutions of the KdV equation close to localized coherent structures such as the solitary wave solutions, namely, solutions of the KdV equation with the traveling-wave profile $u_s(x, t) = \phi_c(x - ct)$, $c > 1$, where the profile ϕ_c is given by

$$\phi_c(\xi) = \frac{3}{2}(c-1)\,\mathrm{sech}^2\left(\frac{\sqrt{c-1}}{2}\xi\right), \quad \xi \in \mathbb{R}. \tag{1.2}$$

The principal study associated with the wave profiles ϕ_c is the so-called *stability in shape or orbital stability*, namely, a slight perturbation of the profile ϕ_c will continue to resemble a solitary wave over the time, rather than evolving into some other wave

Fig. 1 A $\mathcal{Y}$-junction graph



form (see Boussinesq 1877; Benjamin 1972; Bona et al. 1987; Pego and Weinstein 1994. See also Angulo 2009 for a comprehensive description of these results).

Studies for the KdV equation on networks or branched structures have drawn attention in recent years. In Ammari and Crépeau (2018) (see also Cerpa et al. 2020), a control theory for the KdV equation on a finite star-shaped network was established, which is in connection with the mathematical modeling of the human cardiovascular system. Namely, they considered a system formed by N-KdV equations in variable u_j in (1.1) posed on bounded intervals $(0, L_j)$, $j = 1, ..., N$, and with specific boundary conditions at the vertex-graph $\nu = 0$ and on the external nodes L_j. Moreover, recently in Cavalcante (2018), the local well-posedness problem for the KdV equation in Sobolev spaces $H^s(\mathcal{Y})$ with low regularity was studied on a $\mathcal{Y}$-junction graph with three semi-infinite edges, given by one negative half-line and two positive half-lines attached to a common vertex $\nu = 0$ (Fig. 1).

Very recently, Mugnolo et al. (2018) obtained a characterization of all boundary conditions under which the Airy-type evolution equation

$$\partial_t u_{\mathbf{e}}(x,t) = \alpha_{\mathbf{e}} \partial_x^3 u_{\mathbf{e}}(x,t) + \beta_{\mathbf{e}} \partial_x u_{\mathbf{e}}(x,t), \quad t \in \mathbb{R}, \; x = x_{\mathbf{e}} \in \mathbf{e}, \; \mathbf{e} \in \mathbf{E}, \tag{1.3}$$

generates either a semigroup or a unitary group on a metric star graph $\mathcal{G}$. Here $\mathcal{G}$ is a structure represented by the set $\mathbf{E} \equiv \mathbf{E}_- \cup \mathbf{E}_+$ where $\mathbf{E}_+$ and $\mathbf{E}_-$ are finite or countable collections of semi-infinite edges $\mathbf{e}$ parametrized by $(-\infty, 0)$ or $(0, +\infty)$, respectively. The half-lines are connected at a unique vertex $\nu = 0$. In (1.3) we are using $u_{\mathbf{e}}(x,t)$ in the sense that $u_{\mathbf{e}} : I \times [0, T] \to \mathbb{R}$ for $x = x_{\mathbf{e}} \in I$ and I is the half-line determined by $\mathbf{e}$, and by abusing notation, we are using $x_{\mathbf{e}} \in \mathbf{e}$. Here $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ and $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ are two sequences of real numbers.

Thus, one of the objectives of this work is to shed light on the local well-posedness problem for the KdV equation on a metric graph of balanced type, in other words, we are interested in the case of the Airy operator

$$A_0 : (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \mapsto \left(\alpha_{\mathbf{e}} \frac{d}{dx^3} u_{\mathbf{e}} + \beta_{\mathbf{e}} \frac{d}{dx} u_{\mathbf{e}} \right)_{\mathbf{e}\in\mathbf{E}} \tag{1.4}$$

defined on a metric graph $\mathcal{G}$ when the ingoing half-lines equal to outgoing half-lines ($|\mathbf{E}_+| = |\mathbf{E}_-| = n$, see Fig. 2). Thus, from Mugnolo et al. (2018) there are many possible skew-symmetric extensions A_{ext} of A_0 and so by Stone's Theorem the solution of the following linear equation

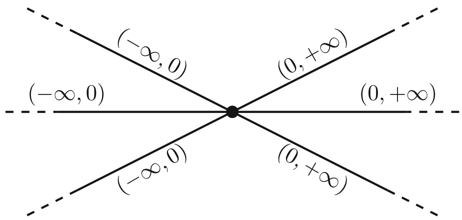


Fig. 2 A balanced star graph with 6 edges

$$\begin{cases} z_t = A_{ext} z, & t \in \mathbb{R} \\ z(0) = z_0 \in D(A_{ext}) \end{cases} \tag{1.5}$$

will be given by a C_0-unitary group $z(t) = e^{tA_{ext}} z_0$.

In the following, we define the pair $(A_{ext}, D(A_{ext}))$ which will be of interest here. By convenience of the reader, we begin with some basic notation. For $\mathbf{u} = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$, we denote $\mathbf{u}(0-)$ and $\mathbf{u}(0+)$ as

$$\mathbf{u}(0-) = (u_{\mathbf{e}}(0-))_{\mathbf{e}\in\mathbf{E}_-} \text{ and } \mathbf{u}(0+) \equiv (u_{\mathbf{e}}(0+))_{\mathbf{e}\in\mathbf{E}_+}.$$

Similarly, we define $\mathbf{u}'(0-)$, $\mathbf{u}'(0+)$, $\mathbf{u}''(0-)$, and $\mathbf{u}''(0+)$. Therefore, for $(A_0, D(A_0))$ with

$$D(A_0) \equiv \bigoplus_{\mathbf{e}\in\mathbf{E}_-} C_c^\infty(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} C_c^\infty(0, +\infty),$$

iA_0 is a densely defined symmetric operator on the Hilbert space $L^2(\mathcal{G})$, and for $Z \in \mathbb{R} - \{0\}$ we obtain a family $(A_Z, D(A_Z))$ of skew-self-adjoint extension of $(A_0, D(A_0))$ parametrized by Z, where

$$\begin{cases} A_Z \mathbf{u} &= \left(\alpha_{\mathbf{e}} \frac{d^3}{dx^3} u_{\mathbf{e}} + \beta_{\mathbf{e}} \frac{d}{dx} u_{\mathbf{e}}\right)_{\mathbf{e}\in\mathbf{E}}, \qquad \mathbf{u} = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \\ D(A_Z) = \{\mathbf{u} \in H^3(\mathcal{G}) : \mathbf{u}(0-) = \mathbf{u}(0+), & \mathbf{u}'(0+) - \mathbf{u}'(0-) = Z\mathbf{u}(0-), \\ \qquad \frac{Z^2}{2}\mathbf{u}(0-) + Z\mathbf{u}'(0-) = \mathbf{u}''(0+) - \mathbf{u}''(0-)\}, \end{cases} \tag{1.6}$$

with

$$H^3(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} H^3(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} H^3(0, +\infty).$$

Moreover, for $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\alpha_-, \alpha_+)$ ($\alpha_- = (\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_-}$, $\alpha_+ = (\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}_+}$) and $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\beta_-, \beta_+)$ we need to have $\alpha_- = \alpha_+$ and $\beta_- = \beta_+$. In particular, the system of boundary condition at the vertex $\nu = 0$ of the graph $\mathcal{G}$,

$$\mathbf{u}(0-) = \mathbf{u}(0+) \quad \text{and} \quad \mathbf{u}'(0+) - \mathbf{u}'(0-) = Z\mathbf{u}(0-)$$

are called δ*-interaction type on each two oriented half-lines*. A formula for the unitary-group generated by $(A_Z, D(A_Z))$ was obtained in Angulo and Cavalcante (2021). We recall that as iA_0 is a symmetric operator with deficiency indices $(3n, 3n)$, it follows from the classical von Neumann-Krein extension theory that the operator $(A_0, D(A_0))$ will admit a $9n^2$-parameter family of skew-self-adjoint extensions generating each one a unitary dynamics on $L^2(\mathcal{G})$ associated to the linear problem (1.5). Our interest in the operator A_Z defined above is due to the study of the non-linear instability of specific stationary solutions for the KdV with a profile in the domain $D(A_Z)$ (see Angulo and Cavalcante 2021).

The discussion is now turning more directly to the contributions in the present manuscript. We begin with the Cauchy problem. We note first that ‘a space-time function $\mathbf{u}(x,t)$ on $\mathcal{G}\times[0,T]$ will be denoted as $\mathbf{u}(x,t) = (u_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}}$. Thus, we are interested in the local well-posedness theory for the following Cauchy problem for the KdV model on $\mathcal{G}$,

$$\begin{cases}\partial_t u_{\mathbf{e}}(x,t) = \alpha_{\mathbf{e}}\partial_x^3 u_{\mathbf{e}}(x,t) + \beta_{\mathbf{e}}\partial_x u_{\mathbf{e}}(x,t) + 2u_{\mathbf{e}}(x,t)\partial_x u_{\mathbf{e}}(x,t), \quad \mathbf{e}\in\mathbf{E},\ t\in(0,T)\\ \mathbf{u}(x,0) = \mathbf{u}_0(x)\in H^s(\mathcal{G})\cap\mathcal{N}_{0,Z}(s),\end{cases} \tag{1.7}$$

with $s \geqq 1$, and $\mathbf{u}$ satisfying,

$$\begin{cases}\mathbf{u}(0-,t) = \mathbf{u}(0+,t), \quad \text{in the sense of } H^{\frac{s+1}{3}}(0,T);\\ \mathbf{u}'(0+,t) - \mathbf{u}'(0-,t) = Z\mathbf{u}(0-,t), \quad \text{in the sense of } H^{\frac{s}{3}}(0,T);\\ \frac{Z^2}{2}\mathbf{u}(0-,t) + Z\mathbf{u}'(0-,t) = \mathbf{u}''(0+,t) - \mathbf{u}''(0-,t), \quad \text{in the sense of} H^{\frac{s-1}{3}}(0,T),\end{cases} \tag{1.8}$$

moreover the set $\mathcal{N}_{0,Z}(s)$ determines the following compatibility conditions on the vertex $\nu = 0$,

$$\mathcal{N}_{0,Z}(s) = \begin{cases}\{\mathbf{v}:\mathcal{G}\to\mathbb{R}:\mathbf{v}(0-) = \mathbf{v}(0+)\}, & \text{if } s\in(\frac{1}{2},\frac{3}{2})\\ \{\mathbf{v}:\mathcal{G}\to\mathbb{R}:\mathbf{v}(0-) = \mathbf{v}(0+),\ \mathbf{v}'(0+) - \mathbf{v}'(0-) = Z\mathbf{v}(0-)\}, & \text{if } s\in(\frac{3}{2},\frac{5}{2}]\\ \{\mathbf{v}:\mathcal{G}\to\mathbb{R}:\mathbf{v}(0-) = \mathbf{v}(0+),\ \mathbf{v}'(0+) - \mathbf{v}'(0-) = Z\mathbf{v}(0-), & \\ \frac{Z^2}{2}\mathbf{v}(0-) + Z\mathbf{v}'(0-) = \mathbf{v}''(0+) - \mathbf{v}''(0-)\}, & \text{if } s > \frac{5}{2}\end{cases} \tag{1.9}$$

We note by (1.6), that $D(A_Z)\subset\mathcal{N}_{0,Z}(s)$, for all $s\geqq 1$.

For solving the Cauchy problem (1.7)–(1.8) we will consider the following functional space $X_{s,b,\beta,\sigma}(T)$, $T>0$, $s\geqq 1$,

$$\begin{aligned}X_{s,b,\beta,\sigma}(T) = \Big\{ &\mathbf{w}:\mathcal{G}\times[0,T]\to\mathbb{R}:\mathbf{w}\in C\left([0,T];H^s(\mathcal{G})\right)\\ &\cap C\left(\mathcal{G};H^{\frac{s+1}{3}}([0,T])\right)\cap\mathcal{X}^{s,b,\beta,\sigma},\\ &\partial_x\mathbf{w}\in C\left(\mathcal{G};H^{\frac{s}{3}}([0,T])\right),\partial_x^2\mathbf{w}\in C\left(\mathcal{G};H^{\frac{s-1}{3}}([0,T])\right)\Big\},\end{aligned} \tag{1.10}$$

with $\mathcal{X}^{s,b,\beta,\sigma}$ being the local Bourgain's space in (2.4), and for $r \geqq 0$ we define $C\left(\mathcal{G}; H^r([0,T])\right)$ as

$$C\left(\mathcal{G}; H^r([0,T])\right) = \{\mathbf{u} : \mathcal{G} \times [0,T] \to \mathbb{R} : \text{ for } \mathbf{u} = (u_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} \text{ we have for each fixed } x \text{ and } \mathbf{e} \in \mathbf{E},\ u_{\mathbf{e}}(x,\cdot) \in H^r([0,T])\}, \tag{1.11}$$

with the norm given by

$$\|\mathbf{u}\|_{C(\mathcal{G};H^r([0,T]))} = \sum_{\mathbf{e}\in\mathbf{E}^+} \|u_{\mathbf{e}}\|_{C(\mathbb{R}^+;H^s(0,T))} + \sum_{\mathbf{e}\in\mathbf{E}^-} \|u_{\mathbf{e}}\|_{C(\mathbb{R}^-;H^s(0,T))}.$$

Remark 1.1 In the spirit of Faminskii works (Faminskii 2004, 2007) on the context of half-lines, we get the required estimates on the space $X_{s,b,\beta,\sigma} \subset C([0,T]; H^s)$ for some (s,b,β,σ) satisfying $s > 0$, $b < \frac{1}{2}$, $\sigma > \frac{1}{2}$. In order to shorten the notation, in this paper we sometimes denote the space $X_{s,b,\beta,\sigma}$ only by X_s.

We note that $X_s(T)$ becomes a Banach space with the norm

$$\|\mathbf{w}\|_{X_s(T)} := \|\mathbf{w}\|_{C([0,T];H^s(\mathcal{G}))} + \|\mathbf{w}\|_{C\left(\mathcal{G};H^{\frac{s+1}{3}}([0,T])\right)} + \|\partial_x\mathbf{w}\|_{C\left(\mathcal{G};H^{\frac{s}{3}}([0,T])\right)} + \|\partial_x^2\mathbf{w}\|_{C\left(\mathcal{G};H^{\frac{s-1}{3}}([0,T])\right)} + \|\mathbf{w}\|_{\mathcal{X}^{s,b,\beta,\sigma}}.$$

Our local well-posedness theory for the Cauchy problem in (1.7)–(1.8) is the following,

Theorem 1.2 *Let $s \geqq 1$, $Z \neq 0$, and $\mathbf{u}_0 \in H^s(\mathcal{G}) \cap \mathcal{N}_{0,Z}(s)$. Then there exists $T = T(\|\mathbf{u}_0\|_s)$ and a unique solution $\mathbf{u}$ of the IVP (1.7)–(1.8) in the class $X_s(T)$ such that $\mathbf{u}(0) = \mathbf{u}_0$. Furthermore, for any $T_0 \in (0,T)$ there exists a neighborhood $W_0 \subset H^s(\mathcal{G}) \cap \mathcal{N}_{0,Z}(s)$ of $\mathbf{u}_0$ such that the data-to-solution map*

$$\mathbf{w}_0 \in W_0 \mapsto \mathbf{w} \in X_s(T_0)$$

is Lipschitz.

The strategy for proving Theorem 1.2 will be considering the case of only two edges, i.e. $|\mathbf{E}_-| = 1$ and $|\mathbf{E}_+| = 1$, and by using an auxiliary extended problem in the spirit of the paper of the second author Cavalcante (2018) (see Sect. 3 below), where the solution will be obtained in the sense of the distributions and it is the restriction of a convenient extended problem for all line $\mathbb{R}$. It extended problem comes from every edge on the balanced star graph $\mathcal{G}$. The extension of the problem in each edge on all of $\mathbb{R}$ is non trivial since we need to recover the original boundary conditions (1.8). The use of potentials for the linearized KdV equations on the positive and negative half-lines is fundamental here, since the exact formula of these potential permit us to define an appropriate integral equation that solves the problem on the distribution sense and satisfying (1.8).

Remark 1.3 The natural regularity assumptions for the trace of the functions given by (1.8) are motivated by the Kato smoothing effects obtained by Kenig, Ponce and Vega Kenig et al. (1991).

Remark 1.4 The boundary conditions in (1.9) are based in a previous authors paper Angulo and Cavalcante (2021) concern to the instability properties of stationary solutions for KdV equation on balanced graphs with a profile determined by the domain $D(A_Z)$ in (1.6) (see Sect. 6 below).

Remark 1.5 The approach based on the Riemann-Liouville integration operator used in the work Cavalcante (2018) does not apply to the present context, as the boundary condition (1.8) involves interactions between derivatives of different orders, preventing the construction of a integral formula by considering only a inversion of Riemann fractional integration. Thus, the approach considered here viewed as being more general in the sense of applications for more complicated boundary conditions.

Remark 1.6 The result of Theorem 1.2 has the bound $s \geq 1$, which is necessary for ensuring that $\partial_x^2 \mathbf{u}(0,t)$ will belong to $L^2(0,T)$. As a consequence, the boundary condition (1.8) gains a more comprehensible meaning. Since regularity of this trace function is $H^{\frac{s-1}{3}}(0,T)$. It is possible to extend the result to a lower regularity assumption, more precisely, for $s \geq 0$, although the function $\partial_x^2 \mathbf{u}(0,t)$ would have a distributional sense.

Next, we establish that the mapping data-to-solution is not only Lipschitz but at least of class C^2 for $s=1$. This will be sufficient for obtaining our nonlinear instability results for specific stationary solutions of the KdV model on balanced graphs.

Theorem 1.7 *Let $Z \neq 0$, and $\mathbf{u}_0 \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$. Then for $T = T(\|\mathbf{u}_0\|_1) > 0$ given by Theorem 1.2, there is a neighborhood $V_0 \subset H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$ of $\mathbf{u}_0$ such that the data-to-solution map associated to problem in (1.7)–(1.8)*

$$\mathbf{w}_0 \in V_0 \mapsto \mathbf{w} \in X_s(T)$$

is of class C^2.

The proof of Theorem 1.7 is based in the Implicit Function Theorem and the estimates used in the proof of Theorem 1.2. In Sect. 5 below, for completeness in the exposition, we give an idea of the proof.

In the following we describe our second main result in this manuscript which is associated to the nonlinear instability of some specific stationary solutions for the KdV model. We recall that solutions of stationary type for the KdV on a balanced graph $\mathcal{G}$ are determined in the form

$$(u_{\mathbf{e}}(x,t))_{\mathbf{e}\in\mathbf{E}} = (\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}}, \qquad \text{for all} \quad t,$$

where for $e \in \mathbf{E}_-$ the profile $\phi_{\mathbf{e}} : (-\infty, 0) \to \mathbb{R}$ satisfies $\phi_{\mathbf{e}}(-\infty) = 0$, and for $\mathbf{e} \in \mathbf{E}_+$, $\phi_{\mathbf{e}}$: $(0,+\infty) \to \mathbb{R}$ satisfies $\phi_{\mathbf{e}}(+\infty) = 0$. Thus, from the the condition

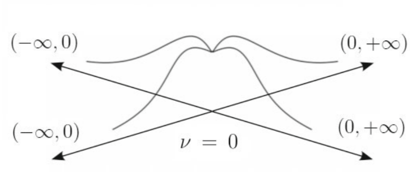


Fig. 3 U_Z, $Z > 0$, bump profiles

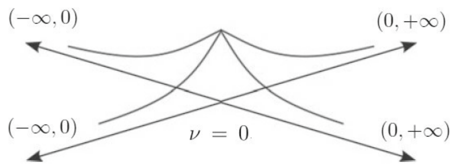


Fig. 4 U_Z, $Z < 0$, tail profiles

$\phi_{\mathbf{e}}(\pm\infty) = 0$ we have that every component $\phi_{\mathbf{e}}$ satisfies *a priori* the following non-linear elliptic equation

$$\alpha_{\mathbf{e}} \frac{d^2}{dx^2}\phi_{\mathbf{e}}(x) + \beta_{\mathbf{e}}\phi_{\mathbf{e}}(x) + \phi^2_{\mathbf{e}}(x) = 0, \qquad \text{for all } \mathbf{e} \in \mathbf{E}.$$

For $\alpha_{\mathbf{e}} > 0$ and $\beta_{\mathbf{e}} < 0$, and for each $\mathbf{e} \in \mathbf{E}$, we can obtain several families of profiles based on the classical soliton of the KdV on the full line,

$$\phi_{\mathbf{e}}(x) = c\,(\alpha_{\mathbf{e}}, \beta_{\mathbf{e}})\,\text{sech}^2\,(d\,(\alpha_{\mathbf{e}}, \beta_{\mathbf{e}})\,x + p_{\mathbf{e}})\,, \quad \mathbf{e} \in \mathbf{E}, \tag{1.12}$$

where the specific values of the shift p_{e} will depend on which other conditions are given for the profile ϕ_{e} on the vertex of $\mathcal{G}$, $\nu = 0$. In Angulo and Cavalcante (2021) was studied the case of the stationary profile $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ belongs to the domain of the family of skew-self-adjoint extension $(A_Z, D(A_Z))$ defined in (1.6) and with the constants sequences $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\alpha_+)$ and $(\beta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = (\beta_+)$, with $\alpha_+ > 0$ and $\beta_+ < 0$. Thus, for $Z \neq 0$, $\omega \equiv -\frac{\beta_+}{\alpha_+} > \frac{Z^2}{4}$ and the half-soliton profiles $\phi_\pm$ defined by (see (1.2))

$$\phi_+(x) = -\frac{3\beta_+}{2} sech^2\Big(\frac{\sqrt{\omega}}{2}\,x - tanh^{-1}\Big(\frac{Z}{2\sqrt{\omega}}\Big)\Big), \quad x > 0 \tag{1.13}$$

and $\phi_-(x) \equiv \phi_+(-x)$ for $x < 0$, we obtain for the constants sequences of functions

$$u_- = (\phi_-)_{\mathbf{e}\in\mathbf{E}_-},\; u_+ = (\phi_+)_{\mathbf{e}\in\mathbf{E}_+}, \tag{1.14}$$

that $U_Z = (u_-, u_+)$ represents one family of stationary profiles for the KdV model on a balanced graph and satisfying $U_Z \in D(A_Z)$. For $Z > 0$, U_Z represents one family of stationary bump profiles (Fig. 3) and for $Z < 0$, U_Z represents a family of stationary tail profiles (Fig. 4).

Now, in Angulo and Cavalcante (2021) was also established a criterion for the linear instability of stationary solutions for the KdV on arbitrary metric star graphs. This criterion was then applied to the case of the profiles of type tail and bump U_Z and so established that they are linearly unstable stationary solutions for the KdV model on balanced graphs (see Theorem 6.2 in Angulo and Cavalcante 2021).

In the following we establish that the linear instability result in Theorem 6.2 of Angulo and Cavalcante (2021) becomes a nonlinear instability behavior. Thus we have the following definition.

Definition 1.8 A stationary solution $\Phi = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_Z)$ is said to be *nonlinearly unstable* in $X \equiv H^1(\mathcal{G})$-norm by the flow of the Korteweg-de Vries model on $\mathcal{G}$ if there is $\epsilon > 0$, such that for every $\delta > 0$ there are a initial data $\mathbf{u}_0$ with $\|\Phi - \mathbf{u}_0\|_X < \delta$ and an instant $t_0 = t_0(\mathbf{u}_0)$ such that $\|\mathbf{u}(t_0) - \Phi\|_X > \epsilon$, where $\mathbf{u} = \mathbf{u}(t)$ is the solution of the Korteweg-de Vries problem in (1.7) with initial data $\mathbf{u}(0) = \mathbf{u}_0$.

Our nonlinear instability result is the following.

Theorem 1.9 *Let $Z \neq 0$, $\alpha_+ > 0$, $\beta_+ < 0$, and $-\frac{\beta_+}{\alpha_+} > \frac{Z^2}{4}$. Then for the profiles $\phi_\pm$ in (1.13) we have that the family of tails and bumps profiles $U_Z = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$ with $\phi_{\mathbf{e}} = \phi_-$ for $\mathbf{e} \in \mathbf{E}_-$ and $\phi_{\mathbf{e}} = \phi_+$ for $\mathbf{e} \in \mathbf{E}_+$, are nonlinearly unstable for the flow associated to the Korteweg-de Vries equation on a balanced graph.*

The strategy for proving Theorem 1.9 is to use the linear instability result in Theorem 6.2 below and to apply the approach by Henry et al. (1982). The key point of this method is to use the fact that if the mapping data-solution associated to the Korteweg-de Vries model is of class C^2 on $H^1(\mathcal{G})$, then we obtain the nonlinear instability from a result of linear instability (see Sect. 6 below).

Lastly, the paper is organized as follows. In the Preliminaries (Sect. 2), we introduce some notations, the functional spaces used in this work, the free group associated with the Airy equation, the Duhamel inhomogeneous operator, and we enunciate some known estimates associated with functional spaces. A review concerning potentials for a linearized KdV equation on the positive and negative half-lines is given on Sect. 3. Sections 4 and 5 are devoted to the proof of Theorems 1.2 and 1.7. Finally, Sect. 6 is devoted to the proof of Theorem of nonlinear instability.

2 Preliminaries

2.1 Sobolev Spaces on $\mathbb{R}^\pm$ and $\mathcal{G}$

We denote by $H^s(\Omega)$ the classical Sobolev spaces on an open set Ω, $\Omega \subset \mathbb{R}$, with the usual norm, and by $\mathcal{G}$ the metric star graph constituted by $|\mathbf{E}_-| + |\mathbf{E}_+|$ half-lines (where $|\mathbf{E}_-|$ denotes the number of half-lines of the form $(-\infty, 0)$ and $|\mathbf{E}_+|$ denotes the number of half-lines of the form $(0, +\infty)$) attached to a common vertex $\nu = 0$.

For $s \geq 0$ we say that $\phi \in H^s(\mathbb{R}^+)$ if exists $\tilde{\phi} \in H^s(\mathbb{R})$ such that $\phi = \tilde{\phi}|_{\mathbb{R}+}$. In this case, we set $\|\phi\|_{H^s(\mathbb{R}^+)} := \inf_{\tilde{\phi}} \|\tilde{\phi}\|_{H^s(\mathbb{R})}$. For $s \geq 0$ define

$$H_0^s(\mathbb{R}^+) = \Big\{\phi \in H^s(\mathbb{R}^+);\ \text{supp}(\phi) \subset [0, +\infty)\Big\}.$$

For $s < 0$, define $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$ as the dual space of $H_0^{-s}(\mathbb{R}^+)$ and $H^{-s}(\mathbb{R}^+)$, respectively.

Also, define

$$C_0^\infty(\mathbb{R}^+) = \Big\{\phi \in C^\infty(\mathbb{R});\ \text{supp}(\phi) \subset [0, +\infty)\Big\}$$

and $C_{0,c}^\infty(\mathbb{R}^+)$ as those members of $C_0^\infty(\mathbb{R}^+)$ with compact support. We recall that $C_{0,c}^\infty(\mathbb{R}^+)$ is dense in $H_0^s(\mathbb{R}^+)$ for all $s \in \mathbb{R}$. A definition for $H^s(\mathbb{R}^-)$ and $H_0^s(\mathbb{R}^-)$ can be given analogous to that for $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$.

On the graph $\mathcal{G}$ we define the classical spaces

$$L^p(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} L^p(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} L^p(0, +\infty), \quad p > 1,$$

and for $s \geq 0$

$$H^s(\mathcal{G}) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} H^s(-\infty, 0) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} H^s(0, +\infty)$$

with the natural norms. We emphasize that in these definitions we are not assuming any condition on the values of the functions at the vertex point $\nu = 0$. For $u = (u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$, $v = (v_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in L^2(\mathcal{G})$, we also define

$$\langle u, v\rangle \equiv \sum_{\mathbf{e}\in\mathbf{E}_-} \int_{-\infty}^{0} u_{\mathbf{e}}(x)\overline{v_{\mathbf{e}}}(x)dx + \sum_{\mathbf{e}\in\mathbf{E}_+} \int_0^\infty u_{\mathbf{e}}(x)\overline{v_{\mathbf{e}}}(x)dx,$$

which turns $L^2(\mathcal{G})$ into a Hilbert space. Depending on the context we will use the following notations for different objects. By $||\cdot||$ we denote the norm in $L^2(\mathbb{R})$ or in $L^2(\mathcal{G})$. By $||\cdot||_p$ we denote the norm in $L^p(\mathbb{R})$ or in $L^p(\mathcal{G})$.

2.2 Bourgain Spaces

Since our approach is based on replacing the original problem in (1.7) with a conveniently extended problem on each edge, extending over all of $\mathbb{R}$, we will need to use appropriate Bourgain spaces defined in all of $\mathbb{R}$. The principal idea is to construct an operator which is a contraction in these functional spaces. Thus, in this section we define and describe the principal properties of these spaces.

Thus, for a function f in the Schwartz space $\mathcal{S}(\mathbb{R})$ we denote the Fourier transform and the inverse transform of f, respectively by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx, \quad \mathcal{F}^{-1}[f](x) = f^{\vee}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} f(\xi)d\xi.$$

The classical Sobolev space based in $L^2(\mathbb{R})$ is given by

$$H^s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}) : \mathcal{F}^{-1}\left[(1+|\xi|)^s \hat{f}(\xi)\right] \in L^2(\mathbb{R}) \right\},$$

where $\mathcal{S}'(\mathbb{R})$ denotes the spaces of tempered distributions.

Now we introduce special modified Bourgain spaces introduced by Faminskii. For $s \geq 0$, $a \in \mathbb{R}$, $b \in (0, 1/2)$, $\sigma \in (1/2, 2/3)$ define the following Bourgain type spaces in the context of Faminskii works Faminskii (2004) and Faminskii (2007)

$$X^{s,b,\beta,\sigma} = \left\{ f(t,x) \in S'\left(\mathbb{R}^2\right) : \right.$$
$$\left. \|f\|_{X^{s,b,\beta,\sigma}} = \left(\iint_{\mathbb{R}^2} \left(1+|\xi|+|\tau|^{1/3}\right)^{2s} \nu^2(\tau,\xi)|\hat{f}(\tau,\xi)|^2 d\xi d\tau \right)^{1/2} < \infty \right\},$$

where the function ν is given by

$$\nu(\tau,\xi) = \nu_{\beta,b,\sigma}(\tau,\xi) \equiv \left(1+\left|\tau-\xi^3+\beta\xi\right|\right)^b + \chi_{[-1,1]}(\xi)(1+|\tau|)^{\sigma}, \tag{2.1}$$

$$Y^{s,b,\beta,\sigma} = \left\{ f(t,x) \in S'\left(\mathbb{R}^2\right) : \right.$$
$$\left. \|f\|_{Y^{s,b,\beta,\sigma}} = \left(\iint_{\mathbb{R}^2} \left(1+|\xi|+|\tau|^{1/3}\right)^{2s} \gamma^2(\tau,\xi)|\hat{f}(\tau,\xi)|^2 d\xi d\tau \right)^{1/2} < \infty \right\}, \tag{2.2}$$

where the function γ is defined as

$$\gamma(\tau,\xi) = \gamma_{b,\beta,\sigma}(\tau,\xi) \equiv \frac{1}{\left(1+\left|\tau-\xi^3+\beta\xi\right|\right)^b} + \frac{\chi_{[-1,1]}(\xi)}{(1+|\tau|)^{1-\sigma}}. \tag{2.3}$$

If Ω is a domain in $\mathbb{R}^2$, then define by $X^{s,b,\beta,\sigma}(\Omega)$ and $Y^{s,b,\beta,\sigma}(\Omega)$ restrictions of $X^{s,b,\beta,\sigma}$ and $Y^{s,b,\beta,\sigma}$ on Ω, respectively, with natural restriction norms.

On the context of a balanced star graph $\mathcal{G}$ we consider the Bourgain spaces $\mathcal{X}^{s,b,\beta,\sigma}(\mathcal{G}\times(0,T))$ by

$$\mathcal{X}^{s,b,\beta,\sigma}(\mathcal{G}\times(0,T)) = \bigoplus_{\mathbf{e}\in\mathbf{E}_-} X^{s,b,\beta,\sigma}((-\infty,0)\times(0,T)) \oplus \bigoplus_{\mathbf{e}\in\mathbf{E}_+} X^{s,b,\beta,\sigma}((0,\infty)\times(0,T)). \tag{2.4}$$

2.3 Linear Group

For convenience in the notation, we will consider the following linearized KdV equation,

$$\partial_t u + \partial_x^3 u + \beta \partial_x u = 0, \quad x \in \mathbb{R} \tag{2.5}$$

for $\beta \in \mathbb{R}$. Then, the linear group $S_\beta(t) := e^{-t(\partial_x^3+\beta\partial_x)} : S'(\mathbb{R}) \to S'(\mathbb{R})$ associated with equation in (2.5) is defined by

$$e^{-t\left(\partial_x^3+\beta\partial_x\right)}\phi(x) = \left(e^{it\left(\xi^3-\beta\xi\right)}\hat{\phi}(\xi)\right)^{\vee}(x),$$

and will satisfy

$$\begin{cases} \left(\partial_t + \partial_x^3 + \beta\partial_x\right) S_\beta(t)\phi(x) = 0 & \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}, \\ S_\beta(0)\phi(x) = \phi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

In the rest of the paper, the function $\psi : \mathbb{R} \to \mathbb{R}$ will denote a cutt off regular function supported in the set $[-2, 2]$ such that $\psi \equiv 1$ on the set $[-1, 1]$. Also we denote $\psi_T(t) = \frac{1}{T}\psi(\frac{t}{T})$.

The next estimates were proven in Faminskii (2004).

Lemma 2.1 *Let $s \geq 0$ and $0 < b < 1$. If $\phi \in H^s(\mathbb{R})$, then hold following estimates*

$$\begin{aligned} &\left\|\psi(t)S_\beta(t)\phi(x)\right\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} + \sum_{j=0}^{2}\left\|\psi(t)\partial_x^j S_\beta(t)\phi(x)\right\|_{C\left(\mathbb{R}_x;H^{(s+1-j)/3}(\mathbb{R}_t)\right)} \\ &\quad + \left\|\psi(t)S_\beta(t)\phi(x)\right\|_{X^{s,b,\beta,\sigma}} \lesssim \|\phi\|_{H^s(\mathbb{R})}. \end{aligned} \tag{2.6}$$

2.4 Duhamel Operator

The known inhomogeneous solution operator $\mathcal{K}$ associated with the KdV equation, for any $w \in L^1_{\text{loc}}\left(\mathbb{R}, S'(\mathbb{R})\right)$ is given by

$$\mathcal{K}w(x,t) = \int_0^t e^{-\left(t-t'\right)\left(\partial_x^3+\beta\partial_x\right)} w\left(x,t'\right)\mathrm{d}t', \tag{2.7}$$

which satisfies in the sense of distributions

$$\begin{cases} \left(\partial_t + \partial_x^3 + \beta\partial_x\right)\mathcal{K}w(x,t) = w(x,t) & \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}; \\ \mathcal{K}w(x,0) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

Now, we summarize some useful estimates for the Duhamel inhomogeneous solution operators $\mathcal{K}$ that will be used later in the proof of the main results and its proof can be seen in Faminskii (2004).

Lemma 2.2 *For all $s \in \mathbb{R}$, we have the following estimate*

$$\|\psi(t)\mathcal{K}w(x,t)\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} + \sum_{j=0}^{2} \left\|\psi(t)\partial_x^j \mathcal{K}w(x,t)\right\|_{C\left(\mathbb{R}_x;H^{(s+1)/3}(\mathbb{R}_t)\right)} + \|\psi(t)\mathcal{K}w(x,t)\|_{X^{s,b,\beta,\sigma}} \lesssim \|w\|_{Y^{s,b,\beta,\sigma}}.$$

The following lemma states an estimate concern the $X^{s,b,\beta,\sigma}$ spaces.

Lemma 2.3 *For $f \in X^{s,b,\beta,\sigma}$, $T > 0$ holds*

$$\|\psi_T(t)f(t,x)\|_{X^{s,b,\beta,\sigma}} \le c(s,b,\beta,\sigma)T^{1/2-\sigma-s/3}\|f(t,x)\|_{X^{s,b,\beta,\sigma}}.$$

Remark 2.4 In our context, we will work with $\sigma > \frac{1}{2}$. Therefore, the exponent $r := \frac{1}{2} - \sigma - \frac{s}{3}$ can become negative depending on the value of s. In the case where this exponent is negative, we can modify the time to $\tilde{T} = T^{-\theta}$ for any $\theta > 0$. Thus, in Lemma 2.3, we obtain the new term $\tilde{T}^{\frac{1}{2}-\sigma-\frac{s}{3}} = T^{-\theta\left(\frac{1}{2}-\sigma-\frac{s}{3}\right)}$. We use this argument occasionally, as in the fixed-point argument, we require a positive power of time.

Next, we state a nonlinear estimate, in the context of the KdV equation equation, for $b < \frac{1}{2}$, which was derived by Faminskii (2004).

Lemma 2.5 *Let $b \in [7/16, 1/2)$, $\sigma \in (1/2, 2/3)$, $s \ge 0$, we have*

$$\|\psi(t)\partial_x(v_1v_2)\|_{Y^{s,b,\beta,\sigma}} \lesssim \|v_1\|_{X^{s,b,\beta,\sigma}}\|v_2\|_{X^{s,b,\beta,\sigma}}.$$

3 A Review Concern Potentials for a Linearized KdV Equation on the Positive and Negative Half-Lines

An important point in this paper is the use of the corresponding linear problems associated with the KdV equation on the positive and negative half-lines. More precisely, we consider the following IBVPs associated with the linearized KdV on the positive half-line

$$\begin{cases} \partial_t v + \partial_x^3 v + \beta\partial_x v = 0, & x \in (0,+\infty), t \in (0,T); \\ v(x,0) = v_0(x), & x \in (0,+\infty); \\ v(0,t) = f(t), & t \in (0,T) \end{cases} \tag{3.1}$$

and the negative half-line

$$\begin{cases} \partial_t w + \partial_x^3 w + \beta\partial_x w = 0, & x \in (-\infty,0), t \in (0,T); \\ w(x,0) = w_0(x), & x \in (-\infty,0); \\ w(0,t) = g(t), & t \in (0,T); \\ \partial_x w(0,t) = h(t), & t \in (0,T). \end{cases} \tag{3.2}$$

Here, we assume that $v_0 \in H^s(\mathbb{R}^+)$, $w_0 \in H^s(\mathbb{R}^-)$, $f, g \in H^{\frac{s+1}{3}}(0, T)$ and $h \in H^{\frac{s}{3}}(0, T)$.

The presence of one boundary condition on the right half-line problem (3.1) versus two boundary conditions on the left half-line problem (3.2) can be motivated by integral identities on smooth solutions to the linearized KdV equation, for details the reader can see Holmer (2006).

For a complete survey about the problems (3.1) and (3.2) the reader can see Cavalcante (2019). Also, results of orbital stability and asymptotic stability can be seen in Cavalcante and Muñoz (2019) and Cavalcante and Muñoz (2023).

In this section, we describe explicitly the solutions for the linearized KdV equation and its properties based in some potentials. For the first time, such potentials were introduced in the paper by Cattabriga (1959). Later, Faminskii (2004) used these potential solutions to solve the nonlinear problems for the KdV on the positive and the negative half-lines. The interesting point in these papers is that the proof of the global existence of solutions for the KdV equations on the half-lines by assuming more natural conditions for the boundary functions.

In the rest of this section we assume $\beta < 0$ in (3.1) and (3.2). In order to get the exact formulas for the IBVPs (3.1) and (3.2), by following the Faminskii works (Faminskii 2004, 2007), we need to study the following algebraic equation

$$z^3 + \beta z + \varepsilon + i\tau = 0. \tag{3.3}$$

If $\varepsilon > 0$, for any fixed τ, then this equation has one root z_0 such that $\operatorname{Re} z_0 < 0$, and two roots z_1 and z_2 such that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Let

$$r_j(\tau) = r_j(\tau; \beta) = \lim_{\varepsilon \to +0} z_j(\varepsilon + i\tau), \quad j = 0, 1, 2.$$

The Faminskii papers (Faminskii 2004, 2007) gives the following exact formula

$$\begin{aligned} r_0(\tau) &= -p(\tau) + iq(\tau), \\ r_1(\tau) &= p(\tau) + iq(\tau), \\ r_2(\tau) &= ik_\beta(\tau), \end{aligned} \tag{3.4}$$

where p and q are real valued functions and k_β is the inverse of function $\phi_\beta = \xi^3 - \beta\xi$.

It is possible to show the following estimate for $j = 0, 1, 2$,

$$\left|r_j(\tau)\right| \le c\left(|\tau|^{1/3} + |\beta|^{1/2}\right). \tag{3.5}$$

For a certain constant $c > 0$

$$p(\tau), q(\tau) \ge c\left(|\tau|^{1/3} + |\beta|^{1/2}\right) \quad \forall \tau \in \mathbb{R}. \tag{3.6}$$

Now, we give the exact formulas for the IBVPs (3.1) and (3.2). We star by denoting $\tilde{h}$, $\tilde{f}$ and $\tilde{g}$ as the extensions of the functions h, f, and g, respectively, on all real line

$\mathbb{R}$, satisfying

$$\|\tilde{f}\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \lesssim \|f\|_{H^{\frac{s+1}{3}}(0,T)}, \ \|\tilde{g}\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \lesssim \|g\|_{H^{\frac{s+1}{3}}(0,T)} \text{ and } \|\tilde{h}\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} \lesssim \|h\|_{H^s(0,T)}.$$

For $\tilde{h} \in S'(\mathbb{R})$, $t \in \mathbb{R}$ and $x > 0$ we define

$$\mathcal{R}_\beta(t, x; \tilde{h}) = \mathcal{F}_t^{-1}\left[e^{r_0(\tau)x}\hat{\tilde{h}}(\tau)\right](t). \tag{3.7}$$

For $\tilde{f}, \tilde{g} \in \mathcal{S}'(\mathbb{R})$ define for $t \in \mathbb{R}$ and $x \le 0$

$$\mathcal{L}^1_\beta(t, x; \tilde{f}) \equiv \mathcal{F}_t^{-1}\left[\frac{r_1(\tau)e^{r_2(\tau)x} - r_2(\tau)e^{r_1(\tau)x}}{r_1(\tau) - r_2(\tau)}\hat{\tilde{f}}(\tau)\right](t) \tag{3.8}$$

and

$$\mathcal{L}^2_\beta(t, x; \tilde{g}) \equiv \mathcal{F}_t^{-1}\left[\frac{e^{r_1(\tau)x} - e^{r_2(\tau)x}}{r_1(\tau) - r_2(\tau)}\hat{\tilde{g}}(\tau)\right](t). \tag{3.9}$$

As the consequence of (3.7), (3.8) and (3.9) we have the following essential traces for the functions $R_\beta\tilde{h}$, $L^1_\beta\tilde{f}$, $L^1_\beta\tilde{g}$ and its spatial derivatives,

$$\begin{cases} \mathcal{R}_\beta\tilde{h}(0,t) = \tilde{h}(t), \quad \mathcal{L}^1_\beta\tilde{f}(0,t) = \tilde{f}(t), \mathcal{L}^2_\beta\tilde{g}(0,t), \ t \in \mathbb{R}; \\ \partial_x(\mathcal{R}_\beta\tilde{h})(0,t) = \mathcal{F}_t^{-1}\left[r_0(\tau)\hat{h}\right](t), \ t \in \mathbb{R}; \\ \partial_x\left(\mathcal{L}^1_\beta\tilde{f}\right)(0,t) = 0, \quad \partial_x\left(\mathcal{L}^2_\beta\tilde{g}\right)(0,t) = \tilde{g}(t), \ t \in \mathbb{R}; \\ \partial_x^2(\mathcal{R}_\beta\tilde{h})(0,t) = \mathcal{F}_t^{-1}\left(r_0^2(\tau)\tilde{h}\right)(t), \ t \in \mathbb{R}; \\ \partial_x^2\left(\mathcal{L}^1_\beta\tilde{f}\right)(0,t) = -\mathcal{F}^{-1}\left(r_1(\tau)r_2(\tau)\hat{\tilde{f}}\right)(t), \ t \in \mathbb{R}; \\ \partial_x^2\left(\mathcal{L}^2_\beta\tilde{g}\right)(0,t) = \mathcal{F}_t^{-1}\left((r_1(\tau)+r_2(\tau))\,\hat{\tilde{g}}\right)(t), \ t \in \mathbb{R};. \end{cases} \tag{3.10}$$

Remark 3.1 The values of the traces given in (3.10) will be essential in the construction of an integral formula equivalent to the problem (1.7) satisfying the boundary condition (1.8).

Lemma 3.2 *For any $h \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ and $s \ge 0$ we have that $\mathcal{R}h \in C(\mathbb{R}^+; H^s(\mathbb{R}^+))$. Moreover, holds the following inequality*

$$\|\mathcal{R}h(\cdot,t)\|_{C(\mathbb{R}^+;H^s(\mathbb{R}+))} \le c(\beta,s)\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}. \tag{3.11}$$

Proof This result was obtained in Faminskii (2004). We only provide a sketch of the proof, as the argument of the proof will be used to make appropriate extensions of the function $\mathcal{R}h$ on the entire plane $\mathbb{R}^2$. The following calculations will be usefull in this work. Without loss of generality, we assume $h \in C_0^\infty(\mathbb{R}^+)$. Let n a nonegative integer number. By using (3.7) for $x > 0, t \geq 0$

$$D_x^n \mathcal{R}_\beta(t, x; h) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda t} (r(\lambda))^n e^{r(\lambda)x} \hat{h}(\lambda) d\lambda,$$

where by abusing notation we consider h the extension for zero on the negative half-line. Now, we consider $\beta > 0$, then

$$\begin{aligned} D_x^n \mathcal{R}_\beta(t, x; h) &= \frac{1}{2\pi} \int_{|\lambda| < 2(\beta/3)^{3/2}} e^{i\lambda t} (r(\lambda))^n e^{r(\lambda)x} \hat{h}(\lambda) d\lambda \\ &\quad + \frac{1}{2\pi} \int_{|\lambda| > 2(\beta/3)^{3/2}} e^{i\lambda t} (r(\lambda))^n e^{r(\lambda)x} \hat{h}(\lambda) d\lambda \\ &\equiv I_1 + I_2. \end{aligned}$$

Consider first I_1 (in this case $r(\lambda) = iq(\lambda)$). Changing variables $\xi = q(\lambda)$ we find

$$I_1 = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\xi^3 - \beta\xi)t} e^{i\xi x} \hat{v}_n(\xi) d\xi = S_\beta(t, x; v_n),$$

where

$$v_n(x) \equiv -\mathcal{F}_x^{-1} \left[\chi\sqrt{\beta/3}(\xi) \left(r\left(\xi^3 - \beta\xi\right)\right)^n \left(3\xi^2 - \beta\right) \hat{h}\left(\xi^3 - \beta\xi\right)\right](x). \tag{3.12}$$

and according to the Parseval equality

$$\|I_1(t, \cdot)\|_{L^2(\mathbb{R}^+)} \leq \|I_1(t, \cdot)\|_{L^2(\mathbb{R})} \leq \left\|\hat{v}_n\right\|_{L^2(\mathbb{R})} \leq c(\beta, n)\|h\|_{L^2(\mathbb{R}^+)}.$$

For I_2 changing variable $\xi = \kappa_\beta(\lambda)$ we find, that

$$I_2 = \frac{1}{2\pi} \int_{|\xi| > 2\sqrt{\beta/3}} e^{i(\xi^3 - \beta\xi)t} \left(r\left(\xi^3 - \beta\xi\right)\right)^n e^{r(\xi^3 - \beta\xi)x} \hat{h}\left(\xi^3 - \beta\xi\right)\left(3\xi^2 - \beta\right) d\xi.$$

By using the following inequality from Bona et al. (2022) if certain continuous function $\gamma(\lambda)$ satisfies an inequality $\Re\gamma(\lambda) \leq -\varepsilon|\lambda|$ for some $\varepsilon > 0$ and all $\lambda \in \mathbb{R}$, then

$$\left\| \int_{\mathbb{R}} e^{\gamma(\lambda)x} f(\lambda) d\lambda \right\|_{L^2(\mathbb{R}^+)} \leq c(\varepsilon)\|f\|_{L^2(\mathbb{R})}.$$

By following Faminskii (2004) we have that

$$\begin{aligned} \|I_2(t, \cdot)\|_{L^2(\mathbb{R}^+)} &\leq \left\| \left(1 - \chi_{2\sqrt{\beta/3}}(\xi)\right)\left(r\left(\xi^3 - \beta\xi\right)\right)^n \hat{h}\left(\xi^3 - \beta\xi\right)\left(3\xi^2 - \beta\right)\right\|_{L^2(\mathbb{R})} \\ &\leq c_1\|h\|_{H_0^{(n+1)/3}(\mathbb{R}^+)} \end{aligned}$$

Combining the obtained estimates we derive (3.29) for $s = n$ and $\beta > 0$. Finally, the case $\beta < 0$ the partition of the integral is unnecessary. □

The following lemma obtained in Faminskii (2004, 2007) gives the solutions for the linearized problems (3.1) and (3.2).

Lemma 3.3 *Let* $f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ *and* $h \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$.

(i) *For any extension* $\tilde{h} \in H^{\frac{s+1}{3}}(\mathbb{R})$ *of* h *on the entire line* $\mathbb{R}$, *the function* $\mathcal{R}_\beta \tilde{h}\Big|_{(0,\infty)\times(0,T)}$ *is the generalized solution of IBVP* (3.1).

(ii) *For any extensions* $\tilde{f}$, $\tilde{g} \in H^{\frac{s+1}{3}}(\mathbb{R})$ *of the functions* f *and* g *on the entire line the function* $\left(\mathcal{L}^1_\beta \tilde{f} + \mathcal{L}^2_\beta \tilde{g}\right)\Big|_{(-\infty,0)\times(0,T)}$ *is the generalized solution of the IBVP* (3.2).

Remark 3.4 By using the result of uniqueness of Faminskii (2004, 2007) we have that the restriction on the set $[0, T]$ of the formulas obtained in Lemma 3.3 for the solutions of (3.1) and (3.2) do not depend on the extensions $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$.

Remark 3.5 (Extensions on all of $\mathbb{R}^2$) Now, to work within the context of Bourgain spaces, we need to extend the formulas (3.7) for $x < 0$ and $t < 0$, as well as extend the formula (3.8) for $x > 0$ and $t < 0$. This extension is performed following the methods outlined in the works by Faminskii (2004, 2007). First, to deal with (3.7), we define the following function,

$$\sigma(x) = \sigma(x;\lambda) \equiv \begin{cases} e^{p(\lambda)x}, & x \geq 0, \\ c_0 e^{-p(\lambda)x} + c_1 e^{-p(\lambda)x/2} + \cdots + c_n e^{-p(\lambda)x/2^n}, & x \leq 0, \end{cases}$$

where the coefficients $c_0, c_1, \ldots c_n$ are chosen such, that

$$\begin{cases} c_0 + c_1 + \cdots + c_n = 1, \\ (-1)\left(c_0 + \frac{1}{2}c_1 + \cdots + \frac{1}{2^n}c_n\right) = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (-1)^n\left(c_0 + \frac{1}{2^n}c_1 + \cdots\cdots + \frac{1}{2^{n^2}}c_n\right) = 1. \end{cases}$$

Note that the system above has a solution, and with these coefficients, the function $\rho \in H^{n+1}(\mathbb{R})$. Thus, we put for any $(x,t) \in \mathbb{R}^2$ in the case $\beta \leq 0$

$$R_\beta(t,x;\tilde{h}) \equiv \mathcal{F}_t^{-1}\left[\sigma(x;\lambda)e^{iq(\lambda)x}\hat{\tilde{h}}(\lambda)\right](t),$$

and in the case $\beta > 0$

$$\begin{aligned} R_\beta(t,x;\mu) &\equiv S_\beta(t,x;v_0) + \mathcal{F}_t^{-1}\left[\sigma(x;\lambda)e^{iq(\lambda)x}\hat{\mu}(\lambda)\left(1 - \chi_{2(\beta/3)^{3/2}}(\lambda)\right)\right](t) \\ &\equiv S_\beta(t,x;v_0) + J_{0\beta}(t,x;\mu), \end{aligned}$$

where v_0 is given by (3.12). An extension of the functions $\mathcal{L}^1_\beta \tilde{f}$ and $\mathcal{L}^2_\beta \tilde{g}$ can be done in a similar way, for more details the reader can see Faminskii (2007).

Remark 3.6 Given any functions $f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ and $h \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, in the rest of the paper we can consider the functions $\mathcal{R}h$, defined on the set $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, as well the functions $\mathcal{L}^1_\beta f + \mathcal{L}^2_\beta g$ on the set $(x,t) \in \mathbb{R}^- \times \mathbb{R}^+$. Sometimes, we will consider its appropriate extensions denoted by $\mathcal{R}\tilde{h}$ and $\mathcal{L}^1_\beta \tilde{f} + \mathcal{L}^2_\beta g$, respectively, in all plane $\mathbb{R}^2$ in the spirit of Remark 3.5.

The following lemma states the fundamental estimates for the potential solutions and was obtained by Faminskii (2004, 2007).

Lemma 3.7 *Let* $s \geq 0$. *For any functions* $f, h \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ *and* $g \in H^{\frac{s}{3}}(\mathbb{R}^+)$ *the following estimates are ensured:*

(a)

$$\|\psi(t)\mathcal{R}_\beta \tilde{h}(x,t)\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} + \left\|\psi(t)\partial_x^n(\mathcal{R}_\beta \tilde{h}(x,t))\right\|_{C\left(\mathbb{R}_x; H^{\frac{s+1-n}{3}}(\mathbb{R}_t)\right)}$$
$$+ \|\psi(t)\mathcal{R}_\beta \tilde{h}(x,t)\|_{X_{s,b,\beta,\sigma}} \lesssim \|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}, \quad (n = 1, 2, \ldots)$$

(b)

$$\left\|\psi(t)\mathcal{L}^1_\beta \tilde{f}(x,t)\right\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} + \left\|\psi(t)\partial_x^n\left(\mathcal{L}^1_\beta \tilde{f}(x,t)\right)\right\|_{C\left(\mathbb{R}_x; H^{\frac{s+1-n}{3}}(\mathbb{R}_t)\right)}$$
$$+ \left\|\psi(t)\mathcal{L}^1_\beta \tilde{f}(x,t)\right\|_{X_{s,b,\beta,\sigma}} \lesssim \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}, \quad (n = 1, 2, \ldots)$$

(c)

$$\left\|(\psi(t))\mathcal{L}^2_\beta \tilde{g}(x,t)\right\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} + \left\|(\psi(t))\partial_x^n\left(\mathcal{L}^2_\beta \tilde{g}(x,t)\right)\right\|_{C\left(\mathbb{R}_x; H^{\frac{s+1-n}{3}}(\mathbb{R}_t)\right)}$$
$$+ \left\|\mathcal{L}^2_\beta \tilde{g}(x,t)\right\|_{X_{s,b,\beta,\sigma}} \lesssim \|g\|_{H^{s/3}(\mathbb{R}^+)}, \quad (n = 1, 2, \ldots).$$

4 An Integral Equation that Solves the Problem on a Balanced Graph

We note that it is sufficient for our proof of Theorem 1.2 to consider the case of only two edges, i.e. $|\mathbf{E}_-| = 1$ and $|\mathbf{E}_+| = 1$, because on the general case of a balanced graph $\mathcal{G}$ the conditions (1.8) in the vectorial notation takes the following form (for $\mathbf{u} = (u_1, ..., u_n, v_1, ..., v_n)$)

$$\begin{cases} (u_1(0-,t), u_2(0-,t), ..., u_n(0-,t)) = (v_1(0+,t), ..., v_n(0+,t)), \text{ in the sense of } H^{\frac{s+1}{3}}(0,T); \\ v_k'(0+,t) - u_k'(0-,t) = Zu_k(0-,t), \ (k = 1, ..., n) \text{ in the sense of } H^{\frac{s}{3}}(0,T); \\ \frac{Z^2}{2}u_k(0-,t) + Zu_k'(0-,t) = v_k''(0+,t) - u_k''(0-,t), \ (k = 1, ..., n) \text{ in the sense of } H^{\frac{s-1}{3}}(0,T). \end{cases} \tag{4.1}$$

Now, fix $s \geq 1$, $b \in [7/16, 1/2)$ and $\alpha \in (1/2, 2/3)$. Our strategy is to extend the problem to all lines $\mathbb{R}$ along each edge, aiming to derive an integral equation that solves the extended problem. This solution should be such that when restricted to each edge, it solves the original problem (1.7)–(1.8) in the sense of distributions. In this context, our solution will be denoted as $\mathbf{u} = (u, v) = \big(\tilde{u}|_{\mathbb{R}^+\times(0,T)}, \tilde{v}|_{\mathbb{R}^-\times(0,T)}\big)$, where $\tilde{u}$ and $\tilde{v}$ are solutions of the corresponding extended problem for all of $\mathbb{R}$ along each edge, in the sense of distributions.

To do this, we start by taking extensions $\widetilde{u}_0$ and $\widetilde{v}_0$ of the initial data u_0 and v_0 satisfying

$$\|\widetilde{u}_0\|_{H^s(\mathbb{R})} \leq c\,\|u_0\|_{H^s(\mathbb{R}^+)} \text{ and } \|\widetilde{v}_0\|_{H^s(\mathbb{R})} \leq c\,\|v_0\|_{H^s(\mathbb{R}^-)}\,.$$

We will find solutions for the Cauchy problem (1.7)–(1.8) as the restriction of the functions satisfying following integral equations

$$\tilde{u}(x,t) = \psi(t)S_\beta(t)\tilde{u}_0 + \psi(t)\mathcal{K}(\psi_T^2(t)\tilde{u}\tilde{u}_x) + \psi(t)\mathcal{R}_\beta h(x,t) \tag{4.2}$$

and

$$\tilde{v}(x,t) = \psi(t)S_\beta(t)\tilde{v}_0 + \psi(t)\mathcal{K}(\psi_T^2(t)\tilde{v}\tilde{v}_x) + \psi(t)\mathcal{L}_\beta^1 f(x,t) + \psi(t)\mathcal{L}_\beta^2 g(x,t), \tag{4.3}$$

where the functions f, g, and h depend on the initial conditions $\widetilde{u}_0$ and $\widetilde{v}_0$, as well as the unknown functions $\tilde{u}$ and $\tilde{v}$. These functions will be chosen in a way that satisfies the boundary conditions (1.8). Here, $\mathcal{K}$ represents the classical Duhamel operator defined in (5.19). With suitable choices for f, g, and h, we will observe that equations (4.2) and (4.3) become integral equations for the Cauchy problem (1.7)–(1.8).

In order to simplify the notation we denote

$$F_1(x,t) := F_1(\tilde{u}_0, \tilde{u}, x, t) = \psi(t)S_\beta(t)\tilde{u}_0 + \psi(t)\mathcal{K}(\psi_T^2(t)\tilde{u}\tilde{u}_x) \tag{4.4}$$

and

$$F_2(x,t) := F_2(\tilde{v}_0, \tilde{v}, x, t) = \psi(t)S_\beta(t)\tilde{v}_0 + \psi(t)\mathcal{K}(\psi_T^2(t)\tilde{v}\tilde{v}_x). \tag{4.5}$$

Now, we start the process of choices of unknown functions by making use of the boundary conditions (1.8). So, by using (4.2), (4.3) and the boundary conditions (1.8) we impose that the functions f, g, and h must satisfy the following relations

$$\tilde{u}(0,t) = \tilde{v}(0,t) \implies h(t) + F_1(0,t) = F_2(0,t) + f(t) \implies \hat{h} - \hat{f} = \hat{F}_2(0,\tau) - \hat{F}_1(0,\tau), \tag{4.6}$$

$$\begin{aligned} &\tilde{u}'(0,t) - \tilde{v}'(0,t) = Z\tilde{\tilde{v}}(0,t) \implies r_0\hat{h} + \partial_x\hat{F}_1(0,t) - \hat{g} - \partial_x\hat{F}_2(0,t) = Z(\hat{f} + \hat{F}_2) \\ &\implies r_0\hat{h} - \hat{g} - Z\hat{f} = -\partial_x\hat{F}_2(0,\tau) + Z\hat{F}_2 + \partial_x\hat{F}_2(0,\tau) \end{aligned} \tag{4.7}$$

and

$$
\begin{aligned}
&\tilde{u}''(0,t) - \tilde{v}''(0,t) = \frac{Z^2}{2} v(0,t) + Z v'(0,t) \\
&\quad \Longrightarrow r_0^2(\tau)\hat{h} + \partial_x^2 \hat{F}_1 + r_1(\tau) r_2(\tau) \hat{f} - (r_1(\tau) + r_2(\tau))\hat{g} - \partial_x^2 \hat{F}_2 \\
&= \frac{Z^2}{2}(\hat{f} + \hat{F}_2) + Z(\hat{g} + \partial_x \hat{F}_2).
\end{aligned} \tag{4.8}
$$

Here " $\hat{}$ " denotes the Fourier transform on the variable t.

The expressions (4.6), (4.7) and (4.8) are written in the context of matrices in the following form:

$$
\begin{aligned}
&\begin{bmatrix} 1 & -1 & 0 \\ r_0(\tau) & -Z & -1 \\ r_0^2(\tau) & r_1(\tau) r_2(\tau) - \frac{Z^2}{2} & -(r_1(\tau) + r_2(\tau) + Z) \end{bmatrix} \begin{bmatrix} \hat{h} \\ \hat{f} \\ \hat{g} \end{bmatrix} \\
&= \begin{bmatrix} \hat{F}_2 - \hat{F}_1 \\ -\partial_x \hat{F}_1 + \partial_x \hat{F}_x + Z\hat{F}_2 \\ -\partial_x^2 \hat{F}_1 + \partial_x^2 \hat{F}_2 + \frac{Z^2}{2}\hat{F}_2 + Z\partial_x \hat{F}_2 \end{bmatrix}.
\end{aligned} \tag{4.9}
$$

We denote this in the reducing form

$$
M\mathbf{f} := M(r_0(\tau), r_1(\tau), r_2(\tau), Z)\mathbf{f} = \hat{\mathbf{F}}, \tag{4.10}
$$

where M denotes the first matrix in (4.9), $\mathbf{f}$ denotes de column vector with coordinates $\hat{h}$, $\hat{f}$, and $\hat{g}$, and $\hat{\mathbf{F}}$ the column vector with entries $\hat{F}_2 - \hat{F}_1$, $-\partial_x \hat{F}_1 + \partial_x \hat{F}_2 + z\hat{F}_2$ and $-\partial_x^2 \hat{F}_1 + \partial_x^2 \hat{F}_2 + \frac{Z^2}{2}\hat{F}_2 + Z\partial_x \hat{F}_2$.

The determinant of M is given by

$$
\det M = \frac{Z^2}{2} + (r_1(\tau) + r_2(\tau) - r_0(\tau)) + r_1(\tau) r_2(\tau) - r_0(\tau) r_1(\tau) - r_0(\tau) r_2(\tau). \tag{4.11}
$$

Now, denotes k_β the inverse of function $\phi_\beta(\xi) = \xi^3 - \beta\xi$. By using (3.4) we have

$$
\begin{aligned}
r_1 + r_2 - r_0 &= 2p + ik_\beta \\
r_1 r_2 &= -qk_\beta + ik_\beta p \\
-r_0 r_1 &= q^2 + p^2, \\
-r_0 r_2 &= qk_\beta - pk_\beta i.
\end{aligned} \tag{4.12}
$$

It follows that

$$
r_1(\tau) r_2(\tau) - r_0(\tau) r_1(\tau) - r_0(\tau) r_2(\tau) = q^2(\tau) + p^2(\tau) \tag{4.13}
$$

and

$$\det M(r_0(\tau), r_1(\tau), r_2(\tau), Z) = \frac{Z^2}{2} + q^2(\tau) + p^2(\tau) + 2p(\tau) + ik_\beta(\tau). \tag{4.14}$$

Now, we will prove that, for any τ the matrix function $M(\tau)$ is invertible. It follows from the fact that function k_β has an unique root $\tau = 0$. Then for $\tau \neq 0$ the determinant of M is nonzero. Now, for $\tau = 0$, we have that

$$\det M(r_0(0), r_1(0), r_2(0), Z) = \frac{Z^2}{2} + q^2(0) + p^2(0) + 2p(0). \tag{4.15}$$

By using (3.6) we have that $p(0) > 0$ under the assumption $\beta < 0$, then the expression (4.15) is nonzero.

Thus, we have proved that the matrix M in invertible for any τ $fixed$, then follows by (4.10) that

$$\mathbf{f} = M^{-1}\hat{\mathbf{F}}, \tag{4.16}$$

where $M^{-1}(\tau)$ denotes the inverse of matrix function $M(\tau)$. Then, formula (4.16) defines the functions f, g, and h in terms of the initial conditions $\widetilde{u}_0$ and $\widetilde{v}_0$, the unknown functions $\tilde{u}$ and $\tilde{v}$ and satisfy the boundary conditions (1.8). Finally, (4.2) and (4.3) joint with (4.16) define the integral equation that solve the Cauchy problem (1.7)–(1.8).

5 Energy Estimates for the Boundary Functions *f*, *g*, and *h*

In this section, we will estimate the boundary vector function $\mathbf{f} = (f, g, h)$. More precisely, we will demonstrate that the previously obtained functions f, g, and h belong to the trace spaces $H^{\frac{s+1}{3}}(\mathbb{R}^+)$, given the assumptions $\widetilde{u}_0, \widetilde{v}_0 \in H^s(\mathbb{R})$, and that the functions $\tilde{u}$ and $\tilde{v}$ are in $C([0, T]; H^s(\mathbb{R}))$. To simplify the computations we will consider $Z = 1$. An elementary computation yields the following formula for the matrix inverse of M

$$M^{-1} = \frac{1}{n_1}\begin{bmatrix} \frac{a_{11}}{n_2} & a_{12} & a_{13} \\ \frac{a_{21}}{n_2} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{5.1}$$

where the entries of the matrix are functions on the variable τ and they depend on the roots of algebraic equation (3.3) and they are given by

$$\begin{cases} n_1 = 2r_0^2 - 2r_0r_1 - 2r_0 - 2r_0r_2 + 2r_1 + 2r_1r_2 + 2r_2 + 1, \\ n_2 = 1 - 2r_0^2 - 2r_1r_2, \\ a_{11} = -4r_0^2r_1 - 4r_0^2r_2 - 4r_0^2r_1r_2 - 2r_0^2 + 2r_1 - 4r_1r_2^2 - 4r_1^2r_2^2 - 4r_1^2r_2 + 2r_2 + 1, \\ a_{21} = -2r_0^3 - 2r_0^2r_1 - 2r_0^2r_2 + 2r_0r_1r_2 - r_0 + r_1 - 2r_1r_2^2 - 2r_1^2 + 2r_2 - 2r_1r_2 + 1, \\ a_{31} = r_0(1 - 2r_0 - 2r_1r_2), \\ a_{12} = a_{22} = 2(r_1 + r_2 + 1), \\ a_{32} = 1 - 2r_0^2 - 2r_1r_2, \\ a_{13} = a_{23} = 2, \\ a_{33} = 2(r_0 - 1). \end{cases} \tag{5.2}$$

Thus, we have the following exact expressions for the functions $\hat{h}$, $\hat{f}$, and $\hat{g}$:

$$\begin{cases} \hat{h}(\tau) = \frac{1}{n_1}\Big(\frac{a_{11}}{n_2}\Big(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{12}\Big(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\Big) \\ \qquad + a_{13}\Big(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\Big)\Big) \\ \hat{f}(\tau) = \frac{1}{n_1}\Big(\frac{a_{21}}{n_2}\Big(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{22}\Big(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\Big) \\ \qquad + a_{23}\Big(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\Big)\Big) \\ \hat{g}(\tau) = \frac{1}{n_1}\Big(a_{31}\Big(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{32}\Big(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\Big) \\ \qquad + a_{33}\Big(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\Big)\Big). \end{cases} \tag{5.3}$$

The following lemma provides the necessary estimates for the functions that appear in (5.2) at high frequencies.

Lemma 5.1 *Let* $n_i = n_i(\tau)$ *given as in* (5.2)*,* $(i = 1, 2)$ *and* $a_{ij} = a_{ij}(\tau)$*,* $(i, j \in \{1, 2, 3\})$ *the functions given by* (5.2) *then hold*

$$\begin{aligned} |n_i(\tau)| &\gtrsim \langle\tau\rangle^{\frac{2}{3}}, (i = 1, 2) \text{ for all } |\tau| > 2; \\ |a_{11}(\tau)| &\lesssim \langle\tau\rangle^{4/3} \text{ for all } |\tau| > 2; \\ |a_{21}(\tau)| + |a_{31}(\tau)| &\lesssim \langle\tau\rangle \text{ for all } |\tau| > 2; \\ |a_{12}(\tau)| + |a_{22}(\tau)| + |a_{33}| &\lesssim \langle\tau\rangle^{1/3} \text{ for all } |\tau| > 2; \\ |a_{32}(\tau)| &\lesssim \langle\tau\rangle^{1/3} \text{ for all } |\tau| > 2; \\ |a_{13}(\tau)| + |a_{23}(\tau)| &\lesssim 1 \text{ for all } \tau \in \mathbb{R}. \end{aligned}$$

Proof The estimates of each a_{ij} follows directly from the explict formulas given by (5.2) and the estimates for r_0, r_1 and r_2 given in (3.5). The proof of n_i term follows from the explicit formulas for r_0, r_1 and r_2 we have that $\operatorname{Re} n_1(\tau) = (2p(\tau) + 1)^2$. Then the estimate (3.6) proves the lemma. □

The next step is to show that the unknown functions f and g are in $H^{\frac{s+1}{3}}(\mathbb{R})$ and h are contained in $H^{\frac{s}{3}}(\mathbb{R})$. To do this we use the decay of functions r_0, r_1, and r_2. By using Lemma 5.1 we have the following lemma.

Lemma 5.2 *Assume that $\tilde{u}$ and $\tilde{v}$ are in $C([0,T];H^s(\mathbb{R}))$, then for the functions f, g, and h given by the formula (5.3) hold the following estimate*

$$\begin{aligned}&\|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}+\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}+\|g\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}\\ &\quad\leq\|F_1(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}+\|F_2(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}\\ &\quad\quad+\|\partial_xF_1(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}+\|\partial_xF_2(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}\\ &\quad\quad+\left\|\partial_x^2F_1(0,t)\right\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}+\left\|\partial_x^2F_2(0,t)\right\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}.\end{aligned}$$

Proof The proof follows from the estimates

$$\begin{gathered}\left|\frac{a_{11}}{n_1n_2}\right|\lesssim 1,\ \left|\frac{a_{12}}{n_1}\right|\lesssim\langle\tau\rangle^{-\frac{1}{3}},\ \left|\frac{a_{13}}{n_1}\right|\lesssim\langle\tau\rangle^{-\frac{2}{3}}\\ \left|\frac{a_{21}}{n_1n_2}\right|\lesssim\langle\tau\rangle^{-1},\ \left|\frac{a_{22}}{n_1}\right|\lesssim\langle\tau\rangle^{-\frac{1}{3}},\ \left|\frac{a_{13}}{n_1}\right|\lesssim\langle\tau\rangle^{-\frac{2}{3}}\\ \left|\frac{a_{31}}{n_1}\right|\lesssim\langle\tau\rangle^{1/3},\ \left|\frac{a_{32}}{n_1}\right|\lesssim 1,\ \left|\frac{a_{33}}{n_1}\right|\lesssim(\tau)^{-\frac{2}{3}},\end{gathered}$$

which is a direct consequence of Lemma 5.1. □

Moreover, as the consequence of Lemmas 2.1 and 2.2, we have the following estimate for the trace functions:

$$\begin{aligned}&\|F_1(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}\\ &\quad+\|\partial_xF_1(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}+\left\|\partial_x^2F_1(0,t)\right\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}\lesssim\|u_0\|_{H^s(\mathbb{R}^+)}+\|u\|^2_{X^1_s};\\ &\|F_2(0,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}+\|\partial_xF_2(0,t)\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}\\ &\quad+\left\|\partial_x^2F_2(0,t)\right\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}\lesssim\|v_0\|_{H^s(\mathbb{R}^-)}+\|v\|^2_{X^2_s}.\end{aligned}\tag{5.4}$$

5.1 Proof of Theorem 1.2

Proof The proof will be based in the Banach's Fixed Point Theorem. By convenience in the exposition, we will consider $|\mathbf{E}_-|=|\mathbf{E}_+|=1$. Let $\mathbf{u}_0=(u_0,v_0)\in H^s(\mathcal{G})\cap\mathcal{N}_{0,Z}(s)$ with $\tilde{\mathbf{u}}_0=(\tilde{u}_0,\tilde{v}_0)\in H^s(\mathbb{R})\times H^s(\mathbb{R})$ such that $\tilde{u}_0|_{\mathbb{R}^+}=u_0$, $\tilde{v}_0|_{\mathbb{R}^-}=v_0$, and $\|\tilde{u}_0\|_{H^s(\mathbb{R})}\leq c\|u_0\|_{H^s(\mathbb{R}^+)}$, $\|\tilde{v}_0\|_{H^s(\mathbb{R})}\leq c\|v_0\|_{H^s(\mathbb{R}^-)}$. We consider the Banach space $X^1_s(\mathbb{R}^2)$

$$X^1_s(\mathbb{R}^2)=\left\{w:\mathbb{R}^2\to\mathbb{R}:\|w\|_{X^1_s}\equiv N(w)<\infty\right\},\tag{5.5}$$

where

$$N(w)=\max\left\{\|w\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))},\|w\|_{C\left(\mathbb{R}_x;H^{\frac{s+1}{3}}(\mathbb{R}_t)\right)},\right.$$

$$\|w\|_{X^{s,b,\beta,\sigma}}, \|w_x\|_{C\left(\mathbb{R}_x;H^{\frac{s}{3}}(\mathbb{R}_t)\right)}, \|w_{xx}\|_{C\left(\mathbb{R}_x;H^{\frac{s-1}{3}}(\mathbb{R}_t)\right)}\Big\}$$

For $X_s = X_s^1(\mathbb{R}^2) \times X_s^1(\mathbb{R}^2)$ and $\mathbf{u} = (\tilde{u}, \tilde{v}) \in X_s$ we will consider $\|\mathbf{u}\|_{X_s} = \|\tilde{u}\|_{X_s^1} + \|\tilde{v}\|_{X_s^1}$. Next, we define $\Lambda_{\tilde{\mathbf{u}}_0} : X_s \to X_s$ as

$$\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u}) = (\Lambda_1(\tilde{u}), \Lambda_2(\tilde{v}))$$

with

$$\Lambda_1(\tilde{u})(x,t) = \psi(t)S_\beta(t)\tilde{u}_0 + \psi(t)\mathcal{K}\left(\psi_T^2(t)\tilde{u}\partial_x\tilde{u}\right) + \psi(t)\mathcal{R}_\beta h(x,t) \tag{5.6}$$

and

$$\begin{aligned}\Lambda_2\tilde{v}(x,t) = \psi(t)S_\beta(t)\tilde{v}_0 + \psi(t)\mathcal{K}\left(\psi_T^2(t)\tilde{v}\partial_x\tilde{v}\right)\\ +\psi(t)\mathcal{L}_\beta^1 f(x,t) + \psi(t)\mathcal{L}_\beta^2 g(x,t),\end{aligned} \tag{5.7}$$

with R_β, L_β^j defined in (3.7)-(3.8)-(3.9), and $h = h(t)$, $f = f(t)$, $g = g(t)$ are given via the Fourier transform with regard to t by the formulas in (5.3), with $\hat{F}_j$ representing the Fourier transform with regard to t of the functions $F_j = F_j(x,t)$, $i = 1, 2$, given by the formulas (4.4) and (4.5).

Remark 5.3 Here, we consider the extended version on all $\mathbb{R}^2$ of the functions $\mathcal{R}_\beta h$, $\mathcal{L}_\beta^1 f$, and $\mathcal{L}_\beta^2 g$ given in Remark 3.5. Also, we note that the operator Λ depends on the extensions $\widetilde{u}_0$ and $\widetilde{v}_0$, but by Remark 3.4 we see that the restriction of the function $\Lambda\mathbf{u}\mid_{\mathcal{G}\times(0,T)}= \left(\Lambda_1(\tilde{u})\mid_{\mathbb{R}^+\times(0,T)}, \Lambda_2(\tilde{v})\mid_{\mathbb{R}^-\times(0,T)}\right)$ does not depend on these extensions. Moreover, the extension of the initial data $\mathbf{u}_0$ is necessary to apply the keys to strategy of Bourgain (1993) on whole the line and of Faminskii (2004, 2007) on half-line, and it do not bring problems of uniqueness of solutions.

Next, we show that there is a $\gamma > 0$ such that for $\mathbf{u} \in \bar{B}_\gamma(\mathbf{0}) = \{\mathbf{w} \in X_s : \|\mathbf{w}\|_{X_s} \leq \gamma\}$, we have $\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u}) \in B_\gamma(\mathbf{0})$ and $\Lambda_{\tilde{\mathbf{u}}_0} : \bar{B}_\gamma(\mathbf{0}) \to \bar{B}_\gamma(\mathbf{0})$ is a contraction.

Firstly, we start by estimating $\|\Lambda_1(\tilde{u})\|_{X_s^1}$. We have from definition of F_i at (4.4) and (4.5), Lemmas 5.1–5.2, and Lemmas 2.1, 2.2, 2.3 and 2.5 that for h in (5.3)

$$\begin{aligned}&\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq D_1 \textstyle\sum_{j=0}^2 \|\partial_x^j F_i(0,\tau)\|_{H^{\frac{s+1-j}{3}}(\mathbb{R}_t)}\\ &\quad\leq D_1 \textstyle\sum_{j=0}^2 \|\partial_x^j F_i(x,\tau)\|_{C(\mathbb{R}_x:H^{\frac{s+1-j}{3}}(\mathbb{R}_t))}\\ &\quad\leq C_0 D_1(\|\tilde{u}_0\|_{H^s(\mathbb{R})} + \|\tilde{v}_0\|_{H^s(\mathbb{R})}) + D_2(T^r)^2(\|\tilde{u}\|^2_{X^{s,b,\beta,\sigma}} + \|\tilde{v}\|^2_{X^{s,b,\beta,\sigma}})\\ &\quad\leq C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})} + D_3(T^r)^2\|\mathbf{u}\|^2_{X_s},\end{aligned} \tag{5.8}$$

with D_i, C_0 and C_3 are generic positive constants depending on Z and s which appear in Lemmas 5.3 and 5.4, and the power $r > 0$ of T is obtained in Lemma 2.3 or Remark

2.4. Thus, from Lemmas 2.1, 2.2, 2.3, Lemma 3.3 and 5.8 we get,

$$\begin{aligned}&\|\psi(t)S_\beta(t)\tilde{u}_0+\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}\tilde{u}))+\psi(t)R_\beta h\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))}\\&\quad\le C_0\|\tilde{u}_0\|_{H^s(\mathbb{R})}+C_1\|\psi_T^2\partial_x(\tilde{u}\tilde{u}))\|_{Y^{s,b,\beta,\sigma}}+C_2\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}\\&\quad\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2.\end{aligned}\tag{5.9}$$

Similarly, we have

$$\begin{aligned}&\sum_{j=0}^2\|\psi(t)\partial_x^jS_\beta(t)\tilde{u}_0+\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{u}\tilde{u}))+\psi(t)\partial_x^jR_\beta h\|_{C(\mathbb{R}_x;H^{\frac{s+1-j}{3}}(\mathbb{R}_t))}\\&\quad\le C_0\|\tilde{u}_0\|_{H^s(\mathbb{R})}+C_1\|\psi_T^2\partial_x(\tilde{u}\tilde{u}))\|_{Y^{s,b,\beta,\sigma}}+C_2\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}\\&\quad\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2\end{aligned}\tag{5.10}$$

and

$$\begin{aligned}&\|\psi(t)S_\beta(t)\tilde{u}_0+\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}\tilde{u}))+\psi(t)R_\beta h\|_{X^{s,b,\beta,\sigma}}\\&\quad\le C_0\|\tilde{u}_0\|_{H^s(\mathbb{R})}+C_1\|\psi_T^2\partial_x(\tilde{u}\tilde{u}))\|_{Y^{s,b,\beta,\sigma}}+C_2\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}\\&\quad\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2.\end{aligned}\tag{5.11}$$

Therefore, from (5.9)–(5.10)–(5.11) follows

$$\|\Lambda_1(\tilde{u})\|_{X_s^1}\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2.$$

Similarly, we obtain $\|\Lambda_2(\tilde{v})\|_{X_s^1}\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2$. Then,

$$\|\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u})\|_{X_s}\le C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}+2D_4(T^r)^2\|\mathbf{u}\|_{X_s}^2.$$

Thus, by choosing first $\gamma=2C_3\|\mathbf{u}_0\|_{H^s(\mathcal{G})}$ and then T such that $2D_4(T^r)^2\gamma<\frac{1}{16}$, we obtain $\|\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u})\|_{X_s}\le\gamma$. Therefore, $\Lambda_{\tilde{\mathbf{u}}_0}(\bar{B}_\gamma(\mathbf{0}))\subset\bar{B}_\gamma(\mathbf{0})$.

Now, to show that $\Lambda_{\tilde{\mathbf{u}}_0}$ is a contraction on $\bar{B}_\gamma(\mathbf{0})$, we argue as above and we get for $\mathbf{u},\mathbf{w}\in\bar{B}_\gamma(\mathbf{0})$

$$\|\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u})-\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{w})\|_{X_s}\le 8\gamma D_4(T^r)^2\|\mathbf{u}-\mathbf{w}\|_{X_s}<\frac{1}{4}\|\mathbf{u}-\mathbf{w}\|_{X_s}.\tag{5.12}$$

By convenience of the reader, we show the former estimate in the case of $\|\Lambda_1(\tilde{u})-\Lambda_1(w_1)\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))}$ with $\mathbf{w}=(w_1,w_2)$. Indeed, initially we have

$$\begin{aligned}&\|\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}^2-{w_1}^2))\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))}\le C_1\|\psi_T^2\partial_x[(\tilde{u}-w_1)(\tilde{u}+w_1)]\|_{Y^{s,b,\beta,\sigma}}\\&\quad\le D_4(T^r)^2\|\tilde{u}-w_1\|_{X^s}\|\tilde{u}+w_1\|_{X^s}\le 2\gamma D_4(T^r)^2\|\mathbf{u}-\mathbf{w}\|_{X_s},\end{aligned}\tag{5.13}$$

and

$$\|\psi(t)R_\beta(h-j)\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} \leq C_2\|h-j\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}$$

with j defined similarly as h in (5.3) with F_1 and F_2 changed, respectively, by

$$F_3(\tilde{u}_0, w_1, x, t) = \psi(t)S_\beta(t)\tilde{u}_0 + \psi(t)\mathcal{K}(\psi_T^2\partial_x(w_1)^2)$$

and

$$F_4(\tilde{u}_0, w_2, x, t) = \psi(t)S_\beta(t)\tilde{v}_0 + \psi(t)\mathcal{K}(\psi_T^2\partial_x(w_2)^2).$$

Thus, by following a similar argument as in (5.8) and (5.13) (with the same generic constants) we obtain for $\mathcal{Z} = C(\mathbb{R}_x; H^{\frac{s+1-j}{3}}(\mathbb{R}_t))$

$$\begin{aligned}\|h-j\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} &\leq D_1\sum_{j=0}^{2}\|\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{u}^2-w_1{}^2))\|_{\mathcal{Z}}\\ &\quad + D_1\sum_{j=0}^{2}\|\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{v}^2-w_2{}^2))\|_{\mathcal{Z}}\\ &\leq D_3(T^r)^2\|\tilde{u}-w_1\|_{X^s}\|\tilde{u}+w_1\|_{X^s} + D_3(T^r)^2\|\tilde{v}-w_2\|_{X^s}\|\tilde{v}+w_2\|_{X^s}\\ &\leq 2\gamma D_3(T^r)^2\|\mathbf{u}-\mathbf{w}\|_{X_s}. \end{aligned} \tag{5.14}$$

Therefore,

$$\|\Lambda_1(\tilde{u})-\Lambda_1(w_1)\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} \leq 4\gamma D_4(T^r)^2\|\mathbf{u}-\mathbf{w}\|_{X_s}.$$

Thus, by (5.12) there is a unique $\mathbf{u} \in \bar{B}_\gamma(\mathbf{0}) \subset X_s$ such that $\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u}) = \mathbf{u}$. Hence, as the linear restriction-mapping

$$\begin{aligned} \mathrm{R} : X_s &\to X_s(T)\\ \mathbf{w} &\mapsto \mathbf{w}|_{\mathcal{G}\times[0,T]} \end{aligned}$$

satisfies $\|\mathrm{R}\mathbf{w}\|_{X_s(T)} \leq 2\|\mathbf{w}\|_{X_s}$, we obtain that

$$(u_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = \mathbf{u}|_{\mathcal{G}\times(0,T)} \in X_s(T) \tag{5.15}$$

with $T < 1$, represents a strong solution of (1.7) and satisfies (1.8), moreover, $(u_{\mathbf{e}}(\cdot,0))_{\mathbf{e}\in\mathbf{E}} = \mathbf{u}_0$. Similarly, using the argument leading to (5.12), we obtain for $T_0 \in (0,T)$ that

$$\|\Lambda_{\tilde{\mathbf{u}}_0}(\mathbf{u})-\Lambda_{\tilde{\mathbf{p}}_0}(\mathbf{p})\|_{X_s} \leq C_3\|\tilde{\mathbf{u}}_0-\tilde{\mathbf{p}}_0\|_{H^s(\mathbb{R})\times H^s(\mathbb{R})} + 8\gamma D_4(T_0^r)^2\|\mathbf{u}-\mathbf{p}\|_{X_s}. \tag{5.16}$$

Thus, for $U_0 = B_\delta(\tilde{\mathbf{u}}_0) \subset H^s(\mathbb{R})\times H^s(\mathbb{R})$ with $\delta > 0$ such that $C_3\delta + 8\gamma D_4(T_0^r)^2 < \frac{1}{4}$, we get for $\tilde{\mathbf{p}}_0 \in U_0$ that $\mathbf{p}$ is a fixed point for $\Lambda_{\tilde{\mathbf{p}}_0}$, and so the map $\tilde{\mathbf{p}}_0 \in U_0 \mapsto \mathbf{p} \in X_s$

is Lipschitz. We also note that by using standard arguments, we can obtain uniqueness of $\mathbf{u}$ in the class X_s (see Kenig, Ponce and Vega Kenig et al. (1991)).

Next, let $T_0 \in (0, T)$ and we consider for $\delta_0 > 0$ (to be chosen) the open ball $W_0 = B_{\delta_0}(\mathbf{u}_0) \cap \mathcal{N}_{0,Z}(s) \subset H^s(\mathcal{G})$. For $\mathbf{p}_0 \in W_0$ we consider a extension $\tilde{\mathbf{p}}_0 \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ of $\mathbf{p}_0$ such that $\|\tilde{\mathbf{u}}_0 - \tilde{\mathbf{p}}_0\|_{H^s(\mathbb{R})\times H^s(\mathbb{R})} \leq 2\|\mathbf{u}_0 - \mathbf{p}_0\|_{H^s(\mathcal{G})}$. Therefore, by choosing δ_0 such that $2\delta_0 < \delta$ we obtain $\tilde{\mathbf{p}}_0 \in U_0$ and for $\mathbf{q} \in X_s$ such that $\mathbf{R}\mathbf{p} \in X_s(T_0)$ is the solution of (1.7) satisfying (1.8) and $\mathbf{R}\mathbf{p}(\cdot, 0) = \mathbf{p}_0$, we have from (5.16)

$$\|\mathbf{R}\mathbf{p} - \mathbf{R}\mathbf{u}\|_{X_s(T_0)} \leq 2\|\mathbf{p} - \mathbf{u}\|_{X_s} \leq 2C_3\|\tilde{\mathbf{u}}_0 - \tilde{\mathbf{p}}_0\|_{H^s(\mathbb{R})\times H^s(\mathbb{R})} \leq 4C_3\|\mathbf{u}_0 - \mathbf{p}_0\|_{H^s(\mathcal{G})}.$$

This shows that the mapping data-solution $\mathbf{p}_0 \in W_0 \mapsto \mathcal{R}\mathbf{p} \in X_s(T_0)$ is Lipschitz. This finishes the proof. □

Remark 5.4 As the consequence of our proof we have an exact formula for the group associated to the Airy operator A_Z. More precisely, we can define $\mathbf{S}(t) : D(A_Z) \subset H^3(\mathcal{G}) \to L^2(\mathcal{G})$ by

$$\mathbf{S}(t)\mathbf{w}_0 = \left((S_\beta(t)\tilde{u}_0 + R_\beta h(x,t))\Big|_{\mathbb{R}^+\times(0,T)}, (S_\beta(t)\tilde{v}_0 + L^1_\beta f(x,t) + L^2_\beta g(x,t))\Big|_{\mathbb{R}^-\times(0,T)} \right), \tag{5.17}$$

where $\mathbf{w}_0 = (u_0, v_0)$ and

$$\begin{cases} \hat{h}(\tau) = \frac{1}{n_1}\left(\frac{a_{11}}{n_2}\left(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{12}\left(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\right)\right.\right. \\ \qquad \left.\left. + a_{13}\left(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\right)\right)\right) \\ \hat{f}(\tau) = \frac{1}{n_1}\left(\frac{a_{21}}{n_2}\left(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{22}\left(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\right)\right.\right. \\ \qquad \left.\left. + a_{23}\left(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\right)\right)\right) \\ \hat{g}(\tau) = \frac{1}{n_1}\left(a_{31}\left(\hat{F}_1(0,\tau) - \hat{F}_2(0,\tau) + a_{32}\left(-\partial_x\hat{F}_1(0,\tau) + \partial_x\hat{F}_2(0,\tau)\right)\right.\right. \\ \qquad \left.\left. + a_{33}\left(-\partial_x^2\hat{F}_1(0,\tau) + \partial_x^2\hat{F}_1(0,\tau) + \frac{Z^2}{2}\hat{F}_2(0,\tau) + Z\partial_x\hat{F}_2(0,\tau)\right)\right)\right), \end{cases} \tag{5.18}$$

where, the functions F_1 and F_2 are given by

$$F_1(\tilde{u}_0, x, t) = S_\beta(t)\tilde{u}_0(x)$$

and

$$F_2(\tilde{v}_0, x, t) = S_\beta(t)\tilde{v}_0(x).$$

Remark 5.5 Since the Airy operator A_Z is skew self adjoint on $D(A_Z)$ follows from the theory of semigroups that A_Z is the generator of a unique unitary group $\{\mathbf{S}(t)\}_{t\in\mathbb{R}}$ on $L^2(\mathcal{G})$. Then, we have that $\mathbf{S}(t)$ defined in (5.17) is the only group associated to Airy operator A_Z.

Also, combining (5.6) and (5.7) with (5.17) we point out that it is possible to rewrite the Duhamel integral formula associated to (1.7), for $t \in [0, T]$ as the following

$$\mathbf{u}(t) = \mathbf{S}(t)\mathbf{u}_0 + \int_0^t \mathbf{S}(t-t')(\mathbf{u}\mathbf{u}_x)dt', \tag{5.19}$$

where we denote $\mathbf{u} := (u, v)$.

5.2 Proof of Theorem 1.7

Proof The proof will be based in the Implicit Function Theorem and our estimates used in the proof of Theorem 1.2. By convenience in the exposition, we will consider $|\mathbf{E}_-| = |\mathbf{E}_+| = 1$. Let $\mathbf{u}_0 = (u_0, v_0) \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$, and $\mathbf{u} \in X_1(T)$, $T = T(\|\mathbf{u}_0\|_1)$, the unique solution of the IVP (1.7)-(1.8), on the class $X_s(T)$, given by Theorem 1.2 with initial data $\mathbf{u}_0$. Moreover, for $\mathbf{u} = (u, v) = (\tilde{u}|_{\mathbb{R}^+}, \tilde{v}|_{\mathbb{R}^-})$, where $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ is the fixed point obtained for $\Lambda = (\Lambda_1, \Lambda_2)$ on X_1 and Λ_i determined by (5.6)–(5.7).

Next, let $\mathcal{W} \subset H^1(\mathbb{R}) \times H^1(\mathbb{R})$ be a neighborhood of the extension $\tilde{\mathbf{u}}_0 = (\tilde{u}_0, \tilde{v}_0)$ of $\mathbf{u}_0$ such that $\tilde{u}_0|_{\mathbb{R}^+} = u_0$, $\tilde{v}_0|_{\mathbb{R}^-} = v_0$, and $\|\tilde{u}_0\|_{H^1(\mathbb{R})} \leq c\|u_0\|_{H^1(\mathbb{R}^+)}$, $\|\tilde{v}_0\|_{H^1(\mathbb{R})} \leq c\|v_0\|_{H^1(\mathbb{R}^-)}$. For $X_1 = X_1^1(\mathbb{R}^2) \times X_1^1(\mathbb{R}^2)$ with $X_1^1(\mathbb{R}^2)$ determined in (5.5), we define the mapping

$$\mathcal{H} : \mathcal{W} \times X_1 \to X_1$$

for $(\mathbf{w}_0, \mathbf{w}) \in \mathcal{W} \times X_1$ as

$$\mathcal{H}(\mathbf{w}_0, \mathbf{w})(t) \equiv \mathbf{w}(t) - \Big(\psi(t)\mathbf{S}_\beta(t)\mathbf{w}_0 + \psi(t)D(\mathbf{w})\Big),$$

where for $\mathbf{w}_0 = (\phi_0, \psi_0)$ we have

$$\mathbf{S}_\beta(t)\mathbf{w}_0 = \Big(S_\beta(t)\phi_0, S_\beta(t)\psi_0\Big), \tag{5.20}$$

and $D(\mathbf{w})$ is defined for $\mathbf{w} = (w_1, w_2) \in X_1$ by

$$D(\mathbf{w}) = \Big(\mathcal{K}(\psi_T^2(t)w_1\partial_x w_1) + R_\beta H, \mathcal{K}(\psi_T^2(t)w_2\partial_x w_2) + L_\beta^1 F + L_\beta^2 G\Big) \tag{5.21}$$

with R_β, L_β^j defined in (3.7)–(3.8)–(3.9), with $H = H(t), F = F(t), G = G(t)$ given via the Fourier transform with regard to t by the formulas in (5.3), with $\hat{F}_j$ representing the Fourier transform with regard to t of the function $F_j = F_j(x, t)$, $i = 1, 2$, determined for $(x, t) \in \mathbb{R}$ by

$$\begin{cases} F_1(x,t) \equiv F_1(\phi_0, w_1, x, t) = \psi(t)S_\beta(t)\phi_0(x) + \psi(t)\mathcal{K}(\psi_T^2 w_1\partial_x w_1)(x,t), \\ F_2(x,t) \equiv F_2(\psi_0, w_2, x, t) = \psi(t)S_\beta(t)\psi_0(x) + \psi(t)\mathcal{K}(\psi_T^2 w_2\partial_x w_2)(x,t). \end{cases} \tag{5.22}$$

Thus, by the analysis in the proof of Theorem 1.2 we get for the fixed point $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ of Λ that $\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}) = \mathbf{0}$.

Next, we show that the linear application $D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}) \equiv \partial_{\mathbf{w}}\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})$ is invertible. Indeed, we start by determining one formula for the expresssion

$$D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})(\mathbf{w}) = \frac{d}{d\epsilon}\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}} + \epsilon\mathbf{w})|_{\epsilon=0}$$

with $\mathbf{w} \in X_1$. Thus, we get

$$D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})(\mathbf{w}) = \mathbf{w} - \psi(t)\frac{d}{d\epsilon}D(\tilde{\mathbf{u}} + \epsilon\mathbf{w})|_{\epsilon=0}. \tag{5.23}$$

Next, by (5.21) we get the following first relation for $\mathbf{w} = (w_1, w_2)$

$$\begin{aligned}&\frac{d}{d\epsilon}(\frac{1}{2}\mathcal{K}(\psi_T^2\partial_x(\tilde{u} + \epsilon w_1)^2), \frac{1}{2}\mathcal{K}(\psi_T^2\partial_x(\tilde{v} + \epsilon w_2)^2)|_{\epsilon=0}\\ &= (\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)), \mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2)).\end{aligned} \tag{5.24}$$

Now, by (5.22) we get for $F_{1,\epsilon} = F_1(\phi_0, \tilde{u} + \epsilon w_1, x, t)$ and $F_{2,\epsilon} = F_2(\psi_0, \tilde{v} + \epsilon w_2, x, t)$ that

$$\begin{cases}\frac{d}{d\epsilon}\hat{F}_{1,\epsilon}(x, \tau)|_{\epsilon=0} = [\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1))]^\wedge(x, \tau) \equiv \widehat{M}_1(x, \tau)\\ \frac{d}{d\epsilon}\hat{F}_{2,\epsilon}(x, \tau)|_{\epsilon=0} = [\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2))]^\wedge(x, \tau) \equiv \widehat{M}_2(x, \tau)\end{cases} \tag{5.25}$$

Thus, we establish that

$$\begin{cases}\widehat{M}_1(0, \tau) = [\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1))]^\wedge(0, \tau)\\ \widehat{M}_2(0, \tau) = [\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2))]^\wedge(0, \tau).\end{cases} \tag{5.26}$$

Similarly, we obtain from the notation $\partial_x^j J(0, \tau) = \lim_{x\to 0}\partial_x^j J(x, \tau)$, that for $j = 1, 2$,

$$\begin{cases}\widehat{\partial_x^j M_1}(0, \tau) = \frac{d}{d\epsilon}\partial_x^j\hat{F}_{1,\epsilon}(0, \tau)|_{\epsilon=0} = [\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1))]^\wedge(0, \tau)\\ \widehat{\partial_x^j M_2}(0, \tau) = \frac{d}{d\epsilon}\partial_x^j\hat{F}_{2,\epsilon}(0, \tau)|_{\epsilon=0} = [\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2))]^\wedge(0, \tau),\end{cases} \tag{5.27}$$

we recall that "$\widehat{\ }$" represents the Fourier transform with regard to the time. Therefore, for $M = M(t)$ we obtain

$$\frac{d}{d\epsilon}R_\beta H|_{\epsilon=0}(x, t) = R_\beta M(x, t),$$

where

$$\begin{aligned}\widehat{M}(\tau) &= \tfrac{1}{n_1}\Big(\tfrac{a_{11}}{n_2}\big(\widehat{M}_1(0, \tau) - \widehat{M}_2(0, \tau) + a_{12}\big(-\widehat{\partial_x M_1}(0, \tau) + \widehat{\partial_x M_2}(0, \tau)\big)\\ &\quad + a_{13}\big(-\widehat{\partial_x^2 M_1}(0, \tau) + \widehat{\partial_x^2 M_2}(0, \tau) + \tfrac{Z^2}{2}\widehat{M}_2(0, \tau) + Z\widehat{\partial_x M_2}(0, \tau)\big)\big)\Big).\end{aligned} \tag{5.28}$$

Similarly, for $N = N(t)$ and $P = P(t)$ we have

$$\frac{d}{d\epsilon}\mathcal{L}^1_\beta F|_{\epsilon=0}(x,t) = \mathcal{L}^1_\beta N(x,t), \quad \frac{d}{d\epsilon}\mathcal{L}^2_\beta G|_{\epsilon=0}(x,t) = \mathcal{L}^2_\beta P(x,t)$$

where

$$\begin{aligned}
\widehat{N}(\tau) &= \tfrac{1}{n_1}\big(\tfrac{a_{21}}{n_2}\big(\widehat{M}_1(0,\tau) - \widehat{M}_2(0,\tau) + a_{22}\big(-\widehat{\partial_x M}_1(0,\tau) + \widehat{\partial_x M}_2(0,\tau)\big) \\
&\quad + a_{23}\big(-\widehat{\partial_x^2 M}_1(0,\tau) + \widehat{\partial_x^2 M}_2(0,\tau) + \tfrac{Z^2}{2}\widehat{M}_2(0,\tau) + Z\widehat{\partial_x M}_2(0,\tau)\big)\big), \\
\widehat{P}(\tau) &= \tfrac{1}{n_1}\big(a_{31}\big(\widehat{M}_1(0,\tau) - \widehat{M}_2(0,\tau) + a_{32}\big(-\widehat{\partial_x M}_1(0,\tau) + \widehat{\partial_x M}_2(0,\tau)\big) \\
&\quad + a_{33}\big(-\widehat{\partial_x^2 M}_1(0,\tau) + \widehat{\partial_x^2 M}_2(0,\tau) + \tfrac{Z^2}{2}\widehat{M}_2(0,\tau) + Z\widehat{\partial_x M}_2(0,\tau)\big)\big).
\end{aligned} \tag{5.29}$$

Therefore, from (5.23)–(5.24) and (5.28)–(5.29) we get

$$\begin{aligned}
D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})(\mathbf{w}) = \mathbf{w} - \psi(t)(\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)) \\
+ R_\beta M, \mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2)) + \mathcal{L}^1_\beta N + \mathcal{L}^2_\beta P).
\end{aligned} \tag{5.30}$$

Now, we show that $\|D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}) - I\|_{X_1} < 1$ and therefore $D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})$ will be invertible. Thus, we need to estimate every component in (5.30) in the norm of $X_1^1(\mathbb{R}^2)$. For the first component, by (5.28), definition of M_i at (5.25), Lemmas 5.1, 5.2, and Lemmas 2.2, 2.3, and (2.5) we obtain for $s = 1$

$$\begin{aligned}
\|M\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} &\le D_1 \textstyle\sum_{j=0}^2 \|\partial_x^j M_i(0,\tau)\|_{H^{\frac{s+1-j}{3}}(\mathbb{R}_t)} \\
&\le D_1 \textstyle\sum_{j=0}^2 \|\partial_x^j M_i(x,\tau)\|_{C(\mathbb{R}_x : H^{\frac{s+1-j}{3}}(\mathbb{R}_t))} \\
&\le D_2(T^r)^2\|\tilde{\mathbf{u}}\|_{X^{s,b,\beta,\sigma}}\|\mathbf{w}\|_{X^{s,b,\beta,\sigma}} \le D_2(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.
\end{aligned} \tag{5.31}$$

with D_i generic positive constants depending on Z and from the estimative-constants in Lemmas 2.2, 2.3, and 2.5. Thus, from Lemmas 2.2, 2.3, 2.5, Lemma 3.3, and (5.31) we get for $s = 1$,

$$\begin{aligned}
&\|\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)) + \psi(t)R_\beta M\|_{C(\mathbb{R}_t;H^s(\mathbb{R}_x))} \\
&\le C_1\|\psi_T^2\partial_x(\tilde{u}w_1))\|_{Y^{s,b,\beta,\sigma}} + C_2\|M\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \\
&\le D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.
\end{aligned} \tag{5.32}$$

Similarly we obtain for $s = 1$ that

$$\begin{aligned}
&\textstyle\sum_{j=0}^2 \|\psi(t)\partial_x^j\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)) + \psi(t)\partial_x^j R_\beta M\|_{C(\mathbb{R}_x;H^{\frac{s+1-j}{3}}(\mathbb{R}_t))} \\
&\le C_1\|\psi_T^2\partial_x(\tilde{u}w_1))\|_{Y^{s,b,\beta,\sigma}} + C_2\|M\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \\
&\le D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.
\end{aligned} \tag{5.33}$$

Next, we obtain again by Lemmas 2.2,2.3, 3.3, and (5.31) that for $s = 1$

$$\begin{aligned}&\|\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)) + \psi(t)R_\beta M\|_{X^{s,b,\beta,\sigma}}\\&\leq C_1\|\psi_T^2\partial_x(\tilde{u}w_1))\|_{Y^{s,b,\beta,\sigma}} + C_2\|M\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}\\&\leq D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.\end{aligned} \tag{5.34}$$

Therefore from (5.32)-(5.33)-(5.34) we have for the first component,

$$\|\psi(t)\mathcal{K}(\psi_T^2\partial_x(\tilde{u}w_1)) + \psi(t)R_\beta M\|_{X_1^1} \leq D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.$$

Similarly for the second component,

$$\|\mathcal{K}(\psi_T^2\partial_x(\tilde{v}w_2)) + \mathcal{L}_\beta^1 N + \mathcal{L}_\beta^2 P\|_{X_1^1} \leq D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1}.$$

Now, by the proof of Theorem 1.1, we know $\tilde{\mathbf{u}} \in B_\gamma(\mathbf{0}) = \{\mathbf{p} \in X_1 : \|\mathbf{p}\|_{X_1} < \gamma\}$ ($\gamma = 2c\|\mathbf{u}_0\|_{H^1(\mathcal{G})}$) and $2D_3(T^r)^2\gamma < \frac{1}{16}$. Therefore,

$$\|D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})(\mathbf{w}) - \mathbf{w}\|_{X_1} \leq 2D_3(T^r)^2\|\tilde{\mathbf{u}}\|_{X_1}\|\mathbf{w}\|_{X_1} < \frac{1}{16}\|\mathbf{w}\|_{X_1}.$$

Therefore, from operator's theory we conclude that $D_2\mathcal{H}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}})$ is invertible.

Next, it is not difficult to show that $\mathcal{H}$ is of class C^2. Therefore, there exists a unique continuous map of class C^2, $\Phi : \mathcal{W}_0 \to X_1$, defined on an open ball $\mathcal{W}_0 = B_\delta(\tilde{\mathbf{u}}_0)$ of $\tilde{\mathbf{u}}_0$ such that $\Phi(\tilde{\mathbf{u}}_0) = \tilde{\mathbf{u}}$ and $\mathcal{H}(\mathbf{z}_0, \Phi(\mathbf{z}_0)) = \mathbf{0}$ for all $\mathbf{z}_0 = (p_0, q_0) \in \mathcal{W}_0$. Hence, for $\mathbf{z} = \Phi(\mathbf{z}_0)$ it follows from the definition of the functional $\mathcal{H}$ that $\mathbf{z}$ satisfies the equation

$$\mathbf{z}(t) = \psi(t)\mathbf{S}_\beta(t)\mathbf{z}_0 + \psi(t)D(\mathbf{z}), \quad t \in \mathbb{R}$$

with $D(\mathbf{z})$ defined for $\mathbf{z} = (z_1, z_2) \in X_1$ by (5.21), and H, F, G by the formulas in (5.3), with $\hat{F}_j$ determined by $F_1(p_0, z_1, x, t)$ and $F_2(q_0, z_2, x, t)$ in (5.22).

In the following we show that the following mapping data-solution associated to (1.7)–(1.8)

$$\begin{aligned}\Psi : B_{\frac{\delta}{2}}(\mathbf{u}_0) \subset H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1) &\to X_1(T)\\ \mathbf{j} = (f_0, g_0) &\mapsto \mathbf{z} = \Phi(\mathbf{z}_0)|_{\mathcal{G}\times[0,T]}\end{aligned} \tag{5.35}$$

is of class C^2, where $\mathbf{z}_0 = (p_0, q_0) \in B_\delta(\tilde{\mathbf{u}}_0)$ represents the even-extension of $\mathbf{j}$, namely, p_0, q_0 are even functions on whole of the line with $p_0|_{H^1(\mathbb{R}^+)} = f_0$, $q_0|_{H^1(\mathbb{R}^-)} = g_0$ and $\|p_0\|_{H^1(\mathbb{R})} \leq 2\|f_0\|_{H^1(\mathbb{R}^+)}$, $\|q_0\|_{H^1(\mathbb{R})} \leq 2\|g_0\|_{H^1(\mathbb{R}^-)}$. We note that we can choose without loss of generality that $\tilde{\mathbf{u}}_0$ is the par-extension of $\mathbf{u}_0$. For showing that Ψ in (5.35) is of class C^2 we will write it as a composition of C^2-maps. We start by considering the even-extension mapping

$$\mathcal{E} : B_{\frac{\delta}{2}}(\mathbf{u}_0) \subset H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1) \to B_\delta(\tilde{\mathbf{u}}_0)$$

which is well defined because $\|\mathcal{E}(\mathbf{j})-\tilde{\mathbf{u}}_0\|_{H^s(\mathbb{R})\times H^s(\mathbb{R})} = \|\mathcal{E}(\mathbf{j}-\mathbf{u}_0)\|_{H^s(\mathbb{R})\times H^s(\mathbb{R})} \leq \sqrt{2}\|\mathbf{j}-\mathbf{u}_0\|_{H^s(\mathcal{G})} < \sqrt{2}\frac{\delta}{2} < \delta$. Moreover, it is not difficult to see that $\mathcal{E}$ is a Lipschitz mapping of class C^∞. Next, we consider the linear restriction-mapping

$$\begin{aligned}\mathcal{R} : X_1 &\to X_1(T)\\ \mathbf{w} &\mapsto \mathbf{w}|_{\mathcal{G}\times[0,T]}\end{aligned}$$

which is well-defined and continuous. Indeed, we have $\|\mathcal{R}\mathbf{w}\|_{X_1(T)} \leq 2\|\mathbf{w}\|_{X_1}$. The proof of the former inequality follows by using the restriction-norm associated to every norm in X_1. For instance, in the case of the norm $\|\cdot\|_{C([0,T];H^1(\mathcal{G}))}$ we have for $\mathbf{w}=(f,g)\in X_1$ and for every $t\in[0,T]$ fixed,

$$\|f(t)\|_{H^1(\mathbb{R}^+)} = inf\{\|q\|_{H^1(\mathbb{R}_x)} : q|_{\mathbb{R}^+} = f(t)\} \leq \|f(t)\|_{H^1(\mathbb{R}_x)} \leq \|\mathbf{w}\|_{C(\mathbb{R}_t;H^1(\mathbb{R}_x))}$$

and

$$\|g(t)\|_{H^1(\mathbb{R}^-)} \leq \|\mathbf{w}\|_{C(\mathbb{R}_t;H^1(\mathbb{R}_x))}.$$

Therefore, $\|\mathcal{R}\mathbf{w}\|_{C([0,T];H^1(\mathcal{G}))} \leq 2\|\mathbf{w}\|_{X_1}$. Moreover, $\mathcal{R}$ is a C^∞-mapping. Lastly, since $\Psi = \mathcal{R}\circ\Phi\circ\mathcal{E}$ follows that Ψ is of class C^2. This finishes the proof. □

6 Nonlinear Instability

The focus of this section is to show the nonlinear instability result in Theorem 1.9. For convenience of the reader we give a brief review of the results in Angulo and Cavalcante (2021) which will be sufficient in our proof.

We consider the family of stationary profiles for the KdV model on a balanced graph given by $(\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}} = U_Z = (u_-, u_+) \in D(A_Z)$ with $u_- = (\phi_-)_{\mathbf{e}\in\mathbf{E}_-}$, $u_+ = (\phi_+)_{\mathbf{e}\in\mathbf{E}_+}$ defined in (1.14) and $Z \neq 0$. Next, we suppose for $\mathbf{e}\in\mathbf{E}$, that $u_{\mathbf{e}}$ satisfy formally the KdV equation in (1.7) and define

$$v_{\mathbf{e}}(x,t) \equiv u_{\mathbf{e}}(x,t) - \phi_{\mathbf{e}}(x). \tag{6.1}$$

Then, for each $\mathbf{e}\in\mathbf{E}$ we have the equation

$$\partial_t v_{\mathbf{e}} = \alpha_{\mathbf{e}}\partial_x^3 v_{\mathbf{e}} + \beta_{\mathbf{e}}\partial_x v_{\mathbf{e}} + 2\partial_x(v_{\mathbf{e}}\phi_{\mathbf{e}}) + \partial_x(v_{\mathbf{e}}^2). \tag{6.2}$$

Thus, we have that the following system (where we are abusing notation)

$$\partial_t v_{\mathbf{e}}(x,t) = \alpha_{\mathbf{e}}\partial_x^3 v_{\mathbf{e}}(x,t) + \beta_{\mathbf{e}}\partial_x v_{\mathbf{e}}(x,t) + 2\partial_x(v_{\mathbf{e}}(x,t)\phi_{\mathbf{e}}(x)), \tag{6.3}$$

represents the linearized equation for the KdV in (1.7) around $(\phi_{\mathbf{e}}(x))_{\mathbf{e}\in\mathbf{E}}$. Next, we looking for a *growing mode solution* of (6.3) with the form

$$v_{\mathbf{e}}(x,t) = e^{\lambda t}\psi_{\mathbf{e}} \text{ and } \mathrm{Re}(\lambda) > 0.$$

In other words, we need to solve the formal system for $\mathbf{e} \in \mathbf{E}$ (we recall $(\alpha_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} = (\alpha_+)$, $(\beta_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} = (\beta_+)$, $\alpha_+ > 0$ and $\beta_+ < 0$),

$$\lambda\psi_\mathbf{e} = -\partial_x \mathcal{L}_\mathbf{e}\psi_\mathbf{e}, \qquad \mathcal{L}_\mathbf{e} = -\alpha_+\partial_x^2 - \beta_+ - 2\phi_\mathbf{e}, \tag{6.4}$$

Next, we write our eigenvalue problem in (6.4) in an Hamiltonian matrix. Indeed, for $\psi = (\psi_-, \psi_+)$ with $\psi_- = (\psi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}_-}$ and $\psi_+ = (\psi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}_+}$, we set $(\mathcal{L}_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} = (\mathcal{L}_-, \mathcal{L}_+)$ where

$$\begin{aligned} \mathcal{L}_-\psi_- &= (-\alpha_+\partial_x^2\psi_\mathbf{e} - \beta_+\psi_\mathbf{e} - 2\phi_\mathbf{e}\psi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}_-}, \\ \mathcal{L}_+\psi_+ &= (-\alpha_+\partial_x^2\psi_\mathbf{e} - \beta_+\psi_\mathbf{e} - 2\phi_\mathbf{e}\psi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}_+}. \end{aligned} \tag{6.5}$$

Thus, eigenvalue problem in (6.4) can be written in a Hamiltonian vectorial form

$$\lambda \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\partial_x\mathcal{L}_- & 0 \\ 0 & -\partial_x\mathcal{L}_+ \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \equiv NE \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{6.6}$$

where we are identifying $\mathcal{L}_\pm$ as a $n \times n$-diagonal matrix and N, E are $2n \times 2n$-diagonal matrix defined by

$$N = \begin{pmatrix} -\partial_\mathbf{x} I & 0 \\ 0 & -\partial_\mathbf{x} I \end{pmatrix}, \quad E = \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix}, \tag{6.7}$$

with I being the $n \times n$-identity matrix and $\partial_\mathbf{x}$ the $n \times n$-diagonal matrix $\partial_\mathbf{x} = \text{diag}\Big(\partial_x, ..., \partial_x)$

If we denote by $\sigma(NE) = \sigma_p(NE) \cup \sigma_{ess}(NE)$ the spectrum of NE (namely, $\lambda \in \sigma_p(NE)$ if λ is isolated with finite multiplicity, and $\sigma_{ess}(NE)$ is the essential spectrum), the later discussion suggests the usefulness of the following definition:

Definition 6.1 The stationary vector solution $(\phi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} \in D(H_Z)$ is said to be *linearly stable* for model (1.7) if the spectrum of NE, $\sigma(NE)$, satisfies $\sigma(NE) \subset i\mathbb{R}$. Otherwise, the stationary solution $(\phi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}}$ is said to be *linearly unstable*.

It is standard to show that $\sigma(NE)$ is symmetric with respect to both the real and imaginary axes and $\sigma_{ess}(NE) \subset i\mathbb{R}$ by supposing N skew-symmetric and E self-adjoint (see, for instance, (Grillakis et al. 1990 Lemma 5.6 and Theorem 5.8)). Thus, by considering for $\mathbf{u} = (u_\mathbf{e})_{\mathbf{e}\in\mathbf{E}}$ the notation

$$\mathbf{u} = (u_{1,-}, ..., u_{n,-}, u_{1,+}, ..., u_{n,+}),$$

we define the set of elements of $L^2(\mathcal{G})$ continuous at the graph-vertex $\nu = 0$ as

$$\mathcal{C} = \{\mathbf{u} \in L^2(\mathcal{G}) : u_{1,-}(0-) = ... = u_{n,-}(0-) = u_{1,+}(0+) = ... = u_{n,+}(0+)\}. \tag{6.8}$$

Then from Lemma 6.4 and Proposition 7.4 in Angulo and Cavalcante (2021), we obtain that for the following domain

$$D(E) = \left\{u \in H^2(\mathcal{G}) \cap \mathcal{C} : \sum_{\mathbf{e}\in\mathbf{E}_+} u'_{\mathbf{e}}(0+) - \sum_{\mathbf{e}\in\mathbf{E}_-} u'_{\mathbf{e}}(0-) = Znu_{1,+}(0+)\right\}, \quad (6.9)$$

E in (6.7) is a self-adjoint operator. Moreover, N in (6.7) will be skew-symmetric in the domain $H^1(\mathcal{G}) \cap \mathcal{C}$. Hence, by the comments above follows that will be equivalent to say that $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_Z)$ is *linearly stable* if $\sigma_p(NE) \subset i\mathbb{R}$, and it is linearly unstable if $\sigma_p(NE)$ contains points λ with $\mathrm{Re}(\lambda) > 0$. We note that $D(A_Z) \cap \mathcal{C} \subset D(E)$ and $(\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} = U_Z \in D(E)$.

Thus from Theorems 4.4, 4.6, and 6.1 in Angulo and Cavalcante (2021), we have the following linear instability result.

Theorem 6.2 *Let $Z \neq 0$. For $\alpha_+ > 0$, $\beta_+ < 0$, and $-\frac{\beta_+}{\alpha_+} > \frac{Z^2}{4}$, we consider the profiles $\phi_\pm$ in (1.13). Define $U_Z = (\phi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_Z)$ with $\phi_{\mathbf{e}} = \phi_-$ for $\mathbf{e} \in \mathbf{E}_-$ and $\phi_{\mathbf{e}} = \phi_+$ for $\mathbf{e} \in \mathbf{E}_+$. Then,*

$$\Phi_Z(x,t) = U_Z(x)$$

defines a family of linearly unstable stationary solutions for the Korteweg–de Vries equation in (1.7). Moreover, the operator NE in (6.7) has a real positive eigenvalue, namely, there is $\lambda > 0$, $\Psi_\lambda \in D(A_Z) \cap \mathcal{C}$, $\Psi_\lambda \neq 0$, such that $NE\Psi_\lambda = \lambda\Psi_\lambda$.

The strategy of proof for Theorem 1.9 is to use the linear instability result in Theorem 6.2, the approach by Henry et al. (1982), and that the mapping data-solution associated to the IVP in (1.7) is of class C^2 on $H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$ (see Theorems 1.7 and 1.9). For convenience of the reader, we establish the following theorem which is the link to obtain nonlinear instability from a linear instability result (see Henry et al. (1982); Angulo and Natali (2016); Angulo et al. (2008)).

Theorem 6.3 (Henry et al. 1982) *Let Y be a Banach space and $\Omega \subset Y$ an open set containing 0. Suppose $\mathcal{T} : \Omega \to Y$ has $\mathcal{T}(0) = 0$, and for some $p > 1$ and continuous linear operator $\mathcal{L}$ with spectral radius $r(\mathcal{L}) > 1$ we have $\|\mathcal{T}(x) - \mathcal{L}x\|_Y = O(\|x\|_Y^p)$ as $x \to 0$. Then 0 is unstable as a fixed point of $\mathcal{T}$.*

Remark 6.4 The statement in Theorem 6.3 establishes the instability of 0 as a fixed point of $\mathcal{T}$; in other words, it shows the existence of points moving away from 0 under successive applications of $\mathcal{T}$

Theorem 6.3 can be recast in a more suitable form for applications to nonlinear wave instability (see also Angulo and Natali (2014, 2016)).

Corollary 6.5 *Let $S : \Omega_1 \subset Y \to Y$ be a C^2 map defined in an open neighborhood of a fixed point ϕ of S. If there is an element $\mu \in \sigma(S'(\phi))$ with $|\mu| > 1$ then ϕ is an unstable fixed point of S.*

Proof For $x \in \Omega \equiv \{y - \phi : y \in \Omega_1\}$ we consider the mapping $\mathcal{T}(x) \equiv S(x+\phi) - \phi$. Then, clearly, $\mathcal{T}(0) = 0$ and $\mathcal{T}$ is of class C^2 in Ω. Define $\mathcal{D} \equiv S'(\phi)$. Then, by hypothesis, there is an eigenvalue $\mu \in \sigma(\mathcal{D})$ with $1 < |\mu| \leq r(\mathcal{D})$. By Taylor's formula

$$\mathcal{T}(x) = \mathcal{T}(0) + \mathcal{T}'(0)x + O(\|x\|_Y^2) = \mathcal{D}x + O(\|x\|_Y^2)$$

provided that $\|x\|_Y \ll 1$. Then Theorem 6.3 implies the existence of $\epsilon_0 > 0$ such that, for any ball $B_\eta(\phi)$, with radius $\eta > 0$ and arbitrarily large $N_0 \in \mathbb{N}$, there exists $n \geq N_0$ and $y \in B_\eta(\phi)$ such that $\|S^n(y) - \phi\|_Y \geq \epsilon_0$. This completes the proof. □

.

Before proving our Theorem 1.9, we need to specify the particular mapping S in Corollary 6.5 suitable for our needs. We start by introducing the following notation: The unique solution $\mathbf{u}$ to the Cauchy problem (1.7)–(1.8) with initial datum $\varphi \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$ given by Theorem 1.2 will denoted as $\mathbf{u} = \Upsilon(\varphi) \in X_1(T) \subset C([0,T] : H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1))$. The former relation follows from the first relation in (1.8) and Sobolev's embedding.

Let us now define a mapping which plays the role of the operator S in the abstract Corollary 6.5. Let $\Phi = (\phi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}}$ be a stationary profile for equation (1.7), and B the ball $B = B_\epsilon(\Phi) = \{\varphi \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1) : \|\varphi - \Phi\|_1 < \epsilon\}$ with $\epsilon > 0$. For each $\varphi \in B$, set

$$\begin{aligned} S &: B \to H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1), \\ S(\varphi) &:= \Upsilon(\varphi)(1) \end{aligned} \tag{6.10}$$

where $\Upsilon(\varphi) \in C([0,1]; H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1))$ is the unique solution to the Cauchy problem (1.7)-(1.8) with $\Upsilon(\varphi)(0) = \varphi$. We are choosing $T = 1$ in Theorem 1.2 without loss of generality. We recall that by the continuity property of the mapping data-to-solution given in Theorem 1.2, we can choose ϵ small enough such that the solution $\Upsilon(\varphi)$ for every $\varphi \in B_\epsilon(\Phi)$ can be defined on $[0,1]$ because the stationary solution Φ is defined for all $t \in \mathbb{R}$. Thus, S is a well-defined mapping.

Lemma 6.6 (properties of S) *Let $\Phi = (\phi_\mathbf{e})_{\mathbf{e}\in\mathbf{E}} \in D(A_Z)$ be a stationary profile for equation the KdV model in* (1.7). *The mapping S defined in* (6.10) *satisfies:*

(a) $S(\Phi) = \Phi$.
(b) *S is twice Fréchet differentiable in an open neighborhood of Φ.*
(c) *For every $\varphi \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1))$ there holds*

$$S'(\Phi)\varphi = \mathbf{V}_\varphi(1), \tag{6.11}$$

where $\mathbf{V}_\varphi \in X_1(1)$, denotes the unique solution to the following linear Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\mathbf{V}(t) = NE\mathbf{V}(t), \quad t \in (0,1), \\ \mathbf{V}(0) = \varphi, \end{cases} \tag{6.12}$$

with N and E defined in (6.5)–(6.7).

Proof First, it is obvious that $S(\Phi) = \Phi$. Now, from Theorem 1.7 (by choosing ϵ small enough again) we know that the data-to-solution map $\varphi \mapsto \Upsilon(\varphi) \in X_1(1)$ is of class C^2. Hence, S is twice Fréchet differentiable on $B_\epsilon(\Phi)$. This proves (b).

Next, we obtain the Fréchet derivative's formula in (6.11) by computing the Gâteaux derivative, namely,

$$S'(\Phi)\varphi = \frac{d}{d\epsilon} S(\Phi + \epsilon\varphi)\big|_{\epsilon=0} = \mathbf{V}_\varphi(1), \tag{6.13}$$

for any arbitrary $\varphi \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$. The proof of the equality in (6.13) is standard and it is essentially based on a good representation of the solution $\Upsilon(\varphi)$. Some calculations made *a priori* and based in the representation induced by the relations (5.6)–(5.7) show us that is possible to obtain the relation in (6.13), but will require long calculations based on strategics of extensions and restrictions of solutions. Thus, by convenience of the reader, we will take advantage of the results established in Propositions 7.8 and 7.10 in Angulo and Cavalcante (2021) about the unitary group $\{W(t)\}_{t\in\mathbb{R}}$ generated by $(A_Z, D(A_Z))$ in (1.6). In particular, we get the invariance of $H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$ by $W(t)$. Now, we observe by definition that $S(\Phi + \epsilon\varphi) = \Upsilon(\Phi + \epsilon\varphi)(1)$ and since Υ is of class C^2 around Φ on the Banach space $X_1(1)$ we make the Taylor's expansion

$$\Upsilon(\Phi + \epsilon\varphi) = \Upsilon(\Phi) + \epsilon\Upsilon'(\Phi)\varphi + O(\epsilon^2). \tag{6.14}$$

Next, we determine an expression for $\Upsilon'(\Phi)\varphi$ (we recall that $\Upsilon'(\Phi) \in \mathcal{L}(H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1); X_1(1)))$. Indeed, by using the Duhamel's integral representation for $\Upsilon(\Phi + \epsilon\varphi)$ we know that

$$\begin{aligned}\Upsilon(\Phi + \epsilon\varphi)(t) = W(t)(\Phi + \epsilon\varphi) + 2\int_0^t W(t-\tau)\Upsilon(\Phi + \epsilon\varphi)(\tau)\partial_x \\ \Upsilon(\Phi + \epsilon\varphi)(\tau)\, d\tau, \qquad \text{for } t \in [0,1]\end{aligned}$$

then from (6.14)

$$\begin{aligned}\Upsilon(\Phi + \epsilon\varphi)(t) &= W(t)\Phi + 2\int_0^t W(t-\tau)[\Upsilon(\Phi)\partial_x\Upsilon(\Phi)](\tau)\, d\tau + \epsilon V_{\Phi,\varphi}(t) \\ &\quad + O(\epsilon^2), \qquad \text{for } t \in [0,1]\end{aligned} \tag{6.15}$$

where

$$V_{\Phi,\varphi}(t) = W(t)\varphi + 2\int_0^t W(t-\tau)[\partial_x(\Upsilon(\Phi)(\Upsilon'(\Phi)\varphi)(\tau))]\, d\tau, \qquad \text{for } t \in [0,1].$$

$$V_{\Phi,\varphi}(t) = (\Upsilon'(\Phi)\varphi)(t) \text{ for all } t \in [0,1], \tag{6.16}$$

and so $V_{\Phi,\varphi} \in X_1(1)$ and satisfies the integral equation

$$V_{\Phi,\varphi}(t) = W(t)\varphi + 2\int_0^t W(t-\tau)[\partial_x(\Upsilon(\Phi)V_{\Phi,\varphi}(\tau))]\,d\tau,$$

with $V_{\Phi,\varphi}(0) = \varphi$. Then, since $\Upsilon(\Phi) = \Phi$ and for $\zeta = (\zeta_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}}$

$$NE\zeta = A_Z\zeta + 2\text{diag}((\partial_x(\phi_{\mathbf{e}}\zeta_{\mathbf{e}}))\delta_{ij}), \quad 1 \le i, j \le |\mathbf{E}_-| + |\mathbf{E}_+|,$$

follows that $V_{\Phi,\varphi}$ is the solution to the linearized Cauchy problem (6.12) in the distributional sense. Lastly, from (6.13), (6.14), and (6.16) we get $S'(\Phi)\varphi = \mathbf{V}_{\Phi,\varphi}(1)$, for any $\varphi \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$. This shows (c) and the lemma is proved. □

We are now able to prove our main instability result.

Proof (*Theorem* 1.9) Let us consider the eigenfunction Ψ_λ of the linearized operator NE in (6.7) associated to the positive eigenvalue λ given by Theorem 6.2. Then, since $\Psi_\lambda \in D(NE) \cap \mathcal{C} = D(A_Z) \cap \mathcal{C}$ follows $\Psi_\lambda \in H^1(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1)$. Moreover, for $\mathbf{V}(t) = e^{\lambda t}\Psi_\lambda$, $\mathbf{V} \in C(\mathbb{R}; H^3(\mathcal{G}) \cap \mathcal{N}_{0,Z}(1))$ and satisfies

$$\partial_t \mathbf{V}(t) = \lambda e^{\lambda t}\Psi_\lambda = e^{\lambda t} NE\Psi_\lambda = NE\big(e^{\lambda t}\Psi_\lambda\big) = NE\mathbf{V}(t),$$

with $\mathbf{V}(0) = \Psi_\lambda$. Furthermore, $\mathbf{V} \in X_1(1)$ in (1.10). In fact, we have that for $\Psi_\lambda = (\psi_{\mathbf{e}})_{\mathbf{e}\in\mathbf{E}} \in D(A_Z)$, then $\partial_x^3\psi_{\mathbf{e}} \in L^2(I)$, $I = (-\infty, 0)$ or $I = (0, \infty)$, and also by choosing a cutoff regular function $\phi(t) : \mathbb{R} \to \mathbb{R}$ supported on the set $[-2, 2]$, such that $\phi \equiv 1$ on the set $[-1, 1]$ we have that each coordinate function of $\phi(t)\mathbf{V}$ given by $\phi(t)e^{\lambda t}\psi_{\mathbf{e}}(x)$ is in $X^{1,b,\beta,\sigma}(\mathbb{R}^2)$. It follows immediately that $\mathbf{V} \in X_1(1)$.

Now, define $\mu := e^\lambda$. This yields by Lemma 6.6,

$$S'(\Phi)\Psi_\lambda = \mathbf{V}(1) = e^\lambda\Psi_\lambda = \mu\Psi_\lambda.$$

This shows that $\mu \in \sigma(S'(\Phi))$ with $|\mu| > 1$ because $\lambda > 0$. Thus, the mapping defined in (6.10) on an open neighborhood of Φ satisfies the hypotheses of Corollary 6.5. Therefore, the profiles $\Phi = U_Z$ of either tail or bump type are nonlinearly unstable by the flow of the Korteweg-de Vries model in $H^1(\mathcal{G})$-norm. The proof is complete. □

Acknowledgements J. Angulo was partially funded by CNPq/Brazil Grant and Pronex-FAPERJ. Furthermore, it author would like to thank to Universidade Federal de Alagoas (UFAL), Maceió/Brazil, by the support and the warm stay where part of the present project was developed. M. Cavalcante was supported by CNPq 310271/2021-5 and the Fundação de Amparo à Pesquisa do Estado de Alagoas - FAPEAL, Brazil (Process E:60030.0000000161/2022).

Data availability The data that support the findings of this study are available upon request from the authors.

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