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Introduction.

It is often the case in infinite dimensional dynamical systems that a certain unbounded operator depends upon a parameter ν. Its eigenvalues and eigenfunctions will also depend upon ν. In many cases (see for example Hale [1986], Hale and Rocha [1987a,b], Fusco [1987], Hale and Carvalho [1991], Carvalho and Pereira [1991] and Carvalho [1992]) there is a finite number of eigenvalues of the unbounded operator that stay bounded as the parameter varies, the remaining eigenvalues blow up at a certain rate. In this case one expects that the dynamics be dictated by a system of ordinary differential equations. This is indeed the case in several applications and its proof strongly uses the invariant manifold theory.

In the cases considered in this paper we study the dynamics of infinite dimensional dynamical systems for which a countable number of eigenvalues of the unbounded operator converges to the eigenvalues of an unbounded operator in a lower dimensional domain, whereas the remaining blow up. This is the case for example in parabolic equations in thin domains studied by Hale and Raugel [1988]. We will not be able to address the general case of thin domains, but only the case of a thin straight channel. However, the property that allows this theory to work is present in the general case; that is, there is a sequence of eigenvalues that stay bounded and a sequence of eigenvalues that blow up (see also Arrieta [1991]).

The theory of invariant manifolds will no longer work in this case since it is centered around existence of backward solutions which can not be expected in this situation. The main tools in this work is a converse theorem on existence of compact attractors and an eigenvalue problem.

A physical interpretation of a parabolic problem in a thin straight channel is that there is a very high diffusion in one of the directions from which our intuition would say that, in that direction, spatial changes will not be observed.

The same interpretation can be given to a damped wave problem in a thin straight channel. The wave speed can be interpreted as being very large in one direction. This would intuitively imply that there would be no spatial changes for the amplitude of the wave, in that direction.

We also consider the problem called Shadow Systems where the diffusion is very large for some of the substances in the reaction and not so large for the remaining substances. In this case, the only possible spatial change for the substances with large diffusion coefficient would come from the reaction and not from the diffusion. This problem has been studied before in Hale and Sakamoto [1989] (see also Nishiura [1982]). However, due to their choice of space, they only give information about the local dynamics of the problem.

The results are presented for damped hyperbolic delay-partial differential equations and parabolic delay-partial differential equations in an abstract setting. We give a class of delay-partial differential equations for which the hypothesis of our general result can be verified.

We remark that if one is interested only in local dynamics, the dissipativeness hypothesis can be replaced by a local condition (e.g., existence of local attractors). Also in this case, the growth conditions can be dropped if we work in fractional power spaces which are embedded in L^∞.

Observe that in proving upper semicontinuity it is natural to assume that the union of all the attractors as the parameter varies is a bounded set; however, this hypothesis is not always easy to be verified. The method employed in this paper to obtain such a priori estimates is the more or less classical one of using a Liapunov functional. If the a priori bounds can be obtained in any other manner the results would still apply.

This paper is divided into five sections. In Section 1 we prove upper semicontinuity of attractors for abstract semilinear parabolic and damped hyperbolic delay-partial differential equations using converse theorems as in Yoshizawa [1966]. Section 2 is devoted to parabolic delay-partial problems on thin straight (two or three dimensional) channels. In Section 3 we consider damped hyperbolic delay-partial problems on thin straight (two or three dimensional) channels. In Section 4 we consider systems called Shadow systems as an application of the results in Section 1. In Section 5 we consider an example of cooperative systems (see Kishimoto and Weinberger [1985]) which arise as a reaction diffusion problem in a thin domain around a point (see Hale and Raugel [1992]).

1. Upper semicontinuity of Attractors for Delay-Partial Differential Equations

In this section we introduce the basic concepts and prove results that will be used throughout this paper. In some cases existing theorems had to be reproved to adapt to the new situation.

Let X be a Banach space and $A : D(A) \subset X \to X$ be the generator of a strongly continuous semigroups of bounded linear operators on X. If the operator A generates an analytic semigroup, we can define the fractional powers A^α of A and the associated fractional power spaces X^α (see Henry [1981]).

To simplify the presentation of the results we make the convention that $\alpha = 0$ for all results in which A is the generator of a strongly continuous semigroup.

Let $r > 0$ be a given real number. We denote by $C = C([-r,0],X^\alpha)$ the space of continuous function $\phi : [-r,0] \to X^\alpha$ endowed with the uniform norm. For any continuous function $x : [-r,T) \to X^\alpha$ and any $0 \le t < T$, we let x_t be the function in C defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

Let $\{T(t), t \geq 0\}$ be a semigroup (usually nonlinear) on X. A set $B \subset X$ is said to attract a set $C \subset X$ under $T(t)$ if $\operatorname{dist}(T(t)C, B) \to 0$ as $t \to \infty$. A set $S \subset X$ is said to be invariant if $T(t)S = S$ for $t \geq 0$. An invariant set A is said to be a *global attractor* if A is a maximal compact invariant set which attracts each bounded set $B \subset X$.

Our first result is a converse theorem on existence of a compact attractor for semilinear equations in Banach spaces. Results of this type are standard in the theory of stability and we only give its proof for the sake of completeness

Theorem 1.1. *Consider the functional differential equation*

$$\dot{x}(t) = Ax(t) + f(x_t) \tag{1.1}$$

where A the generator of a semigroup on a Banach space X and $f : C([-r,0], X^{\alpha}) \to X$ is Lipschitz continuous on bounded sets of $C([-r,0], X^{\alpha})$. Suppose that (1.1) has a global attractor $\mathcal{A}$. Then, there is a function $\Sigma : C([-r,0], X^{\alpha}) \to \mathbb{R}^+$ which is Lipschitz in bounded sets and satisfy

i) $\Sigma(\phi) = 0, \forall\, \phi \in \mathcal{A}$,

ii) $a(d(\phi, \mathcal{A})) \leq \Sigma(\phi) \leq b(d(\phi, \mathcal{A}))$, *where a is continuous nondecreasing, $a(s) > 0$ if $s > 0$, $a(s) \to \infty$ as $s \to \infty$ and $b(s)$ is continuous with $b(0) = 0$,*

iii) $\dot{\Sigma}_{(1.1)}(\phi) \leq -\Sigma(\phi)$, *where $\dot{\Sigma}_{(1.1)}$ is the right hand derivative of Σ along solutions of (1.1).*

Proof. Let $\phi \in C([-r,0], X^{\alpha})$, $\|\phi\|_{C([-r,0],X^{\alpha})} \leq R$. If $x_t(\cdot, \phi)$ is the solution of (1.1) satisfying $x_0(\cdot, \phi) = \phi$, there exists a function $\theta(t, R)$ such that

$$d(x_t(\cdot, \phi), \mathcal{A})) \leq \theta(t, R)$$

where $\theta(t, R)$ is a strictly decreasing in t and a C^1 function. Let $T(\epsilon)$ be its inverse, that is $T(\theta(t, R)) = t$. So $T(\epsilon)$ is continuous on $0 < \epsilon < \theta(0)$ and $T(\epsilon) \to \infty$ as $\epsilon \to 0^+$.

From the hypothesis on f, the map $\varphi \mapsto x_t(\cdot, \phi)$ has Lipschitz constant $\leq Le^{Mt}$ for some L, M depending only upon R.

Define

$$g(\epsilon) = \epsilon^{-(\beta+M)T(\epsilon)}, \quad g(0) = 0.$$

for some $\beta > 0$. Also

$$G_k(z) = \max\left\{0, z - \frac{1}{k}\right\}, \quad k \geq 1.$$

For $k = 1, 2, 3, \cdots$

$$\Sigma_k(\phi) = g\left(\frac{1}{k+1}\right) \sup_{t \geq 0}\{\epsilon^{\beta t} G_k(d(x_t(\cdot, \phi), \mathcal{A}))\}.$$

Observe that the sup is taken only on $0 \leq t \leq T_k = T\left(\frac{1}{k+1}\right)$ and so

$$0 \leq \Sigma_k(\phi) \leq g\left(\frac{1}{k+1}\right) \epsilon^{\beta T_k} \theta(0) \leq \theta(0).$$

$$\begin{aligned}|\Sigma_k(\phi) - \Sigma_k(\psi)| &\leq g\left(\frac{1}{k+1}\right) \sup_{0 \leq t \leq T_k} \left\{L\epsilon^{(\beta+M)t} \|\phi - \psi\|_{C([-r,0],X^{\alpha})}\right\} \\ &\leq g\left(\frac{1}{k+1}\right) L\epsilon^{(\beta+M)T_k} \|\phi - \psi\|_{C([-r,0],X^{\alpha})} \leq L\|\phi - \psi\|_{C([-r,0],X^{\alpha})}.\end{aligned}$$

Finaly,

$$\begin{aligned}\Sigma_k(x_h(\cdot, \phi)) &= \epsilon^{-\beta h} g\left(\frac{1}{k+1}\right) \sup_{t \geq h}\{\epsilon^{\beta t} G_k(d(x_t(\cdot, \phi), \mathcal{A}))\} \\ &\leq \epsilon^{-\beta h} \Sigma_k(\phi).\end{aligned}$$

so $\dot{\Sigma}_k(\phi) \leq -\beta \Sigma_k(\phi)$.

If we let

$$\Sigma(\phi) = \sum_{k=1}^{\infty} 2^{-k}\Sigma_k(\phi),$$

we have that

$$|\Sigma(\phi) - \Sigma(\psi)| \leq L\|\phi - \psi\|_{C([-r,0],X^{\alpha})}, \qquad \dot{\Sigma}(\phi) \leq -\beta\Sigma(\phi), \qquad \Sigma(\phi) = 0,\ \forall\phi \in \mathcal{A},$$

and

$$\Sigma(\phi) \geq \sum_{k=1}^{\infty} 2^{-k} g\left(\frac{1}{k+1}\right) G_k(d(\phi,\mathcal{A})) = a(d(\phi,\mathcal{A})).$$

with a Lipschitz continuous, strictly increasing, $a(s) > 0$ if $s > 0$ and $a(s) \to \infty$ as $s \to \infty$. This concludes the proof.

Let $\nu > 0$ be a positive parameter, X_ν be a Banach space and $A_\nu : D(A_\nu) \subset X_\nu \to X_\nu$ be the generator of a semigroup. Let A_ν^α denote the fractional power of A_ν and X_ν^α the associated fractional power spaces.

Let Y be a Banach space and $B : D(B) \subset Y \to Y$ be the generator of a semigroup. Let B^α denote the fractional power of B and Y^α the associated fractional power spaces.

Assume that the semigroup $\{T_\nu(t), t \geq 0\}$ generated by A_ν satisfies

$$\|T_\nu(t)w\|_{X_\nu^\alpha} \leq Me^{-\beta(\nu)t}\|w\|_{X_\nu^\alpha}, \quad t \geq 0$$
$$\|T_\nu(t)w\|_{X_\nu^\alpha} \leq Mt^{-\alpha}e^{-\beta(\nu)t}\|w\|_{X_\nu}, \quad t > 0,$$

for any $w \in X_\nu^\alpha$, where $\beta(\nu)$ may depend upon the parameter ν and $M \geq 1$ is a constant.
Consider the weakly coupled system

$$\left.\begin{aligned} \dot{x}(t) &= A_\nu x(t) + f_\nu(x_t, y_t) \\ \dot{y}(t) &= By(t) + g_\nu(x_t, y_t). \end{aligned}\right\} \tag{1.2}$$

Let $g : C([-r,0],Y^\alpha) \to Y$ be Lipschitz continuous in bounded sets of $C[-r,0],Y^\alpha)$ and assume that

$$\dot{y}(t) = By(t) + g(y_t) \tag{1.3}$$

has a global attractor $\mathcal{A}$ in $C([-r,0],Y^\alpha)$. Suppose that there exists a constant $\mathcal{M} > 0$, independent of ν, such that the set

$$\mathcal{B} = \{u \in C([-r,0],X_\nu^\alpha \times Y^\alpha) : \|u\|_{C([-r,0],X_\nu^\alpha \times Y^\alpha)} \leq \mathcal{M}\} \tag*{[H]}$$

attracts bounded sets of $C([-r,0],X_\nu^\alpha \times Y^\alpha)$ under the flow defined by (1.2).
Let $R > 0$, $(\phi,\psi) \in C([-r,0],X_\nu^\alpha \times Y^\alpha)$, $\|(\phi,\psi)\|_{C([-r,0],X_\nu^\alpha \times Y^\alpha)} \leq R$. Suppose that there exist nonnegative constants M_f, L_f, depending only on R, such that

$$\|f_\nu(\phi,\psi)\|_{X_\nu} \leq L_f\|\phi\|_{C([-r,0],X_\nu^\alpha)} + M_f, \quad \|P_\nu(\phi,\psi)\|_Y \leq L_P(\nu)\|\phi\|_{C([-r,0],X_\nu^\alpha)} + M_P(\nu). \tag{1.4}$$

where $P_\nu(\phi,\psi) = g_\nu(\phi,\psi) - g(\psi)$, and $L_P(\nu)$, $M_P(\nu) \to 0$ as $\nu \to 0$.
Assume also that either of the following conditions is satisfied
a) The flow defined by (1.2) is asymptotically smooth.
b) $M_f = 0$ and the flow defined by $\dot{y} = By + g_\nu(0, y_t)$ is asymptotically smooth.
c) $M_f = 0$ and $M_P \equiv 0$.

Theorem 1.2. *Assume that A is a sectorial operator and that $\beta(\nu) \to \infty$ as $\nu \to 0$. Assume also that* [H] *and (1.4) are satisfied and that either a), b) or c) above is satisfied. Then, there exists $\nu_0 > 0$ such that , for $0 < \nu \leq \nu_0$, the problem (1.2) has a global attractor $\mathcal{A}_\nu$ and the family of attractors $\{\mathcal{A}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is upper semicontinuous at zero, where $\mathcal{A}_0 := \mathcal{A}$. Furthermore, if c) is satisfied, there exists $\nu_0 > 0$ such that $\mathcal{A}_\nu = \mathcal{A}$, $0 < \nu \leq \nu_0$.*

Proof. Since (1.3) has a global attractor $\mathcal{A}$ there exists a locally Lipschitz continuous function $\Sigma : C([-r, 0], Y^\alpha) \to \mathrm{IR}^+$ such that for any $\psi \in C([-r, 0], Y^\alpha)$,

i) $\Sigma(\psi) = 0$ if $\psi \in \mathcal{A}$,

ii) $a(d(\psi, \mathcal{A})) \leq \Sigma(\psi) \leq b(d(\psi, \mathcal{A}))$, where a is continuous nondecreasing, $a(s) > 0$ if $s > 0$, $a(s) \to \infty$ as $s \to \infty$ and $b(s)$ is continuous with $b(0) = 0$.

iii) $\dot{\Sigma}_{(1.3)}(\psi) \leq -\Sigma(\psi)$, where $\dot{\Sigma}_{(1.3)}$ is the right hand derivative of Σ along the solutions of (1.3).

For any $c > 0$, let $\mathcal{B}_c = \{\psi \in Y^\alpha : \Sigma(\psi) < c\}$. Property *ii)* implies that $\mathcal{B}_c$ is bounded.

Suppose (x_t, y_t) is a solution for (1.2) with initial data (ϕ, ψ). Using the variation of constants formula. (1.2) can be rewritten as

$$\frac{dy}{dt} = By + g(y_t) + P_\nu(x_t, y_t), \tag{1.5}$$

$$x(t) = T_\nu(t)\phi(0) + \int_0^t T_\nu(t-s) f_\nu(x_s, y_s)\,ds. \tag{1.6}$$

for $t > 0$ and $x(t) = \phi(t)$ and $y(t) = \psi(t)$ for $-r \leq t \leq 0$.

Given $c > 0$, $\eta > 0$, there exists $\nu_0 > 0$ such that. for all $\nu \in (0, \nu_0]$.

$$c - LL_P(\nu)\eta - LM_P(\nu) > 0.$$

where L is the Lipschitz constant of Σ on $\mathcal{B}_c$.

If $y_s \in \mathcal{B}_c$ and $\|x_s\|_{C([-r,0],X^\alpha_\nu)} < \eta$ for $0 \leq s \leq t$, then

$$\begin{aligned}\dot{\Sigma}(y_t) \leq& -\Sigma(y_t) + L\|P_\nu(x_t, y_t)\|_Y \\ \leq& -\Sigma(y_t) + LL_P(\nu)\|x_t\|_{C([-r,0],X^\alpha_\nu)} + LM_P(\nu) < -\Sigma(y_t) + LL_P(\nu)\eta + LM_P(\nu)\end{aligned}$$

and

$$\|x(t)\|_{X^\alpha_\nu} \leq M\epsilon^{-\beta(\nu)t}\|\phi(0)\|_{X^\alpha_\nu} + \int_0^t M(t-s)^{-\alpha}\epsilon^{-\beta(\nu)(t-s)}[L_f\|x_s\|_{C([-r,0],X^\alpha_\nu)} + M_f]ds.$$

Fix $\sigma > 0$ and let $\nu_0 > 0$ be such that $0 < \nu \leq \nu_0$ implies $\beta(\nu) - \sigma > 0$. Then. for $t > 0$. we have

$$\|x(t)\|_{X^\alpha_\nu}\epsilon^{\sigma t} \leq M\epsilon^{(-\beta(\nu)+\sigma)t}\|\phi(0)\|_{X^\alpha_\nu} + ML_f\int_0^t (t-s)^{-\alpha}\epsilon^{(-\beta(\nu)+\sigma)(t-s)}\epsilon^{\sigma s}\|x_s\|_C ds + \frac{MM_f\Gamma(1-\alpha)}{\beta(\nu)^{1-\alpha}}\epsilon^{\sigma t}.$$

Let $z : [-r, \infty) \to \mathrm{IR}^+$ be defined by

$$z(t) = \begin{cases} \epsilon^{\sigma t}\|x_t\|_C \text{ , if } t > 0 \\ \|\phi\|_C \qquad \text{, if } -r \leq t \leq 0\end{cases}$$

and let $v(t) = \sup_{-r\leq s\leq t} z(s)$.

If $t > 0$ and $0 \leq \tau \leq t$, we have

$$\|x(\tau)\|_{X^\alpha_\nu}\epsilon^{\sigma\tau} \leq M\|\phi\|_C + \frac{ML_f\Gamma(1-\alpha)}{(\beta(\nu)-\sigma)^{1-\alpha}}v(t) + \frac{MM_f\Gamma(1-\alpha)}{\beta(\nu)^{1-\alpha}}\epsilon^{\sigma t}.$$

On the other hand, if $-r \leq \theta \leq 0$. we have

$$\|x(\theta)\|_{X^\alpha_\nu}\epsilon^{\sigma\theta} \leq \|x(\theta)\|_{X^\alpha_\nu} \leq \|\phi\|_C.$$

Therefore, for $-r \leq \tau \leq t$, we have

$$\|x(\tau)\|_{X^\alpha_\nu} \epsilon^{\sigma\tau} \leq M\|\phi\|_C + \frac{ML_f\Gamma(1-\alpha)}{(\beta(\nu)-\sigma)^{1-\alpha}} v(t) + \frac{MM_f\Gamma(1-\alpha)}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t},$$

and therefore

$$\sup_{-r\leq\tau\leq t} \|x(\tau)\|_{X^\alpha_\nu} \epsilon^{\sigma\tau} \leq M\|\phi\|_C + \frac{ML_f\Gamma(1-\alpha)}{(\beta(\nu)-\sigma)^{1-\alpha}} v(t) + \frac{MM_f\Gamma(1-\alpha)}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t}.$$

But

$$\epsilon^{\sigma t}\|x_t\|_C = \sup_{-r\leq\theta\leq 0} \{\epsilon^{-\sigma\theta}\epsilon^{\sigma(t+\theta)}\|x(t+\theta)\|_{X^\alpha_\nu}\} \leq \epsilon^{\sigma r} \sup_{t-r\leq s\leq t} \{\epsilon^{\sigma s}\|x(s)\|_{X^\alpha_\nu}\} \leq \epsilon^{\sigma r} \sup_{-r\leq s\leq t} \{\epsilon^{\sigma s}\|x(s)\|_{X^\alpha_\nu}\},$$

for all $t > 0$. Therefore

$$\epsilon^{\sigma t}\|x_t\|_C \leq M\|\phi\|_C\epsilon^{\sigma r} + \frac{ML_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{(\beta(\nu)-\sigma)^{1-\alpha}} v(t) + \frac{MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t}.$$

It follows that

$$v(t) \leq M\|\phi\|_C\epsilon^{\sigma r} + \frac{ML_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{(\beta(\nu)-\sigma)^{1-\alpha}} v(t) + \frac{MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t}.$$

Let ν_0 be such that $0 < \nu \leq \nu_0$ implies

$$1 - \frac{ML_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{(\beta(\nu)-\sigma)^{1-\alpha}} \geq \frac{1}{2}.$$

Then

$$v(t) \leq 2[M\|\phi\|_C\epsilon^{\sigma r} + \frac{MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t}]$$

for all $t \geq 0$. In particular, we have

$$\epsilon^{\sigma t}\|x_t\|_C \leq 2[M\|\phi\|_C\epsilon^{\sigma r} + \frac{MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}} \epsilon^{\sigma t}]$$

and therefore

$$\|x_t\|_C \leq 2[M\|\phi\|_C\epsilon^{-\sigma(t-r)} + \frac{MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}}] \tag{1.7}$$

for all $t \geq 0$.

For $\|\phi\|_C < \frac{\eta}{4M}$, (1.7) implies $\|x_s\|_C < \eta$ and $y_s \in \mathcal{B}_c$ whenever defined.

Therefore, for every $t \geq 0$, $\|\phi\|_C < \frac{\eta}{4M}$, $\psi \in \mathcal{B}_c$, (1.7) is satisfied and

$$\dot{\Sigma}(y_t) \leq -\Sigma(y_t) + 2LL_P(\nu)M\left[\|\phi\|_C\epsilon^{-\sigma(t-r)} + \frac{M_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}}\right] + LM_P(\nu) \tag{1.8}$$

for any $t \geq 0$.

This implies that the ω-limit set $\mathcal{A}_\nu$ of $\mathcal{B}_{\eta,c} = \{(x,y) \in X^\alpha_\nu \times Y : \|x\|_{X^\alpha_\nu} < \frac{\eta}{4M},\ y \in \mathcal{B}_c\}$ is a local attractor for (1.2). Since η and c can be chosen arbitrarily, assume that $\mathcal{B} \subset \mathcal{B}_{\eta,c}$ and $\mathcal{A}_\nu$ is a global attractor for (1.2). If c) holds, the above computations show that $\omega(\mathcal{B}_{\eta,c}) \subset \mathcal{A}$.

It remains to show that, in cases a) and b) , the family of attractors $\{\mathcal{A}_\nu, 0 \leq \nu \leq \nu_0\}$ is upper semicontinuous at zero.

Consider (1.8) for $\|\phi\|_{C([-r,0],X_\nu^\circ)} < \frac{\eta}{4M}$ and $\psi \in \mathcal{B}_c$. Then,

$$\begin{aligned}\frac{d}{dt}(\epsilon^t\Sigma(y_t)) \leq& \epsilon^t\dot{\Sigma}(y_t) + \epsilon^t\Sigma(y_t)\\ \leq& 2LL_P(\nu)M\left[\|\phi\|_{C([-r,0],X_\nu^\circ)}\epsilon^{-\sigma(t-r)}\epsilon^t + \frac{M_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}}\epsilon^t\right] + LM_P(\nu)\epsilon^t\end{aligned}$$

and

$$\begin{aligned}\Sigma(y_t) \leq 2LL_P(\nu)M&\left[\epsilon^{\sigma r}\left(\frac{\epsilon^{-\sigma t}-\epsilon^{-t}}{1-\sigma}\right)\|\phi\|_{C([-r,0],X_\nu^\circ)} + \frac{M_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}}(1-\epsilon^{-t})\right]\\ &+ LM_P(\nu)(1-\epsilon^{-t}) + \Sigma(\psi)\epsilon^{-\sigma t}.\end{aligned} \tag{1.9}$$

From (1.7) and (1.9), for every $(\phi,\psi) \in \mathcal{A}_\nu$,

$$\|\phi\|_{C([-r,0],X_\nu^\circ)} \leq \frac{2MM_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}}, \quad \Sigma(\psi) \leq 2LL_P(\nu)M\frac{M_f\Gamma(1-\alpha)\epsilon^{\sigma r}}{\beta(\nu)^{1-\alpha}} + LM_P(\nu)$$

and from property *ii*) of Σ

$$\lim_{\nu\to 0}\ \sup_{(\phi,\psi)\in\mathcal{A}_\nu} \mathrm{dist}((\phi,\psi),(0,\mathcal{A})) = 0,$$

and the Theorem 1.2 is proved.

The hypothesis $\beta(\nu) \to \infty$ as $\nu \to 0$ strongly relates the results of Theorem 1.2 to parabolic problems for which we can prove that the eigenvalues of A_ν diverge to $-\infty$. If we want to consider damped delay-hyperbolic problems we must eliminate this hypothesis. Our next result is an attempt to overcome this difficulty.

Let $R > 0$, $(\phi,\psi) \in C([-r,0],X_\nu \times Y)$, $\|(\phi,\psi)\|_{C([-r,0],X_\nu\times Y)} \leq R$. Suppose that

$$\|f_d(\phi,\psi)\|_{X_\nu} \leq L_f(\nu)\|\phi\|_{C([-r,0],X_\nu)} + M_f(\nu), \quad \|P_d(\phi,\psi)\|_Y \leq L_P(\nu)\|\phi\|_{C([-r,0],X_\nu)} + M_P(\nu), \tag{1.10}$$

where $P_d(\phi,\psi) = g_\nu(\phi,\psi) - g(\psi)$, $M_f(\nu)$, $M_P(\nu)$, $L_f(\nu)$, $L_P(\nu) \to 0$ as $d \to \infty$.

Assume also that either of the following conditions is satisfied

a') The flow defined by $\dot{y}(t) = By(t) + g_\nu(0,y_t)$ is asymptotically smooth or

b') $M_f(\nu) \equiv 0$ and $M_P(\nu) \equiv 0$.

The proof of our next result follows the steps in the proof of Theorem 1.2. However, there are some qualitative differences that require special attention and we chose to provide a complete proof for completeness.

Theorem 1.3. *Assume that $\beta(\nu) =: \sigma > 0$ is a constant and that A is the generator of a strongly continuous semigroup. Suppose that $[H]$, (1.10) and either of the conditions a') or b') is satisfied. Then, there exists a $\nu_0 > 0$ such that, for $0 < \nu \leq \nu_0$, the problem (1.2) has a global attractor $\mathcal{A}_\nu$ and the family of attractors $\{\mathcal{A}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is upper semicontinuous at zero, where $\mathcal{A}_0 := \mathcal{A}$. Furthermore, if b') is satisfied we have that $\mathcal{A}_\nu = \mathcal{A}$ for $0 < \nu \leq \nu_0$.*

Proof. Since (1.3) has a global attractor $\mathcal{A}$ there exists a locally Lipschitz continuous function $\Sigma : C([-r,0],Y) \to \mathbb{R}^+$ such that for any $\psi \in C([-r,0],Y)$,

i) $\Sigma(\psi) = 0$ if $\psi \in \mathcal{A}$.

ii) $a(d(\psi,\mathcal{A})) \leq \Sigma(\psi) \leq b(d(\psi,\mathcal{A}))$, where a is continuous nondecreasing, $a(s) > 0$ if $s > 0$, $a(s) \to \infty$ as $s \to \infty$ and $b(s)$ is continuous with $b(0) = 0$.

iii) $\dot{\Sigma}_{(1.3)}(\psi) \leq -\Sigma(\psi)$, where $\dot{\Sigma}_{(1.3)}$ is the right hand derivative of Σ along the solutions of (1.3).

For any $c > 0$, let $\mathcal{B}_c = \{\psi \in Y : \Sigma(\psi) < c\}$; the property *ii*) implies that $\mathcal{B}_c$ is bounded.

Suppose $(x(t),y(t))$ is a solution for (1.10), (1.10) with initial data (ϕ,ψ). Using the variation of constants formula (1.11) can be rewritten as

$$\frac{dy}{dt} = By + g(y_t) + P_\nu(x_t,y_t). \tag{1.11}$$

$$x(t) = T_\nu(t)\phi(0) + \int_0^t T_\nu(t-s)f_\nu(x_s,y_s)ds. \tag{1.12}$$

Given $c > 0$, $\eta > 0$, there exists $\nu_0 > 0$ such that, for all $\nu \in (0, \nu_0]$,

$$c - LL_P(\nu)\eta - LM_P(\nu) > 0,$$

where L is the Lipschitz constant of Σ on B_c. If $y_s \in B_c$ and $\|x_s\|_C < \eta$ for $0 \le s \le t$, then

$$\begin{aligned}\dot{\Sigma}(y_t) \le & -\Sigma(y_t) + L\|P_\nu(x_t, y_t)\|_Y \\ \le & -\Sigma(y_t) + LL_P(\nu)\|x_t\|_{C([-r,0,X_\nu)} + LM_P(\nu) < -\Sigma(y_t) + LL_P(\nu)\eta + LM_P(\nu)\end{aligned}$$

and

$$\|x(t)\|_{X_\nu} \le M\epsilon^{-\sigma t}\|\phi\|_C + M\int_0^t \epsilon^{-\sigma(t-s)}\left[L_f(\nu)\|x_s\|_C + M_f(\nu)\right] ds.$$

Let $z : [-r, \infty) \to X_\nu$ be defined by

$$z(t) = \begin{cases} M\epsilon^{-\sigma t}\|\phi(0)\|_{X_\nu} + M\displaystyle\int_0^t \epsilon^{-\sigma(t-s)}\left[L_f(\nu)\|x_s\|_C + M_f(\nu)\right] ds, & \text{if } t > 0 \\ \|\phi\|_C \quad , \text{if } -r \le t \le 0 \end{cases}$$

Then, z is a nonnegative function satisfying

$$\dot{z}(t) \le -\sigma z(t) + ML_f(\nu) \sup_{-r \le \theta \le 0} z(t+\theta) + MM_f(\nu)$$

for $t > 0$ and $z(t) \le \|\phi\|_C$ for $-r \le t \le 0$.

Letting $z(t) = w(t) + \frac{MM_f(\nu)}{\sigma - ML_f(\nu)}$, we have

$$\dot{w}(t) \le -\sigma w(t) + ML_f(\nu) \sup_{-r \le \theta \le 0} w(t+\theta)$$

The reader is referred to Halanay [1966] for the proof of the following result

Lemma 1.4. *Suppose $x : [-r, \infty) \to \mathbf{R}$ is a non-negative function satisfying*

$$\dot{x}(t) = -\alpha x(t) + \beta \sup_{-r \le \theta \le 0} x(t+\theta)$$

for $t > 0$ and assume $\alpha > \beta > 0$. Then, $0 \le x(t) \le \epsilon^{-kt}\|\phi\|$, for all $t \ge 0$, where $k > 0$ is the unique real solution of the equation $k - \alpha + \beta\epsilon^{kr} = 0$.

Therefore, for $\sigma > ML_f(\nu)$, we have

$$z(t) \le \epsilon^{-\lambda t}\|\phi\|_C + (1 - \epsilon^{-\lambda t})\frac{MM_f(\nu)}{\sigma - ML_f(\nu)},$$

where $\lambda > 0$ is the unique real solution of the equation

$$\lambda - \sigma + ML_f(\nu)\epsilon^{\lambda r} = 0.$$

It follows that

$$\|x(t)\|_{X_\nu} \le \epsilon^{-\lambda t}\|\phi\|_C + \frac{MM_f(\nu)}{\sigma - ML_f(\nu)},$$

for all $t > 0$.

Assume that ν_0 is such that $0 < \nu \le \nu_0$ implies $2ML_f(\nu) \le \sigma$ and $4MM_f(\nu) \le \sigma\eta$. Under this conditions

$$\|x(t)\|_{X_\nu} \le \epsilon^{-\lambda t}\|\phi\|_C + \frac{2MM_f(\nu)}{\sigma}. \tag{1.13}$$

For $\|\phi\|_C < \frac{\eta}{2}$ and $t \ge 0$, (1.13) implies $\|x_s\|_C < \eta$ and $y_s \in B_c$ whenever defined.

Therefore, for every $t \geq 0$, $\|\phi\|_{X_\nu} < \frac{\eta}{2}$, $\psi \in \mathcal{B}_c$, (1.13) is satisfied and

$$\dot{\Sigma}(y(t)) \leq -\Sigma(y(t)) + LL_P(\nu)\left[\|\phi\|_C e^{-\lambda t} + \frac{2MM_f(\nu)}{\sigma}\right] + LM_P(\nu) \tag{1.14}$$

for any $t \geq 0$.

This implies that the ω-limit set $\mathcal{A}_\nu$ of the set $\mathcal{B}_{\eta,c} = \{(\phi,\psi) \in C([-r,0],X_\nu) \times C([-r,0],Y) : \|\phi\|_C < \frac{\eta}{2},\ \psi \in \mathcal{B}_c\}$ is a local attractor for (1.2). Since η and c can be chosen arbitrarily, assume that $\mathcal{B} \subset \mathcal{B}_{\eta,c}$ and $\mathcal{A}_\nu$ is a global attractor for (1.2). If b') is satisfied, the above computations show that $\omega(\mathcal{B}_{\eta,c}) \subset \mathcal{A}$.

It remains to show that, in the case a'), the family of attractors $\{\mathcal{A}_\nu, 0 < \nu \leq \nu_0\}$ is upper semicontinuous at infinity.

Consider (1.14) for $\|\phi\|_{C([-r,0],X)} < \frac{\eta}{4}$ and $\psi \in \mathcal{B}_c$. Then,

$$\begin{aligned}\frac{d}{dt}(e^t\Sigma(y_t)) =& e^t\dot{\Sigma}(y_t) + e^t\Sigma(y_t)\\ \leq& LL_P(\nu)\|\phi\|_{C([-r,0],X)}e^{-(\lambda-1)t} + \left[\frac{2LL_P(\nu)MM_f(\nu)}{\sigma} + LM_P(\nu)\right]e^t\end{aligned}$$

and

$$\begin{aligned}\Sigma(y_t) \leq& LL_P(\nu)\left[\frac{e^{-\lambda t} - e^{-t}}{1-\lambda}\right]\|\phi\|_{C([-r,0],X)}\\ &+ \left[\frac{2LL_P(\nu)MM_f(\nu)}{\sigma} + LM_P(\nu)\right](1-e^{-t}) + \Sigma(\psi)e^{-t}.\end{aligned} \tag{1.15}$$

From (1.13) and (1.15), for every $(\phi,\psi) \in \mathcal{A}_\nu$,

$$\|\phi\|_{C([-r,0],X)} \leq \frac{2MM_f(\nu)}{\sigma}, \quad \Sigma(\psi) \leq \frac{2LL_P(\nu)MM_f(\nu)}{\sigma} + LM_P(\nu)$$

and from property ii) of Σ

$$\lim_{\nu\to 0}\ \sup_{(\phi,\psi)\in\mathcal{A}_\nu} \operatorname{dist}((\phi,\psi),(0,\mathcal{A})) = 0$$

and the proof is complete.

2. Upper semicontinuity of Attractors for Parabolic Delay-Partial Differential Equations on Thin Channels

As an application of Theorem 1.2 we consider a parabolic delay-partial differential equation in $\Omega = (0,1) \times \mathcal{O} \subset \mathbb{R}^{n+1}$, $n \leq 2$, where $\mathcal{O}$ is a smooth domain in $\mathbb{R}^n$. Denote by (x,y) a generic element of Ω, where $x \in (0,1)$ and $y \in \mathcal{O}$ and by Δ_y the Laplacian operator in $\mathcal{O}$. Consider

$$\begin{aligned}&\frac{\partial u}{\partial t} = d^2\frac{\partial^2 u}{\partial x^2} + \Delta_y u + f\left(u, \int_{-r}^{0} g(u(x,y,t+\theta))d\theta\right) \qquad \text{in} \quad \Omega := [0,1]\times\mathcal{O}\\ &\frac{\partial u}{\partial n} = 0 \qquad \text{in} \quad \partial\Omega.\end{aligned} \tag{2.1}$$

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$|f(u,s) - f(v,t)| \leq L\left(|u-v|(1 + e^{|u|^r} + e^{|v|^r}) + |s-t|\right), \quad \text{if } n = 1 \tag{2.2}$$

$$|f(u,s) - f(v,t)| \leq L\left(|u-v|(1 + u^2 + v^2) + |s-t|\right), \quad \text{if } n = 2 \tag{2.2'}$$

for some $L > 0$, $p < 1$ and all $(u,s),(v,t) \in \mathbb{R}^2$ and

$$\limsup_{|u|\to\infty} \frac{f(u,0)}{u} \leq -\delta. \tag{2.3}$$

for some $\delta > 0$. Assume also that $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous and satisfies $|g(u)| \leq B$ for all $u \in \mathbb{R}$ and some $B \geq 0$.

It is our purpose to use Theorem 1.2 to prove that, if $d > 0$ is sufficiently large, then the dynamics of (2.1) can be described by

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta_y v + f\left(v(y,t), \int_{-r}^{0} g(v(y,t+\theta))d\theta\right) \quad \text{in} \quad \mathcal{O} \\ \frac{\partial v}{\partial n} &= 0 \quad \text{in} \quad \partial\mathcal{O}. \end{aligned} \tag{2.4}$$

To better present the problem we introduce some terminology. Let $X = L^2(\Omega)$ and $A_d : D(A_d) \subset X \to X$ be the self adjoint operator defined by

$$D(A_d) = \left\{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n} = 0\right\}, \qquad A_d\phi = -d^2\frac{\partial^2 \phi}{\partial x^2} - \Delta_y \phi \quad \forall \phi \in D(A_d). \tag{2.5}$$

Let X_d^α denote the fractional power spaces associated to A_d endowed with the graph norm. Equation (2.1) can be written as

$$\dot{u}(t) = -A_d u(t) + f\left(u(t), \int_{-r}^{0} g(u(t+\theta))d\theta\right) \tag{2.6}$$

in the space $C([-r,0], X_d^\alpha)$.

In what follows we obtain some embeddings necessary to handle the growth condition (2.2). The next lemma follows an argument used in Carvalho [1992] where such embeddings are obtained for a fixed energy space.

Let $W_\alpha = \{\phi \in X_d^\alpha : \phi \text{ is independent of } x\}$ and $W_\alpha^\perp$ its orthogonal complement in X_d^α.

Lemma 2.1. *For any $p \geq 1$ we have*

$$X_d^{\frac{1}{2}} \subset L^p(\Omega),$$

with embedding constant proportional to p. Furthermore, there exists a constant $K > 0$, independent of d, such that

$$\|\phi\|_{L^p(\Omega)} \leq K\, p\, d^{-\frac{1}{2}} \|\phi\|_{X_d^{\frac{1}{2}}}, \tag{2.7}$$

for all $\phi \in W_{\frac{1}{2}}^\perp$

Proof. Let $\phi \in C^1(\bar{\Omega})$, $\phi \in W_{\frac{1}{2}}^\perp$. Then, for any $x_1 \in [0,1]$, there exist $c \in [0,1]$ such that $\phi(x_1, c) = 0$. This implies the following inequalities

$$|\phi(x_1,x_2)| = \left|\int_c^{x_2} \frac{\partial \phi}{\partial x_2}(x_1,s)ds\right| \leq \int_0^1 |\frac{\partial \phi}{\partial x_2}(x_1,s)|ds = f_1(x_1).$$

$$|\phi(x_1,x_2)| \leq |\phi(0,x_2)| + \int_0^1 |\frac{\partial \phi}{\partial x_1}(s,x_2)|ds = f_2(x_2).$$

$$|\phi(0,x_2)| \leq |\phi(x_1,x_2)| + \left|\int_0^{x_1} \frac{\partial \phi}{\partial x_1}(s,x_2)ds\right|.$$

$$\int_0^1 |\phi(0,x_2)|dx_2 \leq \|\phi\|_{L^1(\Omega)} + \|\frac{\partial \phi}{\partial x_1}\|_{L^1(\Omega)}.$$

Therefore,

$$|\phi(x_1,x_2)|^2 \leq f_1(x_1) f_2(x_2)$$

and

$$\|\phi\|_{L^2(\Omega)} \leq \left(\int_0^1 |\phi(0,x_2)|dx_2 + \|\frac{\partial \phi}{\partial x_1}\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \|\frac{\partial \phi}{\partial x_2}\|_{L^1(\Omega)}^{\frac{1}{2}}.$$

Let $t \geq 1$. If we apply the above inequality to $\phi = |u|^{t-1}u$ we have that (observe that, if $u \in C^1(\bar{\Omega}) \cap W^1_{\frac{1}{2}}$, then ϕ satisfy the above inequalities)

$$\begin{aligned}
\|u\|^t_{L^{2t}(\Omega)} &\leq t\left(\int_0^1 |u(0,x_2)|^t dx_2 + \||u|^{t-1}\frac{\partial u}{\partial x_1}\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \|\,|u|^{t-1}\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^1(\Omega)} \\
&\leq t\left(\|u^t\|_{L^1(\Omega)} + 2\||u|^{t-1}\frac{\partial u}{\partial x_1}\|_{L^1(\Omega)}\right)^{\frac{1}{2}} \|\,|u|^{t-1}\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^1(\Omega)} \\
&\leq t\,\|u\|^{t-1}_{L^{p'(t-1)}}\left(\|u\|_{L^p(\Omega)} + 2\|\frac{\partial u}{\partial x_1}\|_{L^p(\Omega)}\right)^{\frac{1}{2}} \|\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^p(\Omega)}.
\end{aligned}$$

Choose $2t = p'(t-1)$. Then, $t = \frac{p'}{p'-2}$ and

$$\|u\|_{L^{\frac{2p'}{p'-2}}(\Omega)} \leq \frac{p'}{p'-2}\left(\|u\|_{L^p(\Omega)} + 2\|\frac{\partial u}{\partial x_1}\|_{L^p(\Omega)}\right)^{\frac{1}{2}} \|\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^p(\Omega)}.$$

Since $\|\phi\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{2}}\|\phi\|_{L^2(\Omega)}$ for all $\phi \in L^2(\Omega)$, we have that

$$\begin{aligned}
\|u\|_{L^{\frac{2p'}{p'-2}}(\Omega)} &\leq \frac{p'}{p'-2}|\Omega|^{\frac{1}{p}-\frac{1}{2}}\left(\|u\|_{L^2(\Omega)} + 2\|\frac{\partial u}{\partial x_1}\|_{L^2(\Omega)}\right)^{\frac{1}{2}} \|\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\leq \frac{p'}{p'-2}|\Omega|^{\frac{1}{p}-\frac{1}{2}}\,d^{-\frac{1}{2}}\left(\|\frac{\partial u}{\partial x_2}\|_{L^2(\Omega)} + 3\|d\frac{\partial u}{\partial x_1}\|_{L^2(\Omega)}\right)^{\frac{1}{2}} \|\frac{\partial u}{\partial x_2}\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\leq 3\frac{p'}{p'-2}\,d^{-\frac{1}{2}}\,\frac{|\Omega|^{\frac{1}{p}-\frac{1}{2}}}{2}\left[\|d\frac{\partial u}{\partial x_1}\|_{L^2(\Omega)} + \|\frac{\partial u}{\partial x_2}\|_{L^2(\Omega)}\right],
\end{aligned}$$

where we use that, for $\phi \in W^1_{\frac{1}{2}}$,

$$\|\phi\|^2_{L^2(\Omega)} \leq d\left(\|d\frac{\partial \phi}{\partial x_1}\|^2_{L^2(\Omega)} + \|\frac{\partial \phi}{\partial x_2}\|^2_{L^2(\Omega)}\right).$$

The result now follows from a density argument.

The following lemma guarantees that the problem (2.1) is locally well defined.

Lemma 2.2. *Under the previous growth assumption on f, the function $f^\epsilon : C([-r,0], X^{\frac{1}{2}}_d) \to X$, given by*

$$f^\epsilon(\phi)(x) = f\left(\phi(0)(x), \int_{-r}^0 g(\phi(\theta)(x))d\theta\right)$$

is Lipschitz continuous in bounded sets of $C([-r,0], X^{\frac{1}{2}}_d)$; that is, given a ball of radius ρ in $C([-r,0], X^{\frac{1}{2}}_d)$, there exists a constant $L_f = L_f(\rho)$ such that

$$\|f^\epsilon(\phi) - f^\epsilon(\psi)\|_X \leq L_f\|\phi - \psi\|_C.$$

for $\|\phi\|_{C([-r,0],X_d^{\frac{1}{2}})} \leq \rho$, $\|\psi\|_{C([-r,0],X_d^{\frac{1}{2}})} \leq \rho$.

Proof. The proof is based on the embeddings (2.7). If $\|\phi\|_{C([-r,0],X_d^{\frac{1}{2}})} \leq r$, $\|\psi\|_{C([-r,0],X_d^{\frac{1}{2}})} \leq r$, it follows that

$$\begin{aligned}
\|f^\epsilon(\phi) - f^\epsilon(\psi)\|^2_{L^2(\Omega)} \leq{}& L \int_\Omega [\epsilon^{\theta|\phi(0)(x)|^p} + \epsilon^{\theta|\psi(0)(x)|^p}]\, |\phi(0)(x) - \psi(0)(x)|^2 dx + LL_g r \|\phi - \psi\|_{C([-r,0],X)} \\
\leq{}& L \left(\int_\Omega [\epsilon^{\theta|\phi(0)(x)|^p} + \epsilon^{\theta|\psi(0)(x)|^p}]^2\, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\phi(0)(x) - \psi(0)(x)|^4\, dx \right)^{\frac{1}{2}} + LL_g r \|\phi - \psi\|_{C([-r,0],X)} \\
\leq{}& L\, \|\phi(0) - \psi(0)\|^2_{L^4(\Omega)} \left(\int_\Omega [\epsilon^{\theta|\phi(0)(x)|^p} + \epsilon^{\theta|\psi(0)(x)|^p}]^2\, dx \right)^{\frac{1}{2}} + LL_g r \|\phi - \psi\|_{C([-r,0],X)} \\
\leq{}& 16\, K^2 L\, \|\phi(0) - \psi(0)\|^2_{W^{1,2}(\Omega)} \left(\int_\Omega [\epsilon^{\theta|\phi(0)(x)|^p} + \epsilon^{\theta|\psi(0)(x)|^p}]^2\, dx \right)^{\frac{1}{2}} + LL_g r \|\phi - \psi\|_{C([-r,0],X)}.
\end{aligned}$$

Therefore, we only need to prove that

$$\|\epsilon^{\theta|\phi(0)(\cdot)|^p}\|_{L^2(\Omega)} \leq C(r),$$

for $\|\phi(0)\|_{X_\nu^{\frac{1}{2}}} \leq r$, where $C(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and increasing. This follows from the following

$$\begin{aligned}
\|\epsilon^{\theta|\phi(0)(x)|^p}\|_{L^2(\Omega)} &\leq \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \|\, |\phi(0)|^{np}\, \|_{L^2(\Omega)} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \|\phi(0)\|^{np}_{L^{2np}(\Omega)} \\
&\leq \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left(2npK \|\phi(0)\|_{H^1(\Omega)}\right)^{np} \leq \sum_{n=0}^{\infty} \frac{n^{np}}{n!} \left(2rp\theta^{\frac{1}{p}} K\right)^{np}
\end{aligned}$$

and we only need to prove that the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n^{np}}{n!} x^n$$

is infinity. This follows from the fact that, if $a_k = \frac{k^{kp}}{k!}$,

$$\frac{a_n}{a_{n+1}} = (n+1)^{1-p} \left(1 + \frac{1}{n}\right)^{-np} \to \infty$$

as $n \to \infty$. The lemma is now proved.

Consider the Liapunov functional $V : C([-r,0], X_d^{\frac{1}{2}}) \to \mathbb{R}$

$$V(\phi) = \frac{1}{2} \|A_d^{\frac{1}{2}} \phi(0)\|^2_X + \frac{1}{2} \|\phi(0)\|^2_X - \int_\Omega F(\phi(0)) dx\,.$$

where $F(u) = \int_0^u f(s,0) ds$.

We use this Liapunov functional to prove that hypothesis [H] of Theorem 1.2 is satisfied. We prove the result for the case $n = 1$ since in this case the estimates are more complicated and one could easily reproduce the proof for the case $n = 2$ from the case $n = 1$.

Lemma 2.3. *There exists a constant $M > 0$, independent of d, such that for any $d \geq d_0$ and any bounded set B in $C([-r,0],X_d^{\frac{1}{2}})$, there exists $t_0 = t_0(d,B)$ such that, for $t \geq t_0$, $u \in B$*

$$u(t,u_0) \in \{u \in C([-r,0],X_d^{\frac{1}{2}}) : \|u\|_{C([-r,0],X_d^{\frac{1}{2}})} \leq M\},$$

where $u(t,u_0)$ is the solution of (2.1) satisfying $u(0,u_0) = u_0$.

Proof. Hypothesis (2.3) implies that there exist positive constants $c_{\frac{\epsilon}{2}}$ and m such that $sf(s,0) \leq \frac{-\delta}{2}s^2 + c_{\frac{\epsilon}{2}}$, $F(s) \leq \frac{-\delta}{2}s^2 + c_{\frac{\epsilon}{2}}$ and $|F(s)| \leq m e^{\theta|s|^p}$, for all $s \in \mathbb{R}$. Therefore, we have the following estimates

$$\begin{aligned} V(\phi) &\geq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}(1+\delta)\|\phi(0)\|_X^2 - c_{\frac{\epsilon}{2}}|\Omega| \\ &\geq c_1\|\phi(0)\|^2_{X_d^{\frac{1}{2}}} - c_{\frac{\epsilon}{2}}|\Omega|. \end{aligned}$$

On the other hand,

$$\begin{aligned} V(\phi) &\leq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}\|\phi(0)\|_X^2 + m\|\epsilon^{\theta|\phi(0)|^p}\|_X^2 \\ &\leq C(\|\phi(0)\|_{X_d^{\frac{1}{2}}}), \end{aligned}$$

where $C : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. If $u(\cdot,t)$ is a solution of (2.1), then we have

$$\begin{aligned} \frac{dV}{dt}(u_t) &= -\|\frac{\partial u}{\partial t}(\cdot,t)\|_X^2 - \langle u(\cdot,t), Au\rangle + \langle u, f(u(\cdot,t),0)\rangle \\ &\quad + \left\langle f\left(u(\cdot,t), \int_{-r}^{0} g(u(\cdot,t+\theta))d\theta\right) - f(u(\cdot,t),0), u(\cdot,t) + \frac{\partial u}{\partial t}(\cdot,t)\right\rangle \\ &\leq -\|\frac{\partial u}{\partial t}(\cdot,t)\|_X^2 - \langle u(\cdot,t), Au\rangle + \langle u, f(u(\cdot,t),0)\rangle + BLr|\Omega|^{\frac{1}{2}}\left(\|u(\cdot,t)\|_X + \|\frac{\partial u}{\partial t}(\cdot,t)\|_X\right) \\ &\leq -\|\frac{\partial u}{\partial t}(\cdot,t)\|_X^2 - \langle u(\cdot,t), Au\rangle - \frac{\delta}{2}\|u\|_X^2 + BLr|\Omega|^{\frac{1}{2}}\left(\|u(\cdot,t)\|_X + \|\frac{\partial u}{\partial t}(\cdot,t)\|_X\right) + c_{\frac{\epsilon}{2}}|\Omega|^{\frac{1}{2}} \\ &\leq -c_2\|u(\cdot,t)\|_{X_d^{\frac{1}{2}}} + c_3. \end{aligned}$$

which implies the result.

Let us denote by W and W_α the closed subspaces of X and X_d^α, respectively, whose elements are functions independent on the variable x. Let $W^\perp$ and $W_\alpha^\perp$ be the orthogonal complement of W and W_α in X and X_d^α, respectively. Observe that W is isometric to $L^2(\mathcal{O})$ and W_α is isometric to the fractional power of the restriction of A_d to the subspace $W \cap D(A_d)$. This decomposition induces the decomposition

$$C([-r,0],X_\nu^\alpha) = C([-r,0],W_\alpha) \oplus C([-r,0],W_\alpha^\perp).$$

Writting $u(x,y,t) = v(y,t) + w(x,y,t)$, where $v(y,t) = \int_0^1 u(x,y,t)dx$, and $w = u - v$, the equation (2.1) can be written as

$$\begin{aligned} &\frac{\partial v}{\partial t} = \Delta_y v + \int_0^1 f\left(v(y,t) + w(s,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(s,y,t+\theta))d\theta\right) ds \qquad \text{in } \mathcal{O} \\ &\frac{\partial v}{\partial y} = 0 \qquad \text{in } \partial\mathcal{O} \\ &\frac{\partial w}{\partial t} = d^2\frac{\partial^2 w}{\partial x^2} + \Delta_y w + f\left(v(y,t) + w(x,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(x,y,t+\theta))d\theta\right) \\ &\qquad\qquad - \int_0^1 f\left(v(y,t) + w(s,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(s,y,t+\theta))d\theta\right) ds \qquad \text{in } \Omega \\ &\frac{\partial w}{\partial n} = 0 \qquad \text{in } \partial\Omega \end{aligned}$$

Theorem 2.4. *There exists a $d_0 > 0$ such that, for $d \geq d_0$ the problem (2.1) has a global attractor $\mathcal{A}_d$ which coincide with the embedding of $\mathcal{A}$ into $C([-r,0], X_d^{\frac{1}{2}})$.*

Let $T_d(t)$ denote the semigroup generated by $-A_d$. Then, there exist constants $K \geq 1$ and $\beta(d) \to \infty$ such that

$$\|T_d(t)w\|_{W^{\perp}_{\frac{1}{2}} \times W^{\perp}} \leq K e^{-\beta(d)t} \|w\|_{W^{\perp}_{\frac{1}{2}} \times W^{\perp}}.$$

Let $\pi : X \to W$ be the projection

$$\pi(u) = \int_0^1 u(s,y)ds.$$

Let $P_d : C([-r,0], X_d^{\alpha}) \to W$ be given by

$$P_d(\phi, \tilde{\phi}) = f\left(\phi(0), \int_{-r}^{0} g(\phi(\theta))d\theta\right) - \pi\left[f\left(\phi(0) + \tilde{\phi}(0), \int_{-r}^{0} g(\phi(\theta) + \tilde{\phi}(\theta))d\theta\right)\right] \tag{2.8}$$

The variable w satisfy

$$\begin{cases} \dfrac{\partial w}{\partial t} = d^2 \dfrac{\partial^2 w}{\partial x^2} + \Delta_y w + Q_d(v_t, \xi_t, w_t, \eta_t) \\ \dfrac{\partial w}{\partial n} = 0, \end{cases} \tag{2.9}$$

where $Q_d : C([-r,0], X_d^{\alpha}) \to W^{\perp}$ is given by where

$$Q_d(\phi, \tilde{\phi}) = f(\phi(0) + \tilde{\phi}(0), \int_{-r}^{0} g(\phi(\theta) + \tilde{\phi}(\theta))d\theta) - \pi(f(\phi(0) + \tilde{\phi}(0), \int_{-r}^{0} g(\phi(\theta) + \tilde{\phi}(\theta))d\theta)) \tag{2.10}$$

The following results can easily be obtained using the embeddings (2.7) and the growth assumption (2.2).

Lemma 2.5. *The function $Q_d : C([-r,0], X_d^{\frac{1}{2}}) \to W^{\perp}$ defined by (2.10) is Lipschitz continuous on bounded sets of $C([-r,0], X_d^{\frac{1}{2}})$. Furthermore, given $\rho > 0$, there exists constant $L_Q = L_Q(\rho)$ such that*

$$\|Q_d(\phi, \tilde{\phi})\|_{W^{\perp}} \leq L_Q\, d^{-\frac{1}{2}}\, \|\tilde{\phi}\|_{C([-r,0], W^{\perp}_{\frac{1}{2}})},$$

for $\|(\phi, \tilde{\phi})\|_{C([-r,0], X_d^{\frac{1}{2}})} \leq \rho$.

Lemma 2.6. *The function $P_d : C([-r,0], X_d^{\frac{1}{2}}) \to W$ given by (2.8) is Lipschitz continuous in bounded sets of $C([-r,0], X_d^{\frac{1}{2}})$. Furthermore, given $\rho > 0$, there exists constant $L_P = L_P(\rho)$ such that*

$$\|P_d(\phi, \tilde{\phi})\|_{W} \leq L_P\, d^{-\frac{1}{2}}\, \|\tilde{\phi}\|_{C([-r,0], W^{\perp}_{\frac{1}{2}})},$$

for $\|(\phi, \tilde{\phi})\|_{C([-r,0], X_d^{\frac{1}{2}})} \leq \rho$.

Remark 1. *It has been shown (see Moser [1971]) that f^{ϵ} is a bounded operator from H^1 into L^2 for $p < 2$. Using Moser's inequality and the techniques in this section, we can prove that the same result holds for $p < 2$. Indeed, it has been shown (see Hale and Raugel [1992]) that the operator f^{ϵ} is a bounded operator from H^1 into $W^{1,s}$ for $s < 2$. Their proof strongly uses the results of Moser [1971].*

Remark 2. *The above results can be easily proved for the equations*

$$\begin{aligned} \frac{\partial u}{\partial t} &= d^2 \frac{\partial^2 u}{\partial x^2} + \Delta_y u + f(u, u(t-r)) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where f satisfies the conditions (2.2) and (2.3). If $g \equiv 0$, problem (2.1) reduces to a parabolic partial differential equation and all the results hold with the same proof.

3. Upper Semicontinuity of Attractors for Damped Hyperbolic Delay-Partial Differential Equations on Thin Channels

As an application of Theorem 1.3 we consider a damped hyperbolic-delay equation in $\Omega = (0,1) \times \mathcal{O}$ (see Section 2). Denote by (x,y) a generic element of Ω, where $x \in (0,1)$ and $y \in \mathcal{O}$ and by Δ_y the Laplacian operator in $\mathcal{O}$. Let $\beta > 0$ and consider

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = d^2 \frac{\partial^2 u}{\partial x^2} + \Delta_y u + f\left(u, \int_{-r}^{0} g(u(x,y,t+\theta))d\theta\right) \qquad \text{in} \quad \Omega := [0,1] \times \mathcal{O} \\ &\frac{\partial u}{\partial n} = 0 \qquad \text{in} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

To simplify the notations, we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is globally Lipschitz continuous and satisfies the dissipation condition (2.3). Assume also that $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous and satisfies $|g(u)| \leq B$ for all $u \in \mathbb{R}$ and some $B \geq 0$. If $g \equiv 0$ then (3.1) becomes the equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = d^2 \frac{\partial^2 u}{\partial x^2} + \Delta_y u + h(u) \qquad \text{in} \quad \Omega := [0,1] \times \mathcal{O} \\ &\frac{\partial u}{\partial n} = 0 \qquad \text{in} \quad \partial\Omega. \end{aligned} \tag{3.2}$$

where $h(u) = f(u,0)$.

As in Section 2, it is our purpose to prove that if $d > 0$ is sufficiently large, then the dynamics of (3.1) can be described by

$$\begin{aligned} &\frac{\partial^2 v}{\partial t^2} + \beta \frac{\partial v}{\partial t} = \Delta_y v + f\left(v(y,t), \int_{-r}^{0} g(v(y,t+\theta))d\theta\right) \qquad \text{in} \quad \mathcal{O} \\ &\frac{\partial v}{\partial y} = 0 \qquad \text{in} \quad \partial\mathcal{O}. \end{aligned} \tag{3.3}$$

Using the notation introduced in the previous section, Equation (3.1) can be written as the following system

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -A_d u(t) - \beta v(t) + f\left(u(t), \int_{-r}^{0} g(u(t+\theta))d\theta\right) \end{cases} \tag{3.4}$$

in the space $C([-r,0], X_d^{\frac{1}{2}} \times X)$.

Under the above conditions the problem (3.3) has a global attractor $\mathcal{A}$ and by considering the Liapunov functional $V : C([-r,0], X_d^{\frac{1}{2}} \times X) \to \mathbb{R}$ given by

$$V(\phi,\psi) = \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}\|\psi(0)\|_X^2 + b\langle\phi(0),\psi(0)\rangle - \int_\Omega F(\phi(0))dx\,.$$

where b is a positive constant and $F(u) = \int_0^u f(s,0)ds$, it follows from computations similar to the computations in the proof of Lemma 2.3 that

Lemma 3.1. *There exists a constant $M > 0$, independent of d, such that for any $d \geq d_0$ and any bounded set B in $C([-r,0], X_d^{\frac{1}{2}} \times X)$, $\exists\, t_0 = t_0(d,B)$ such that, for $t \geq t_0$, $u \in B$*

$$u(t,u_0) \in \{u \in X_d^{\frac{1}{2}} : \|u\|_{X_d^{\frac{1}{2}}} \leq M\},$$

where $u(t,u_0)$ is the solution of (3.1) satisfying $u(0,u_0) = u_0$.

Proof. Hypothesis (2.3) implies that there exist positive constants c_δ and m such that $sf(s,0) \leq \frac{-\delta}{2}s^2 + c_\delta$, $F(s) \leq \frac{-\delta}{2}s^2 + c_\delta$ and $|F(s)| \leq m(1+s^2)$, for all $s \in \mathbb{R}$. Therefore, we have the following estimate

$$V(\phi,\psi) \geq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}(1-b)\|\psi(0)\|_X^2 + \frac{1}{2}(\delta - b)\|\phi(0)\|_X^2 - c_\delta|\Omega|.$$

Choosing $0 < b < \min\{1,\delta\}$ and letting $c_1 = \min\{\frac{1}{2}, 1-b, \delta - b\}$, we have

$$V(\phi,\psi) \geq c_1(\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \|\phi(0)\|_X^2 + \|\psi(0)\|_X^2) - c_\delta|\Omega| = c_1\|(\phi(0),\psi)(0)\|^2_{X_d^{\frac{1}{2}}\times X} - c_\delta|\Omega|. \tag{3.5}$$

On the other hand,

$$\begin{aligned}
V(\phi,\psi) &\leq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}\|\psi(0)\|_X^2 + \frac{b}{2}\|\phi(0)\|_X^2 + m\|\phi(0)\|_X^2 + m|\Omega| \\
&\leq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + (\frac{b}{2} + m)\|\phi(0)\|_X^2 + \frac{1}{2}(b+1)\|\psi(0)\|_X^2 + m|\Omega| \\
&\leq \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + (\frac{1}{2} + m)\|\phi(0)\|_X^2 + \|\psi(0)\|_X^2 + m|\Omega|.
\end{aligned}$$

Letting $c_2 = \max\{1, \frac{1}{2} + m\}$ we have

$$V(\phi,\psi) \leq c_2\left(\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \|\phi(0)\|_X^2 + \|\psi(0)\|_X^2\right) + m|\Omega|. \tag{3.6}$$

Consider the functional $V_1 : C([-r,0], X_d^{\frac{1}{2}} \times X) - \mathbb{R}$ given by

$$V_1(\phi,\psi) = \frac{1}{2}\|A_d^{\frac{1}{2}}\phi(0)\|_X^2 + \frac{1}{2}\|\psi(0)\|_X^2 - \int_\Omega F(\phi(0))dx\,.$$

If $(u(\cdot,t), v(\cdot,t))$ is a solution of (3.4), then we have

$$\begin{aligned}
\frac{dV_1}{dt}(u_t,v_t) &= -\beta\|v(\cdot,t)\|_X^2 + \langle v(\cdot,t), f\left(u(\cdot,t), \int_{-r}^0 g(u(\cdot,t+\theta))d\theta\right) - f(u(\cdot,t),0)\rangle \\
&\leq -\beta\|v(\cdot,t)\|_X^2 + BrL|\Omega|^{\frac{1}{2}}\|v(\cdot,t)\|_X \\
&\leq -\beta\|v(\cdot,t)\|_X^2 + \frac{\beta}{2}\|v(\cdot,t)\|_X^2 + \frac{2(BrL)^2}{\beta}|\Omega|.
\end{aligned}$$

and therefore

$$\frac{dV_1}{dt}(u_t,v_t) \leq -\frac{\beta}{2}\|v(\cdot,t)\|_X^2 + \frac{2(BrL)^2|\Omega|}{\beta}.$$

On the other hand, if $V_2(\phi,\psi) = \langle\phi(0), \psi(0)\rangle$, we have

$$\begin{aligned}
\frac{dV_2}{dt}(u_t,v_t) =& \|v(\cdot,t)\|_X^2 - \langle u, A_d u(\cdot,t) + \beta v(\cdot,t) - f\left(u(\cdot,t), \int_{-r}^0 g(u(\cdot,t+\theta))d\theta\right)\rangle \\
=& \|v(\cdot,t)\|_X^2 - \langle u, A_d u(\cdot,t) + \beta v(\cdot,t) - f(u(\cdot,t),0)\rangle \\
&- \langle u(\cdot,t), f(u(\cdot,t),0) - f\left(u(\cdot,t), \int_{-r}^0 g(u(\cdot,t+\theta))d\theta\right)\rangle.
\end{aligned}$$

and therefore

$$\frac{dV_2}{dt}(u_t, v_t) \leq \|v(\cdot,t)\|_X^2 - \|A_d^{\frac{1}{2}} u(\cdot,t)\|_X^2 - \beta\langle u(\cdot,t), v(\cdot,t)\rangle - \frac{\delta}{2}\|u(\cdot,t)\|_X^2 + c_\delta|\Omega| + BrL|\Omega|^{\frac{1}{2}}\|u(\cdot,t)\|_X .$$

Since $\beta\langle u(\cdot,t), v(\cdot,t)\rangle \leq \epsilon\|u(\cdot,t)\|_X^2 + \frac{\beta^2}{4\epsilon}\|v(\cdot,t)\|_X^2$ and $BrL|\Omega|^{\frac{1}{2}}\|u(\cdot,t)\|_X \leq \epsilon\|u(\cdot,t)\|_X^2 + \frac{(BrL)^2}{4\epsilon}|\Omega|$ for any $\epsilon > 0$, we have

$$\frac{dV_2}{dt}(u_t, v_t) \leq (1 + \frac{\beta^2}{4\epsilon})\|v(\cdot,t)\|_X^2 - \|A_d^{\frac{1}{2}} u(\cdot,t)\|_X^2 + (2\epsilon - \frac{\delta}{2})\|u(\cdot,t)\|_X^2 + c_\delta|\Omega| + \frac{(BrL)^2}{4\epsilon}|\Omega|.$$

and, by taking $\epsilon = \frac{\delta}{8}$, we have

$$\frac{dV_2}{dt}(u_t, v_t) \leq (1 + \frac{2\beta^2}{\delta})\|v(\cdot,t)\|_X^2 - \|A_d^{\frac{1}{2}} u(\cdot,t)\|_X^2 - \frac{\delta}{4}\|u(\cdot,t)\|_X^2 + (c_\delta + \frac{2(BrL)^2}{\delta})|\Omega|.$$

Since $V = V_1 + bV_2$, we have

$$\begin{aligned}\frac{dV}{dt}(u_t, v_t) \leq &-(\frac{\beta}{2} - b(1 + \frac{2\beta^2}{\delta}))\|v(\cdot,t)\|_X^2 - b\|A_d^{\frac{1}{2}} u(\cdot,t)\|_X^2 - \frac{b\delta}{4}\|u(\cdot,t)\|_X^2 \\ &+ b\left(c_\delta + \frac{2(BrL)^2}{\delta} + \frac{2(BrL)^2}{\beta}\right)|\Omega|.\end{aligned}$$

Now, we fix $b > 0$ sufficiently small so that $\frac{\beta}{2} - b(1 + \frac{2\beta^2}{\epsilon}) > 0$ and we conclude that there are positive constants c_3 and c_4 such that

$$\frac{dV}{dt}(u_t, v_t) \leq -c_3\left(\|A_d^{\frac{1}{2}} u(\cdot,t)\|_X^2 + \|u(\cdot,t)\|_X^2 + \|v(\cdot,t)\|_X^2\right) + c_4|\Omega|.$$

It follows from (3.8) that

$$\frac{dV}{dt}(u_t, v_t) \leq -\frac{c_3}{c_2}V(u_t, v_t) + \left(\frac{mc_3}{c_2} + c_4\right)|\Omega|,$$

which implies that

$$V(u_t, v_t) \leq V(u_0, v_0)\epsilon^{-\frac{c_3}{c_2}t} + \left(m + \frac{c_2c_4}{c_3}\right)|\Omega||,$$

for all $t \geq 0$. This estimate together with (3.5) and (3.6) imply the result.

Let us denote by W and W_α the closed subspaces of X and X_d^α, respectively, whose elements are functions independent on the variable x. Let $W^\perp$ and $W_\alpha^\perp$ be the orthogonal complement of W and W_α in X and X_d^α, respectively. Observe that W is isometric to $L^2([0,1])$ and W_α is isometric to the fractional power of the restriction of A_d to the subspace $W \cap D(A_d)$. We will use the identification

$$X_d^\alpha \times X = W_\alpha \oplus W \oplus W_\alpha^\perp \oplus W^\perp.$$

as well the identification

$$C([-r,0], X_d^\alpha \times X) = C([-r,0], W_\alpha \oplus W) \times C([-r,0], W_\alpha^\perp \oplus W^\perp).$$

Writing $u(x,y,t) = v(y,t) + w(x,y,t)$, where $v(y,t) = \int_0^1 u(x,y,t)dx$ and $w = u - v$, the equation (6.10) can be written as

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \xi \\
\frac{\partial \xi}{\partial t} &= \Delta_y v - \beta\xi + \int_0^1 f\left(v(y,t) + w(s,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(s,y,t+\theta))d\theta\right) ds \qquad \text{in } \mathcal{O} \\
\frac{\partial v}{\partial y} &= 0 \qquad \text{in } \partial\mathcal{O} \\
\frac{\partial w}{\partial t} &= \eta \\
\frac{\partial \eta}{\partial t} &= d^2\frac{\partial^2 w}{\partial x^2} + \Delta_y w - \beta\eta + f\left(v(y,t) + w(x,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(x,y,t+\theta))d\theta\right) \\
&\quad - \int_0^1 f\left(v(y,t) + w(s,y,t), \int_{-r}^0 g(v(y,t+\theta) + w(s,y,t+\theta))d\theta\right) ds \qquad \text{in } \Omega \\
\frac{\partial w}{\partial n} &= 0 \qquad \text{in } \partial\Omega
\end{aligned}
$$

Theorem 3.2. *There exists a $d_0 > 0$ such that, for $d \geq d_0$ the problem (6.10) has a global attractor $\mathcal{A}_d$ which coincide with the embedding of $\mathcal{A}$ into $C([-r,0], X_d^{\frac{1}{2}} \times X)$.*

Let $T_d(t)$ denote the C_0 semigroup generated by $C_d = \begin{pmatrix} 0 & I \\ -A_d & -\beta \end{pmatrix}$. Then, there exist constants $K \geq 1$ and $\sigma > 0$ such that

$$\|T_d(t)w\|_{W_{\frac{1}{2}}^{\perp} \times W^{\perp}} \leq Ke^{-\sigma t}\|w\|_{W_{\frac{1}{2}}^{\perp} \times W^{\perp}}.$$

Let $\pi : X \to W$ be the projection

$$\pi(u) = \int_0^1 u(s,y)ds.$$

Let $P_d : C([-r,0], X_d^{\alpha} \times X) \to W$ be given by

$$P_d(\phi, v, \tilde{\phi}, \tilde{v}) = f\left(\phi(0), \int_{-r}^0 g(\phi(\theta))d\theta\right) - \pi\left[f\left(\phi(0) + \tilde{\phi}(0), \int_{-r}^0 g(\phi(\theta) + \tilde{\phi}(\theta))d\theta\right)\right] \tag{3.7}$$

The variable w satisfy

$$
\begin{cases}
\dfrac{\partial w}{\partial t} = \eta \\
\dfrac{\partial \eta}{\partial t} = d^2\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - \beta\eta + Q_d(v_t, \xi_t, w_t, \eta_t) \\
\dfrac{\partial w}{\partial n} = 0.
\end{cases}
\tag{3.8}
$$

where $Q_d : C([-r,0], X_d^{\alpha} \times X) \to W^{\perp}$ is given by where

$$Q_d(\phi, v, \tilde{\phi}, \tilde{v}) = f(\phi(0) + \tilde{\phi}(0), \int_{-r}^0 g(\phi(\theta) + \tilde{\phi}(\theta))d\theta) - \pi(f(\phi(0) + \tilde{\phi}(0), \int_{-r}^0 g(\phi(\theta) + \tilde{\phi}(\theta))d\theta)) \tag{3.9}$$

Lemma 3.3. *The function $Q_d : C([-r,0], X_d^{\frac{1}{2}} \times X) \to W^{\perp}$ defined by (3.9) is Lipschitz continuous in bounded sets of $C([-r,0], X_d^{\frac{1}{2}} \times X)$. Furthermore, given $\rho > 0$, there exists a constant $L_Q = L_Q(\rho)$ such that*

$$\|Q_d(\phi, v, \tilde{\phi}, \tilde{v})\|_{W^{\perp}} \leq L_Q\, d^{-\frac{1}{2}}\, \|(\tilde{\phi}, \tilde{v})\|_{C([-r,0], W_{\frac{1}{2}}^{\perp} \oplus W^{\perp})}.$$

for $\|(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|_{C([-r,0], X_d^{\frac{1}{2}} \times X)} \leq \rho.$

Lemma 3.4. *The function* $P_d : C([-r,0], X_d^{\frac{1}{2}} \times X) \to W$ *given by (3.7) is Lipschitz continuous in bounded sets of* $C([-r,0], X_d^{\frac{1}{2}} \times X)$*. Furthermore, given* $\rho > 0$*, there exists a constant* $L_P = L_P(\rho)$ *such that*

$$\|P_d(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|_W \leq L_P\, d^{-\frac{1}{2}}\, \|(\tilde{\phi}, \tilde{\psi})\|_{C([-r,0], W_{\frac{1}{2}}^{\perp} \oplus W^{\perp})},$$

for $\|(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|_{C([-r,0], X_d^{\frac{1}{2}} \times X)} \leq \rho.$

Remark 1. Proceeding as in Section 2, we could consider the less restrictive conditions (2.2), (2.2)' on f.

Remark 2. The above results can be easily proved for equations

$$\frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = d^2 \frac{\partial^2 u}{\partial x^2} + \Delta_y u + f(u, u(t-r)) \qquad \text{in} \quad \Omega := [0,1] \times \mathcal{O}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{in} \quad \partial\Omega,$$

where f satisfies the conditions (2.3) and (2.2) or (2.2)'. In this case, we should use the following Liapunov funtional

$$V(\phi, \psi) = \frac{1}{2}\|A_d^{\frac{1}{2}} \phi(0)\|_X^2 + \frac{1}{2}\|\psi(0)\|_X^2 + b\langle \phi(0), \psi(0)\rangle + \frac{\beta}{2r}\int_{-r}^{0}\left(\int_{\tau}^{0} \|\psi(\theta)\|_X^2 d\theta\right) d\tau - \int_\Omega F(\phi(0))dx,$$

to obtain the *a priori* bounds on the attractors.

4. Application of Theorem 1.2 to a Class of Cooperative System Arising in Parabolic Problems on Thin Domains Around a Point

Suppose $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and consider the scalar parabolic partial differential equation

$$u_t = (a(x,\nu)u_x)_x + u_{yy} + f(x,u), \qquad (x,y) \in \Omega \tag{4.1}$$

$$\frac{\partial u}{\partial n} = 0, \qquad (x,y) \in \partial\Omega. \tag{4.2}$$

This problem arises as a limiting problem for

$$u_t = \Delta u + f(u), \qquad (x,y) \in \Omega_{\epsilon,\nu}$$

$$\frac{\partial u}{\partial n} = 0, \qquad (x,y) \in \Omega_{\epsilon,\nu},$$

where $\Omega_{\epsilon,\nu} = \{(x,y) \in \mathbb{R}^2 : 0 < x < \nu,\ 0 < y < \epsilon(a(\frac{x}{\nu}, \nu))^{\frac{1}{2}}\}$, (see Hale and Raugel [1992]).

To proceed describing the results some additional notation is needed.

Let $X = L^2(\Omega)$; then the operator $A_\nu : D(A_\nu) \subset X \to X$, defined by $A_\nu \varphi = -(a(x,\nu)\varphi_x)_x - \varphi_{yy}$, $\forall \varphi \in D(A_\nu)$ where

$$D(A_\nu) = \{\varphi \in H^2(\Omega) : \frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\},$$

is a sectorial operator. Therefore, the fractional power spaces X_ν^α associated with A_ν can be defined; that is, for $\alpha \geq 0$, $X_\nu^\alpha = D(A_\nu^\alpha)$ endowed with the graph norm. It is assumed throughout that $\alpha = \frac{1}{2}$, unless stated otherwise. Then, for any $p \geq 1$,

$$X_\nu^{\frac{1}{2}} \subset L^p(\Omega) \tag{4.3}$$

with continuous inclusion (see Friedman [1983], Theorem 10.1).

Consider the eigenvalue problem

$$\left.\begin{aligned} &(a\varphi_x)_x + \varphi_{yy} = -\lambda\varphi, \qquad x \in \Omega \\ &\frac{\partial\varphi}{\partial n} = 0, \qquad (x,y) \in \partial\Omega. \end{aligned}\right\} \tag{4.4}$$

The following eigenvalue problems

$$\left.\begin{aligned} &(a\psi_x)_x = -\mu\psi, \qquad x \in (0,1) \\ &\frac{d\psi}{dx} = 0, \quad x = 0,1. \end{aligned}\right\} \tag{4.5}$$

and

$$\left.\begin{aligned} &\theta_{yy} = -\xi\theta, \qquad y \in (0,1) \\ &\frac{d\theta}{dy} = 0, \quad y = 0,1. \end{aligned}\right\} \tag{4.6}$$

are associated to (4.4) by the following result

Theorem 4.1. *Let $\{\mu_m,\ \psi_m\}$ be the solutions of the eigenvalue problem (4.5) and $\{\xi_n,\ \theta_n\}$ be the solutions of the eigenvalue problem (4.6). Then,*

$$\lambda_{mn} := \mu_m + \xi_n, \qquad \varphi_{mn}(x,y) := \psi_m(x)\theta_n(y) \tag{4.7}$$

are the eigenvalues and associated eigenfunctions for the problem (4.4).

Assume that $a(\cdot,\nu) \in C^2([0,1],\mathbb{R})$, $a(x,\nu) > 0$, $x \in [0,1]$. We assume that a is large except in a neighborhood of a point where it becomes small. To describe the coefficient a, let $0 = x_0 < x_1 < x_2 = 1$ be a partition of the interval [0,1] and let l_1, a_1 be two positive constants and l_1', a_1' be functions of a positive parameter ν that approach l_1, a_1 from above as $\nu \to 0$. Then if ϵ_1, ϵ_2 are other positive constants (and ν is sufficiently small), let a be such that

$$\left.\begin{aligned} &a(x,\nu) \geq \frac{\epsilon_1}{\nu}, && \text{for} \quad x_0 \leq x \leq x_1 - \nu l_1', \\ &a(x,\nu) \geq \frac{\epsilon_2}{\nu}, && \text{for} \quad x_1 + \nu l_1' \leq x \leq x_2, \\ &a(x,\nu) \geq \nu a_1, && \text{for} \quad x_1 - \nu l_1' \leq x \leq x_1 + \nu l_1', \\ &a(x,\nu) \leq \nu a_1', && \text{for} \quad x_1 - \nu l_1 \leq x \leq x_1 + \nu l_1. \end{aligned}\right\} \tag{4.8}$$

We assume that

$$\left.\begin{aligned} &l_i' - l_i = o(1), \\ &a_i' - a_i = o(1). \end{aligned}\right\} \tag{4.9}$$

as $\nu \to 0$.

The following result plays an essential role in this example and its proof can be found in Carvalho and Pereira [1991].

Lemma 4.2. *Under these hypotheses, $\mu_1 = 0$, $\mu_2 \to \frac{a_1}{2l_1}\frac{1}{x_1(1-x_1)}$ as $\nu \to 0$ and there exists $m > 0$ such that, $\mu_3 \geq \frac{m}{\nu}$. Furthermore, we can choose $\psi_1 \equiv 1$ and ψ_2 to satisfy*

$$\psi_2(x) = \begin{cases} -\sqrt{\dfrac{1-x_1}{x_1}} + o(1), & x \in [0, x_1 - \nu l_1] \\ O(1), & x \in [x_1 - \nu l_1, x_1 + \nu l_1] \\ \sqrt{\dfrac{x_1}{1-x_1}} + o(1), & x \in [x_1 + \nu l_1, 1] \end{cases}$$

With this information we proceed to guess the differential equation that carries the asymptotic behavior.

Let $W_1^o = \overline{\text{span}[\theta_1,\theta_2,\theta_3,\cdots]}$, $W_2^o = \overline{\text{span}[\psi_2\theta_1,\psi_2\theta_2,\psi_2\theta_3,\cdots]}$ and $W_\alpha^\perp = [W_1^o \oplus W_2^o]^\perp$ its orthogonal complement in X^α. If $u \in X_\nu^{\frac{1}{2}}$ it can be written as

$$u = v_1\psi_1 + v_2\psi_2 + w$$

where hereafter we denote ψ_1 by 1, ψ_2 by ψ and

$$\begin{aligned} v_1 &= \int_0^1 u(x,y)dx, \\ v_2 &= \int_0^1 u(x,y)\psi(x)dx, \\ w &= u - v_1 - v_2\psi. \end{aligned}$$

This decomposition of the space induces a decomposition in the equations (4.1), (4.2) as follows. Suppose $u(t,x,y)$ is a solution of (4.1), (4.2); then,

$$u(t,x,y) = v_1(t,y) + v_2(t,y)\psi(x) + w(t,x,y)$$

and

$$\frac{\partial v_1}{\partial t}(t,y) = \int_0^1 u_t(t,s,y)ds = \frac{\partial^2 v_1}{\partial y^2} + \int_0^1 f(u(t,s,y))ds. \tag{4.10}$$

$$\frac{\partial v_2}{\partial t}(t,y) = \int_0^1 u_t(t,s,y)\psi(s)ds = \frac{\partial^2 v_2}{\partial y^2} - \mu_2 v_2 + \int_0^1 f(u(t,s,y))\psi(s)ds. \tag{4.11}$$

$$\left.\begin{aligned} w_t &= (aw_x)_x + w_{yy} + f(u(t,x,y)) - \int_0^1 f(u(t,s,y))ds - \int_0^1 f(u(t,s,y))\psi(s)ds\ \psi(x) \\ \frac{\partial w}{\partial n} &= 0. \end{aligned}\right\} \tag{4.12}$$

We know from Lemma 4.2 that

$$\psi(x) \to \begin{cases} -\sqrt{\dfrac{1-x_1}{x_1}}, & x \in (0,x_1) \\ \sqrt{\dfrac{x_1}{1-x_1}}, & x \in (x_1,1) \end{cases}$$

as $\nu \to 0$.

Also from Lemma 4.2 we know that $\mu_3 \to \infty$ as $\nu \to 0$. Therefore, one expects that the component w of u does not play any role in the large time behavior. Thus,

$$\begin{aligned} \frac{\partial v_1}{\partial t}(t,y) &\sim \frac{\partial^2 v_1}{\partial y^2} + \int_0^1 f(v_1(t,y) + v_2(t,y)\psi(s))ds \\ &\sim \frac{\partial^2 v_1}{\partial y^2} + x_1 f(v_1(t,y) - k_1 v_2(t,y)) + (1-x_1)f(v_1(t,y) + k_2 v_2(t,y)). \end{aligned}$$

$$\begin{aligned} \frac{\partial v_2}{\partial t}(t,y) &\sim \frac{\partial^2 v_2}{\partial y^2} - \mu v_2(t,y) + \int_0^1 f(v_1(t,y) + v_2(t,y)\psi(s))\psi(s)ds \\ &\sim \frac{\partial^2 v_2}{\partial y^2} - \mu v_2(t,y) + \int_0^{x_1} f(v_1(t,y) - k_1 v_2(t,y))(-k_1)ds + \int_{x_1}^1 f(v_1(t,y) + k_2 v_2(t,y))k_2 ds \\ &\sim \frac{\partial^2 v_2}{\partial y^2} - \mu v_2(t,y) - x_1 k_1 f(v_1(t,y) - k_1 v_2(t,y)) + (1-x_1)k_2 f(v_1(t,y) + k_2 v_2(t,y)). \end{aligned}$$

where $k_1 = \sqrt{\frac{1-x_1}{x_1}}$, $k_2 = \sqrt{\frac{x_1}{1-x_1}}$ and $\mu = \frac{a_1}{2l_1}\frac{1}{x_1(1-x_1)}$.

Therefore, we guess that the limiting differential equations are,

$$\left.\begin{aligned} \frac{\partial v_1}{\partial t}(t,y) &= \frac{\partial^2 v_1}{\partial y^2} + g_1(v_1(t,y), v_2(t,y)), \\ \frac{\partial v_2}{\partial t}(t,y) &= \frac{\partial^2 v_2}{\partial y^2} - \mu v_2(t,y) + g_2(v_1(t,y), v_2(t,y)), \\ \frac{\partial v_1}{\partial y} &= \frac{\partial v_2}{\partial y} = 0, \qquad y = 0,1 \end{aligned}\right\} \tag{4.13}$$

where

$$g_1(v_1, v_2) = x_1 f(v_1 - k_1 v_2) + (1 - x_1) f(v_1 + k_2 v_2)$$
$$g_2(v_1, v_2) = -x_1 k_1 f(v_1 - k_1 v_2) + (1 - x_1) k_2 f(v_1 + k_2 v_2).$$

We now exhibt a change of variables that simplify equations (4.13). Consider the average on each of the intervals; that is,

$$z_1(t,y) = \frac{1}{x_1}\int_0^{x_1} u(t,x,y)dx,$$
$$z_2(t,y) = \frac{1}{(1-x_1)}\int_{x_1}^{1} u(t,x,y)dx;$$

this suggests that the variables in (4.13) are

$$v_1(t,y) = x_1 z_1(t,y) + (1 - x_1) z_2(t,y).$$
$$v_2(t,y) = -k_1 x_1 z_1(t,y) + k_2(1 - x_1) z_2(t,y).$$

In the variables z_1 and z_2 the equations (4.13) become

$$\left.\begin{aligned} \frac{\partial z_1}{\partial t}(t,y) &= \frac{\partial^2 z_1}{\partial y^2} + \frac{a_1}{2\,l_1\,x_1}(z_2 - z_1) + f(z_1(t,y)), \\ \frac{\partial z_2}{\partial t}(t,y) &= \frac{\partial^2 z_2}{\partial y^2} - \frac{a_1}{2\,l_1\,(1-x_1)}(z_2 - z_1) + f(z_2(t,y)), \\ \frac{\partial z_1}{\partial y} &= \frac{\partial z_2}{\partial y} = 0, \qquad y = 0,1. \end{aligned}\right\} \tag{4.14}$$

The techniques in section 1 are applied to prove that the dynamics of (4.1), (4.2) in $X_\nu^{\frac{1}{2}}$ is given by (4.14). To accomplish this, the class of nonlinearities in consideration must be restricted to those that satisfy some dissipativeness condition and for which the above problem is locally well posed.

Assume that $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous function with Lipschitz constant L_1. Also, assume that f satisfies the dissipativeness condition (2.3).

The following result is an easy consequence of the above hypothesis.

Lemma 4.3. *Under the previous assumptions on f, the function $f^e : X_\nu^{\frac{1}{2}} \to X$, given by*

$$f^e(\phi)(x) = f(\phi(x))$$

is globally Lipschitz continuous in $X_\nu^{\frac{1}{2}}$ with Lipschitz constant L_1.

This implies that (4.1), (4.2) is globally well posed. By standard arguments one can prove that the semigroup associated to (4.1), (4.2) is bounded dissipative; that is, there exists a set $\mathcal{O}_\nu$ in $X_\nu^{\frac{1}{2}}$ that attracts bounded sets. The aim is to find a constant $M > 0$, independent of ν, such that, for some $\nu_0 > 0$,

$$\|u\|_{X_\nu^{\frac{1}{2}}} \leq M$$

for every $u \in \mathcal{O}_\nu$, $0 < \nu \leq \nu_0$.

Once the above constant is found, by Theorem 1.2, the following result follows.

Theorem 4.4. *Denote the attractor of (4.1), (4.2) by $\mathcal{A}_\nu$ and the attractor of (4.13) (equivalently (4.14)) by $\mathcal{A}$. Let $\tilde{\mathcal{A}}_\nu = \{(v_1, v_2) : v_1 + v_2\psi + w \in \mathcal{A}_\nu\}$. Then, the family of attractors $\{\tilde{\mathcal{A}}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is upper semicontinuous at zero.*

To apply Theorem 1.2 we have to prove that the constant M can be chosen independently of $\nu \leq \nu_0$. To accomplish this the standard proof that the semigroup associated to (4.1), (4.2) is bounded dissipative is repeated, keeping track of the constants involved.

In what follows, the existence of a uniform bound for the attractors is shown. Consider the Liapunov functional $E : X_\nu^{\frac{1}{2}} \to \mathbb{R}$ defined by

$$E_\nu(\phi) = \int_\Omega \left[\frac{a(x)}{2} \left|\frac{\partial \phi}{\partial x}(x,y)\right|^2 + \frac{1}{2}\left|\frac{\partial \phi}{\partial y}(x,y)\right|^2 + \frac{\delta}{2}\phi(x,y)^2 - F(\phi(x,y)) \right] dx\,dy,$$

where $F(u) = \int_0^u f(s)ds$.

From the dissipativeness condition it follows that $sf(s) \leq -\frac{\delta}{2}s^2 + c_\delta$ and $F(s) \leq -\frac{\delta}{2}s^2 + c_\delta$ for $s \in \mathbb{R}$ and from the growth condition that $|F(s)| \leq r(1 + |s|^2)$ for $s \in \mathbb{R}$.

With these estimates on $F(s)$

$$\begin{aligned}E_\nu(\phi) \geq& \int_\Omega \left[\frac{a(x)}{2} \left|\frac{\partial \phi}{\partial x}(x,y)\right|^2 + \frac{1}{2}\left|\frac{\partial \phi}{\partial y}(x,y)\right|^2 \right] dx\,dy + \delta\|\phi\|_{L^2}^2 - c_\delta|\Omega| \\ \geq& c_1 \|\phi\|^2_{X_\nu^{\frac{1}{2}}} - c_\delta|\Omega|.\end{aligned}$$

On the other hand,

$$\begin{aligned}E_\nu(\phi) \leq& \int_\Omega \left[\frac{a(x)}{2} \left|\frac{\partial \phi}{\partial x}(x,y)\right|^2 + \frac{1}{2}\left|\frac{\partial \phi}{\partial y}(x,y)\right|^2 \right] dx\,dy + \frac{\delta}{2}\|\phi\|_{L^2(\Omega)}^2 + r\|\phi\|_{L^2(\Omega)}^2 + r|\Omega| \\ \leq& c_2\|\phi\|^2_{X_\nu^{\frac{1}{2}}} + r|\Omega|.\end{aligned}$$

Also, if $u(t)$ is the solution of (4.1), (4.2) satisfying $u(0) = u_0 \in X_\nu^{\frac{1}{2}}$.

$$\begin{aligned}\frac{d}{dt}E_\nu(u(t)) =& -\|u_t\|_{L^2} - \delta \int_\Omega \left[\frac{a(x)}{2} \left|\frac{\partial u}{\partial x}(x,y)\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial y}(x,y)\right|^2 \right] dx\,dy + \delta \int_\Omega uf(u)dx\,dy \\ \leq& -\delta \int_\Omega \left[\frac{a(x)}{2} \left|\frac{\partial u}{\partial x}(x,y)\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial y}(x,y)\right|^2 \right] dx\,dy - \frac{\delta^2}{2}\|u\|_{L^2} + \delta c_\delta|\Omega| \leq -\delta c_1\|u\|^2_{X_\nu^{\frac{1}{2}}} + \delta c_\delta|\Omega|.\end{aligned}$$

Observe that all the constants involved in the above estimates do not depend on ν. Therefore, the above inequalities imply the following result.

Lemma 4.5. *There exists a constant $M > 0$, independent of ν, such that for any $0 < \nu \leq \nu_0$ and bounded set B in $X_\nu^{\frac{1}{2}}$ $\exists\, t_0 = t_0(\nu, B)$ such that, for $t \geq t_0$, $u_0 \in B$,*

$$u(t, u_0) \in \{u \in X_\nu^{\frac{1}{2}} : \|u\|_{X_\nu^{\frac{1}{2}}} \leq M\},$$

where $u(t, u_0)$ is the solution of (4.1), (4.2) satisfying $u(0, u_0) = u_0$.

This proves that (4.1), (4.2) defines a global semigroup which is bounded dissipative.

The fractional power space X_ν^α can be decomposed as

$$X_\nu^\alpha = W_1 \oplus W_2 \oplus W_\alpha^\perp. \tag{4.15}$$

where W_1 is the subspace functions that do not depend upon x in X_ν^α, $W_2 = \{\eta \in X_\nu^{\frac{1}{2}} : \eta = \psi_2(x)v(y),\ v \in W_1\}$, $0 \leq \alpha \leq 1$.

Let $T_\nu(t)$ denote the analytic semigroup generated by A_ν. Then,

$$\|T_\nu(t)w\|_{W^\perp_{\frac{1}{2}}} \leq K\epsilon^{-m\nu^{-\frac{1}{2}}t}\|w\|_{W^\perp_{\frac{1}{2}}}$$

$$\|T_\nu(t)w\|_{W^\perp_{\frac{1}{2}}} \leq Kt^{-\frac{1}{2}}\epsilon^{-m\nu^{-\frac{1}{2}}t}\|w\|_{W^\perp_0}.$$

Let

$$P(v_1, v_2, w) = \begin{bmatrix} \int_0^1 f(v_1(y) + v_2(y)\psi_2(s) + w(s,y))ds \\ -\mu_2(\nu)v_2 + \int_0^1 f(v_1(y) + v_2(y)\psi_2(s) + w(s,y))\psi_2(s)ds \end{bmatrix} \tag{4.16}$$

and

$$g(v_1, v_2) = \begin{bmatrix} g_1(v_1, v_2) \\ -\mu v_2 + g_2(v_1, v_2) \end{bmatrix}.$$

The variable w satisfy

$$\left.\begin{aligned} w_t &= (aw_x)_x + u_{yy} + Q(v_1, v_2, w) \\ \frac{\partial w}{\partial n} &= 0, \end{aligned}\right\} \tag{4.17}$$

where

$$Q(v_1, v_2, w(x)) = f(v_1+v_2\psi_2(x)+w(x)) - \int_0^1 f(v_1+v_2\psi_2(s)+w(s))ds - \int_0^1 f(v_1+v_2\psi_2(s)+w(s))\psi_2(s)ds\,\psi_2(x)$$

Let $Q^\epsilon : X_\nu^{\frac{1}{2}} \to X$ be defined by

$$Q^\epsilon(v_1, v_2, w)(x, y) = Q(v_1(y), v_2(y), w(x, y)). \tag{4.18}$$

The following results can be easily proved using growth assumptions on f and from the variational characterization of the eigenvalues.

Lemma 4.6. *The function* $Q^\epsilon : W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}} - X$ *defined by (4.12) is Lipschitz continuous in bounded sets of* $W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}$. *Furthermore, given* $r > 0$, *there exists constants* $L_Q = L_Q(r)$ *and* $M_Q = M_Q(r)$ *such that*

$$\|Q^\epsilon(v_1, v_2, w)\|_X \leq L_Q\,\nu^{\frac{1}{2}}\,\|w\|_{W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}} + M_Q.$$

for $\|(v_1, v_2, w)\|_{W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}} \leq r.$

Lemma 4.7. *If* P *is given by (4.16), the function* $\tilde{P} : W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}} - W_1^0 \oplus W_2^0$ *defined by* $\tilde{P}(v_1, v_2, w) = P(v_1, v_2, w) - g(v_1, v_2)$ *is Lipschitz continuous in bounded sets of* $W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}$. *Furthermore, given* $r > 0$. *there exists constant* $L_P = L_P(r)$ *and function* $M_P = M_P(r, \nu) - 0$ *as* $\nu \to 0$ *such that*

$$\|\tilde{P}(v_1, v_2, w)\|_{W_1^0 \oplus W_2^0} \leq L_P\,\nu^{\frac{1}{2}}\,\|w\|_{W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}} + M_P(r, \nu).$$

for $\|(v_1, v_2, w)\|_{W_1^{\frac{1}{2}} \oplus W_2^{\frac{1}{2}} \oplus W^\perp_{\frac{1}{2}}} \leq r.$

Theorem 4.4 now follows from Theorem 1.2.

In what follows we consider a family of problems that include the problem (4.14). That is, consider the system of parabolic partial differential equations

$$\left.\begin{aligned} \frac{\partial z_1}{\partial t} &= d_1\frac{\partial^2 z_1}{\partial y^2} + p(z_2 - z_1) + f(z_1, z_2), \\ \frac{\partial z_2}{\partial t} &= d_2\frac{\partial^2 z_2}{\partial y^2} - q(z_2 - z_1) + g(z_1, z_2), \\ \frac{\partial z_1}{\partial y} &= \frac{\partial z_2}{\partial y} = 0, \qquad y = 0, 1 \end{aligned}\right\} \tag{4.19}$$

where p, q, d_1 and d_2 are positive parameters and f, g are smooth functions.

Define

$$F(z_1, z_2) := \begin{pmatrix} p(z_2 - z_1) + f(z_1, z_2) \\ -q(z_2 - z_1) + g(z_1, z_2) \end{pmatrix}.$$

Next we introduce the notion of invariant region as in Smoller [1982].

Definition 4.8. *A closed subset $\Sigma \subset \mathbb{R}^2$ is called a positively invariant region for the local solution defined by (4.19), if any solution $(z_1(t, y), z_2(t, y))$ having initial value in Σ, satisfies $(z_1(t, y), z_2(t, y)) \in \Sigma$ for all $y \in (0, 1)$ and t for which the solution is defined.*

The following result is quoted from Smoller [1982] (page 202) and it is due to Chueh, Conley and Smoller [1977].

Proposition 4.9. *The following holds*

(a) Any region of the form

$$\Sigma = \cap_{i=1}^{2}\{(z_1, z_2) \in \mathbb{R}^2 : a_i \le z_i \le \beta_i\}$$

is an invariant region for (4.19), provided that F points into Σ on $\partial\Sigma$.

(b) If $d_1 = d_2 = 1$, then any convex region Σ, in which F points strictly into Σ on $\partial\Sigma$, is an invariant region for (4.19).

Proposition (4.9) implies the following result.

Corollary 4.10. *Assume that there are real numbers a, b, c and d such that $f(s, t) > 0$ $\forall s \le a$, $f(s, t) < 0$ $\forall s \ge b$, $g(s, t) > 0$ $\forall t \le c$ and $g(s, t) < 0$ $\forall t \ge d$. Then, the rectangle $[a, b] \times [c, d]$ is an invariant region for (4.19).*

Proof. Under these hypotheses, it is easy to see that, on the boundary of the rectangle $[a, b] \times [c, d]$, the flow F points into $[a, b] \times [c, d]$. The proof of the result now follows from Proposition 4.9.

Next we introduce the notion of cooperative systems in convex domains with no flux boundary condition (see Kishimoto and Weinberger [1985]). Let Ω be a bounded convex smooth domain in $\mathbb{R}^n$ and consider the weakly coupled system of parabolic equations

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \sigma_i \Delta u_i + f_i(u_1, \cdots, u_k), \qquad 1 \le i \le k, \quad x \in \Omega \\ \frac{\partial u_i}{\partial \vec{n}} &= 0, \qquad x \in \partial\Omega \end{aligned} \tag{4.20}$$

where $\vec{n}$ is the outward normal and $\sigma_i > 0$, $1 \le i \le k$. This system is said to be cooperative if

$$\frac{\partial f_i}{\partial u_j} > 0, \qquad i \ne j. \tag{4.21}$$

Kishimoto and Weinberger [1985] prove that the following result holds.

Theorem 4.11. *Let $\bar{u}$ be a nonconstant equilibrium solution of (4.20). Suppose that (4.21) holds on the range of $\bar{u}$. Then $\bar{u}$ is unstable.*

Even though the equivalent system (4.3) is not necessarily cooperative it is equivalent (by a change of coordinates) to a cooperative system. This ensures that the above theorem also holds for (4.3).

The condition (4.21) is immediately satisfied for the system (4.14). Therefore, the systems in consideration in this section are cooperative systems and the only possible stable equilibrium points are the constant equilibrium points.

Now we consider some generalizations of the problem (4.1), (4.2). We introduce some notation (see also Carvalho [1992]) to describe the diffusion coefficients that we are going to consider.

Let $\mathcal{N}^n \subset \mathbb{R}^n$ be the cone defined by

$$\mathcal{N}^n = \{\bar{\epsilon} = (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{R}^n : \epsilon_i > 0,\ 1 \leq i \leq n\},$$

with the partial ordering defined by $\bar{\epsilon} \leq \bar{f}$ iff $\epsilon_i \leq f_i$, $1 \leq i \leq n$ and let $\mathcal{N}_0^{n+1} \subset \mathbb{R}^{n+1}$ be the cone defined by

$$\mathcal{N}_0^{n+1} = \{\bar{\epsilon} = (\epsilon_1, \cdots, \epsilon_{n+1}) \in \mathbb{R}^{n+1} : 0 = \epsilon_1 < \epsilon_2 < \cdots \epsilon_n < \epsilon_{n+1} = 1\}.$$

For $\bar{\epsilon} \in \mathcal{N}^n$, $\bar{x} \in \mathcal{N}_0^{n+1}$ and continuous increasing functions $\vec{l'}, \vec{a'} : [0, \infty) \to \mathcal{N}^{n+1}$, we say that a function $a : [0,1] \times (0, \infty) \to (0, \infty)$ is a $C_n^1(\bar{\epsilon}, \vec{l'}, \vec{a'}, \bar{x}, \nu)$ function if, for each ν, $a(\cdot, \nu) \in C^1[0,1]$ and the following is satisfied

$$\left.\begin{aligned} &a(x,\nu) \geq \frac{\epsilon_i}{\nu}, && \text{for} \quad x_i + \nu l'_i \leq x \leq x_{i+1} - \nu l'_{i+1} && 1 \leq i \leq n \\ &a(x,\nu) \geq \nu a_i, && \text{for} \quad x_i - \nu l'_i \leq x \leq x_i + \nu l'_i && 1 \leq i \leq n+1 \\ &a(x,\nu) \leq \nu a'_i, && \text{for} \quad x_i - \nu l_i \leq x \leq x_i + \nu l_i && 1 \leq i \leq n+1 \end{aligned}\right\} \tag{4.22}$$

where $\vec{a} = \vec{a'(0)}$, $\vec{l} = \vec{l'(0)}$, $\vec{a'} = \vec{a'(\nu)}$ and $\vec{l'} = \vec{l'(\nu)}$.

Consider the scalar parabolic problem

$$u_t = (a u_x)_x + u_{yy} + f(u), \quad (x,y) \in \Omega \tag{4.23}$$

$$\rho u + (1 - \rho)(a u_x, u_y) \cdot \vec{n} = 0, \quad (x,y) \in \partial\Omega \tag{4.24}$$

where $0 \leq \rho, \sigma \leq 1$.

Carvalho and Pereira [1991] studied the eigenvalue problem for the operator

$$A_\nu : D(A_\nu) \subset L^2(0,1) \to L^2(0,1)$$

defined by

$$\begin{aligned} &D(A_\nu) = \left\{u \in H^2(0,1) : \rho u(0) - (1-\rho)a(0)u_x(0) = u(1) - (1-\sigma)a(1)u_x(1) = 0\right\}, \\ &A_\nu u = (a(x,\nu)u_x)_x, \quad u \in D(A_\nu), \end{aligned}$$

where $0 \leq \rho, \sigma \leq 1$, $a \in C_n^1(\bar{\epsilon}, \vec{l'}, \vec{a'}, \bar{x}, \nu)$. More specificaly to study the behavior of the eigenvalues and eigenfunctions of A_ν as the parameter ν tends to zero.

Let B be the tridiagonal symmetric matrix

$$B = \begin{bmatrix} m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & r_{n-3} & m_{n-2} & r_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{n-2} & m_{n-1} & r_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & r_{n-1} & m_n \end{bmatrix}$$

where

$$m_1 = \frac{a_1\rho}{\rho l_1 + a_1(1-\rho)} + \frac{a_2}{2l_2}, \quad m_n = \frac{a_n}{2l_n} + \frac{a_{n+1}\sigma}{\sigma l_{n+1} + a_{n+1}(1-\sigma)},$$

$$m_k = \frac{a_k}{2l_k} + \frac{a_{k+1}}{2l_{k+1}}, \quad k = 2, \cdots, n-1,$$

$$r_k = \frac{-a_{k+1}}{2l_{k+1}}, \quad k = 1, \cdots, n-1.$$

If $\mathcal{M}$ is the matrix $\mathcal{M} = \operatorname{diag}(L_1, \cdots, L_n)$, $L_i = x_{i+1} - x_i$, $1 \leq i \leq n$, consider the inner product

$$\langle y, z\rangle_{\mathcal{M}} = \langle \mathcal{M}y, z\rangle$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^n$.

Consider the matrix $\tilde{A} := \mathcal{M}^{-1}B$. Since $\tilde{A}$ is a tridiagonal matrix with non-zero product of the off-diagonal entries, it has n distinct eigenvalues . In addition, its eigenvectors have the Sturm property; that is, the first eigenvector can be chosen with all components positive, the second has one change of sign in the components, the third has two changes of sign and so forth (see Gantmacher [1959]).

We need the following result of Carvalho and Pereira [1990] which is stated without proof.

Theorem 4.12. *Suppose the above hypotheses are satisfied. Let $\tilde{A}$ be the matrix defined above, $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ the sequence of its eigenvalues, $z_1, z_2, \cdots, z_n$ the corresponding sequence of eigenvectors. Writing $z_i := (z_{i1}, \cdots, z_{in})$ we have:*

$$\mu_k = \sigma_k + O(\nu^{1/2}),$$

$$\mu_{n+1} \geq \tfrac{\tau}{\nu},$$

$$\psi_k(x) = \begin{cases} z_{ki} + O(\nu^{q/2}), & x \in [x_i + \nu l_i, x_{i+1} - \nu l_{i+1}], \quad i = 1, \cdots, n \\ \\ O(1), & x \in [0, 1]. \end{cases}$$

where (μ_k, ψ_k) is the k^{th} eigenpair of A_ν. $1 \leq k \leq n$.

Observe that Theorem 4.1 holds. We use the same notation for eigenvalues and eigenfunctions as in Theorem 4.1.

Let A^α_ν be the fractional power of A_ν and X^α be $D(A^\alpha)$ endowed with the graph norm. Let

$$W^i_{\frac{1}{2}} = \overline{\operatorname{span}[\psi_i\theta_1, \psi_i\theta_2, \psi_i\theta_3, \cdots]}, \qquad W_{\frac{1}{2}} = \oplus^n_{i=1} W^i_{\frac{1}{2}} \quad \text{and} \quad W^\perp_\alpha = \{\phi \in X^\alpha : \langle \phi, \psi\rangle = 0.\ \forall\psi \in W\}....$$

Then if $u \in X^{\frac{1}{2}}_\nu$ it can be decomposed as

$$u = \sum_{i=1}^{n} v_i\psi_i + w$$

where

$$v_i = \int_0^1 u(x, y)\psi_i(x)dx. \quad 1 \leq i \leq n$$

$$w = u - \sum_{i=1}^{n} v_i\psi_i.$$

This decomposition of the space induces a decomposition in the equations (4.23), (4.24) as follows. Suppose $u(t, x, y)$ is a solution of (4.23). (4.24); then.

$$u(t, x, y) = \sum_{i=1}^{n} v_i(t, y)\psi_i(x, y) + w(t, x, y)$$

and

$$\begin{aligned}
&\frac{\partial v_i}{\partial t}(t,y) = \frac{\partial^2 v_i}{\partial y^2} - \mu_i v_i + P_i(v_1,\cdots,v_n,w), \quad 1 \le i \le n \\
&\frac{\partial w}{\partial t} = (aw_x)_x + w_{yy} + Q(v_1,\cdots,v_n,w) \\
&\rho w - (1-\rho)(aw_x,u_y)\cdot\vec{n} = 0, \quad (x,y)\in\partial\Omega
\end{aligned}$$

where

$$P_i(v_1,\cdots,v_n,\varphi)(y) = \int_0^1 f\left(\sum_{i=1}^n v_i(y)\psi_i(s) + \phi(s,y)\right)\psi_i(s)ds, \quad 1\le i\le n$$

$$Q(v_1,\cdots,v_n,\phi)(x,y) = f\left(\sum_{i=1}^n v_i(y)\psi_i(x) + \phi(x,y)\right) - \sum_{i=1}^n \int_0^1 f\left(\sum_{i=1}^n v_i(y)\psi_i(s) + \phi(s,y)\right)\psi_i(s)ds\ \psi_i(x).$$

With this information it is possible to guess that the system of partial differential equations that determines the asymptotic behavior is,

$$\left.\begin{aligned}
\frac{\partial v_1}{\partial t} &= \frac{\partial^2 v_1}{\partial y^2} - \sigma_1 v_1 + f_1(v), \\
&\vdots \\
\frac{\partial v_n}{\partial t} &= \frac{\partial^2 v_n}{\partial y^2} - \sigma_n v_n + f_n(v).
\end{aligned}\right\} \tag{4.25}$$

where $v = (v_1,\cdots,v_n)$ and $f_j(v) = \sum_{i=1}^n [x_{i+1} - x_i] f(v_1(t)z_{1i} + v_2(t))z_{2i} + \cdots + v_n(t)z_{ni})z_{ji}$

Under the assumptions on f, it is not hard to see that the system of parabolic equations (4.25) has a global attractor $\mathcal{A}$. For example one could consider the Liapunov function

$$V(v_1,\cdots,v_n) = \int_0^1 \left\{ \frac{1}{2}\sum_{i=1}^n \left[\left(\frac{\partial v_i}{\partial y}\right)^2 + [1+\sigma_i]v_i^2\right] - \sum_{j=1}^n (x_j - x_{j-1})F\left(\sum_{i=1}^n v_i z_{ij}\right)\right\} dy$$

to prove the existence of such global attractor.

Define $Q^\epsilon : X_\nu^{\frac{1}{2}} \to L^2(\Omega)$ by $Q^\epsilon(v,w)(x) = Q(v,w(x))$ and $\tilde{P}_j : X_\nu^{\frac{1}{2}} \to L^2(0,1)$ by $\tilde{P}_j(v,w) = P_j(v,w) - f_j(v)$, $1\le j\le n$. To apply Theorem 1.4 the following lemmas are needed.

Lemma 4.13. *For* $(v,w)\in W^{1}_{\frac{1}{2}}$, $\|(v,w)\|_{X_\nu^{\frac{1}{2}}} \le r$ *there exist* $L_{P_j}(r,\nu)$. $M_{P_j}(r,\nu)$ *such that*

$$\|\tilde{P}_j(v,w)\|_{H^1(0,1)} \le L_{P_j}\|w\|_{X_\nu^{\frac{1}{2}}} + M_{P_j}(r,\nu)$$

where $L_{P_j}(r,\nu)$, $M_{P_j}(r,\nu) \to 0$ *as* $\nu \to 0$.

Lemma 4.14. *For* $(v,w)\in W^{1}_{\frac{1}{2}}$, $\|(v,w)\|_{X_\nu^{\frac{1}{2}}} \le r$ *there exist* $L_Q(r,\nu)$. $M_Q(r)$ *such that*

$$\|\tilde{Q}(v,w)\|_{L^2(\Omega)} \le L_Q(r,\nu)\|w\|_{X_\nu^{\frac{1}{2}}} + M_Q(r)$$

where $L_Q(r,\nu)\to 0$ *as* $\nu\to 0$.

The proof of these lemmas can be reproduced from Carvalho [1992] and we omit them.

From Theorem 1.2 the following result holds.

Theorem 4.14. *Denote by $\mathcal{A}_\nu$ the global attractor for the problem (4.23), (4.24). Let $\hat{\mathcal{A}}_\nu = \{(v_1, \cdots, v_n) : \sum_{i=1}^n v_i \psi_i + w \in \mathcal{A}_\nu\}$. Then, the family of attractors $\{\hat{\mathcal{A}}_\nu,\ 0 \le \nu \le \nu_0\}$ is upper semicontinuous at zero.*

5. Applications of Theorem 1.2 to Shadow Systems

Several models in applied sciences involve systems of reaction-diffusion equations in a bounded smooth domain with homogeneous Neumann boundary conditions. In such problems, the study of pattern formation is a subject of great interest to mathematicians and applied scientists.

Negative results are important to rule out the cases where stable patterns are not observed. Some negative results when the diffusion coefficients are large are proved in Conway, Hoff and Smoller [1978] for the case when invariant regions exist and later in Hale [1986], Hale and Rocha [1987a,b] and Hale and Carvalho [1991] for more general situations.

The next step towards understanding the formation of stable patterns seems to be to consider the case where some of the diffusion coefficients are large and the remaining are kept fixed. This situation has been addressed by Hale and Sakamoto [1989], among others.

Our purpose is to consider the question of upper semicontinuity of attractors as in Hale and Sakamoto [1989]. Our primary goal is to give a simple proof that allows us to consider global attractors. Hale and Sakamoto [1989] are only able to consider local attractors, except in the one dimensional case, due to their choice of spaces. We apply the techniques of Section 1 to obtain the results.

We now introduce some terminology necessary to precisely state the results. Let $\Omega \subset \mathbb{R}^n$, $n \le 3$, be a bounded smooth domain (for convenience assume that $|\Omega| = 1$) and consider the system of reaction-diffusion equations

$$\begin{aligned} u_t &= \Delta u + h(u, v) \qquad x \in \Omega \\ v_t &= D\Delta u + g(u, v) \qquad x \in \Omega \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \qquad x \in \partial\Omega \end{aligned} \tag{5.1}$$

where $u \in \mathbb{R}^m$, $v \in \mathbb{R}^k$ are vectors, $D = \mathrm{diag}(d_1, \cdots, d_k)$, $d_j > 0$, $1 \le j \le k$ and $h : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ are continuous. Let $f := \begin{pmatrix} h \\ g \end{pmatrix} : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$ and assume that

$$|f(p) - f(q)| \le c\,|p - q|(1 + |p|^2 + |q|^2), \qquad \forall p, q \in \mathbb{R}^m \times \mathbb{R}^k$$

for all $p, q \in \mathbb{R}^m \times \mathbb{R}^k$ and

$$\limsup_{|p_i| \to \infty} \frac{f(p)}{p_i} \le -\delta, \quad \frac{\partial f_j(p)}{\partial p_i} = \frac{\partial f_i(p)}{\partial p_j} \quad \forall p \in \mathbb{R}^m \times \mathbb{R}^k \qquad 1 \le i, j \le m + k$$

for some $\delta > 0$.

Let $X = L^2(\Omega, \mathbb{R}^k)$ and $Y = L^2(\Omega, \mathbb{R}^m)$. Let $A_D : D(A_D) \subset X \to X$ and $B : D(B) \subset Y \to Y$ be the sectorial operators defined by

$$D(A_D) = \{\phi \in H^2(\Omega, \mathbb{R}^k) : \frac{\partial \phi}{\partial n} = 0\}, \qquad A_D\phi = -D\Delta\phi \quad \forall \phi \in D(A_D)$$

$$D(B) = \{\phi \in H^2(\Omega, \mathbb{R}^m) : \frac{\partial \phi}{\partial n} = 0\}, \qquad B\phi = -\Delta\phi \quad \forall \phi \in D(B).$$

Let X_D^α, Y^α be the fractional power spaces associated to A_D and B, respectively. If $d := \min\{d_1, \cdots, d_k\}$ is viewed as a large parameter, there is a large gap between the eigenvalue zero and the remaining eigenvalues of A_D. This suggests that in the process of obtaining a limiting equation for (5.1) we decompose the space X_D^α into the subspace of constant functions W and its orthogonal complement $W_\alpha^\perp$; that is, $\phi \in X_D^\alpha$ can be written as

$$\phi = \xi + \psi, \tag{5.2}$$

where $\xi = \int_\Omega \phi(x)dx \in W$ and $\psi = \phi - \xi \in W_\alpha^\perp$.

Let $(u(t),v(t))$ be a solution of (5.1), using the decomposition (5.2), this solution can be written as $(u(t),\xi(t)+w(t))$ where $\xi(t)\in W$ and $w(t)\in W_{\alpha}^{\perp}$ and the equations (5.1) become

$$
\begin{aligned}
&u_t = \Delta u + h(u(x),\xi + w(x))\\
&\frac{\partial u}{\partial n} = 0\\
&\dot{\xi} = \int_{\Omega} g(u(y),\xi + w(y))dy\\
&w_t = D\Delta w + g(u(x),\xi + w(x)) - \int_{\Omega} g(u(y),\xi + w(y))dy\\
&\frac{\partial w}{\partial n} = 0.
\end{aligned}
\tag{5.3}
$$

Since the linear part of the equation for w has a strong exponential decay we expect it not to play an important role in the asymptotic behavior and the limiting equation should be

$$
\begin{aligned}
&u_t = \Delta u + h(u(x),\xi)\\
&\frac{\partial u}{\partial n} = 0\\
&\dot{\xi} = \int_{\Omega} g(u(y),\xi)dy.
\end{aligned}
\tag{5.4}
$$

The equation (5.4) has been called a *Shadow System* for (5.1), in the previous works, since it partially describe the dynamics of (5.1).

Under the above conditions on f the problem (5.4) has a global attractor $\mathcal{A}$ and by considering the Liapunov functional $V : Y^{\frac{1}{2}} \times X_D^{\frac{1}{2}} \to \mathbb{R}^+$

$$V(\phi,\psi) = \int_{\Omega}\left[\frac{1}{2}|\nabla\phi(x)|^2 + \frac{d}{2}|\nabla\psi(x)|^2 + \frac{1}{2}|\phi(x)|^2 + \frac{1}{2}|\psi(x)|^2 - F(\phi(x),\psi(x))\right]dx,$$

where F is defined by

$$F(p_1,\cdots,p_{m+k}) = \frac{1}{m+k}\sum_{i=1}^{m+k}\int_0^{p_i} f_i(p_1,\cdots,p_{i-1},s,p_{i+1},\cdots,p_{m+k})ds,$$

it follows from computations similar to the computations in the proof of Lemma 2.5 that

Lemma 5.1. *There exists a constant $M>0$, independent of d, such that for any $d\geq d_0$ and bounded set B in $Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}}$ $\exists\, t_0 = t_0(d,B)$ such that, for $t\geq t_0$, $u\in B$*

$$u(t,u_0)\in\{u\in Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}} : \|u\|_{Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}}}\leq M\},$$

where $u(t,u_0)$ is the solution of (5.1) satisfying $u(0,u_0)=u_0$.

Theorem 5.2. *There exists a $d_0>0$ such that the family of attractors $\{\mathcal{A}_d . d\geq d_0\}$ is upper semicontinuous at infinity, where $\mathcal{A}_{\infty}=\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ is the embedding of $\mathcal{A}$ into $Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}}$.*

Let $T_D(t)$ denote the analytic semigroup generated by $C_D = \begin{pmatrix} B & 0\\ 0 & A_D\end{pmatrix}$. Then, for any $w\in W_{\frac{1}{2}}^{\perp}$,

$$\|T_d(t)(0,0,w)\|_{Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}}}\leq Ke^{-dt}\|w\|_{W_{\frac{1}{2}}^{\perp}}, \qquad t\geq 0$$

$$\|T_d(t)(0,0,w)\|_{Y^{\frac{1}{2}}\times X_D^{\frac{1}{2}}} \leq Kt^{-\frac{1}{2}}\epsilon^{-dt}\|w\|_{W_0^{\perp}}, \qquad t>0$$

Let

$$P_d(u,\xi,w)(x) = \begin{pmatrix} h(u(x),\xi+w(x)) - h(u(x),\xi) \\ \int_\Omega [g(u(y),\xi+w(y)) - g(u(y),\xi)]\,dy \end{pmatrix} \tag{5.5}$$

The variable w satisfy

$$\left.\begin{aligned} w_t &= D\Delta w + Q(u,\xi,w) \\ \frac{\partial w}{\partial n} &= 0, \end{aligned}\right\} \tag{5.6}$$

where

$$Q_D(u,\xi.w) = g(u(x).\xi+w(x)) - \int_\Omega g(u(y).\xi+w(y))\,dy. \tag{5.7}$$

The following results strongly use the growth condition of f.

Lemma 5.3. *The function $Q_D : Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp} \to X\times Y$ defined by (5.7) is Lipschitz continuous in bounded sets of $Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}$. Furthermore, given $r>0$ $(u,\xi,w)\in Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}$, there exists constant $L_Q = L_Q(r)$ and $M_Q = M_Q(r)$ such that*

$$\|Q_D(u,\xi,w)\|_{W_0^{\perp}} \leq L_Q\, d^{-\frac{1}{2}}\, \|w\|_{W_{\frac{1}{2}}^{\perp}} + M_Q,$$

for $\|(u,\xi,w)\|_{Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}} \leq r$.

Proof. Let $(u.\xi,w)\in Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}$ such that $\|(u,\xi,w)\|_{Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}} \leq r$. Then,

$$\begin{aligned} \|Q_D(u,\xi.w)\|_{W_0^{\perp}}^2 &= \int_\Omega |Q_D(u,\xi,w)(x)|^2 dx \\ &= \int_\Omega \left| \int_\Omega [g(u(y),\xi+w(y)) - g(u(x),\xi+w(x))]\,dy \right|^2 dx \\ &\leq \int_\Omega \int_\Omega [g(u(y).\xi+w(y)) - g(u(x).\xi+w(x))]^2\,dy\,dx \\ &\leq 4c^2 \left[\left(\int_\Omega |w(x)|^6 dx \right)^{\frac{1}{3}} + \left(\int_\Omega |u(x)|^6 dx \right)^{\frac{1}{3}} \right] \left(\int_\Omega (1+|(u(x).\xi+w(x)|^2)^3 dx \right)^{\frac{2}{3}} \\ &\leq L_Q(r)^2 \left(\int_\Omega |w(x)|^6 dx \right)^{\frac{1}{3}} + M_Q(r) \leq L_Q(r)^2\, d^{-\frac{1}{2}} \|w\|_{X_D^{\frac{1}{2}}}^2 + M_Q(r) \end{aligned}$$

and the result is proved.

Lemma 5.4. *If $P_d : Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp} - X$ is given by (5.5), is Lipschitz continuous in bounded sets of $Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}$. Furthermore, given $r>0$ $(u,\xi,w)\in Y^{\frac{1}{2}}\times W\oplus W_{\frac{1}{2}}^{\perp}$. there exists constant $L_P = L_P(r)$ such that*

$$\|P_d(u.\xi,w)\|_{Y\times W} \leq L_P\, d^{-\frac{1}{2}}\, \|w\|_{W_{\frac{1}{2}}^{\perp}}.$$

for $\|(u,\xi,w)\|_{Y^{\frac{1}{2}}\times W\oplus W^{\perp}_{\frac{1}{2}}} \leq r.$

Proof. Let $(u,\xi,w)\in Y^{\frac{1}{2}}\times W\oplus W^{\perp}_{\frac{1}{2}}$ such that $\|(u,\xi,w)\|_{Y^{\frac{1}{2}}\times W\oplus W^{\perp}_{\frac{1}{2}}} \leq r$. Then.

$$\begin{aligned}
\|P_d(u,\xi,w)\|^2_{Y\times W} &= \int_\Omega |P_d(u,\xi,w)(x)|^2dx = \int_\Omega \left|\begin{pmatrix} h(u(x),\xi+w(x)) - h(u(x),\xi) \\ \int_\Omega [g(u(y),\xi+w(y))dx - g(u(y),\xi)]dy \end{pmatrix}\right|^2 dx \\
&\leq \int_\Omega |h(u(x),\xi+w(x)) - h(u(x),\xi)|^2dx + \int_\Omega\int_\Omega |g(u(y),\xi+w(y))dx - g(u(y),\xi)|^2dy\,dx \\
&\leq 6c^2\int_\Omega (1+|(u(x),\xi+w(x))|^2)^2|w(x)|^2dx \\
&\leq 6c^2\left(\int_\Omega |w(x)|^6dx\right)^{\frac{1}{3}}\left(\int_\Omega (1+|(u(x),\xi+w(x))|^2)^3dx\right)^{\frac{2}{3}} \\
&\leq L_p(r)^2\left(\int_\Omega |w(x)|^6dx\right)^{\frac{1}{3}} \leq K^2L_p(r)^2\,d^{-1}\|w\|^2_{X_d^{\frac{1}{2}}}
\end{aligned}$$

and the result is proved.

Theorem 5.2 now follows from Theorem 1.2.

Remark. *The above results also hold for systems of reaction diffusion delay-partial differential equations. The proofs are similar and are left to the reader.*

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