

RT-MAE 9530

**BOUNDS FOR THE GAP OF THE
METROPOLIS DYNAMICS FOR
THE RANDOM ENERGY MODEL**

by

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Key words: Disordered systems, random energy model, Metropolis dynamics, spectral gap.

Classificação AMS: 60K35, 82B44.
(AMS Classification)

Bounds for the Gap of the Metropolis Dynamics for the Random Energy Model

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Abstract

We derive upper and lower bounds of the same (exponential) order for the gap of the Metropolis dynamics for the Random Energy Model (REM). The upper bounds are based on the Poincaré inequality derived recently by Diaconis and Stroock.

Mathematics Subject Classification 1991: 60K35, 82B44.

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1 Introduction

The Random Energy Model (REM) is a disordered hamiltonian spin system designed by Derrida ([1] and [2]) as a caricature of the Sherrington-Kirkpatrick (SK) Spin-Glass Model. Both models are mean-field ones. While in the SK model one has (gaussian) pair interactions, in the REM there are (all sizes) multibody interactions. That is, the hamiltonian for the SK model is (for σ a given configuration of spins ± 1 in a volume Λ)

$$H_{SK}(\sigma) = |\Lambda|^{-1/2} \sum_{i,j \in \Lambda} J_{ij} \sigma_i \sigma_j, \quad (1.1)$$

where the sum is over all pairs of distinct sites in Λ and $\{J_{ij}, i, j\}$ is a family of i.i.d. standard gaussians, whereas that for the REM is

$$H(\sigma) = \frac{|\Lambda|^{1/2}}{2^{|\Lambda|/2}} \sum_{\alpha \subset \Lambda} J_\alpha \sigma_\alpha, \quad (1.2)$$

where the sum is over all subsets of Λ , $\{J_\alpha, \alpha\}$ is a family of i.i.d. standard gaussians and $\sigma_\alpha = \prod_{i \in \alpha} \sigma_i$ ($\sigma_\emptyset = 1$).

We observe that for the REM the hamiltonians of all configurations form a family of i.i.d. gaussians with mean zero and variance $|\Lambda|$ (A proof of this elementary fact is provided in an appendix to the paper). That could be taken as an alternative description of this model, indeed it is the usual one. We will adopt in the next session.

The REM (in equilibrium) has been much studied ([1], [2], [3], [4]). In this work we consider the dynamical model, that is, the REM undergoing a Glauber dynamics. We want to study the speed of convergence to equilibrium, the exponential rate of which will be given by the *spectral gap* of the dynamics, which is the difference between the first and second eigenvalues of the transition probability matrix of the Markov chain giving the dynamics (see [5] Proposition 3).

To get bounds for the (inverse of the) gap, we make use of the technology developed by Diaconis and Stroock relying on a variational characterization of the gap and on a bound based on a geometric Poincaré inequality ([5]).

Recently, Cassandro et al. have studied a disordered dynamical model, the random walk with random traps, with a different approach, via coupling techniques, to get the order of the speed of convergence to equilibrium ([6]).

The rest of the paper is organized as follows. In the next section we describe the model in detail. In Section 3, the main one, we derive upper and lower bounds for the inverse of the gap. Then we make some concluding remarks on Section 4. Section 5 is devoted to a brief discussion on existing work in dynamical random spin systems.

2 The Model

Throughout we consider the Random Energy Model (REM), which is a disordered Hamiltonian spin system, as a non-equilibrium system undergoing

Metropolis type (discrete) dynamics in finite volume. We want to study the behavior of the gap between the first and second eigenvalues of this process, which is a Markov chain, as the volume goes to infinity.

Let Λ be a nonempty set with $|\Lambda| = N$ and let Ω denote $\{-1, 1\}^\Lambda$. Let $\mathcal{H} = \{H(\sigma), \sigma \in \Omega\}$ be an independent family of Gaussian random variables with common mean zero and common variance N . \mathcal{H} will play the role here of the random hamiltonian. We consider a Markov chain in the state space Ω with transition probabilities reversible with respect to the Gibbs measure μ_N on Ω obtained from \mathcal{H} and the inverse temperature parameter β in the usual way, that is

$$\mu_N(\sigma) = \frac{1}{Z_N} \exp\{-\beta H(\sigma)\}, \quad \sigma \in \Omega,$$

where Z_N is the usual normalization factor known as partition function.

More specifically, we have Metropolis type transition probabilities given by

$$P(\sigma, \sigma') = \begin{cases} \frac{1}{N} \exp\{-\beta(H(\sigma') - H(\sigma))^+\}, & \text{if } |\sigma' - \sigma| = 1, \\ 0, & \text{if } |\sigma' - \sigma| > 1, \\ 1 - \sum_{\sigma'' \neq \sigma} P(\sigma, \sigma''), & \text{if } \sigma' = \sigma, \end{cases}$$

where $a^+ = (a + |a|)/2$.

3 Upper and Lower Bounds

Let ϕ be a vector in the 2^N dimensional unit sphere indexed by Ω with non-identical entries. Define

$$\text{Var}(\phi) = \sum_{\sigma, \sigma'} (\phi(\sigma) - \phi(\sigma'))^2 \mu_N(\sigma) \mu_N(\sigma') \quad (3.1)$$

$$= \frac{1}{Z_N^2} \sum_{\sigma, \sigma'} (\phi(\sigma) - \phi(\sigma'))^2 \exp\{-\beta(H(\sigma) + H(\sigma'))\}, \quad (3.2)$$

$$\mathcal{E}(\phi, \phi) = \sum_{\sigma, \sigma'} (\phi(\sigma) - \phi(\sigma'))^2 Q(\sigma, \sigma'), \quad (3.3)$$

where $Q(\sigma, \sigma') = \mu_N(\sigma) P(\sigma, \sigma')$.

$$\tau(\phi) = \frac{\text{Var}(\phi)}{\mathcal{E}(\phi, \phi)}, \quad (3.4)$$

Then the inverse of the gap between the first and second eigenvalues of the transition matrix is given by ([5])

$$\tau = \sup_{\phi} \tau(\phi). \quad (3.5)$$

Remark 3.1 We speed up time by a factor of N , so that the final expression for τ is (3.5) divided by N .

We derive now upper and lower bounds for τ . For the lower bounds, we maximize over ϕ 's which are indicator functions of a given coordinate.

Lower bounds.

Define ϕ_{σ} by $\phi_{\sigma}(\sigma') = \delta(\sigma, \sigma')$, where $\delta(\cdot, \cdot)$ is the Kronecker delta. We have

$$\tau(\phi_{\sigma}) = \frac{\exp\{-\beta H(\sigma)\}}{NZ_N \sum_{(\sigma, \sigma')} Q(\sigma, \sigma')} \quad (3.6)$$

For the Metropolis dynamics, $Q(\sigma, \sigma') = (NZ_N \exp\{\beta(H(\sigma) \vee H(\sigma'))\})^{-1}$ and thus (3.6) equals

$$\frac{\exp\{-\beta H(\sigma)\}}{\sum_{(\sigma, \sigma')} \exp\{-\beta(H(\sigma) \vee H(\sigma'))\}}. \quad (3.7)$$

Let $\underline{\sigma}$ be the configuration of spins σ for which $H(\sigma)$ is minimal. Then we get the bound

$$\tau \geq \max_{\sigma} \tau(\phi_{\sigma}) \geq \tau(\phi_{\underline{\sigma}}) = \quad (3.8)$$

$$= \frac{\exp\{-\beta H(\underline{\sigma})\}}{\sum_{(\underline{\sigma}, \sigma')} \exp\{-\beta H(\sigma')\}} \quad (3.9)$$

Our upper bounds are based on the *Poincaré inequality* derived in [5].

Upper bounds.

Let \mathcal{C}_N denote the hypercube in N dimensions obtained from Ω by adding nearest neighbor bonds. Let Γ_N be the following set of paths in \mathcal{C}_N .

For η and η' in \mathcal{C}_N , choose $\gamma_{\eta, \eta'}$ by changing the coordinates where η differs from η' to their opposite, working left to right, one coordinate at a time. Make $\Gamma_N = \{\gamma_{\eta, \eta'}, \eta, \eta' \in \mathcal{C}_N\}$.

Remark 3.2 Γ_N is a complete set of paths (in the sense that every two distinct sites of \mathcal{C}_N are extremes of a path in the Γ_N) of length at most N . For any given bond e in \mathcal{C}_N there are 2^{N-1} elements of Γ_N traversing e ([5], Example 2.2).

From Proposition 1' in [5] one gets the bound

$$\tau \leq \max_{\epsilon} Q(\epsilon)^{-1} \sum_{\gamma_{\eta, \eta'} \ni \epsilon} |\gamma_{\eta, \eta'}| \mu_N(\eta) \mu_N(\eta'), \quad (3.10)$$

where the max is over nearest neighbor bonds of \mathcal{C}_N and the summation is over all paths (indexed by their endpoints η and η') through ϵ .

From (3.10) we derive our bound

$$N2^N \max_{\epsilon} Q(\epsilon)^{-1} \max_{\sigma} \mu_N(\sigma)^2 \quad (3.11)$$

$$= N2^N \max_{\epsilon} Q(\epsilon)^{-1} Z_N^{-2} \exp\{-2\beta H(\underline{\sigma})\}. \quad (3.12)$$

where $N2^N$ is justified from Remark 3.2.

After taking into account the change in time scale, it takes the form

$$N \exp\{\beta H(\bar{\sigma})\} \exp\{-2\beta H(\underline{\sigma})\} / \bar{Z}_N, \quad (3.13)$$

where $\bar{\sigma}$ is the configuration σ for which $H(\sigma)$ is maximal and $\bar{Z}_N = 2^{-N} Z_N$.

By applying these bounds, we get the following result.

Theorem 1 *There exist positive finite constants c_1 and c_2 such that*

$$\liminf_{N \rightarrow \infty} \frac{\log \tau}{N} \geq c_1, \quad \limsup_{N \rightarrow \infty} \frac{\log \tau}{N} \leq c_2. \quad (3.14)$$

Proof.

We use here and elsewhere the basic fact of gaussian distributions that the maximum (resp. minimum) of n independent standard gaussian random variables is of the order of $\sqrt{2 \log n}$ (resp. $-\sqrt{2 \log n}$) with probability 1 (that is, the maximum (resp. minimum) divided by $\sqrt{2 \log n}$ (resp. $-\sqrt{2 \log n}$) goes to 1 with probability 1).

The log of the denominator in (3.8) is of order $\sqrt{N \log N}$ since the sum can be bounded below by the maximum of the summands and above by N times this maximum.

Only the log of the numerator contributes then. When divided by N it converges to $\beta \sqrt{2 \log 2}$ as $N \rightarrow \infty$, so we take this expression for c_1 .

For the upper bound, we use the (exact) asymptotic expression for $N^{-1} \log Z_N$ derived in [4], namely

$$\begin{aligned} \beta^2/2 + \log 2, & \quad \text{if } \beta \leq \sqrt{2 \log 2} \\ \sqrt{2 \log 2} \beta, & \quad \text{if } \beta \geq \sqrt{2 \log 2}. \end{aligned}$$

This combined with the numerator allows us to take c_2 as

$$\begin{aligned} 3\beta\sqrt{2 \log 2} - \beta^2/2 & \quad \text{if } \beta \leq \sqrt{2 \log 2} \\ 2\beta\sqrt{2 \log 2} + \log 2 & \quad \text{if } \beta \geq \sqrt{2 \log 2} \quad \square \end{aligned}$$

4 Concluding Remarks

One interesting feature of the REM is its (third order) phase transition at $\beta = \sqrt{2 \log 2}$. This should affect the asymptotic behavior of the gap. Our bounds of course do not say anything about that. One should instead get exact asymptotic expressions for $\log \tau/N$. Partial information as to the existence of such expressions for all β and their nonrandomness would be already of interest (we conjecture that this is the case). The methods used in this paper do not seem to apply for these questions, although a more careful, random choice of the complete set of paths entering the upper bound, or a more careful estimation with the considered set might narrow the gap with the lower bound.

5 The Dynamics of Random Spin Systems

Dynamics of spin glasses was studied (non-rigorously) by Sompolinsky and Zippelius ([7]) which considered a *soft spin* approach so that they could work with quantities varying continuously. This has been made rigorous by Ben Arous and Guionnet ([10]). Some recent work ([8] and [9]) shows that non exponential relaxation is possible in infinite volume for models with short

range interaction. This is of course not in contradiction with our result for a (large) finite sample. We plan to investigate infinite volume behaviour and the relationship with the short models results in a forthcoming paper.

A Appendix

We establish here the microscopic representation for the REM hamiltonian mentioned in the introduction.

Proposition A.1 *Let $\mathcal{H} = \{H(\sigma), \sigma \in \Omega\}$ be defined by Equation 1.2, where $\{J_\alpha, \alpha \subset \Lambda\}$ is a family of i.i.d. standard gaussian random variables. Then \mathcal{H} is a family of i.i.d. gaussian random variables with mean 0 and variance N .*

Proof

The gaussianness, correct marginal mean and variance are clear. It suffices thus to establish independence. It is enough to show that the matrix

$$\{\sigma_\alpha, \sigma \in \Omega, \alpha \subset \Lambda\} \quad (\text{A.1})$$

is an orthogonal one, that is, that

$$\sum_{\alpha \subset \Lambda} \sigma_\alpha \sigma'_\alpha = 0$$

for all $\sigma, \sigma' \in \Omega$ with $\sigma \neq \sigma'$.

Given $\sigma, \sigma' \in \Omega$ with $\sigma \neq \sigma'$, let Δ denote the (nonempty) set where σ and σ' disagree, that is

$$\Delta = \{i \in \Lambda : \sigma_i \neq \sigma'_i\}.$$

We have

$$\sum_{\alpha \subset \Lambda} \sigma_\alpha \sigma'_\alpha = \sum_{\alpha \subset \Lambda} (-1)^{|\alpha \cap \Delta|}.$$

The last sum can be rewritten as

$$\sum_{k=0}^M \sum_{\alpha \subset \Lambda: |\alpha \cap \Delta|=k} (-1)^k.$$

where $M = |\Delta|$. Which equals

$$\sum_{k=0}^M g_{\Delta}(k)(-1)^k,$$

where $g_{\Delta}(k)$ is the number of distinct subsets α of Λ intersecting Δ at exactly k points. There are k choices out of M in Δ , which yields $\binom{M}{k}$ possibilities for $\alpha \cap \Delta$, and total freedom in $\Lambda \setminus \Delta$, which yields 2^{N-M} possibilities for $\alpha \setminus \Delta$. Thus

$$g_{\Delta}(k) = \binom{M}{k} 2^{N-M}.$$

We finally have

$$\begin{aligned} \sum_{\alpha \subset \Lambda} \sigma_{\alpha} \sigma'_{\alpha} &= 2^{N-M} \sum_{k=0}^M \binom{M}{k} (-1)^k \\ &= 2^{N-M} (1-1)^M = 0, \end{aligned}$$

since $M > 0$. The result is proven. \square

Acknowledgements.

We thank M. Cassandro, A. Galves and A. Pellegrinotti for stimulating discussions concerning this work, which was partially supported by CNPq grant no. 453865/93-3, Fapesp grant no. 94/4303-5 and the reasearch agreement between the Universities of Rome "La Sapienza" and São Paulo. L.F. thanks the Mathematics Department of the former university for warm hospitality. M.I. acknowledges hospitality in São Paulo.

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