PERIODIC TRAJECTORY TRACKING FOR CONTROL-AFFINE DRIFTLESS SYSTEMS ON COMPACT LIE GROUPS

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Abstract. We treat the *periodic trajectory tracking problem*: given a periodic trajectory of a control-affine, left-invariant driftless system in a compact and connected Lie group G and an initial condition in G, find another trajectory of the system satisfying the initial condition given and that asymptotically tracks the periodic trajectory. We solve this problem locally (for initial conditions in a neighborhood of some point of the periodic trajectory) when G is semisimple and the system is Lie-determined (i.e. controllable), and only for a class of periodic trajectories (which we call *regular*). Finally we present a set of sufficient conditions to ensure the existence of such trajectories.

1. Introduction

The present work addresses the problem of periodic trajectory tracking for control-affine driftless systems, specifically in the case when the ambient manifold is a Lie group G (which we will further assume to be compact and connected) and the system is left-invariant (see below). It is heavily inspired by [7] (see also its first author's PhD thesis [6]), in which the problem is studied in $\mathrm{SU}(n)$ aiming applications to Quantum Computing, and can indeed be considered as a (tentative) extension of their methods to abstract Lie groups. We do not, however, rely on any of their results or even notations directly, but rather on their ideas; nor we aim at any applications whatsoever.

In Section 2 we describe the periodic trajectory tracking problem (PTTP) for our system (2.1) and reduce it to the problem of stabilization of an auxiliary system (2.3). The main conclusion here is that if the identity element of e of G is a critical point, and moreover a local attractor of this new system, then one can solve the PTTP locally i.e. for initial conditions close to the reference periodic trajectory. This leads us to investigate some aspects of the stability of time-dependent vector fields on compact Riemannian manifolds, which we do in Section 3, and then apply our conclusions to characterize the ω-limit points of an auxiliary vector field W associated to (2.3): they are precisely the critical points of W. We also conclude that every central point of G is critical to W, so a necessary condition for our approach to work is that G is semisimple e.g. $\mathrm{SU}(n)$.

In Section 4 we restrict our attention to a class of periodic trajectories, which we call *regular*, for which an even simpler characterization of the critical points of the associated W is achieved: they are the critical points of a Lyapunov-like function V; and moreover central points of G are non-degenerate critical points of V provided G is semisimple. A little more effort then allows us to conclude that, in that case, the latter points are also local attractors of W, and since e is obviously central we solve the PTTP locally. We close this work (Section 5) discussing a condition (5.1) that ensures the existence of regular periodic trajectories, including a more or less concrete construction of them, as well as providing examples in which such condition holds.

We refer the reader to [5] and [1] for the basics of Control Theory on Lie groups. For more sophisticate aspects of Lie group theory – notably some results regarding the adjoint representation of G, to which we are naturally led by the change of variables that produces the auxiliary system (2.3) and that stalks us until the end, revealing how semisimplicity is an essential feature of the problem – the reader is referred to a less introductory text on the subject e.g. [3]; more paramount results and definitions, as well as possibly non-standard notation, are also briefly explained in the footnotes.

1991 *Mathematics Subject Classification.* 93B27, 93D15.

Key words and phrases. Trajectory tracking, control-affine driftless systems, compact Lie groups.

This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, grants 131876/2010-4 and 140838/2012-0) and the São Paulo Research Foundation (FAPESP, grant 2018/12273-5).

Acknowledgments. I wish to thank H. B. Silveira and P. A. Tonelli for reading the original manuscript and making invaluable suggestions, and the latter also for many long discussions and for proposing the problem. I'm also indebted to an anonymous reviewer for many relevant suggestions.

2. The periodic trajectory tracking problem

Let G be a compact, connected Lie group, whose Lie algebra of left-invariant vector fields we denote by $\mathfrak{g}$. Given $X_1, \dots, X_m \in \mathfrak{g}$ we consider the left-invariant driftless system

$$\mathbf{x}' = \sum_{k=1}^{m} u_k X_k(\mathbf{x}) \tag{2.1}$$

where $u_1, \dots, u_m \in \mathbb{R}$ are controls. We shall work exclusively with smooth trajectories: $(m+1)$-uples $(\mathbf{x}, u_1, \dots, u_m)$ where $u_1, \dots, u_m : \mathbb{R} \to \mathbb{R}$ are smooth (i.e. C^∞) functions – the controls – and $\mathbf{x} : \mathbb{R} \to G$ is an integral curve of the time-dependent vector field

$$X(t, x) \doteq \sum_{k=1}^{m} u_k(t) X_k(x).$$

The trajectory is said to be T-*periodic* ($T > 0$) provided $\mathbf{x}, u_1, \dots, u_m$ are T-periodic functions.

For simplicity, we shall assume that $\Gamma \doteq \operatorname{span}\{X_1, \dots, X_m\}$ is bracket-generating i.e. the Lie algebra generated by Γ is $\mathfrak{g}$, and hence Γ has a single orbit thanks to Sussmann's Theorem.

Definition 2.1. The *periodic trajectory tracking problem* (PTTP) for system (2.1) is stated as follows: given a T-periodic reference trajectory $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ and an initial state $x_0 \in G$, find another (non-periodic) trajectory $(\mathbf{x}, u_1, \dots, u_m)$ of (2.1) such that $\mathbf{x}(0) = x_0$ and[1]

$$\lim_{t \to +\infty} \mathbf{x}(t) \cdot \mathbf{x}_r(t)^{-1} = e. \tag{2.2}$$

We call $\mathbf{x} \cdot \mathbf{x}_r^{-1}$ the *tracking error* between the two trajectories.

Remark 2.2. When G is a subgroup of $\mathrm{GL}(n, \mathbb{F})$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$), and $\mathbf{x}, \mathbf{y} : \mathbb{R} \to G$ are curves then

$$\lim_{t \to +\infty} \mathbf{x}(t) \cdot \mathbf{y}(t)^{-1} = e \iff \lim_{t \to +\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0$$

where $\|\cdot\|$ is any matrix norm. This is what motivates our definition of the tracking error, and we thus interpret condition (2.2) as the curve $\mathbf{x}$ tracking the reference curve $\mathbf{x}_r$.

Consider the following asymptotic controllability problem (also sometimes called the T-*sampling stabilization problem*, see for instance [8]) for system (2.1):

> Given an initial state $x_0 \in G$ and a target state $x_\infty \in G$, find a trajectory $(\mathbf{x}, u_1, \dots, u_m)$ of (2.1) such that, for some $T > 0$, we have
>
> $$\lim_{k \to +\infty} \mathbf{x}(kT) = x_\infty.$$

It is clear that this problem can be solved if we are able to find

(1) $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ a periodic reference trajectory for (2.1) with $\mathbf{x}_r(0) = x_\infty$ and
(2) $(\mathbf{x}, u_1, \dots, u_m)$ a trajectory of (2.1) that tracks $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ i.e. solving the PTTP.

While the second question above is the main subject of the present paper, we shall discuss the first one – the existence of periodic reference trajectories passing through arbitrary points of G – in Section 5.

The very definition of the tracking error suggests that we can reduce the PTTP associated to a given reference trajectory $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ to a stabilization problem, via a change of coordinates that we describe below. From now on we denote

$$x_\infty \doteq \mathbf{x}_r(0).$$

[1] e: the identity element of G.

Proposition 2.3. *Assume that* $(\mathbf{z}, v_1, \ldots, v_m)$ *is a trajectory of the system*[2]

$$\mathbf{z}' = \sum_{k=1}^{m} v_k \operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{z}) \tag{2.3}$$

such that $\mathbf{z}(0) = x_0 \cdot x_\infty^{-1}$ *and* $\lim_{t\to+\infty} \mathbf{z}(t) = e$. *If we define*

$$\mathbf{x} \doteq \mathbf{z} \cdot \mathbf{x}_r, \tag{2.4}$$

$$u_k \doteq v_k + u_k^r, \quad k \in \{1, \ldots, m\} \tag{2.5}$$

then $(\mathbf{x}, u_1, \ldots, u_m)$ *is a trajectory of* (2.1) *solving the PTTP i.e.* $\mathbf{x}(0) = x_0$ *and* $\lim_{t\to+\infty} \mathbf{x}(t) \cdot \mathbf{x}_r(t)^{-1} = e$.

By (2.3) we mean, of course,

$$\mathbf{z}'(t) = \sum_{k=1}^{m} v_k(t) [\operatorname{Ad}(\mathbf{x}_r(t)) X_k](\mathbf{z}(t)), \quad \forall t \in \mathbb{R}.$$

We shall often omit the time parameter t in order to lighten the notation. For the same reason we shall frequently omit parentheses whenever this does not introduce ambiguities: for instance, given $x, y \in G$ and $X \in \mathfrak{g}$

$$\operatorname{Ad}(x) X(y) \text{ means } [\operatorname{Ad}(x) X](y)$$

which is a vector in $T_y G$.

Proof of Proposition 2.3. It is essentially based on the following simple fact – a kind of Leibniz rule for curves on G – which the reader can easily verify: given $\mathbf{x}_1, \mathbf{x}_2 : \mathbb{R} \to G$ two smooth curves we have[3]

$$(\mathbf{x}_1 \cdot \mathbf{x}_2)' = \mathrm{d}R_{\mathbf{x}_2} \mathbf{x}_1' + \mathrm{d}L_{\mathbf{x}_1} \mathbf{x}_2'.$$

Let $(\mathbf{x}, u_1, \ldots, u_m)$ be defined by (2.4)-(2.5). Then

$$\mathbf{x}' = (\mathbf{z} \cdot \mathbf{x}_r)' = \mathrm{d}R_{\mathbf{x}_r} \mathbf{z}' + \mathrm{d}L_{\mathbf{z}} \mathbf{x}_r' = \mathrm{d}R_{\mathbf{x}_r} \sum_{k=1}^{m} v_k \operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{z}) + \mathrm{d}L_{\mathbf{z}} \sum_{k=1}^{m} u_k^r X_k(\mathbf{x}_r)$$

where the first sum can be rewritten as

$$\begin{aligned}
\mathrm{d}R_{\mathbf{x}_r} \sum_{k=1}^{m} v_k \operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{z}) &= \mathrm{d}R_{\mathbf{x}_r} \sum_{k=1}^{m} v_k (R_{\mathbf{x}_r^{-1}})_* X_k(\mathbf{z}) \\
&= \mathrm{d}R_{\mathbf{x}_r} \sum_{k=1}^{m} v_k \mathrm{d}R_{\mathbf{x}_r^{-1}} X_k(R_{\mathbf{x}_r} \mathbf{z}) \\
&= \mathrm{d}R_{\mathbf{x}_r} \mathrm{d}R_{\mathbf{x}_r^{-1}} \sum_{k=1}^{m} v_k X_k(\mathbf{z} \cdot \mathbf{x}_r) \\
&= \sum_{k=1}^{m} v_k X_k(\mathbf{x})
\end{aligned}$$

while the second is

$$\mathrm{d}L_{\mathbf{z}} \sum_{k=1}^{m} u_k^r X_k(\mathbf{x}_r) = \sum_{k=1}^{m} u_k^r \mathrm{d}L_{\mathbf{z}} X_k(\mathbf{x}_r) = \sum_{k=1}^{m} u_k^r X_k(L_{\mathbf{z}} \mathbf{x}_r) = \sum_{k=1}^{m} u_k^r X_k(\mathbf{x})$$

Summing it up and using (2.5) we conclude that $\mathbf{x}$ solves (2.1). Moreover

$$\mathbf{x}(0) = \mathbf{z}(0) \cdot \mathbf{x}_r(0) = x_0 \cdot x_\infty^{-1} \cdot x_\infty = x_0$$

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[2]The adjoint map $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the group homomorphism that associates to each $x \in G$ an invertible linear map $\operatorname{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$ as follows: if I_x stands for the map $y \in G \mapsto x \cdot y \cdot x^{-1} \in G$ then $\operatorname{Ad}(x)$ corresponds to $\mathrm{d}(I_x)_e : T_e G \to T_e G$ via the canonical isomorphism $\mathfrak{g} \cong T_e G$.

[3]For $x \in G$ we denote by L_x (resp. R_x) the left (resp. right) translation map $y \in G \mapsto x \cdot y \in G$ (resp. $y \in G \mapsto y \cdot x \in G$).

and

$$\lim_{t\to+\infty}\mathbf{x}(t)\cdot\mathbf{x}_r(t)^{-1}=\lim_{t\to+\infty}\mathbf{z}(t)=e.$$

□

Thanks to Proposition 2.3, in order to solve the PTTP our main concern shall be, from now on, to find a trajectory $(\mathbf{z},v_1,\dots,v_m)$ of system (2.3) satisfying $\mathbf{z}(0)=x_0\cdot x_\infty^{-1}$ and $\lim_{t\to+\infty}\mathbf{z}(t)=e$: the solution $(\mathbf{x},u_1,\dots,u_m)$ of the PTTP for (2.1) can thus be recovered from our knowledge of $(\mathbf{z},v_1,\dots,v_m)$ and $(\mathbf{x}_r,u_1^r,\dots,u_m^r)$.

We define a *Lyapunov-like function* $V:G\to\mathbb{R}$ by

$$V(x)\doteq\operatorname{trace}\operatorname{Ad}(x),\quad x\in G, \tag{2.6}$$

and an *auxiliary vector field* $W:\mathbb{R}\times G\to TG$ by

$$W(t,w)\doteq\sum_{k=1}^m a_k(t,w)\operatorname{Ad}(\mathbf{x}_r(t))X_k(w),\quad (t,w)\in\mathbb{R}\times G, \tag{2.7}$$

where

$$a_k(t,w)\doteq\mathrm{d}V\left(\operatorname{Ad}(\mathbf{x}_r(t))X_k(w)\right),\quad k\in\{1,\dots,m\}. \tag{2.8}$$

Notice that W is a time-dependent vector field which is *not* left-invariant. The main reason for introducing it is the following: if $\mathbf{w}:\mathbb{R}\to G$ is one of its integral curves and if we define

$$v_k(t)\doteq a_k(t,\mathbf{w}(t)),\quad t\in\mathbb{R},\ k\in\{1,\dots,m\}, \tag{2.9}$$

then $(\mathbf{w},v_1,\dots,v_m)$ is a trajectory of our modified system (2.3). Moreover, let us denote by $\mathcal{C}_W$ the set of critical points of W, that is:

$$\mathcal{C}_W\doteq\{w\in G\ ;\ W(t,w)=0,\ \forall t\in\mathbb{R}\}.$$

Recall that one such critical point $w\in\mathcal{C}_W$ is a *local attractor* if there exists $U\subset G$ a neighborhood of w such that given any initial condition $(t_0,w_0)\in\mathbb{R}\times U$ and $\mathbf{w}:\mathbb{R}\to G$ the unique integral curve of W satisfying $\mathbf{w}(t_0)=w_0$ then $\lim_{t\to+\infty}\mathbf{w}(t)=w$.

Example (Systems on matrix groups). Suppose that G is a compact subgroup of $\mathrm{GL}(n,\mathbb{F})$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Then we may rewrite system (2.1) explicitly in matrix form:

$$\mathbf{x}'=\sum_{k=1}^m u_kX_k(\mathbf{x})=\sum_{k=1}^m u_kX_k(\mathbf{x}\cdot e)=\sum_{k=1}^m u_k\mathrm{d}L_{\mathbf{x}}X_k(e)=\sum_{k=1}^m u_k\mathbf{x}\cdot X_k=\mathbf{x}\cdot\left(\sum_{k=1}^m u_kX_k\right).$$

Notice that since we are dealing with left-invariant vector fields (instead of right-invariant ones), elements in $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{F})$ act on G by multiplication on the right (as matrices). Moreover, in this case, given $X\in\mathfrak{g}$ we have, as matrices [3, pp. 79]:

$$\operatorname{Ad}(x)X=x\cdot X\cdot x^{-1},\quad\forall x\in G.$$

Hence by computations similar to the ones above we may write our modified system (2.3) as

$$\mathbf{z}'=\mathbf{z}\cdot\mathbf{x}_r\cdot\left(\sum_{k=1}^m v_kX_k\right)\cdot\mathbf{x}_r^{-1}$$

as well as the auxiliary vector field (2.7)

$$W(t,w)=w\cdot\mathbf{x}_r(t)\cdot\left(\sum_{k=1}^m a_k(t,w)X_k\right)\cdot\mathbf{x}_r(t)^{-1}$$

where $a_k(t,w)=\mathrm{d}V(w\cdot\mathbf{x}_r(t)\cdot X_k\cdot\mathbf{x}_r(t)^{-1})$.

The next result tells us that if the identity element of G is a local attractor of the auxiliary vector field W then we can solve the PTTP *locally* near the target state $x_\infty=\mathbf{x}_r(0)$, and also provides a recipe to obtain the tracking trajectory $(\mathbf{x},u_1,\dots,u_m)$.

Proposition 2.4. *Suppose that $e \in \mathcal{C}_W$ and is a local attractor for W. Then there exists $U_\infty \subset G$ a neighborhood of x_∞ enjoying the following property: for every $x_0 \in U_\infty$ there exists $(\mathbf{x}, u_1, \dots, u_m)$ a trajectory of (2.1) such that $\mathbf{x}(0) = x_0$ and $\lim_{t\to+\infty} \mathbf{x}(t) \cdot \mathbf{x}_r(t)^{-1} = e$. The trajectory $(\mathbf{x}, u_1, \dots, u_m)$ can be obtained as follows: for $\mathbf{w} : \mathbb{R} \to G$ the unique integral curve of W satisfying $\mathbf{w}(0) = x_0 \cdot x_\infty^{-1}$, define*

$$\mathbf{x}(t) \doteq \mathbf{w}(t) \cdot \mathbf{x}_r(t),$$
$$u_k(t) \doteq a_k(t, \mathbf{w}(t)) + u_k^r(t), \quad k \in \{1, \dots, m\}.$$

Proof. Let $U \subset G$ be an attractive neighborhood of e. Then $U_\infty \doteq U \cdot x_\infty$ is clearly a neighborhood of x_∞, and if $x_0 \in U_\infty$ then $x_0 \cdot x_\infty^{-1} \in U$, hence $\lim_{t\to+\infty} \mathbf{w}(t) = e$. If $v_1, \dots, v_m$ are as in (2.9) then $(\mathbf{w}, v_1, \dots, v_m)$ is a trajectory of the modified system (2.3), so the conclusion follows from Proposition 2.3. □

3. Some results on stability

In this section we shall depart a little from the original setting for the PTTP and establish some technical results on the stability of time-dependent vector fields that will be needed in the next sections. Since the group structure here plays no role, we shall take a step back and work in the more general framework of smooth manifolds.

Remark 3.1. As pointed out by H. B. Silveira in personal communication, our approach in this section (see especially Proposition 3.6) holds some connections with the periodic version of LaSalle's Invariance Principle [4] (for its use in a similar context see [8]). The proofs we present here are, nevertheless, self-contained.

Let M be a smooth manifold, which for simplicity we assume to be compact, and $X : \mathbb{R} \times M \to TM$ a time-dependent vector field. Recall that given $(t_0, x_0) \in \mathbb{R} \times M$ its *ω-limit set*, $\omega_X(t_0, x_0)$, is the set of all $x \in M$ enjoying the following property: there exists an increasing sequence $t_n \to +\infty$ such that $\mathbf{x}(t_n) \to x$, where $\mathbf{x} : \mathbb{R} \to M$ is the unique integral curve of X satisfying $\mathbf{x}(t_0) = x_0$. Of course the compactness of M ensures that the ω-limit sets of X are never empty.

Definition 3.2. A continuous function $V : M \to \mathbb{R}$ is said to be *non-decreasing along* X if for every integral curve $\mathbf{x} : \mathbb{R} \to M$ of X the function $V \circ \mathbf{x}$ is non-decreasing.

For instance, if $V \in C^\infty(M)$ satisfies $\mathrm{d}V(X(t, x)) \geq 0$ everywhere then clearly V is non-decreasing along X.

Proposition 3.3. *Let $V : M \to \mathbb{R}$ be continuous and non-decreasing along X. Then V is constant on $\omega_X(t_0, x_0)$ for any $(t_0, x_0) \in \mathbb{R} \times M$.*

Proof. Let $\mathbf{x} : M \to \mathbb{R}$ be the unique integral curve of X satisfying $\mathbf{x}(t_0) = x_0$. For $j = 1, 2$ let $x_j \in \omega_X(t_0, x_0)$ and take increasing sequences $\{t_n^j\}_{n\in\mathbb{N}}$, $t_n^j \to +\infty$, such that $\mathbf{x}(t_n^j) \to x_j$ as $n \to +\infty$. By continuity, $V(\mathbf{x}(t_n^j)) \to V(x_j)$ and since $V \circ \mathbf{x}$ is non-decreasing we must have

$$V(\mathbf{x}(t_n^j)) \leq V(x_j), \quad \forall n \in \mathbb{N},\ j = 1, 2.$$

We first extract a subsequence $\{t_{n_k}^2\}_{k\in\mathbb{N}}$ of $\{t_n^2\}_{n\in\mathbb{N}}$ with the property that $t_k^1 \leq t_{n_k}^2$ for every $k \in \mathbb{N}$: again, since V is non-decreasing along X one gets

$$V(\mathbf{x}(t_k^1)) \leq V(\mathbf{x}(t_{n_k}^2)) \leq V(x_2), \quad \forall k \in \mathbb{N}.$$

By letting $k \to +\infty$ in the left-hand side of the inequality above we conclude that $V(x_1) \leq V(x_2)$, and hence the equality holds. □

Corollary 3.4. *If $V : M \to \mathbb{R}$ be continuous and non-decreasing along X then*

$$\lim_{t\to+\infty} V(\mathbf{x}(t)) = V(x), \quad \forall x \in \omega_X(t_0, x_0),$$

where $\mathbf{x} : M \to \mathbb{R}$ be the unique integral curve of X satisfying $\mathbf{x}(t_0) = x_0$.

Proof. It suffices to prove that any increasing sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$, admits a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ such that $V(\mathbf{x}(t_{n_k})) \to V(x)$ as $k \to +\infty$. And indeed, by compactness of M there exist $\{t_{n_k}\}_{k\in\mathbb{N}}$ subsequence of $\{t_n\}_{n\in\mathbb{N}}$ and $y \in M$ such that $\mathbf{x}(t_{n_k}) \to y$, and by continuity $V(\mathbf{x}(t_{n_k})) \to V(y)$, as $k \to +\infty$. Since obviously $y \in \omega_X(t_0, x_0)$ we have $V(x) = V(y)$ by the previous proposition, and the conclusion follows. ☐

The last two results in this section do not assume compactness of M. We do, however, endow it with a Riemannian metric: below we denote by $\|\cdot\|$ the induced norm on each tangent space.

Lemma 3.5. *Let $\mathbf{x} : \mathbb{R} \to M$ be a smooth curve such that*

$$\lim_{t\to+\infty} \|\mathbf{x}'(t)\| = 0$$

and $\{t_n\}_{n\in\mathbb{N}}$ be an increasing sequence such that $t_n \to +\infty$ and $\mathbf{x}(t_n) \to x \in M$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \mathbf{x}(t_n + \epsilon) = x, \quad \forall \epsilon \in \mathbb{R}.$$

Proof. We may assume w.l.o.g. that M is connected, and let $d : M \times M \to \mathbb{R}$ be the distance function on M induced by the Riemannian metric[4]: we then must prove that

$$\lim_{n\to+\infty} d(\mathbf{x}(t_n + \epsilon), x) = 0$$

whatever $\epsilon \in \mathbb{R}$. If we denote by $I(a, b) \subset \mathbb{R}$ the closed interval with endpoints $a, b \in \mathbb{R}$ then by definition of d we have

$$d(\mathbf{x}(t_n + \epsilon), \mathbf{x}(t_n)) \le \left| \int_{I(t_n, t_n+\epsilon)} \|\mathbf{x}'(t)\| \mathrm{d}t \right| \le \left(\sup_{t\in I(t_n,t_n+\epsilon)} \|\mathbf{x}'(t)\| \right) |\epsilon|$$

which, we claim, goes to zero as $n \to +\infty$. Indeed, given $\delta > 0$ there exists $R > 0$ such that $\|\mathbf{x}'(t)\| < \delta$ for every $t > R$. Moreover, since $t_n \to +\infty$ there exists $n_0 \in \mathbb{N}$ such that

$$n \ge n_0 \Longrightarrow \max\{t_n, t_n + \epsilon\} > R \Longrightarrow \sup_{t\in I(t_n,t_n+\epsilon)} \|\mathbf{x}'(t)\| < \delta$$

thus proving our claim. Now for every $n \in \mathbb{N}$

$$d(\mathbf{x}(t_n + \epsilon), x) \le d(\mathbf{x}(t_n + \epsilon), \mathbf{x}(t_n)) + d(\mathbf{x}(t_n), x) \longrightarrow 0$$

since both terms go to zero. ☐

Proposition 3.6. *Let $\mathbf{x} : \mathbb{R} \to M$ be a smooth curve, $\{t_n\}_{n\in\mathbb{N}}$ an increasing sequence and $x \in M$ as in Lemma 3.5. Let also $f : \mathbb{R} \times M \to \mathbb{R}$ be continuous, T-periodic (for some $T > 0$) and such that* $\lim_{t\to+\infty} f(t, \mathbf{x}(t)) = 0$. *Then*

$$f(s, x) = 0, \quad \forall s \in \mathbb{R}.$$

Proof. Let $s \in \mathbb{R}$. For each $n \in \mathbb{N}$ select $l_n \in \mathbb{Z}$ such that

$$s_n \doteq t_n - l_n T \in [0, T)$$

hence the sequence $\{s_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence, say

$$\lim_{k\to+\infty} s_{n_k} = \theta \in [0, T].$$

We define

$$\begin{aligned} s^*_{n_k} &\doteq s_{n_k} - \theta + s \\ t^*_{n_k} &\doteq t_{n_k} - \theta + s = s_{n_k} + l_{n_k}T - \theta + s = s^*_{n_k} + l_{n_k}T. \end{aligned}$$

for each $k \in \mathbb{N}$, so clearly $s^*_{n_k} \to s$. Applying Lemma 3.5 with $\epsilon \doteq -\theta + s$ one gets

$$\mathbf{x}(t^*_{n_k}) = \mathbf{x}(t_{n_k} + \epsilon) \longrightarrow x.$$

Since f is continuous and T-periodic we have

$$f(s, x) = \lim_{k\to+\infty} f(s^*_{n_k}, \mathbf{x}(t^*_{n_k})) = \lim_{k\to+\infty} f(t^*_{n_k} - l_{n_k}T, \mathbf{x}(t^*_{n_k})) = \lim_{k\to+\infty} f(t^*_{n_k}, \mathbf{x}(t^*_{n_k}))$$

[4] I.e. the distance between two given points is the infimum of the lengths of all piecewise smooth curves connecting them.

which is zero thanks to our last hypothesis on f and the fact that $t^*_{n_k} \to +\infty$. □

Now back to the setup of Section 2, we use our results above to prove:

Theorem 3.7. *Let $(t_0, w_0) \in \mathbb{R} \times G$. If $w \in \omega_W(t_0, w_0)$ then*

$$a_k(t, w) = 0, \quad \forall t \in \mathbb{R},\ \forall k \in \{1, \dots, m\}.$$

In particular, every ω-limit point is a critical point of W.

Its proof depends on some auxiliary results that we prove below. Before we go on, however, let us make a small digression about its content.

Remark 3.8 (A note on limit sets). It is certain that any critical point of W is also an ω-limit point of W, for given $x \in \mathcal{C}_W$ we have that the constant curve $\mathbf{x} \equiv x$ is an integral curve of W, hence $\omega_W(t, x) = \{x\}$ for every $t \in \mathbb{R}$. What Theorem 3.7 claims is simply that these are *all* the ω-limit points of W i.e. there are no other, more exotic, limit points of W (say, attractive periodic integral curves, for instance): we are indeed claiming that

$$\{\omega\text{-limit points of } W\} = \{\text{critical points of } W\}.$$

A similar analysis can be carried out for the *α-limit points* of W – which, however, we will not do since it is not relevant for our current purposes. Recall that given $(t_0, x_0) \in \mathbb{R} \times G$ its *α-limit set*, $\alpha_W(t_0, x_0)$, is the set of all $x \in G$ such that there exists a *decreasing* sequence $t_n \to -\infty$ such that $\mathbf{x}(t_n) \to x$, where $\mathbf{x} : \mathbb{R} \to G$ is the unique integral curve of W satisfying $\mathbf{x}(t_0) = x_0$; moreover, we call a point $x \in G$ an *α-limit point of W* if $x \in \alpha_W(t_0, x_0)$ for some $(t_0, x_0) \in \mathbb{R} \times G$. The arguments below, with suitable modifications, will likely show that these are also critical points of W (again, critical points of W are certainly α-limit points of W, and the arguments below prove that no other α-limit points exist).

The reader must keep in mind that nothing prevents a point $x \in G$ from being simultaneously an α-limit point and an ω-limit point of W e.g. if $x \in \mathcal{C}_W$, and the conclusion one will reach is that actually

$$\{\alpha\text{-limit points of } W\} = \{\text{critical points of } W\} = \{\omega\text{-limit points of } W\}. \tag{3.1}$$

Back to our main line of reasoning, first of all we must obtain a more convenient expression for the functions a_k defined in (2.8).

Lemma 3.9. *For $(x, v) \in TG$ we have*[5]

$$\mathrm{d}V_x v = \operatorname{trace}\left\{\mathrm{Ad}(x) \cdot \mathrm{ad}(\mathrm{d}L_{x^{-1}} v)\right\}. \tag{3.2}$$

Also, for each $k \in \{1, \dots, m\}$:

$$a_k(t, w) = \operatorname{trace}\left\{\mathrm{Ad}(w) \cdot \mathrm{ad}\left(\mathrm{Ad}\left(\mathbf{x}_r(t)\right) X_k\right)\right\} \tag{3.3}$$

for $(t, w) \in \mathbb{R} \times G$.

Proof. We start by showing that

$$\mathrm{d}\,\mathrm{Ad}_x\, v = \mathrm{Ad}(x) \cdot \mathrm{ad}(\mathrm{d}L_{x^{-1}} v), \quad \forall (x, v) \in TG. \tag{3.4}$$

Indeed, notice that $\mathrm{d}L_{x^{-1}} v \in T_e G$, which we then identify with an element of $\mathfrak{g}$, thus making sense of (3.4). We consider the map $F \doteq \mathrm{Ad} \circ L_x : G \to \mathrm{GL}(\mathfrak{g})$: by the chain rule we have, on the one hand,

$$\mathrm{d}\,\mathrm{Ad}_x\, v = \mathrm{d}F_e \mathrm{d}L_{x^{-1}} v.$$

On the other hand, we can write

$$F(y) = \mathrm{Ad}(x \cdot y) = \mathrm{Ad}(x) \cdot \mathrm{Ad}(y), \quad y \in G,$$

i.e. $F = \Lambda \circ \mathrm{Ad}$ where

$$\begin{array}{cccc} \Lambda : & \mathfrak{gl}(\mathfrak{g}) & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ & T & \longmapsto & \mathrm{Ad}(x) \cdot T \end{array}$$

[5] Given $X \in \mathfrak{g}$ the adjoint map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is defined by $Y \in \mathfrak{g} \mapsto [X, Y] \in \mathfrak{g}$. By means of the canonical isomorphism $\mathfrak{g} \cong T_e G$ it makes perfect sense to write $\mathrm{ad}(v)$ for $v \in T_e G$ – as we do often throughout the text – which we regard as a linear map $T_e G \to T_e G$. In that sense, $\mathrm{ad} : T_e G \to \mathfrak{gl}(T_e G)$ is precisely the differential of the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$ at $e \in G$ [3, Proposition 1.91].

is a linear map: for $\xi \in T_eG$ we have again by the chain rule

$$\mathrm{d}F_e\xi = \mathrm{d}\Lambda_{\mathrm{Ad}(e)}\mathrm{d}\,\mathrm{Ad}_e\,\xi = \Lambda(\mathrm{ad}(\xi)) = \mathrm{Ad}(x)\cdot\mathrm{ad}(\xi)$$

which for $\xi \doteq \mathrm{d}L_{x^{-1}}v$ proves (3.4) thanks to our previous conclusions.

Now, recalling that the map trace : $\mathfrak{gl}(\mathfrak{g}) \to \mathbb{R}$ is linear and by definition $V = \mathrm{trace}\circ\mathrm{Ad}$, identity (3.2) follows immediately from (3.4) after a third application of the chain rule.

To conclude, it follows from the definition of a_k and from (3.2) that

$$\begin{aligned} a_k(t,w) &= \mathrm{d}V(\mathrm{Ad}(\mathbf{x}_r)X_k(w)) \\ &= \mathrm{trace}\left\{\mathrm{Ad}(w)\cdot\mathrm{ad}\left(\mathrm{d}L_{w^{-1}}\,\mathrm{Ad}(\mathbf{x}_r)X_k(w)\right)\right\} \\ &= \mathrm{trace}\left\{\mathrm{Ad}(w)\cdot\mathrm{ad}\left(\mathrm{Ad}(\mathbf{x}_r)X_k(e)\right)\right\} \\ &= \mathrm{trace}\left\{\mathrm{Ad}(w)\cdot\mathrm{ad}\left(\mathrm{Ad}(\mathbf{x}_r)X_k\right)\right\} \end{aligned}$$

where we used that $\mathrm{Ad}\left(\mathbf{x}_r(t)\right)X_k$ is left-invariant for all $t \in \mathbb{R}$. □

We can now elucidate a couple of questions raised by Proposition 2.4.

Corollary 3.10. *Every*[6] $w \in Z(G)$ *is a critical point of the auxiliary vector field* W*. However, if the identity element is a local attractor of* W *then* G *must be semisimple.*

Proof. Since $Z(G) = \ker\mathrm{Ad}$ we have $\mathrm{Ad}(w) = \mathrm{id}_{\mathfrak{g}}$, and then for each $k \in \{1,\dots,m\}$

$$a_k(t,w) = \mathrm{trace}\left\{\mathrm{Ad}(w)\cdot\mathrm{ad}\left(\mathrm{Ad}(\mathbf{x}_r)X_k\right)\right\} = \mathrm{trace}\,\mathrm{ad}\left(\mathrm{Ad}(\mathbf{x}_r)X_k\right) = 0$$

for every $t \in \mathbb{R}$ since $\mathrm{ad}(X)$ is traceless[7] for all $X \in \mathfrak{g}$. By definition of W we have then $W(t,w) = 0$ for all $t \in \mathbb{R}$ i.e. w is a critical point.

In particular $e \in \mathcal{C}_W$. If G were not semisimple then $Z(G)$ would be a Lie subgroup of G of positive dimension, hence any neighborhood of e would contain infinitely many points in $Z(G)$. Since $Z(G) \subset \mathcal{C}_W$ this proves that e would not be an isolated point of $\mathcal{C}_W$, even less a local attractor. □

The next technical remark will also be needed in Section 5. We define

$$X_r \doteq \sum_{j=1}^{m} u_j^r X_j \tag{3.5}$$

where $u_1^r,\dots,u_m^r$ are the controls of our reference trajectory of system (2.1). We will consider X_r both as a time-dependent vector field on G – of which $\mathbf{x}_r$ is an integral curve – and as a smooth curve in $\mathfrak{g}$.

Lemma 3.11. *Let* $\Lambda^0 : \mathbb{R} \to \mathfrak{g}$ *be any smooth curve and define* $\lambda : \mathbb{R} \to \mathfrak{g}$ *by*

$$\lambda \doteq \mathrm{Ad}(\mathbf{x}_r)\Lambda^0.$$

Then its n*-th derivative is given by*

$$\lambda^{(n)} = \mathrm{Ad}(\mathbf{x}_r)\Lambda^n$$

where $\Lambda^n : \mathbb{R} \to \mathfrak{g}$ *is defined inductively by*

$$\Lambda^{n+1} \doteq (\Lambda^n)' + \mathrm{ad}(X_r)\Lambda^n, \quad n \in \mathbb{N}.$$

Proof. Using the identity $\mathrm{Ad}(\mathbf{x}_r)' = \mathrm{Ad}(\mathbf{x}_r)\cdot\mathrm{ad}(X_r)$, which in turn follows easily from (3.4), we have

$$(\mathrm{Ad}(\mathbf{x}_r)\Lambda^n)' = \mathrm{Ad}(\mathbf{x}_r)'\Lambda^n + \mathrm{Ad}(\mathbf{x}_r)(\Lambda^n)' = \mathrm{Ad}(\mathbf{x}_r)\,\mathrm{ad}(X_r)\Lambda^n + \mathrm{Ad}(\mathbf{x}_r)(\Lambda^n)' = \mathrm{Ad}(\mathbf{x}_r)\Lambda^{n+1}.$$

□

Proposition 3.12. *Let* $\mathbf{w} : \mathbb{R} \to G$ *be an integral curve of* W*. For each* $k \in \{1,\dots,m\}$ *the function* $b_k : \mathbb{R} \to \mathbb{R}$ *defined by*

$$b_k(t) \doteq \frac{\mathrm{d}}{\mathrm{d}t}a_k\left(t,\mathbf{w}(t)\right)$$

is bounded.

[6] $Z(G)$: the center of G i.e. the subgroup of all $x \in G$ such that $x\cdot y = y\cdot x$ for every $y \in G$. Since G is compact, it is semisimple if and only if $Z(G)$ is discrete [3, Corollary 4.25].

[7] See footnote 8.

Proof. We shall write down an explicit expression for b_k, which boils down to computing the partial derivatives of a_k since

$$b_k = \frac{\partial a_k}{\partial t} + \frac{\partial a_k}{\partial w}\mathbf{w}'.$$

First, since $\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism for each $x \in G$, it follows easily that

$$\mathrm{ad}\,(\mathrm{Ad}(x)X) = \mathrm{Ad}(x) \cdot \mathrm{ad}(X) \cdot \mathrm{Ad}(x)^{-1} \tag{3.6}$$

for every $X \in \mathfrak{g}$ (just apply both sides to an arbitrary $Y \in \mathfrak{g}$). Now, by (3.3) we have that $a_k(t, w) = \mathrm{trace}\,\{\mathrm{Ad}(w) \cdot \mathrm{ad}\,(\lambda)\}$ where $\lambda \doteq \mathrm{Ad}(\mathbf{x}_r)X_k$, hence

$$\frac{\partial a_k}{\partial t} = \mathrm{trace}\,\{\mathrm{Ad}(w) \cdot \mathrm{ad}\,(\lambda')\} = \mathrm{trace}\left\{\mathrm{Ad}(w) \cdot \mathrm{Ad}(\mathbf{x}_r) \cdot \mathrm{ad}\,(\mathrm{ad}(X_r)X_k) \cdot \mathrm{Ad}(\mathbf{x}_r)^{-1}\right\}$$

by (3.6), since $\lambda' = \mathrm{Ad}(\mathbf{x}_r)\,\mathrm{ad}(X_r)X_k$ thanks to Lemma 3.11.

Moreover, using (3.4) and taking into account that $\mathbf{w}$ is an integral curve of W

$$\begin{aligned}
\mathrm{d}\,\mathrm{Ad}_{\mathbf{w}}\,\mathbf{w}' &= \mathrm{Ad}(\mathbf{w}) \cdot \mathrm{ad}\,(\mathrm{d}L_{\mathbf{w}^{-1}}\mathbf{w}') \\
&= \mathrm{Ad}(\mathbf{w}) \cdot \mathrm{ad}\left(\mathrm{d}L_{\mathbf{w}^{-1}} \sum_{j=1}^{m} a_j(t, \mathbf{w})\,\mathrm{Ad}(\mathbf{x}_r)X_j(\mathbf{w})\right) \\
&= \mathrm{Ad}(\mathbf{w}) \cdot \sum_{j=1}^{m} a_j(t, \mathbf{w})\ \ \mathrm{ad}\,(\mathrm{Ad}(\mathbf{x}_r)X_j)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\frac{\partial a_k}{\partial w}\mathbf{w}' &= \mathrm{trace}\,\{\mathrm{d}\,\mathrm{Ad}_{\mathbf{w}}\,\mathbf{w}' \cdot \mathrm{ad}\,(\mathrm{Ad}(\mathbf{x}_r)X_k)\} \\
&= \mathrm{trace}\left\{\mathrm{Ad}(\mathbf{w}) \cdot \sum_{j=1}^{m} a_j(t, \mathbf{w})\ \ \mathrm{ad}\,(\mathrm{Ad}(\mathbf{x}_r)X_j) \cdot \mathrm{ad}\,(\mathrm{Ad}(\mathbf{x}_r)X_k)\right\} \\
&= \mathrm{trace}\left\{\mathrm{Ad}(\mathbf{w}) \cdot \mathrm{Ad}(\mathbf{x}_r) \cdot \sum_{j=1}^{m} a_j(t, \mathbf{w})\ \ \mathrm{ad}(X_j) \cdot \mathrm{ad}(X_k) \cdot \mathrm{Ad}(\mathbf{x}_r)^{-1}\right\}
\end{aligned}$$

thanks again to a double application of (3.6)

Summing both derivatives evaluated at $(t, \mathbf{w}(t))$, we conclude that

$$b_k = \mathrm{trace}\left\{\mathrm{Ad}(\mathbf{w}) \cdot \mathrm{Ad}(\mathbf{x}_r) \cdot B_k \cdot \mathrm{Ad}(\mathbf{x}_r)^{-1}\right\}$$

where $B_k : \mathbb{R} \to \mathfrak{gl}(\mathfrak{g})$ is defined given by

$$B_k \doteq \mathrm{ad}(\mathrm{ad}(X_r)X_k) + \sum_{j=1}^{m} a_j(t, \mathbf{w})\,\mathrm{ad}\,(X_j) \cdot \mathrm{ad}(X_k).$$

Denoting by $\|\cdot\|$ any norm in $\mathfrak{gl}(\mathfrak{g})$, it follows from the compactness of G the existence of $M > 0$ such that $\|\,\mathrm{Ad}(x)\| \leq M$ for every $x \in G$, hence for every $t \in \mathbb{R}$ we have

$$|b_k(t)| = \left|\mathrm{trace}\left\{\mathrm{Ad}(\mathbf{w}) \cdot \mathrm{Ad}(\mathbf{x}_r) \cdot B_k(t) \cdot \mathrm{Ad}(\mathbf{x}_r)^{-1}\right\}\right| \leq M^3 \|\,\mathrm{trace}\,\| \|B_k(t)\|$$

where $\|\,\mathrm{trace}\,\|$ stands for the norm of the linear functional $\mathrm{trace} : \mathfrak{gl}(\mathfrak{g}) \to \mathbb{R}$: in order to finish the proof, it suffices to show that B_k is bounded. But

$$\|B_k(t)\| \leq \|\,\mathrm{ad}(\mathrm{ad}(X_r(t))X_k)\| + \sum_{j=1}^{m} |a_j(t, \mathbf{w})|\ \|\,\mathrm{ad}\,(X_j) \cdot \mathrm{ad}(X_k)\|$$

and while the first term is clearly bounded for the map

$$t \in \mathbb{R} \longmapsto \mathrm{ad}(\mathrm{ad}(X_r(t))X_k) \in \mathfrak{gl}(\mathfrak{g}),$$

is T-periodic, the second term is bounded because $a_j : \mathbb{R} \times G \to \mathbb{R}$ is T-periodic for each $j \in \{1, \dots, m\}$ and hence $a_j(\mathbb{R} \times G) = a_j([0, T] \times G)$ is a compact set. ☐

Corollary 3.13. *If $\mathbf{w} : \mathbb{R} \to G$ is an integral curve of W then*

$$\lim_{t \to +\infty} a_k\left(t, \mathbf{w}(t)\right) = 0, \quad \forall k \in \{1, \dots, m\}. \tag{3.7}$$

In particular

$$\lim_{t \to +\infty} \|\mathbf{w}'(t)\| = 0$$

where $\|\cdot\|$ is the norm associated to any left-invariant Riemannian metric on G.

Proof. Let $\alpha \doteq V \circ \mathbf{w}$ where V is our Lyapunov-like function (2.6). Its first derivative is

$$\alpha' = \mathrm{d}V(\mathbf{w}') = \mathrm{d}V\left(\sum_{k=1}^{m} a_k(t, \mathbf{w}) \operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{w})\right) = \sum_{k=1}^{m} a_k(t, \mathbf{w}) \mathrm{d}V(\operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{w})) = \sum_{k=1}^{m} a_k(t, \mathbf{w})^2,$$

by definition of a_k (2.8), and thus non-negative. Differentiating once again yields

$$\alpha'' = 2 \sum_{k=1}^{m} a_k(t, \mathbf{w}) \frac{\mathrm{d}}{\mathrm{d}t} a_k(t, \mathbf{w}) = 2 \sum_{k=1}^{m} a_k(t, \mathbf{w}) b_k(t)$$

where b_k is as in Proposition 3.12, hence bounded, which implies boundedness of α''. In turn, this ensures that α' is uniformly continuous. Now

$$\mathrm{d}V(W(t, w)) = \sum_{k=1}^{m} a_k(t, w)^2 \geq 0, \quad \forall (t, w) \in \mathbb{R} \times G$$

so V is non-decreasing along W, thus thanks to Corollary 3.4 we have

$$\lim_{t \to +\infty} V(\mathbf{w}(t)) = V(w)$$

where $w \in \omega_W(0, \mathbf{w}(0))$ is arbitrary (recall that the latter set is never empty). We have proved that

$$\lim_{t \to +\infty} \int_0^t \alpha'(s) \mathrm{d}s = \lim_{t \to +\infty} \alpha(t) - \alpha(0) = V(w) - V(\mathbf{w}(0))$$

which brings us into position to apply Barbalat's Lemma [2, Lemma 8.2] to α' and conclude that

$$\lim_{t \to +\infty} \alpha'(t) = 0$$

which clearly proves (3.7) thanks to our previous computations. Our second statement now follows:

$$\|\mathbf{w}'(t)\| = \left\|\sum_{k=1}^{m} a_k(t, \mathbf{w}) \operatorname{Ad}(\mathbf{x}_r) X_k(\mathbf{w})\right\| \leq \sum_{k=1}^{m} |a_k(t, \mathbf{w})| \, \|\operatorname{Ad}(\mathbf{x}_r) X_k\| \leq M \sum_{k=1}^{m} |a_k(t, \mathbf{w})| \, \|X_k\| \longrightarrow 0.$$

☐

Proof of Theorem 3.7. Let $\|\cdot\|$ stand for the norm associated to some left-invariant metric on G. For $\mathbf{w} : \mathbb{R} \to G$ the unique integral curve of W satisfying $\mathbf{w}(t_0) = w_0$ we have, thanks to Corollary 3.13, that $\|\mathbf{w}'(t)\| \to 0$ as $t \to +\infty$. Moreover, the function $f : \mathbb{R} \times G \to \mathbb{R}$ defined by $f(t, w) \doteq a_k(t, w)$ is T-periodic and satisfies, again by Corollary 3.13,

$$\lim_{t \to +\infty} f(t, \mathbf{w}(t)) = \lim_{t \to +\infty} a_k(t, \mathbf{w}(t)) = 0.$$

But since $w \in \omega_W(t_0, w_0)$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ increasing to infinity such that $\mathbf{w}(t_n) \to w$ as $n \to +\infty$. The conclusion follows from Proposition 3.6. ☐

4. Regular trajectories

Up to this point, all the results obtained are valid for arbitrary periodic reference trajectories of (2.1). In this section we introduce a special class of trajectories such that the set of critical points of their associated auxiliary vector fields W admit a nice algebraic description: it coincides with the set of critical points of our Lyapunov-like function V. This characterization allows us to show that the identity element is a local attractor for W provided G is semisimple, hence solving the PTTP in a neighborhood of the target state x_∞ by Proposition 2.4.

Definition 4.1. A trajectory $(\mathbf{x}, u_1, \ldots, u_m)$ of (2.1) (not necessarily periodic) is said to be *regular* if

$$\operatorname{span}\{\operatorname{Ad}(\mathbf{x}(t))\, X_k\ ;\ t \in \mathbb{R},\ 1 \le k \le m\} = \mathfrak{g}.$$

In Section 5 we prove the existence of regular periodic trajectories through any initial state $x_0 \in G$ under some extra assumptions on system (2.1), and also provide some simple examples in concrete matrix groups in which these extra assumptions are satisfied. Theorem 3.7 admits the following:

Corollary 4.2. *Assume $(\mathbf{x}_r, u_1^r, \ldots, u_m^r)$ is a regular periodic trajectory of system (2.1) and let W be its auxiliary vector field. If $w \in G$ is any ω-limit point of W then*

$$\operatorname{trace}\{\operatorname{Ad}(w) \cdot \operatorname{ad}(X)\} = 0, \quad \forall X \in \mathfrak{g}.$$

Or, by Lemma 3.9: $\mathrm{d}VX(w) = 0$ *for all* $X \in \mathfrak{g}$.

Proof. By Theorem 3.7 we have $a_k(t, w) = 0$ for every $t \in \mathbb{R}$ and $k \in \{1, \ldots, m\}$, so by Lemma 3.9

$$\operatorname{trace}\{\operatorname{Ad}(w) \cdot \operatorname{ad}(\operatorname{Ad}(\mathbf{x}_r(t))\, X_k)\} = 0, \quad \forall t \in \mathbb{R},\ \forall k \in \{1, \ldots, m\}.$$

But since $(\mathbf{x}_r, u_1^r, \ldots, u_m^r)$ is regular the linear functional $\operatorname{trace}\{\operatorname{Ad}(w) \cdot \operatorname{ad}(\cdot)\}$ vanishes on $\mathfrak{g}$. □

In particular, for a regular periodic trajectory $(\mathbf{x}_r, u_1^r, \ldots, u_m^r)$ the set of critical points of its auxiliary field W can be expressed as

$$\mathcal{C}_W = \{w \in G\ ;\ \operatorname{trace}\{\operatorname{Ad}(w) \cdot \operatorname{ad}(X)\} = 0,\ \forall X \in \mathfrak{g}\}. \tag{4.1}$$

Indeed, if w belongs to the set in the right-hand side of (4.1) then by (3.3) we have $a_k(t, w) = 0$ for all $t \in \mathbb{R}$ and every $k \in \{1, \ldots, m\}$, hence clearly $w \in \mathcal{C}_W$. We have thus equated, in this case:

- the set of critical points of the vector field W,
- the set of ω-limit points of W and
- the set of critical points of the Lyapunov-like function V.

Remark 4.3. Although our result above might seem a little awkward at first, there is no inherent contradiction to it. Actually, it enlightens a very natural question one might ask: what about global minima of V, which are the bona fide candidates for α-limit points of W (since V is non-decreasing along the integral curves of W)? They certainly exist by compactness of G, and are critical points of V, hence ω-limit points of W. But recall (see also Remark 3.8) that nothing prevents a point $x \in G$ from being both an α- and an ω-limit point of W – for instance, if x is a critical point of W, which is precisely the case by our results above.

Also, being a critical point of W could be the only reason to classify a point as an ω-limit point of W (and that would definitely be the case, for instance, if such critical point were repulsive), and does not reflect what kind of critical point of V (maximum, minimum, or otherwise) it is: there is no problem whatsoever in all critical points of V (including minima!) being ω-limit points of W.

Thereby, the full picture is drawn as follows: we further equate the sets in (3.1) to the set of all critical points of V, and these can be categorized into maxima (candidates for attractive equilibria of W, hence "true" ω-limit points – although the latter notion is meaningless in the present context), minima (likely repulsive equilibria of W; "true" α-limit points), and possible saddle-like points of V.

The next result gathers some interesting consequences of (4.1), which, however, we will not use in what follows.

Proposition 4.4. *Let $(\mathbf{x}_r, u_1^r, \ldots, u_m^r)$ be a regular periodic trajectory of (2.1) and W be its auxiliary vector field. Then for $x, y \in G$:*

(1) $x \in \mathcal{C}_W \Longrightarrow y \cdot x \cdot y^{-1} \in \mathcal{C}_W$.
(2) $x \cdot y \in \mathcal{C}_W \Longrightarrow y \cdot x \in \mathcal{C}_W$.
(3) $x \in \mathcal{C}_W \Longrightarrow x^{-1} \in \mathcal{C}_W$.

Moreover, if G is semisimple and $\mathcal{C}_W$ is a subgroup of G then $\mathcal{C}_W$ is finite.

Proof.

(1) For $X \in \mathfrak{g}$ a simple computation shows that

$$\operatorname{trace}\left\{\operatorname{Ad}\left(y \cdot x \cdot y^{-1}\right) \cdot \operatorname{ad}(X)\right\} = \operatorname{trace}\left\{\operatorname{Ad}(x) \cdot \operatorname{ad}\left(\operatorname{Ad}(y)^{-1} X\right)\right\} = 0$$

since x belongs to $\mathcal{C}_W$, hence so does $y \cdot x \cdot y^{-1}$.

(2) Follows from the previous item since $y \cdot (x \cdot y) \cdot y^{-1} = y \cdot x$.

(3) Pick $Y_1, \ldots, Y_n$ an orthonormal basis of $\mathfrak{g}$ w.r.t. some Ad-invariant[8] inner product $\langle \cdot, \cdot \rangle$. Then:

$$\begin{aligned}
\operatorname{trace}\left\{\operatorname{Ad}\left(x^{-1}\right) \cdot \operatorname{ad}(X)\right\} &= \sum_{j=1}^{n} \left\langle \operatorname{Ad}\left(x^{-1}\right) \operatorname{ad}(X) Y_j, Y_j \right\rangle \\
&= -\sum_{j=1}^{n} \left\langle Y_j, \operatorname{ad}(X) \operatorname{Ad}(x) Y_j \right\rangle \\
&= -\operatorname{trace}\left\{\operatorname{Ad}(x) \cdot \operatorname{ad}(X)\right\}
\end{aligned}$$

which equals 0 if $x \in \mathcal{C}_W$. Since $X \in \mathfrak{g}$ is arbitrary, $x^{-1} \in \mathcal{C}_W$.

As for the second part of the statement, since $\mathcal{C}_W$ is a closed set it is a Lie subgroup of G: let $\mathfrak{h} \subset \mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{h}$ and for each $Y \in \mathfrak{g}$ define $f_Y : \mathbb{R} \to \mathbb{R}$ by[9]

$$f_Y(t) \doteq \operatorname{trace}\left\{\operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}(Y)\right\}, \quad t \in \mathbb{R}.$$

Since then $\mathbf{e}^{tX} \in \mathcal{C}_W$ we have $f_Y(t) = 0$ for every $t \in \mathbb{R}$, and thus

$$\begin{aligned}
f_Y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{trace}\left\{\operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}(Y)\right\} \\
&= \operatorname{trace}\left\{\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}(Y)\right\} \\
&= \operatorname{trace}\left\{\operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}\left(\mathrm{d}L_{\mathbf{e}^{-tX}} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{e}^{tX}\right) \cdot \operatorname{ad}(Y)\right\} \\
&= \operatorname{trace}\left\{\operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}\left(\mathrm{d}L_{\mathbf{e}^{-tX}} X\left(\mathbf{e}^{tX}\right)\right) \cdot \operatorname{ad}(Y)\right\} \\
&= \operatorname{trace}\left\{\operatorname{Ad}\left(\mathbf{e}^{tX}\right) \cdot \operatorname{ad}(X) \cdot \operatorname{ad}(Y)\right\}
\end{aligned}$$

also equals 0 for all $t \in \mathbb{R}$, in particular for $t = 0$: we have thus proved that $\operatorname{trace}\{\operatorname{ad}(X) \cdot \operatorname{ad}(Y)\} = 0$ for every $Y \in \mathfrak{g}$. But this is the Killing form of $\mathfrak{g}$, which is non-degenerate since we are assuming G semisimple[10], from which we conclude that $X = 0$. Since $X \in \mathfrak{h}$ is arbitrary we have $\mathfrak{h} = \{0\}$ i.e. $\mathcal{C}_W$ is a discrete subgroup of G. Since G is compact and $\mathcal{C}_W$ is closed the latter must be finite. □

As we have seen, $e \in \mathcal{C}_W$ and if this set is finite then e is an isolated point, which is a necessary condition for e to be a local attractor. Next we shall focus on proving the latter property without relying on the assumption of $\mathcal{C}_W$ being a group.

Proposition 4.5. *If G is semisimple then every $x \in Z(G)$ is a non-degenerate critical point of V, and, in particular, an isolated point in $\mathcal{C}_W$.*

[8] An inner product on $\mathfrak{g}$ is said to be Ad-*invariant* if $\operatorname{Ad}(x)$ is orthogonal w.r.t. it for every $x \in G$. Such an inner product always exists when G is compact, and thanks to the relationship between the adjoint maps (see footnote 5) one also has that $\operatorname{ad}(X)$ is skew-symmetric w.r.t. it for every $X \in \mathfrak{g}$ [3, Proposition 4.24]. In particular, $\operatorname{trace} \operatorname{ad}(X) = 0$ for every $X \in \mathfrak{g}$.

[9] Here and below we denote by $\mathbf{e}^X \in G$ the exponential of $X \in \mathfrak{g}$.

[10] This is Cartan's Criterion for Semisimplicity [3, Theorem 1.45]: the Killing form of $\mathfrak{g}$ is the bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $B(X, Y) \doteq \operatorname{trace}\{\operatorname{ad}(X) \cdot \operatorname{ad}(Y)\}$. It is always negative semidefinite, while non-degenerate precisely when G is semisimple.

Proof. By Corollary 3.10, every $x \in Z(G)$ belongs to $\mathcal{C}_W$, hence is a critical point of V by our characterization of the latter set following Corollary 4.2 (since $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ is regular). As such, we check its non-degeneracy by computing the Hessian matrix of V in convenient coordinates around x.

We denote by B the Killing form of $\mathfrak{g}$. Since G is assumed semisimple, $-B$ is an inner product on $\mathfrak{g}$ and we denote by $Y_1, \dots, Y_n \in \mathfrak{g}$ an orthonormal basis w.r.t. it to introduce the so-called coordinates of second kind: let $\varphi : \mathbb{R}^n \to G$ be defined by

$$\varphi(s_1, \dots, s_n) \doteq x \cdot \mathbf{e}^{s_1 Y_1} \cdot \dots \cdot \mathbf{e}^{s_n Y_n}, \quad (s_1, \dots, s_n) \in \mathbb{R}^n.$$

Simple computations show that

$$\mathrm{d}\varphi \left(\frac{\partial}{\partial s_j}\bigg|_{s=0} \right) = Y_j(x), \quad \forall j \in \{1, \dots, n\},$$

hence φ is a local diffeomorphism near $s = 0$. Moreover

$$\begin{aligned} V(\varphi(s_1, \dots, s_n)) &= \operatorname{trace}\left\{\mathrm{Ad}\left(x \cdot \mathbf{e}^{s_1 Y_1} \cdot \dots \cdot \mathbf{e}^{s_n Y_n}\right)\right\} \\ &= \operatorname{trace}\left\{\mathrm{Ad}(x) \cdot \mathrm{Ad}\left(\mathbf{e}^{s_1 Y_1}\right) \cdot \dots \cdot \mathrm{Ad}\left(\mathbf{e}^{s_n Y_n}\right)\right\} \\ &= \operatorname{trace}\left\{\mathrm{Ad}(x) \cdot \mathbf{e}^{\mathrm{ad}(s_1 Y_1)} \cdot \dots \cdot \mathbf{e}^{\mathrm{ad}(s_n Y_n)}\right\} \\ &= \operatorname{trace}\left\{\mathbf{e}^{s_1 \,\mathrm{ad}(Y_1)} \cdot \dots \cdot \mathbf{e}^{s_n \,\mathrm{ad}(Y_n)}\right\} \end{aligned}$$

where we used two well-known facts: that $Z(G)$ is precisely the kernel of the Ad homomorphism; and the identity [3, Proposition 1.91]:

$$\mathrm{Ad}(\mathbf{e}^X) = \mathbf{e}^{\mathrm{ad}(X)}, \quad \forall X \in \mathfrak{g}.$$

Another simple computation then shows that

$$\frac{\partial^2 (V \circ \varphi)}{\partial s_j \partial s_k}\bigg|_{s=0} = \operatorname{trace}\left\{\mathrm{ad}(Y_j) \cdot \mathrm{ad}(Y_k)\right\} = B(Y_j, Y_k) = -\delta_{jk}, \quad \forall j, k \in \{1, \dots, n\}$$

thus showing that the Hessian matrix of V at $x = \varphi(0)$ is non-degenerate. The last claim follows from Morse Lemma. □

Next we characterize the center of G in terms of V regardless of semisimplicity. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$ and let $\mathrm{Ad}_{\mathbb{C}}(x) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathrm{Ad}(x)$. Therefore

$$\operatorname{trace} \mathrm{Ad}(x) = \operatorname{trace} \mathrm{Ad}_{\mathbb{C}}(x) = \sum_{j=1}^{n} \lambda_j(x)$$

where $\lambda_1(x), \dots, \lambda_n(x) \in \mathbb{C}$ are the eigenvalues of $\mathrm{Ad}_{\mathbb{C}}(x)$, repeated according to their multiplicities. Let $\langle \cdot, \cdot \rangle$ be any Ad-invariant inner product on $\mathfrak{g}$ (recall that G is compact) and let $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ stand for its sesquilinear extension to $\mathfrak{g}_{\mathbb{C}}$, which is then a Hermitian inner product on $\mathfrak{g}_{\mathbb{C}}$. Clearly $\mathrm{Ad}_{\mathbb{C}}(x)$ is unitary w.r.t. $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, in particular it is a diagonalizable map and

$$|\lambda_j(x)| = 1, \quad \forall j \in \{1, \dots, n\}.$$

It follows then that for every $x \in G$ we have

$$V(x) \leq |V(x)| = |\operatorname{trace} \mathrm{Ad}_{\mathbb{C}}(x)| = \left| \sum_{j=1}^{n} \lambda_j(x) \right| \leq \sum_{j=1}^{n} |\lambda_j(x)| = n.$$

Therefore, if $x \in Z(G)$ then $\mathrm{Ad}(x) = \mathrm{id}_{\mathfrak{g}}$ and hence $V(x) = n$, and thus is a global maximum of V.

We claim that the converse is also true i.e. if $V(x) = n$ then $x \in Z(G)$. We must prove that $\lambda_j(x) = 1$ for every $j \in \{1, \dots, n\}$. Indeed we have

$$n = \sum_{j=1}^{n} \lambda_j(x) = \sum_{j=1}^{n} \mathsf{Re}\lambda_j(x) + i \sum_{j=1}^{n} \mathsf{Im}\lambda_j(x)$$

which implies that

$$\sum_{j=1}^{n} \mathsf{Re}\lambda_j(x) = n, \quad \sum_{j=1}^{n} \mathsf{Im}\lambda_j(x) = 0.$$

Moreover

$$\mathsf{Re}\lambda_j(x) \leq |\mathsf{Re}\lambda_j(x)| \leq |\lambda_j(x)| = 1, \quad \forall j \in \{1, \dots, n\},$$

so if $\mathsf{Re}\lambda_k(x) < 1$ for some $k \in \{1, \dots, n\}$ then

$$\sum_{j=1}^{n} \mathsf{Re}\lambda_j(x) = \mathsf{Re}\lambda_k(x) + \sum_{j \neq k} \mathsf{Re}\lambda_j(x) < n$$

which would lead us to a contradiction, hence $\mathsf{Re}\lambda_k(x) = 1$ for every $k \in \{1, \dots, n\}$, and since $|\lambda_k(x)| = 1$ we must also have $\mathsf{Im}\lambda_k(x) = 0$ for every $k \in \{1, \dots, n\}$. In particular, we have proved:

Lemma 4.6. *Any $x \in G$ belongs to $Z(G)$ if and only if $V(x) = \dim \mathfrak{g}$.*

Now we can prove one of the main results of the present work. Recall that W denotes the auxiliary vector field associated to a *regular* periodic reference trajectory.

Theorem 4.7. *If G is semisimple then every $x \in Z(G)$ is a local attractor of W.*

Proof. By Proposition 4.5 we may find a neighborhood $U \subset G$ of x such that $\overline{U}$ is compact and contains no point in $\mathcal{C}_W$ other than x. We may further assume U connected, and define

$$M \doteq \max_{y \in \partial U} V(y).$$

Clearly $M < n$ for $\partial U \cap Z(G) = \emptyset$. Also, since $V(x) = n$ we have by continuity that there exists $U' \subset U$ another neighborhood of x such that $V(y) > M$ for every $y \in U'$.

Now let $(t_0, w_0) \in \mathbb{R} \times U'$ and $\mathbf{w} : \mathbb{R} \to G$ be the unique integral curve of W satisfying $\mathbf{w}(t_0) = w_0$. Since $V \circ \mathbf{w}$ is non-decreasing (for V is non-decreasing along W, as pointed out earlier) we have

$$V(\mathbf{w}(t)) \geq V(\mathbf{w}(t_0)) = V(w_0) > M$$

for every $t \geq t_0$ since $w_0 \in U'$ by hypothesis. Thus $\mathbf{w}(t) \notin \partial U$ and by continuity we have that $\mathbf{w}(t) \in U$ for every $t \geq t_0$.

It remains to show that $\mathbf{w}(t) \to x$ as $t \to +\infty$. Indeed, we will show that any sequence $\{t_n\}_{n \in \mathbb{N}}$ increasing to infinity admits a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ such that $\mathbf{w}(t_{n_k}) \to x$ as $k \to +\infty$. We may assume w.l.o.g. that $t_n \geq t_0$ for every $n \in \mathbb{N}$, hence $\mathbf{w}(t_n) \in U$. Because $\overline{U}$ is compact there exists $w \in \overline{U}$ and a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ of $\{t_n\}_{n \in \mathbb{N}}$ such that $\mathbf{w}(t_{n_k}) \to w$ as $k \to +\infty$. In particular $w \in \omega_W(t_0, w_0) \subset \mathcal{C}_W$, but as we have seen $\overline{U} \cap \mathcal{C}_W = \{x\}$. ☐

In particular e is a local attractor of W, hence Proposition 2.4 solves the PTTP locally.

5. Existence of regular periodic trajectories: sufficient conditions

Recall that we are always assuming that our left-invariant system (2.1) is bracket-generating i.e. the Lie algebra spanned by $X_1, \dots, X_m$ is $\mathfrak{g}$. In this section, we will prove that if moreover

$$\operatorname{span}\{\operatorname{ad}(X_1)^n X_k \ ; \ 1 \leq k \leq m, \ n \in \mathbb{N}\} = \mathfrak{g} \tag{5.1}$$

then given any $x_\infty \in G$ and $T > 0$ there exists $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ a smooth T-periodic trajectory of (2.1) satisfying $\mathbf{x}_r(0) = x_\infty$ and which is regular in the sense of Definition 4.1. Before that, however, we present a few examples in which condition (5.1) can be easily verified, as well as some examples in which condition (5.1) does not hold in spite of being controllable. For the first couple of them, recall that the special unitary group of order $n \geq 1$ is defined by

$$\mathrm{SU}(n) \doteq \{x \in \mathrm{GL}(n, \mathbb{C}) \ ; \ x^* \cdot x = e, \ \det x = 1\}$$

(where e stands for the $n \times n$ identity matrix) which is a compact and connected subgroup of $\mathrm{GL}(n, \mathbb{C})$. Its Lie algebra, realized as a linear subspace of $\mathfrak{gl}(n, \mathbb{C})$, is

$$\mathfrak{su}(n) \doteq \{X \in \mathfrak{gl}(n, \mathbb{C}) \ ; \ X + X^* = 0, \ \operatorname{trace}(X) = 0\}.$$

Moreover, $\mathrm{SU}(n)$ is semisimple (actually simple) for every n.

Example (Systems in SU(2)). The matrices

$$X_1 \doteq \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 \doteq \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 \doteq \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

form a linear basis of $\mathfrak{su}(2)$, and moreover the following commutation relations hold

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$

which characterize the Lie algebra $\mathfrak{su}(2)$ up to isomorphism. Therefore, each one of the families

$$\{X_1, X_2\}, \quad \{X_2, X_3\}, \quad \{X_1, X_3\}$$

is not only bracket-generating on SU(2) but also satisfies (5.1).

Example (System in SU(3)). A linear basis for $\mathfrak{su}(3)$ is given by the matrices

$$X_1 \doteq \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 \doteq \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad X_3 \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad T_1 \doteq \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 \doteq \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 \doteq \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_3 \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 \doteq \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}.$$

They obey the following commutation table

	X_1	X_2	X_3	Y_1	Y_2	Y_3	T_1	T_2
X_1	0	Y_3	Y_2	$2T_1$	$-X_3$	$-X_2$	$-2Y_1$	0
X_2		0	Y_1	$-X_3$	T_1+T_2	X_1	$-Y_2$	$-3Y_2$
X_3			0	X_2	X_1	T_2-T_1	Y_3	$-3Y_3$
Y_1				0	Y_3	$-Y_2$	$2X_1$	0
Y_2					0	Y_1	X_2	$3X_2$
Y_3						0	$-X_3$	$3X_3$
T_1							0	0
T_2								0

where the entries below the diagonal are obtained by skew-symmetry of the commutator bracket. It follows from these relations that the system

$$\{X_1, X_2, X_3\}$$

is bracket-generating, but does not satisfy (5.1) since

$$\begin{aligned} &\mathrm{span}\{X_1, X_2, X_3, Y_2, Y_3\} \text{ is invariant by } \mathrm{ad}(X_1), \\ &\mathrm{span}\{X_1, X_2, X_3, Y_1, Y_3\} \text{ is invariant by } \mathrm{ad}(X_2), \\ &\mathrm{span}\{X_1, X_2, X_3, Y_1, Y_2\} \text{ is invariant by } \mathrm{ad}(X_3). \end{aligned}$$

On the other hand, the enlarged system

$$\{X_1, X_2, X_3, Y_1, T_2\}$$

satisfies (5.1) since $\mathrm{ad}(X_1)X_2 = Y_3$, $\mathrm{ad}(X_1)X_3 = Y_2$, $\mathrm{ad}(X_1)Y_1 = 2T_1$.

Example (Systems in SO(4)). The special orthogonal group of order 4

$$\mathrm{SO}(4) \doteq \{x \in \mathrm{GL}(4, \mathbb{R}) \; ; \; {}^{\mathrm{t}}x \cdot x = e, \; \det x = 1\}$$

is compact, connected and semisimple (but not simple). Its Lie algebra is

$$\mathfrak{so}(4) \doteq \{X \in \mathfrak{gl}(4, \mathbb{R}) \; ; \; X + {}^{\mathrm{t}}X = 0\}$$

and a linear basis for it is given by

$$X_1 \doteq \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 \doteq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_3 \doteq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$Y_1 \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad Y_3 \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which satisfy the commutation table

	X_1	X_2	X_3	Y_1	Y_2	Y_3
X_1	0	$-Y_1$	$-Y_2$	X_2	X_3	0
X_2		0	$-Y_3$	$-X_1$	0	X_3
X_3			0	0	$-X_1$	$-Y_1$
Y_1				0	$-Y_3$	$-Y_2$
Y_2					0	$-Y_1$
Y_3						0

As in the previous example, one can check that while the system $\{X_1, X_2, X_3\}$ is bracket-generating but does not satisfy (5.1), the enlarged system $\{X_1, X_2, X_3, Y_3\}$ does.

Now we proceed to the proof of the existence of regular periodic trajectories provided (5.1) holds. For $j \in \{1, \dots, m\}$ and $p \in \mathbb{Z}$ let

$$J_j^p \doteq \left(\frac{(j-1)T}{m} + pT, \frac{jT}{m} + pT \right) = J_j^0 + pT.$$

Clearly $J_j^p \subset (pT, (p+1)T) = (0,T) + pT$ and

$$J_j^p \cap J_k^q \neq \emptyset \iff j = k, \ p = q. \tag{5.2}$$

Take $\chi_j \in C_c^\infty(J_j^0)$ equal to 1 in some open interval $I_j \subset J_j^0$ and with zero integral, and let $u_j^r : \mathbb{R} \to \mathbb{R}$ be the unique T-periodic function which is equal to χ_j on $[0,T]$: this is clearly smooth.

Since $\operatorname{supp} \chi_j \subset J_j^0$ it is easy to check that

$$\operatorname{supp} u_j^r \subset \bigcup_{p \in \mathbb{Z}} J_j^p \tag{5.3}$$

which, together with (5.2), easily ensures the following:

Proposition 5.1. *Given $t \in \mathbb{R}$, if $u_j^r(t) \neq 0$ for some $j \in \{1, \dots, m\}$ then $u_k^r(t) = 0$ for every $k \neq j$.*

Thus on $I_j^p \doteq I_j + pT$ we have $u_j^r = 1$, while $u_k^r = 0$ for $k \neq j$, identically. Next, define $\xi_j : \mathbb{R} \to \mathbb{R}$ by

$$\xi_j(t) \doteq \int_0^t u_j^r(s) \mathrm{d}s$$

which is obviously smooth and also T-periodic since u_j^r is T-periodic and its integral over $[0,T]$ is zero. Moreover, one may check that

$$\operatorname{supp} \xi_j \subset \bigcup_{p \in \mathbb{Z}} J_j^p. \tag{5.4}$$

We finally define $\mathbf{x}_r : \mathbb{R} \to G$ by

$$\mathbf{x}_r(t) \doteq \begin{cases} \mathbf{e}^{\xi_j(t) X_j}(x_\infty), & \text{if } t \in \bigcup_{p \in \mathbb{Z}} J_j^p, \text{ for } j = 1, \dots, m; \\ x_\infty, & \text{otherwise.} \end{cases}$$

This is well-defined thanks to (5.2), and moreover smooth by (5.4). Also, on I_j^p we have

$$\mathbf{x}_r'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}^{\xi_j(t)X_j}(x_\infty) = \xi_j'(t)X_j(\mathbf{e}^{\xi_j(t)X_j}(x_\infty)) = u_j^r(t)X_j(\mathbf{x}_r(t)) = \sum_{k=1}^m u_k^r(t)X_k(\mathbf{x}_r(t))$$

where the last identity follows from Proposition 5.1. On the other hand, if $t \notin J_j^p$ for any $j \in \{1,\dots,m\}$ and $p \in \mathbb{Z}$ then near t we have $\mathbf{x}_r = x_\infty$ identically, hence $\mathbf{x}_r'(t) = 0$, which also agrees with (2.1) thanks to (5.3): we have proved that $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ is a trajectory of (2.1), which is T-periodic by construction.

Theorem 5.2. *If (5.1) holds then the trajectory $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$ above is regular.*

Proof. We denote by

$$\mathcal{V} \doteq \operatorname{span}\{\operatorname{Ad}(\mathbf{x}_r(t))\, X_k\ ;\ t \in \mathbb{R},\ 1 \le k \le m\}$$

which we must prove that is equal to $\mathfrak{g}$. For each $k \in \{1,\dots,m\}$ let $\Lambda_k^0 \doteq X_k$ and $\lambda_k : \mathbb{R} \to \mathfrak{g}$ be defined by $\lambda_k \doteq \operatorname{Ad}(\mathbf{x}_r)\Lambda_k^0$: this is a smooth curve that lies in $\mathcal{V}$, and since the latter is a linear subspace of $\mathfrak{g}$ the same is true for all of its derivatives. By Lemma 3.11 we have, for every $n \in \mathbb{N}$,

$$\begin{aligned}\lambda_k^{(n)} &= \operatorname{Ad}(\mathbf{x}_r)\Lambda_k^n\\ \Lambda_k^{n+1} &\doteq (\Lambda_k^n)' + \operatorname{ad}(X_r)\Lambda_k^n\end{aligned}$$

where X_r is given by (3.5).

We need the following technical lemma, which does not depend on the construction of $(\mathbf{x}_r, u_1^r, \dots, u_m^r)$.

Lemma 5.3. *For each $n \in \mathbb{N}$ we may write*

$$\Lambda_k^n = \Delta_n X_k + \operatorname{ad}(X_r)^n X_k$$

where $\Delta_0 = \Delta_1 = 0$ and, for $n \ge 2$, Δ_n is a sum of products enjoying the following property: in each summand there is at least one factor that is a derivative of some order of $\operatorname{ad}(X_r)$.

Proof of Lemma 5.3. By induction on n. We start calculating recursively by Lemma 3.11

$$\begin{aligned}\Lambda_k^0 &= X_k\\ \Lambda_k^1 &= \operatorname{ad}(X_r)X_k\\ \Lambda_k^2 &= \operatorname{ad}(X_r)'X_k + \operatorname{ad}(X_r)^2X_k\end{aligned}$$

from which we identify $\Delta_0 = 0$, $\Delta_1 = 0$ and $\Delta_2 = \operatorname{ad}(X_r')$, thus proving our claim for $n = 0, 1, 2$, which we use as basis for induction. Assuming our conclusion for some $n \ge 2$ we have

$$\begin{aligned}\Lambda_k^{n+1} &= (\Lambda_k^n)' + \operatorname{ad}(X_r)\Lambda_k^n\\ &= (\Delta_n X_k + \operatorname{ad}(X_r)^n X_k)' + \operatorname{ad}(X_r)\left(\Delta_n X_k + \operatorname{ad}(X_r)^n X_k\right)\\ &= (\Delta_n)' X_k + (\operatorname{ad}(X_r)^n)' X_k + \operatorname{ad}(X_r)\Delta_n X_k + \operatorname{ad}(X_r)^{n+1}X_k\\ &= \Delta_{n+1}X_k + \operatorname{ad}(X_r)^{n+1}X_k\end{aligned}$$

where obviously

$$\Delta_{n+1} \doteq (\Delta_n)' + (\operatorname{ad}(X_r)^n)' + \operatorname{ad}(X_r)\Delta_n.$$

Since Δ_n is a sum of products in which each summand there is at least one factor that is a derivative of some order of $\operatorname{ad}(X_r)$ then of course the same property holds true for both $(\Delta_n)'$ and $\operatorname{ad}(X_r)\Delta_n$. Moreover

$$(\operatorname{ad}(X_r)^n)' = \sum_{p=1}^n \operatorname{ad}(X_r)^{p-1} \cdot \operatorname{ad}(X_r)' \cdot \operatorname{ad}(X_r)^{n-p}$$

also enjoys the aforementioned property, hence so does Δ_{n+1}. ☐

Now, if $t_1 \in I_1^0$ then $u_1^r(t_1) = 1$, while for $j \neq 1$ we have by Proposition 5.1 that u_j^r vanishes identically near t_1, and therefore

$$X_r(t_1) = \sum_{j=1}^{m} u_j^r(t_1) X_j = X_1$$

while

$$X_r^{(n)}(t_1) = 0, \quad \forall n \geq 1.$$

It follows from Lemma 5.3 that $\Delta_n(t_1) = 0$ (for $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ is linear, hence $\mathrm{ad}(X_r)^{(k)} = \mathrm{ad}\left(X_r^{(k)}\right)$ for every $k \in \mathbb{N}$) for every $n \in \mathbb{N}$, so

$$\Lambda_k^n(t_1) = \Delta_n(t_1) X_k + \mathrm{ad}(X_r(t_1))^n X_k = \mathrm{ad}(X_1)^n X_k$$

for every $n \in \mathbb{N}$. We conclude that

$$\lambda_k^{(n)}(t_1) = \mathrm{Ad}(\mathbf{x}_r(t_1)) \Lambda_k^n(t_1) = \mathrm{Ad}(\mathbf{x}_r(t_1))\, \mathrm{ad}(X_1)^n X_k$$

belongs to $\mathcal{V}$ for every $n \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$; in other words, if we denote

$$\mathcal{W} \doteq \mathrm{span}\{\mathrm{ad}(X_1)^n X_k \ ; \ 1 \leq k \leq m, \ n \in \mathbb{N}\}$$

then $\mathrm{Ad}(\mathbf{x}_r(t_1))\mathcal{W} \subset \mathcal{V}$. But we assumed in (5.1) that $\mathcal{W} = \mathfrak{g}$ and $\mathrm{Ad}(\mathbf{x}_r(t_1))$ is invertible, hence also $\mathcal{V} = \mathfrak{g}$ i.e. $(\mathbf{x}_r, u_1^r, \ldots, u_m^r)$ is regular. □

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