Research Paper

Strong equivalence of graded algebras

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A R T I C L E   I N F O

A B S T R A C T

We introduce the notion of a strong equivalence between graded algebras and prove that any partially-strongly-graded algebra by a group G is strongly-graded-equivalent to the skew group algebra by a product partial action of G. As to a more general idempotent graded algebra B, we point out that the Cohen-Montgomery duality holds for B, and B is graded-equivalent to a global skew group algebra. We show that strongly-graded-equivalence preserves strong gradings and is nicely related to Morita equivalence of product partial actions. Furthermore, we prove that any product partial group action α is globalizable up to Morita equivalence; if such a globalization β is minimal, then the skew group algebras by α and β are graded-equivalent; moreover, β is unique up to Morita equivalence. Finally, we show that strongly-graded-equivalent partially-strongly-graded algebras with orthogonal local units are stably isomorphic as graded algebras.

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Contents

1. Introduction

Two kinds of graded-equivalences are usually considered when dealing with graded rings: one, the graded-equivalence, as defined in [26] and [29], and a stronger one, the graded Morita equivalence, given in [13]. A general description of equivalences of categories of graded modules over unital rings graded by different groups was given in [15]. Morita theory for unital rings was extended to rings with local units in [7] and [8], with graded versions worked out in [27] (graded-equivalence) and [28] (graded Morita equivalence). To a great extent the graded theory is stimulated by the Cohen-Montgomery duality, saying that if B is a unital ring graded by a finite group G of order n, then the skew group ring $(B\#G) \rtimes_\beta G$ and the matrix ring $M_n(B)$ are isomorphic, where β is a certain canonical global action of G on the smash product $B\#G$. Extensions of the above duality theorem were obtained, in particular, in [32], [12], [1], [16] and [27].

Cohen-Montgomery duality was inspired by the use of duality in studying von Neumann algebras and C*-algebras. In C*-theory a duality development based on partial group actions was done in [3], introducing, in particular, the notion of the Morita equivalence of partial actions and stimulating algebraic analogues in [6]. Partial actions on algebras are closely related with graded algebras: on the one hand any partial group action on an algebra gives rise to the corresponding crossed product, and on the other by [19] an algebra partially strongly graded by an arbitrary group G (see Definition 4.5 below) with enough local units is stably isomorphic to a crossed product by a twisted partial action of G. The latter fact is a purely algebraic analogue of a C*-algebraic result from [21] stated in terms of an important concept of a Fell bundle, which is roughly the essential data remaining after disassembling a graded C*-algebra into its homogeneous pieces. Moreover, any Fell bundle carries an associated partial action of the underlying group on the spectrum of the unit fiber, as shown in [2].

More recently, notions of *weak* equivalence and *strong* equivalence between Fell bundles were introduced in [5] and [4], and in the present work we look at these notions from a purely algebraic point of view. In the algebraic case, a Fell bundle is essentially a graded algebra, more precisely a *partially-strongly-graded algebra*, as introduced in Definition 4.5 below, and weak equivalence corresponds to graded-equivalence. However, strong equivalence gives rise to a new kind of relation between partially-strongly-graded algebras, which we call here strong-graded-equivalence.

Crossed products by partial actions on algebras came into algebra from the theory of C*-algebras. Since a C*-algebra enjoys the nice property that the intersection of two closed ideals is equal to their product, the notion of a partial action in this category can be introduced equivalently in terms of intersection or product of ideals. However the situation is very different in the purely algebraic framework, and one has to make a choice. This fact was not explicitly observed in the first algebraic works on partial actions, but it emerged clearly when twisted partial actions were considered in [19]. It appeared again when the question of globalization up to Morita equivalence was considered in [6], giving rise to the notion of a *regular partial action*. The work done in the present article suggests that perhaps the notion of a product partial action is the most convenient choice to translate the concept of a partial action from the C*-algebraic world to the purely algebraic one.

We begin the paper recalling some notions on graded algebras in Section 2 and pointing out in Theorem 2.1 that the above mentioned duality holds for any idempotent algebra B graded by an arbitrary group G, with $M_n(B)$ replaced by the algebra $\mathrm{FMat}_G(B)$ of $G \times G$-matrices over B with only a finite number of non-zero entries. For graded algebras with 1 this is known by [12, Theorem 2.2]. In Section 3 we define graded-equivalence of idempotent graded algebras using graded Morita contexts and prove in Theorem 3.3 that if B is an idempotent algebra graded by a group G, then B and the skew group algebra $(B\#G) \rtimes_{\beta^B} G$ are graded-equivalent, where $B\#G$ is the Beattie's version of the smash product [12] and β^B is the usual global action of G on $B\#G$, sometimes referred to as the *dual* action.

Strong-graded-equivalence is introduced in Section 4, and we deal with it concentrating on *partially-strongly-graded algebras* $A = \oplus_{t\in G}A_t$, i.e. we assume that the equality $A_t = A_t A_{t^{-1}} A_t$ is satisfied for all $t \in G$. The latter condition naturally appeared in a characterization of graded algebras as crossed products by twisted partial group actions in [19]. Section 5 is dedicated to product partial actions, the main result being Theorem 5.7, which says that the skew group algebra $A \rtimes_\alpha G$ by a product partial action α is graded-equivalent to the skew group algebra $B \rtimes_\beta G$, where β is a minimal globalization of α. Given a partially-strongly-graded algebra $B = \oplus_{t\in G}B_t$, the dual action β^B of G on $B\#G$ can be restricted to the ideal I^B of $B\#G$ defined in Section 2 (which is called *partial smash product* in this work), resulting in a product partial action γ^B, called the *canonical partial action* associated to B, such that B and the skew group algebra $I^B \rtimes_{\gamma^B} G$ are strongly-graded-equivalent. Theorem 6.5 states that the (global) skew group algebra $(B\#G) \rtimes_{\beta^B} G$ and the partial one $I^B \rtimes_{\gamma^B} G$ are graded-equivalent. We

begin Section 7 by adapting to product partial actions the Morita equivalence of partial actions considered earlier in [6]. One of the main results of the section is Theorem 7.10, which says that if $B = \oplus_{t \in G} B_t$ is a partially-strongly-graded algebra, then B is strongly-graded-equivalent to $I^B \rtimes_{\gamma^B} G$. As a consequence we obtain in Corollary 7.11 that if B is a strongly-graded algebra, then B is strongly-graded-equivalent to the (global) skew group algebra $(B\#G) \rtimes_{\beta^B} G$. Another consequence states that the crossed product by any twisted partial group action is strongly-graded-equivalent to the skew group algebra of a product partial action (see Corollary 7.12). We also prove in Theorem 7.15 that the canonical partial actions γ^A and γ^B associated to strongly-graded-equivalent partially-strongly-G-graded algebras A and B, are Morita equivalent. We also consider, at the end of Section 7, another notion of equivalence of product partial actions, weaker than Morita equivalence.

The question of whether a partial action is the restriction of a global action, that is, the question of *globalization*, is one of the most important topics in the theory of partial actions [22]. It was initially considered in [3] in the context of C*-algebras, and afterwards in a series of algebraic papers. In Section 8 we deal with globalization up to Morita equivalence. More specifically, we show in Theorem 8.8 that any product partial action α of a group G has a so-called Morita enveloping action β, i.e. β is a minimal globalization of a product partial action which is Morita equivalent to α. Furthermore, β is unique up to Morita equivalence, and the skew group algebras $A \rtimes_\alpha G$ and $B \rtimes_\beta G$ are graded-equivalent. In order to prove Theorem 8.8 we establish several facts, one of them being Theorem 8.3, saying that skew group algebras by product partial actions α and α' are strongly-equivalent if and only if α and α' are Morita equivalent. We then use our results on globalization to give different characterizations of Morita equivalence and weak equivalence of product partial actions. Finally, in Section 9 we employ the technique developed in [19] to prove in Theorem 9.1 that given strongly-graded-equivalent partially-strongly-G-graded algebras A and B with orthogonal local units, there exists a graded isomorphism of algebras $\mathrm{FMat}_{\mathcal{X}}(A) \cong \mathrm{FMat}_{\mathcal{X}}(B)$, where $\mathcal{X}$ is a sufficiently large cardinal.

In what follows G will stand for an arbitrary group and k for an arbitrary commutative associative unital ring, which will be the base ring for our algebras. The latter will be assumed to be associative and non-necessarily unital. Let A and B be algebras. A left module $_A M$ over A is said to be *unital* if $AM = M$. We shall say that an (A, B)-bimodule $_A M_B$ is *unital* if $AM = M = MB$. Given a right A-module M_A, a left A-module $_A N$ and subsets $M' \subseteq M, N' \subseteq N$, we denote by $M' \otimes_A N'$ the k-submodule of the tensor product $M \otimes_A N$ generated by all elements of the form $x \otimes y$ with $x \in M', y \in N'$.

2. Group graded algebras

Let G be a group, and let $B = \oplus_{t \in G} B_t$ be a G-graded algebra, possibly non-unital. We denote by 1 the unit element of G. If $b \in B$, we write b_t for the homogeneous component of b in B_t, so that $b = \sum_{t \in G} b_t$. Note also that b_t may also stand for an element in B_t not

necessarily related to some $b \in B$. This will be clear from the context and no confusion should arise. Consider the algebra $\mathrm{RCFMat}_G(B)$ of all row and column finite $G \times G$-matrices with coefficients in B, with the usual operations of addition and multiplication of matrices. An (r,s)-position of a matrix $d \in \mathrm{RCFMat}_G(B)$ will be denoted by $d(r,s)$. Observe that the algebra $\mathrm{RCFMat}_G(B)$ is unital if so is B. Let $\mathrm{FMat}_G(B)$ be the two-sided ideal of $\mathrm{RCFMat}_G(B)$ whose elements are the matrices with finitely many non-zero entries. If B is unital then $e_{r,s} \in \mathrm{RCFMat}_G(B)$ will denote the matrix unit and $be_{r,s}$, with $b \in B$, will stand for the product of the scalar matrix bI with $e_{r,s}$, where $I \in \mathrm{RCFMat}_G(B)$ is the identity matrix. Then clearly $be_{r,s}$ is the matrix having b at (r,s)-position and 0 at all other positions. By convention we shall denote this matrix by $be_{r,s}$ even if B is non-unital. Then $\mathrm{FMat}_G(B) = \mathrm{span}\{be_{r,s} : r, s \in G, b \in B\}$. We will be interested in the following subalgebra $B\#G$ of $\mathrm{FMat}_G(B)$:

$$B\#G := \mathrm{span}\{be_{r,s} \in \mathrm{FMat}_G(B) : r, s \in G, b \in B_{r^{-1}s}\}.$$

Thus $B\#G = \oplus_{r,s \in G} B_{r^{-1}s} e_{r,s}$. Note that if B is unital, then this algebra is nothing but the smash product in the sense of Beattie [12] (see also [32]), which in the case of a finite G agrees with the smash product in [14]. If $A = \oplus_{t \in G} A_t$ is another G-graded algebra, and $\phi : A \to B$ is a (graded) homomorphism of G-graded algebras, then ϕ induces a homomorphism $\phi^\# : A\#G \to B\#G$ such that $\phi^\#(a_{r^{-1}s}e_{r,s}) = \phi(a_{r^{-1}s})e_{r,s}$. It is easily shown that $B \mapsto B\#G$, $\phi \mapsto \phi^\#$ is a functor from the category of G-graded algebras into the category of algebras. Note that $\phi^\#$ is injective if and only if so is ϕ. Moreover, suppose that $B = \oplus_{t \in G} B_t$ is a G-graded algebra, and that $A = \oplus_{t \in G} A_t$ is a G-graded subalgebra of B. Then, if $\iota : A \to B$ is the natural inclusion, we have that $\iota^\# : A\#G \to B\#G$ is also the natural inclusion.

We may think of B as a subalgebra of $\mathrm{RCFMat}_G(B)$ via the map $\eta : b \mapsto \eta(b)$ such that $\eta(b)(r,s) = b_{r^{-1}s}$, $\forall r, s \in G$. If G is finite, then the map η has its range contained in $B\#G$.

There is a natural action β of G on $\mathrm{RCFMat}_G(B)$, such that $(t \cdot d)(r,s) = d(t^{-1}r, t^{-1}s)$, $\forall r, s, t \in G$, $d \in \mathrm{RCFMat}_G(B)$. Thus $t \cdot (be_{r,s}) = be_{tr,ts}$, $\forall t \in G$, $be_{r,s} \in \mathrm{RCFMat}_G(B)$. Clearly, the subalgebras $\mathrm{FMat}_G(B)$, $B\#G$, and $\eta(B)$ are invariant under β. We denote by β^B the *dual action*, namely restriction of β to $B\#G$. This action is natural with respect to the smash product functor: $\phi^\#\beta_t^A = \beta_t^B\phi^\#$, $\forall t \in G$. Note that each element of $\eta(B)$ is fixed by β and, moreover, if G is finite, then $\eta(B) \subseteq B\#G$ is precisely the subalgebra of fixed points of β^B.

We concentrate now our attention on the smash product $B\#G$ of the G-graded algebra B. The skew group algebra

$$(B\#G) \rtimes_{\beta^B} G = \oplus_{t \in G} (B\#G)\delta_t$$

possesses the usual G-grading, defined by declaring $(B\#G)\delta_t$ to be the t-homogeneous component of $(B\#G) \rtimes_{\beta^B} G$, where $t \in G$. In Theorem 2.1 below we shall consider a

G-grading on $\mathrm{FMat}_G(B)$, defined by setting $R_u = \mathrm{span}\{B_r e_{s,t} : srt^{-1} = u\}$ to be the u-homogeneous component of $\mathrm{FMat}_G(B)$, where $u, r, s, t \in G$. It is readily verified that this indeed defines a G-grading.

Our first remark about $B\#G$ is the following duality theorem, that generalizes [14, Theorem 3.5] and [12, Theorem 2.2] (see also [32, Theorem 1.3]):

Theorem 2.1. *Let B be any G-graded algebra. Then the skew group algebra $(B\#G) \rtimes_{\beta^B} G$ is naturally isomorphic, as a graded algebra, to $\mathrm{FMat}_G(B)$.*

Proof. Let C be a unital algebra that contains B as a two-sided ideal. Thus $\mathrm{RCFMat}_G(B)$ and $\mathrm{FMat}_G(B)$ are two-sided ideals of $\mathrm{RCFMat}_G(C)$. For $t \in G$, let $\Delta_t \in \mathrm{RCFMat}_G(C)$ be such that $\Delta_t(r, s) = [r = ts] \in C$, where the (square) brackets stand for the boolean value. Let $\psi_B : (B\#G) \rtimes_{\beta^B} G \to \mathrm{FMat}_G(B)$ be the $(B\#G)$-module map given by $\psi_B(c\delta_t) = c\Delta_t \in (B\#G)\mathrm{RCFMat}_G(C) \subseteq \mathrm{FMat}_G(B)$. We have

$$(e_{r,s}\Delta_t)(u, v) = \sum_{w \in G} e_{r,s}(u, w)\Delta_t(w, v) = [r = u][s = tv] = e_{r,t^{-1}s}(u, v).$$

Then $e_{r,s}\Delta_t = e_{r,t^{-1}s}$ (similarly one can show that $\Delta_t e_{r,s} = e_{tr,s}$, so $\beta_t(e_{r,s}) = \Delta_t e_{r,s}\Delta_{t^{-1}}$). Now if $c_1 = b_1 e_{r_1,s_1}$, $c_2 = b_2 e_{r_2,s_2}$, $b_i \in B_{r_i^{-1}s_i}$, $r_i, s_i, r, s \in G, i = 1, 2$, then

$$\psi_B(c_1\delta_r)\psi_B(c_2\delta_s) = b_1 e_{r_1,s_1}\Delta_r\, b_2 e_{r_2,s_2}\Delta_s$$
$$= b_1 b_2 e_{r_1,r^{-1}s_1} e_{r_2,s^{-1}s_2}$$
$$= [s_1 = rr_2]b_1 b_2 e_{r_1,s^{-1}s_2}.$$

On the other hand:

$$\psi_B((c_1\delta_r)(c_2\delta_s)) = \psi_B((b_1 e_{r_1,s_1}\delta_r)(b_2 e_{r_2,s_2}\delta_s))$$
$$= \psi_B(b_1 b_2 e_{r_1,s_1} e_{rr_2,rs_2}\delta_{rs})$$
$$b_1 b_2 e_{r_1,s_1} e_{rr_2,rs_2}\Delta_{rs}$$
$$b_1 b_2 e_{r_1,s_1} e_{rr_2,s^{-1}r^{-1}rs_2}$$
$$[s_1 = rr_2]b_1 b_2 e_{r_1,s^{-1}s_2}.$$

Hence ψ_B is a homomorphism of algebras. Moreover, it respects the gradings, since $\psi_B(B_{r^{-1}s}e_{r,s}\delta_t) = B_{r^{-1}s}e_{r,s}\Delta_t = B_{r^{-1}s}e_{r,t^{-1}s}$, whose degree is $r(r^{-1}s)(t^{-1}s)^{-1} = t$. Furthermore, ψ_B is injective, because $\{\Delta_t : t \in G\}$ is $(B\#G)$-linearly independent: if $\sum_{t\in G} c_t\Delta_t$ is a finite sum which is equal to zero, then $\sum_{t\in G} c_t\Delta_t(u, v) = 0$, $\forall u, v \in G$, that is, $\sum_{t\in G} c_t[u = tv] = 0$, $\forall u, v \in G$; fixing t and choosing $u = t$, $v = e$, we conclude $c_t = 0$. Let us compute now the range of ψ_B:

$$\psi_B((B\#G)\rtimes_{\beta^B} G) = \sum_{t\in G}(B\#G)\Delta_t = \sum_{t\in G}\oplus_{r,s\in G}B_{r^{-1}s}e_{r,s}\Delta_t$$

$$= \sum_{t\in G}\sum_{r,s\in G}B_{r^{-1}s}e_{r,t^{-1}s} = \oplus_{r,u\in G}\sum_{t\in G}B_{r^{-1}tu}e_{r,u}$$

$$= \oplus_{r,s\in G}\sum_{v\in G}B_v e_{r,s} = \oplus_{r,s}Be_{r,s} = \mathrm{FMat}_G(B).$$

This ends the proof of $(B\#G)\rtimes_{\beta^B} G \cong \mathrm{FMat}_G(B)$.

We next show the naturality of the isomorphism ψ_B. The definition of ψ_B does not depend on the choice of the unital algebra C containing B, so one may adjoin a unity element to B by one of the most common ways: $C = K\times B$, $(k,b)(k',b') := (kk', kb' + k'b + bb')$. Let $\phi: A\to B$ be a morphism of G-graded algebras, and $C' = K\times B$ the unital algebra obtained same way as C. Then ϕ obviously extends to a morphism of unital algebras $C'\to C$. The homomorphism $\phi^\#: A\#G\to B\#G$ between the smash products induces a G-graded homomorphism $\tilde\phi: (A\#G)\rtimes_{\beta^A} G \to (B\#G)\rtimes_{\beta^B} G$, determined by

$$\tilde\phi(c^A\delta_t) = \phi^\#(c^A)\delta_t = \sum_{r,s\in G}\phi^\#(c^A_{r^{-1}s}e_{r,s})\delta_t = \sum_{r,s\in G}\phi(c^A_{r^{-1}s})e_{r,s}\delta_t.$$

The map $\phi\mapsto\tilde\phi$ is the morphism level of the functor $A\mapsto A\#G\rtimes_{\beta^A} G$. Similarly, $\phi^{\mathrm{fin}}: \mathrm{FMat}_G(A)\to\mathrm{FMat}_G(B)$ given by $\phi^{\mathrm{fin}}(d)(r,s) = \phi(d(r,s))$, is the morphism level of the functor $A\mapsto\mathrm{FMat}_G(A)$. Clearly, $\phi^{\mathrm{fin}}(d)$ is also a G-graded map. A direct computation shows that the diagram below commutes:

$$
\begin{array}{ccc}
(A\#G)\rtimes_{\beta^A} G & \xrightarrow[\cong]{\psi_A} & \mathrm{FMat}_G(A)\\[4pt]
\tilde\phi\downarrow & & \downarrow\phi^{\mathrm{fin}}\\[4pt]
(B\#G)\rtimes_{\beta^B} G & \xrightarrow[\psi_B]{\cong} & \mathrm{FMat}_G(B)
\end{array}
$$

Now the proof is complete. $\quad\square$

2.1. Multipliers

We recall that the multiplier algebra $\mathfrak{M}(A)$ of an algebra A is the set

$$\mathfrak{M}(A) = \{(L,R)\in\mathrm{End}(A_A)\times\mathrm{End}(_A A): R(a)b = aL(b)\text{ for all } a,b\in A\}$$

with component-wise addition, and multiplication given by

$$(L,R)(L',R') := (LL', R'R), \quad \forall(L,R), (L',R')\in\mathfrak{M}(A).$$

See, for example, [17] or [23] for details. For a multiplier $w = (L, R) \in \mathfrak{M}(A)$ and $a \in A$ we set $aw = R(a)$ and $wa = L(a)$, so that one always has $(aw)b = a(wb)$ $(a, b \in A)$. The first (resp. second) components of the elements of $\mathfrak{M}(A)$ are called left (resp. right) multipliers of A.

Consider a graded algebra $B = \oplus_{t \in G} B_t$ over the group G. We denote by $\mu : B \to \mathfrak{M}(B)$ the natural map, that is, $\mu(b) = (L_b, R_b)$, where L_b and R_b are respectively the maps of left and right multiplication by b.

A multiplier $w = (L, R)$ of B is said to have degree $t \in G$ if $wB_s \subseteq B_{ts}$ and $B_s w \subseteq B_{st}$, $\forall s \in G$. For instance, the multiplier $\mu(b_t)$ defined by $b_t \in B_t$, is a multiplier of B of degree t.

Let $\mathfrak{M}_t(B) := \{w \in \mathfrak{M}(B) : w$ is of degree $t\}$. It is not hard to see that $\mathfrak{M}_s(B)\mathfrak{M}_t(B) \subseteq \mathfrak{M}_{st}(B)$, $\forall s, t \in G$, from which it easily follows that $\mathfrak{M}_1(B)$ is a unital algebra and each $\mathfrak{M}_t(B)$ is a bimodule over $\mathfrak{M}_1(B)$. On the other hand, since the family $\{B_t\}_{t \in G}$ is linearly independent, it follows that the family $\{\mathfrak{M}_t(B)\}_{t \in G}$ is linearly independent as well. Thus we get a graded algebra $\mathfrak{M}_\mathfrak{g}(B) = \oplus_{t \in G} \mathfrak{M}_t(B)$, which will be called the *graded multiplier algebra* of B. Note that the natural map $\mu : B \to \mathfrak{M}_\mathfrak{g}(B)$ is now a homomorphism of graded algebras, and $\mu(B)$ is a graded ideal in $\mathfrak{M}_\mathfrak{g}(B)$.

3. Graded-equivalence

Suppose A is an associative idempotent algebra. Consider, in the category of all right A-modules, the full subcategory mod-A of the unital and torsion-free modules. Thus a right module M over A is in mod-A if and only if $MA = M$ and $mA = 0$ implies $m = 0$. This is a Grothendieck category, that is, an abelian category which is cocomplete and such that direct limits are exact and there exist generators. In [25] the authors characterized the equivalence of the categories mod-A and mod-B for idempotent algebras A and B in terms of Morita-type theorems: they proved that these categories are equivalent if and only if there exists a Morita context $(A, B, {}_AX_B, {}_BY_A, \tau_A, \tau_B)$, where the modules ${}_AX$, X_B, ${}_BY$, Y_A are unital and the bimodule maps $\tau_A : {}_AX \otimes_B Y_A \to A$ and $\tau_B : {}_BY \otimes_A X_B \to B$ are surjective (Proposition 2.6 and Theorem 2.7 of [25]).

Since we are working with graded algebras, we are interested in graded Morita contexts. By a graded Morita context between two idempotent G-graded algebras $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ we mean a Morita context $(A, B, {}_AX_B, {}_BY_A, \tau_A, \tau_B)$, where $X = \oplus_{t \in G} X_t$ and $Y = \oplus_{t \in G} Y_t$ are G-graded bimodules, and $\tau_A(X_r \otimes_B Y_s) \subseteq A_{rs}$ and $\tau_B(Y_r \otimes_A X_s) \subseteq B_{rs}$, $\forall r, s, t \in G$. Notice that this extends the concept of a graded Morita context given for the case of unital rings in [13]. We say that a bimoldule ${}_AX_B$ is unital if both ${}_AX$, X_B are unital modules. Equivalently, $AXB = X$.

Definition 3.1. Let $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ be two idempotent G-graded algebras. We say that they are *graded-equivalent* if there exists a graded Morita context $(A, B, X, Y, \tau_A, \tau_B)$ with unital bimodules ${}_AX_B$, ${}_BY_A$ for which τ_A and τ_B are surjective.

It follows from [28, Theorem 2.6] that for graded rings with graded local units the concept of a graded equivalence in Definition 3.1 is tantamount to that considered earlier.

In general we will work with Morita contexts that are contained in a graded algebra, that is, A, B, X and Y will be contained (as graded objects) in a certain graded algebra C, and all the algebraic operations of the context will be inherited from the algebra structure of C (Proposition 3.4 below shows that we do not lose generality in doing so). In particular τ_A and τ_B will be determined by the product of C, and we will omit to mention them. We will refer to $\mathsf{M} := (A, B, X, Y)$ as a Morita context in C.

Proposition 3.2. *Let $B = \oplus_{t \in G} B_t$ be a graded algebra. Then B is idempotent if and only if $B_r = \sum_{s \in G} B_s B_{s^{-1}r}$, $\forall r \in G$.*

Proof. Just note that

$$B^2 = \sum_{s,t \in G} B_s B_t = \sum_{r \in G} \sum_{st=r} B_s B_t = \bigoplus_{r \in G} \sum_{s \in G} B_s B_{s^{-1}r}. \quad \square$$

Theorem 3.3. *If $B = \oplus_{t \in G} B_t$ is an idempotent G-graded algebra, then B and $(B\#G) \rtimes_{\beta^B} G$ are graded-equivalent.*

Proof. Given $r, t \in G$, consider the following subsets of $B\#G$: $X_t(r) = B_{r^{-1}t} e_{r,t}$, $X_t = \oplus_{r \in G} X_t(r)$, $Y_t(r) = B_r e_{1,r}$, and $Y_t = \oplus_{r \in G} Y_t(r)$ (so $Y_t(r)$ and Y_t do not really depend on t). Define now the following subsets of $(B\#G) \rtimes_{\beta^B} G$: $X := \oplus_{t \in G} X_t \delta_t$, $Y = \oplus_{t \in G} Y_t \delta_t$, and $B' := \oplus_{t \in G} B_t e_{1,t} \delta_t$. Note that the map $B' \to B$ given by $b_t e_{1,t} \delta_t \mapsto b_t$ is an isomorphism of graded algebras, because

$$(b_s e_{1,s} \delta_s)(b_t e_{1,t} \delta_t) = b_s e_{1,s} \beta_s(b_t e_{1,t}) \delta_{st} = b_s b_t e_{1,st} \delta_{st} \mapsto b_s b_t.$$

So it is enough to prove that $\mathsf{M} = ((B\#G) \rtimes_{\beta^B} G, B', X, Y)$ is a graded-equivalence, which implies that $(B\#G) \rtimes_{\beta^B} G$ and B are graded-equivalent. Since X, Y, and B' are graded according to the grading of the crossed product $(B\#G) \rtimes_{\beta^B} G$, to see that M is a graded-equivalence is enough to show that it is a Morita equivalence.

We show first that $XY = (B\#G) \rtimes_{\beta^B} G$. For all u, r, s, t in G we have:

$$(X_u(r)\delta_u)(Y_{u^{-1}t}(s)\delta_{u^{-1}t}) = X_u(r)\beta_u(Y_{u^{-1}t}(s))\delta_t = (B_{r^{-1}u} e_{r,u})(B_s e_{u,us})\delta_t$$

$$= B_{r^{-1}u} B_s e_{r,us}\delta_t = B_{r^{-1}u} B_{u^{-1}s'} e_{r,s'}\delta_t \subseteq (B\#G)\delta_t,$$

where $s' = us$. Therefore:

$$(XY)_t = \sum_{u \in G} (X_u \delta_u)(Y_{u^{-1}t}\delta_{u^{-1}t}) = \sum_{u,r,s} (X_u(r)\delta_u)(Y_{u^{-1}t}(s)\delta_{u^{-1}t}).$$

Hence

$$XY = \oplus_{t \in G}(XY)_t \delta_t = \oplus_{t \in G} \sum_{u \in G} \oplus_{r,s \in G} B_{r^{-1}u} B_{u^{-1}s} e_{r,s} \delta_t$$

$$= \oplus_{t \in G} \oplus_{r,s \in G} \sum_{u \in G} B_{r^{-1}u} B_{u^{-1}s} e_{r,s} \delta_t = \oplus_{t \in G}(B\#G)\delta_t = (B\#G) \rtimes_\beta G,$$

in view of Proposition 3.2. We now show that $((B\#G) \rtimes_{\beta^B} G)X = X$. Given $u, v, r, s, t \in G$:

$$(B_{r^{-1}s} e_{r,s} \delta_u)(X_{u^{-1}t}(v)\delta_{u^{-1}t}) = (B_{r^{-1}s} e_{r,s})\beta_u(B_{v^{-1}u^{-1}t} e_{v,u^{-1}t})\delta_t$$

$$= (B_{r^{-1}s} B_{v^{-1}u^{-1}t} e_{r,s} e_{uv,t})\delta_t = [s = uv]B_{r^{-1}s} B_{s^{-1}t} e_{r,t}\delta_t.$$

Thus, using again Proposition 3.2, we see that

$$((B\#G) \rtimes_{\beta^B} G)X = \oplus_{t \in G}\Big(\sum_u \sum_{r,s,v} [s = uv]B_{r^{-1}s} B_{s^{-1}t} e_{r,t}\Big)\delta_t$$

$$= \oplus_{t \in G}\Big(\sum_r B_{r^{-1}t} e_{r,t}\Big)\delta_t = \oplus_t X_t \delta_t = X.$$

Let us show that $YX = B'$. Let $u, r, t, s \in G$. Then:

$$(Y_u(r)\delta_u)(X_{u^{-1}t}(s)\delta_{u^{-1}t}) = (B_r e_{1,r})\beta_u(B_{s^{-1}u^{-1}t} e_{s,u^{-1}t})\delta_t$$

$$= (B_r e_{1,r})(B_{s^{-1}u^{-1}t} e_{us,t})\delta_t = [r = us]B_r B_{r^{-1}t} e_{1,t}\delta_t.$$

Hence $YX = \oplus_t \big(\sum_u (\sum_{r,s}[r = us]B_r B_{r^{-1}t} e_{1,t})\big)\delta_t = \oplus_t B_t e_{1,t}\delta_t = B'$.
 Now it is easy to check that $XB' = X$:

$$XB' = X(YX) = (XY)X = ((B\#G) \rtimes_\beta G)X = X.$$

For the equality $B'Y = Y$ notice first that

$$(B_u e_{1,u}\delta_u)(Y_{u^{-1}t}(r)\delta_{u^{-1}t}) = (B_u e_{1,u}\delta_u)(B_r e_{1,r}\delta_{u^{-1}t})$$

$$= (B_u e_{1,u})\beta_u(B_r e_{1,r}\delta_t) = B_u e_{1,u} B_r e_{u,ur}\delta_t = B_u B_r e_{1,ur}\delta_t.$$

Then $B'Y = \oplus_t(\sum_u \sum_r B_u B_r e_{1,ur})\delta_t = \oplus_t(\sum_{s,u} B_u B_{u^{-1}s} e_{1,s})\delta_t$, where $s = ur$. With one more use of Proposition 3.2 this shows that $B'Y = Y$. Finally,

$$Y((B\#G) \rtimes_{\beta^B} G) = Y(XY) = (YX)Y = B'Y = Y,$$

which ends the proof. $\square$

If C is an algebra and $e \in \mathfrak{M}(C)$ is an idempotent element, then $C' := eCe$ is clearly an algebra. Suppose in addition that $C = \oplus_{t \in G} C_t$ is graded over G, and e is a multiplier of C of degree 1. Define $C'_t := eC_t e$. Then $C'_s C'_t = eC_s eC_t e \subseteq eC_s C_t e \subseteq eC_{st} e = C'_{st}$, so it follows that C' is also a graded algebra, because $C' = \oplus_{t \in G} C'_t$.

Proposition 3.4. *Let $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ be idempotent algebras graded over the group G. Then the following statements are equivalent:*

(1) *The algebras A and B are graded-equivalent.*
(2) *There exist an idempotent graded algebra $C = \oplus_{t \in G} C_t$ and $e = e^2 \in \mathfrak{M}_1(C)$ (the algebra of multipliers of degree 1 of C), such that: $A \cong eCe$ and $B \cong (1-e)C(1-e)$ as graded algebras, and $CeC = C = C(1-e)C$.*

Proof. Suppose first that $(A, B, X, Y, \tau_A, \tau_B)$ is a graded Morita context between A and B. Let $\mathbb{L} := \{\begin{pmatrix} a & x \\ y & b \end{pmatrix} : a \in A, b \in B, x \in X, y \in Y\}$ with entry-wise addition, and the product given by the Morita context, as follows:

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + \tau_A(x \otimes y) & ax' + xb' \\ ya' + by' & \tau_B(y \otimes x') + bb' \end{pmatrix}.$$

These operations give an associative algebra structure on $\mathbb{L}$, as it is easy to check. Moreover, since the algebras A and B are idempotent, and the modules of the Morita context are unital, it follows that $\mathbb{L}$ is also an idempotent algebra. If

$$\mathbb{L}_t := \{\begin{pmatrix} a_t & x_t \\ y_t & b_t \end{pmatrix} : a_t \in A_t, b_t \in B_t, x_t \in X_t, y_t \in Y_t\},$$

then the fact that the Morita context is graded implies $\mathbb{L}_s \mathbb{L}_t \subseteq \mathbb{L}_{st}$, $\forall s, t \in G$. On the other hand it is clear that $\mathbb{L} = \oplus_{t \in G} \mathbb{L}_t$, so $\mathbb{L}$ is a graded algebra over G.

Now let $L, R : \mathbb{L} \to \mathbb{L}$ be the maps given by $L(\begin{pmatrix} a & x \\ y & b \end{pmatrix}) = \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix}$ and $R(\begin{pmatrix} a & x \\ y & b \end{pmatrix}) = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}$. Then $L(\mathbb{L}_t) \subseteq \mathbb{L}_t$, $R(\mathbb{L}_t) \subseteq \mathbb{L}_t$, and routine matrix computations show that $L(cc') = L(c)c'$, $R(cc') = cR(c')$, and $cL(c') = R(c)c'$, $\forall c, c' \in \mathbb{L}$. That is, $e := (L, R)$ is a multiplier of $\mathbb{L}$, of degree 1. It is clear that $e^2 = e$. Note that we may conveniently think of e as the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ acting in the obvious way on $\mathbb{L}$: L is multiplication on the left by this matrix, while R corresponds to multiplication on the right. Now, $e\mathbb{L}_t e = R(L(\mathbb{L}_t)) = \begin{pmatrix} A_t & 0 \\ 0 & 0 \end{pmatrix}$, and $(1-e)\mathbb{L}_t(1-e) = (1-R)((1-L)(\mathbb{L}_t)) = \begin{pmatrix} 0 & 0 \\ 0 & B_t \end{pmatrix}$, which are obviously isomorphic to A_t and B_t respectively. On the other hand:

$$\mathbb{L}e\mathbb{L} = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}\begin{pmatrix} A & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & AX \\ YA & \tau_B(_BY \otimes_A X_B) \end{pmatrix} = \mathbb{L},$$

where the latter equality is due to the facts that A is idempotent, $_AX$ and Y_A are unital, and τ_B is surjective. In a similar way we conclude that $\mathbb{L}(1-e)\mathbb{L} = \mathbb{L}$.

Conversely, suppose $C = C^2$ and $e \in \mathfrak{M}_1(C)$ are such that $A' := eCe \cong A$ and $B' := (1-e)C(1-e) \cong B$ as graded algebras, and $CeC = C = C(1-e)C$. Let

$X := eC(1-e)$, $X_t := eC_t(1-e)$, $Y := (1-e)Ce$, and $Y_t := (1-e)C_t e$. It is clear that X is an (A', B')-bimodule. Moreover:

$$A'_s X_t = (eC_s e)(eC_t(1-e)) \subseteq eC_s C_t(1-e) \subseteq X_{st},$$
$$A'X = (eCe)(eC(1-e)) = e(Ce^2 C)(1-e) = eC(1-e) = X.$$

Similarly we have $X_s B'_t \subseteq X_{st}$ and $X B' = X$, so $_A X$ and X_B are unital and graded modules. In the same way we conclude that $_B Y$ and Y_A are unital and graded modules. Since

$$XY = (eC(1-e))((1-e)Ce) = e(C(1-e)C)e = eCe = A',$$

and $YX = B'$ after a similar computation, we have that the natural maps $\tau_{A'} : X \otimes_{B'} Y \to A'$ and $\tau_{B'} : Y \otimes_{A'} X \to B'$ associated to the multiplication on C are both surjective, and it is easily seen that $\tau_{A'}(X_s \otimes_{B'} Y_t) \subseteq A'_{st}$ and $\tau_{B'}(Y_s \otimes_{A'} X_t) \subseteq B'_{st}$, $\forall s, t \in G$. Then $(A', B', X, Y, \tau_{A'}, \tau_{B'})$ is a graded-equivalence between A' and B'. Since $A \cong A'$ and $B \cong B'$ as graded algebras, we are done. $\quad\square$

Definition 3.5. Let $\mathsf{M} = (A, B, X, Y, \tau_A, \tau_B)$ be a graded Morita context giving a graded-equivalence between the idempotent graded algebras A and B. The graded algebra $\mathbb{L}$ constructed out of this Morita context as in the first part of the proof of Proposition 3.4 will be called the *graded Morita algebra* (or the *graded linking algebra*) of M. We will write $\mathbb{L}(\mathsf{M})$ if we need to stress the dependence of $\mathbb{L}$ on the Morita context M.

We end the section by showing that graded-equivalence is an equivalence relation:

Proposition 3.6. *Graded-equivalence is an equivalence relation for graded idempotent algebras.*

Proof. Suppose $\mathsf{M} = (A, A', X, Y, \tau_A, \tau_{A'})$ and $\mathsf{N} = (A', B, X', Y', \tau'_{A'}, \tau_B)$ are graded Morita contexts giving graded-equivalences between A and A' and between A' and B respectively. As in [30, p. 30] we consider $_A\bar{X}_B = X \otimes_{A'} X'$ as a G-graded bimodule, whose t-homogeneous component $\bar{X}_t$, $(t \in G)$ is the K-submodule of $\bar{X}$ generated by all elements $x \otimes x'$, $x \in X_r, x' \in X'_s$, such that $rs = t$. Thus, for $t \in G$, $\bar{X}_t = \sum_{s \in G} X_s \otimes_{A'} X'_{s^{-1}t}$. Similarly, $_B\bar{Y}_A = Y' \otimes_{A'} Y$ is a G-graded bimodule, with $\bar{Y}_t = \sum_{s \in G} Y'_s \otimes_{A'} Y_{s^{-1}t}$. Obviously, $_A\bar{X}_B$ and $_B\bar{Y}_A$ are unital bimodules. Denote by ρ_A the following composition of surjective (A, A)-bimodule maps:

$$\bar{X} \otimes_B \bar{Y} = (X \otimes_{A'} X') \otimes_B (Y' \otimes_{A'} Y)$$

$$\cong X \otimes_{A'} (X' \otimes_B Y') \otimes_{A'} Y \overset{id \otimes \tau'_{A'} \otimes id}{\longrightarrow} X \otimes_{A'} A' \otimes_{A'} Y$$

$$\longrightarrow X \otimes_{A'} Y \overset{\tau_A}{\longrightarrow} A.$$

Note that $\rho_A(\bar{X}_s \otimes_B \bar{Y}_t) \subseteq A_{st}$, $\forall s, t \in G$. In fact, if $u, v \in G$ and $x \in X_u$, $x' \in X'_{u^{-1}s}$, $y' \in Y'_v$, $y \in Y_{v^{-1}t}$, we have:

$$\rho_A((x \otimes x') \otimes (y' \otimes y)) = \tau_A((x \otimes \tau'_{A'}(x' \otimes y')y) \in \tau_A(X_u \otimes Y_{u^{-1}st}) \subseteq A_{st}.$$

In the same way we construct a surjective (B, B)-bimodule map

$$\rho_B : \bar{Y} \otimes_A \bar{X} \to B,$$

such that $\rho_B(\bar{Y}_s \otimes_A \bar{X}_t) \subseteq B_{st}$, $\forall s, t \in G$. It then follows that $\mathsf{MN} := (A, B, \bar{X}, \bar{Y}, \rho_A, \rho_B)$ is a graded-equivalence between A and B. $\square$

Remark 3.7. Note that combining Proposition 3.6 with Theorem 2.1 and Theorem 3.3, we obtain that any G-graded idempotent algebra B is graded-equivalent to $\mathrm{FMat}_G(B)$.

4. Strong-graded-equivalence

If $A = \oplus_{t \in G} A_t$ is a G-graded algebra, then for each $t \in G$ the set $D_t^A := A_t A_{t^{-1}}$ is a two-sided ideal of A_1.

Suppose that $(A, B, X, Y, \tau_A, \tau_B)$ is a graded Morita context between idempotent G-graded algebras A and B. Since $D_{t^{-1}}^B$ is a subalgebra of B, we have a natural map $\mu_A^t : X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}} \to X \otimes_B Y$ (observe that whenever X_t or $Y_{t^{-1}}$ are unital $D_{t^{-1}}^B$-modules, then $Y_t \otimes_{D_{t^{-1}}^B} X_{t^{-1}} = Y_t \otimes_{B_1} X_{t^{-1}}$). Thus composing this map with τ_A we obtain an A_1-bimodule map $\tau_A^t := \tau_A \mu_A^t : X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}} \to A_1$. Similarly, we have a natural map $\mu_B^t : Y_t \otimes_{D_{t^{-1}}^A} X_{t^{-1}} \to Y \otimes_A X$, and also a B_1-bimodule map $\tau_B^t := \tau_B \mu_B^t : Y_t \otimes_{D_{t^{-1}}^A} X_{t^{-1}} \to B_1$.

Suppose that D_t^A and D_t^B are idempotent, $\forall t \in G$. We will say that the Morita context above is *strong*, if each X_t is a unital $(D_t^A, D_{t^{-1}}^B)$-bimodule and each Y_t is a unital $(D_t^B, D_{t^{-1}}^A)$-bimodule. Under this condition observe that, for each $t \in G$, the ranges of τ_A^t and τ_B^t are contained in D_t^A and D_t^B respectively. Indeed,

$$\tau_A^t(X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}}) = \tau_A^t(D_t^A X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}}) =$$
$$= D_t^A \tau_A^t(X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}}) \subseteq D_t^A A_1 \subseteq D_t^A,$$

and similarly for the range of τ_B^t.

The following definition is an algebraic adaptation of [4, Definition 2.6], which in turn is a generalization of the notion of Morita-Rieffel equivalence of partial actions introduced in [3]. As we shall see later in Theorem 8.3, Definition 4.1 is a generalization of Morita equivalence of *product partial actions*, which are defined in Section 5.

Definition 4.1. Let $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ be two idempotent graded algebras such that D_t^A and D_t^B are idempotent, $\forall t \in G$. We say that A and B are *strongly-graded-equivalent* if there exists a strong-graded Morita context $(A, B, X, Y, \tau_A, \tau_B)$ with

surjective τ_A and τ_B, such that $\tau_A^t(X_t \otimes_{D_{t-1}^B} Y_{t^{-1}}) = D_t^A$ and $\tau_B^t(Y_t \otimes_{D_{t-1}^A} X_{t^{-1}}) = D_t^B$, $\forall t \in G$.

Evidently, if $(A, B, X, Y, \tau_A, \tau_B)$ is a strong-graded Morita context, then the bimodules ${}_A X_B$, ${}_B Y_A$ are unital. Hence, strongly-graded-equivalent algebras are graded-equivalent. Moreover, notice that if D_t^A is idempotent then D_t^A is a unital (A_1, A_1)-bimodule, since, for example, $D_t^A = (D_t^A)^2 \subseteq D_t^A A_1 \subseteq D_t^A$.

Proposition 4.2. *Let* $\mathsf{M} = (A, B, X, Y, \tau_A, \tau_B)$ *be a strong-graded Morita context. Then:*

(1) $A_r X_1 = X_r = X_1 B_r$ *and* $B_r Y_1 = Y_r = Y_1 A_r$, $\forall r \in G$.
(2) $D_r^A X_1 = X_1 D_r^B$ *and* $D_r^B Y_1 = Y_1 D_r^A$, $\forall r \in G$.
(3) $\tau_A(X_r \otimes_B Y_s) = A_r \tau_A(X_1 \otimes_B Y_1) A_s \subseteq A_r A_s$ *and* $\tau_B(Y_r \otimes_A X_s) = B_r \tau_B(Y_1 \otimes_A X_1) B_s \subseteq B_r B_s$, $\forall r, s \in G$.
(4) *If the context* M *is a strong-graded-equivalence and each* A_t *is a unital* A_1-*bimodule and each* B_t *is a unital* B_1-*bimodule, then* $\tau_A(X_r \otimes_B Y_s) = A_r A_s$ *and* $\tau_B(Y_r \otimes_A X_s) = B_r B_s$.

Proof. Since X_r is a unital left D_r^A-module, we have

$$A_r X_1 \subseteq X_r = D_r^A X_r = A_r A_{r^{-1}} X_r \subseteq A_r X_1,$$

so $A_r X_1 = X_r$. The other identities in (1) are proved similarly, and (2) follows at once from (1). Now, since τ_A is an A-bimodule map:

$$\tau_A(X_r \otimes Y_s) = \tau_A(A_r X_1 \otimes Y_1 A_s) = A_r \tau_A(X_1 \otimes Y_1) A_s \subseteq A_r A_1 A_s \subseteq A_r A_s.$$

An analogous argument proves the second claim of (3). Finally, (4) follows from (3), because the assumption implies $\tau_A(X_1 \otimes_B Y_1) = A_1$ and $\tau_B(Y_1 \otimes_A X_1) = B_1$. $\quad\square$

Note that, if in Proposition 4.2 the context $(A_1, B_1, X_1, Y_1, \tau_A^1, \tau_B^1)$ is a Morita equivalence, and each A_t is a unital A_1-bimodule and each B_t is a unital B_1-bimodule, then both of the inclusions in (3) above are actually equalities. So we have:

Corollary 4.3. *Let* $A = \oplus_{t \in G} A_t$ *and* $B = \oplus_{t \in G} B_t$ *be idempotent* G-*graded algebras such that each* A_t *is a unital* A_1-*bimodule and each* B_t *is a unital* B_1-*bimodule. Then a strongly-graded Morita context* $(A, B, X, Y, \tau_A, \tau_B)$ *is a strong-graded-equivalence between* A *and* B *if and only if the Morita context* $(A_1, B_1, X_1, Y_1, \tau_A^1, \tau_B^1)$ *is a Morita equivalence.*

Proof. The 'only if' part is evident and for the 'if' part it remains to show that if $(A_1, B_1, X_1, Y_1, \tau_A^1, \tau_B^1)$ is a Morita equivalence then τ_A and τ_B are surjective. Indeed, using (1) of Proposition 4.2 we see, for instance, that τ_A is surjective as follows:

$$\tau_A(X \otimes_B Y) = \sum_{r,s \in G} \tau_A(X_r \otimes_B Y_s) = \sum_{r,s \in G} A_r \tau_A^1(X_1 \otimes_{B_1} Y_1) A_s$$

$$= \sum_{r,s \in G} (A_r A_1)(A_1 A_s) = A^2 = A,$$

thanks to the unital condition on each A_s. Analogously, we show that $\tau_A^t(X_t \otimes_{D_{t^{-1}}^B} Y_{t^{-1}}) = D_t^A$ and $\tau_B^t(Y_t \otimes_{D_{t^{-1}}^A} X_{t^{-1}}) = D_t^B$, $\forall t \in G$. $\quad\square$

Corollary 4.4. *If A and B are strongly-graded-equivalent, then the algebras $D_t^A, D_{t^{-1}}^A, D_t^B$ and $D_{t^{-1}}^B$ are Morita equivalent to each other.*

Proof. Note that if $(A, B, X, Y, \tau_A, \tau_B)$ gives a strong-graded-equivalence between the graded algebras A and B, then by definition the context $(D_t^A, D_{t^{-1}}^B, X_t, Y_{t^{-1}}, \tau_A^t, \tau_B^{t^{-1}})$ is a Morita equivalence, $\forall t \in G$. Since Morita equivalence of idempotent algebras is transitive, we only need to show that D_t^A and D_t^B are Morita equivalent, $\forall t \in G$. But Proposition 4.2 shows that if $(A, B, X, Y, \tau_A, \tau_B)$ is a strong-graded-equivalence between A and B, then $D_t^A X_1$ is a unital (D_t^A, D_t^B)-bimodule and $D_t^B Y_1$ is a unital (D_t^B, D_t^A)-bimodule. Moreover:

$$\tau_A(D_t^A X_1 \otimes_B D_t^B Y_1) = \tau_A(D_t^A X_1 D_t^B \otimes_B Y_1)$$

$$= D_t^A \tau_A(X_1 \otimes_B Y_1) = D_t^A.$$

Similar computations show that also $\tau_B(D_t^B Y_1 \otimes_A D_t^A X_1) = D_t^B$. Thus $(D_t^A, D_t^B, D_t^A X_1, D_t^B Y_1, \tilde{\tau}_A, \tilde{\tau}_B)$ defines a Morita equivalence between D_t^A and D_t^B, where $\tilde{\tau}_A$ is the composition of the natural map $D_t^A X_1 \otimes_{D_t^B} D_t^B Y_1 \to X \otimes_B Y$ with τ_A, and $\tilde{\tau}_B$ is defined symmetrically. $\quad\square$

Observe that every graded algebra $A = \oplus_{t \in G} A_t$ defines a trivial Morita context, namely, the context (A, A, A, A), where the range maps are given by the product of A. Note that this context is a strong-graded-equivalence between A and itself if and only if A satisfies the following property:

$$A_r = A_r A_{r^{-1}} A_r, \quad \forall r \in G. \tag{1}$$

In particular, (1) implies that A is an idempotent algebra, as $A^2 \supseteq A_t A_{t^{-1}} A_t = A_t$ for each $t \in G$. Moreover, it also follows from (1) that the algebra A_1 is idempotent, as well as each ideal D_t^A of A_1, $(t \in G)$. In addition, $A_1 A_t \supseteq A_t A_{t^{-1}} A_t = A_t$ and consequently, $A_1 A_t = A_t$, $t \in G$. Similarly, $A_t A_1 = A_t$, so that each A_t is a unital A_1-bimodule. As a consequence, we see that each D_t^A is also a unital A_1-bimodule.

Observe that if A satisfies (1) then A satisfies each of the following two conditions

$$A_{r^{-1}} A_r A_s = A_{r^{-1}} A_{rs} \quad \text{and} \quad A_s A_r A_{r^{-1}} = A_{sr} A_{r^{-1}}, \quad \forall r, s \in G. \tag{2}$$

Indeed, if A satisfies (1), then:

$$A_{r^{-1}}A_{rs} = A_{r^{-1}}A_r A_{r^{-1}}A_{rs} \subseteq A_{r^{-1}}A_r A_{r^{-1}rs} = A_{r^{-1}}A_r A_s \subseteq A_{r^{-1}}A_{rs}.$$

The second equality is proved similarly. Observe that if each A_t is a unital left (or right) A_1-module, then (2) implies (1).

Notice that equalities of the form (2) appear in the definition of the concept of a partial representation of a group (see for example [18, Definition 2.1]). As an effect, if A satisfies (1) then, for all $t, s \in G$ we have:

$$A_t D_s^A = D_{ts}^A A_t, \qquad D_t^A D_s^A = D_s^A D_t^A, \quad \forall t, s \in G, \tag{3}$$

which are the analogues of useful properties of partial representations (see (2) and (3) in [18]). One can readily adapt the easy computations in [18], and for an illustration we prove the first equality, from which the second one follows easily:

$$A_t D_s^A = A_t A_s A_{s^{-1}} = A_{ts} A_{s^{-1}} = A_{ts} A_{s^{-1}t^{-1}} A_{ts} A_{s^{-1}} =$$
$$A_{ts} A_{s^{-1}t^{-1}} A_t = D_{ts}^A A_t.$$

Note also that (1) is one of the key properties which characterize graded algebras as crossed products by twisted partial actions (see [19, Theorem 6.1]). Recall that $A = \oplus_{t \in G} A_t$ is called strongly-graded if $A_r A_t = A_{rt}$ for all $r, s, \in G$. Obviously, each strongly-graded algebra satisfies (1), so we give the next:

Definition 4.5. A graded algebra $A = \oplus_{t \in G} A_t$ over the group G satisfying (1) is said to be *partially-strongly-graded*.

For instance, the so called epsilon-strongly-graded algebras considered in [31] are partially-strongly-graded.

A graded algebra A is strongly-graded precisely when $D_t^A = A_1$ and $A_1 A_t = A_t$ $\forall t \in G$, that is, whenever each $(A_t, A_{t^{-1}})$ defines a Morita autoequivalence of A_1. Similarly, A is partially-strongly-graded exactly when each $(D_t^A, D_{t^{-1}}^A, A_t, A_{t^{-1}})$ is a Morita equivalence between D_t^A and $D_{t^{-1}}^A$ (we omitted the trace maps, which are given by the product of A).

The next result extends formulæ (2) and (3) above for graded modules over partially-strongly-graded algebras:

Proposition 4.6. *Let $B = \oplus_{t \in G} B_t$ be a partially-strongly-graded algebra, and $X = \oplus_{t \in G} X_t$ a graded module over B. Then:*

(1) *X is partially-strongly-graded, in the sense that for all $r, s \in G$ we have $X_r B_s B_{s^{-1}} = X_{rs} B_{s^{-1}}$, if X is a right B-module, and $B_{r^{-1}} B_r X_s = B_{r^{-1}} X_{rs}$ if X is a left B-module*

(2) *If X is a right module, then X_r is a unital right $D^B_{r^{-1}}$-module if and only if $X_r = X_1 B_r$, and if X is a left module, then X_r is a unital left D^B_r-module if and only if $X_r = B_r X_1$.*

(3) *If $A = \oplus_{t \in G} A_t$ is a partially-strongly-graded algebra and $X = \oplus_{t \in G} X_t$ is a graded (A, B)-bimodule such that each X_r is a unital $(D^A_r, D^B_{r^{-1}})$-bimodule, $\forall r \in G$, then $X_r D^B_s = D^A_{rs} X_r$, $\forall r \in G$.*

Proof. Suppose that X is a graded right B-module. Since B is strongly-graded we have $X_{rs} B_{s^{-1}} = X_{rs} B_{s^{-1}} B_s B_{s^{-1}} \subseteq X_r B_s B_{s^{-1}} \subseteq X_{rs} B_{s^{-1}}$. In particular we get $X_r D^B_{r^{-1}} = X_1 B_r$, so the first two claims are proved for a right B-module X. A similar argument for a left module X concludes the proof of (1) and (2). As for (3) note that by (1) we have $X_r D^B_s = X_{rs} B_{s^{-1}}$. On the other hand, by (2) we have $A_r X_1 = X_r = X_1 B_r$, $\forall r \in G$. Thus $X_{rs} B_{s^{-1}} = A_{rs} X_1 B_{s^{-1}} = A_{rs} A_{s^{-1}} X_1 = D^A_{rs} A_{rs} A_{s^{-1}} X_1 \subseteq D^A_{rs} X_r$. Hence $X_r D^B_s \subseteq D^A_{rs} X_r$. A symmetric argument shows that $D^A_t X_r \subseteq X_r D_{r^{-1}t}$, $\forall r, t \in G$. This ends the proof, since in particular we get $D^A_{rs} X_r \subseteq X_r D^B_{r^{-1}rs} = X_r D^B_s$. $\square$

Proposition 4.7. *Let $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ be partially-strongly-graded algebras. Then the following statements are equivalent:*

(1) *The algebras A and B are strongly-graded-equivalent.*
(2) *There exist a partially-strongly-graded algebra $C = \oplus_{t \in G} C_t$ and $e = e^2 \in \mathfrak{M}_1(C)$ as in (2) of Proposition 3.4, which moreover satisfy, $\forall t \in G$:*
  (a) $e D^C_t e \cong D^A_t$, $(1 - e) D^C_t (1 - e) \cong D^B_t$
  (b) $D^C_t e D^C_t = D^C_t = D^C_t (1 - e) D^C_t$.

Proof. Let $\mathbb{L} = \oplus_{t \in G} \mathbb{L}_t$ be the Morita algebra of the strong-graded-equivalence $(A, B, X, Y, \tau_A, \tau_B)$ between A and B, and let $e \in \mathfrak{M}_1(\mathbb{L})$ be the multiplier defined in the first part of the proof of Proposition 3.4. Then the pair $\mathbb{L}$ and e satisfy (2) of Proposition 3.4. Moreover, using Proposition 4.2:

$$D^{\mathbb{L}}_t = \begin{pmatrix} A_t & X_t \\ Y_t & B_t \end{pmatrix} \begin{pmatrix} A_{t^{-1}} & X_{t^{-1}} \\ Y_{t^{-1}} & B_{t^{-1}} \end{pmatrix} = \begin{pmatrix} D^A_t & D^A_t X_1 \\ D^B_t Y_1 & D^B_t \end{pmatrix}.$$

Thus $e D^{\mathbb{L}}_t e \cong D^A_t$, $(1 - e) D^{\mathbb{L}}_t (1 - e) \cong D^B_t$. Now, in order to compute $D^{\mathbb{L}}_t e D^{\mathbb{L}}_t$, note first that it follows from (2) of Proposition 4.2 that $D^B_t Y_1 D^A_t = D^B_t Y_1$ and $\tau_B(D^B_t Y_1 \otimes_A D^A_t X_1) = D^B_t B_1 = D^B_t$. Therefore:

$$D^{\mathbb{L}}_t e D^{\mathbb{L}}_t = \begin{pmatrix} D^A_t & D^A_t X_1 \\ D^B_t Y_1 D^A_t & \tau_B(D^B_t Y_1 \otimes_A D^A_t X_1) \end{pmatrix} = D^{\mathbb{L}}_t.$$

The equality between $D^{\mathbb{L}}_t (1 - e) D^{\mathbb{L}}_t$ and $D^{\mathbb{L}}_t$ is proved in a similar way. Finally, let us see that $\mathbb{L}$ is partially-strongly-graded. Since both A and B are partially-strongly-graded, we have:

$$D_t^{\mathbb{L}}\mathbb{L}_t = \begin{pmatrix} A_t + D_t^A\tau_A(X_1 \otimes Y_t) & D_t^A X_t + D_t^A X_1 B_t \\ D_t^B Y_1 A_t + D_t^B Y_t & D_t^B\tau_B(Y_1 \otimes X_t) + B_t \end{pmatrix}$$

Now, using Proposition 4.2 and the fact that X_t is unital as a left D_t^A-module, we have:

$$A_t + D_t^A\tau_A(X_1 \otimes Y_t) = A_t + D_t^A A_1 A_t = A_t + A_t = A_t,$$

$$D_t^A X_t + D_t^A X_1 B_t = X_t + D_t^A X_t = X_t,$$

and similarly we see that $D_t^B Y_1 A_t + D_t^B Y_t = Y_t$ and $D_t^B\tau_B(Y_1 \otimes X_t) + B_t = B_t$. Thus $\mathbb{L}_t\mathbb{L}_t\mathbb{L}_{t^{-1}} = \mathbb{L}_t$.

Suppose conversely that C and e satisfy statement (2), and let $A' = \oplus_{t \in G}A_t'$, $B' = \oplus_{t \in G}B_t'$, $X = \oplus_{t \in G}X_t$, $Y = \oplus_{t \in G}Y_t$ and $\tau_{A'}$, $\tau_{B'}$ as in the proof of (2)→(1) in Proposition 3.4. Since $A \cong A'$ and $B \cong B'$ as graded algebras, then both A' and B' are partially-strongly-graded, so in particular $D_t^{A'}$ and $D_t^{B'}$ are idempotent. Notice that

$$D_t^{A'} = A_t' A_{t^{-1}}' = eC_t e C_{t^{-1}} e = eC_t D_{t^{-1}}^C e D_{t^{-1}}^C C_{t^{-1}} e =$$
$$eC_t D_{t^{-1}}^C C_{t^{-1}} e = e D_t^C e.$$

Let us see that the graded Morita context $(A', B', X, Y, \tau_{A'}, \tau_{B'})$ is strong. In first place, since C is partially-strongly-graded and satisfies (2)(b), we have

$$D_t^{A'} X_t = (eC_t C_{t^{-1}} e)(eC_t(1 - e))$$
$$= eC_t C_{t^{-1}} D_t^C e D_t^C C_t(1 - e) = eC_t C_{t^{-1}} D_t^C C_t(1 - e)$$
$$= eC_t(1 - e) = X_t,$$

showing that $D_t^{A'} X_t = X_t$. Similar computations yield $X_t D_{t^{-1}}^{B'} = X_t$, $D_t^{B'} Y_t = Y_t$, and $Y_t D_{t^{-1}}^{A'} = Y_t$.

On the other hand:

$$X_t Y_{t^{-1}} = (eC_t(1 - e))((1 - e)C_{t^{-1}} e) = eC_t D_{t^{-1}}^C(1 - e)D_{t^{-1}}^C C_{t^{-1}} e$$
$$= eC_t D_{t^{-1}}^C C_{t^{-1}} e = e(D_t^C)^2 e = e D_t^C e = D_t^{A'}.$$

The equality $Y_{t^{-1}} X_t = D_{t^{-1}}^B$ is proved in the same way. Thus we conclude that $(A', B', X, Y, \tau_{A'}, \tau_{B'})$ is a strong-graded-equivalence between A' and B', which are isomorphic to A and B respectively. Hence A and B are strongly-graded-equivalent. □

Proposition 4.8. *Strong-graded-equivalence is an equivalence relation in the class of partially-strongly-graded algebras.*

Proof. We have already seen that every partially-strongly-graded algebra is a strongly-graded autoequivalence. Since the symmetric property of strongly-graded-equivalence is clear, we prove that it is transitive. To this end consider strong-graded-equivalences $\mathsf{M} =$

$(A, A', X, Y, \tau_A, \tau_{A'})$ and $\mathsf{N} = (A', B, X', Y', \tau'_{A'}, \tau_B)$ between A and A' and between A' and B respectively. Let $\mathsf{MN} = (A, B, \bar{X}, \bar{Y}, \rho_A, \rho_B)$ be the graded-equivalence constructed in the proof of Proposition 3.6. We will show that MN is in fact a strong equivalence. First note that each $\bar{X}_t$ is a unital $(D_t^A, D_{t^{-1}}^B)$-bimodule and each $\bar{Y}_t$ is a unital $(D_t^B, D_{t^{-1}}^A)$-bimodule. Indeed, using Proposition 4.2 and (3), we see that

$$X_s \otimes_{A'} X'_{s^{-1}t} = X_s \otimes_{A'} D_{s^{-1}t}^{A'} X'_{s^{-1}t} = X_s D_{s^{-1}t}^{A'} \otimes_{A'} X'_{s^{-1}t}$$

$$= X_1 A'_s D_{s^{-1}t}^{A'} \otimes_{A'} X'_{s^{-1}t} = X_1 D_t^{A'} A'_s \otimes_{A'} X'_{s^{-1}t} = D_t^A X_1 A'_s \otimes_{A'} X'_{s^{-1}t}$$

$$= D_t^A X_s \otimes_{A'} X'_{s^{-1}t} \subseteq D_t^A \bar{X}_t,$$

which shows that $\bar{X}_t = D_t^A \bar{X}_t$. Similarly, $\bar{X} = \bar{X} D_{t^{-1}}^B$, $\bar{Y}_t = D_t^B \bar{X}_t$, and $\bar{Y} = \bar{Y} D_{t^{-1}}^A$. We have, for $t \in G$:

$$\rho_A^t(\bar{X}_t \otimes_{D_{t^{-1}}^B} \bar{Y}_{t^{-1}}) = \sum_{r,s \in G} \rho_A\big((X_r \otimes_{A'} X'_{r^{-1}t}) \otimes_B (X_{t^{-1}s^{-1}} \otimes_{A'} Y_s)\big)$$

$$= \sum_{r,s \in G} \tau_A\big((X_r \otimes_{A'} \tau'_{A'}(X'_{r^{-1}t} \otimes_B Y'_{t^{-1}s^{-1}})Y_s\big).$$

Now using again Proposition 4.2 we obtain:

$$\tau_A\big((X_r \otimes_{A'} \tau'_{A'}(X'_{r^{-1}t} \otimes_B Y'_{t^{-1}s^{-1}})Y_s\big) = \tau_A\big(X_1 A'_r \otimes_{A'} A'_{r^{-1}t} A'_{t^{-1}s^{-1}} A'_s Y_1\big)$$

$$= \tau_A(X_1 \otimes_{A'} Y_1) A_r A_{r^{-1}t} A_{t^{-1}s^{-1}} A_s = A_1 D_r^A D_t^A D_{s^{-1}}^A = D_r^A D_t^A D_{s^{-1}}^A.$$

Therefore $\rho_A^t(\bar{X}_t \otimes_{D_{t^{-1}}^B} \bar{Y}_{t^{-1}}) = \sum_{r,s \in G} D_r^A D_t^A D_{s^{-1}}^A = D_t^A$, $\forall t \in G$. Analogous computations show that $\rho_B^t(\bar{Y}_t \otimes_{D_{t^{-1}}^A} \bar{X}_{t^{-1}}) = D_t^B$, $\forall t \in G$, which concludes the proof. $\square$

We end this section by showing that the property of having a strong-grading is invariant by strongly-graded-equivalence, unlike the situation with graded-equivalence (see for instance Theorem 3.3 or Theorem 5.7).

Proposition 4.9. *Let $A = \oplus_{t \in G} A_t$ and $B = \oplus_{t \in G} B_t$ be partially-strongly-graded algebras that are strongly-graded-equivalent algebras. Then A is strongly-graded if and only if so is B.*

Proof. Let $(A, B, X, Y, \tau_A, \tau_B)$ be a strong-graded-equivalence. By Proposition 4.7, we may suppose that there exist a partially-strongly-graded algebra $C = \oplus_{t \in G} C_t$ and $e = e^2 \in \mathfrak{M}_1(C)$, such that $A = eCe$ and $B = (1 - e)C(1 - e)$ are graded subalgebras of C, $X = eC(1 - e)$, $Y = (1 - e)Ce$, $eD_t^C e = D_t^A$, $(1 - e)D_t^C(1 - e) = D_t^B$, $D_t^C e D_t^C = D_t^C = D_t^C(1 - e)D_t^C$, and τ_A and τ_B are given by the product of C. Suppose that B is strongly-graded. Then $Y_t X_{t^{-1}} = D_t^B = B_1$, $\forall t \in G$. Now, (1) and (2) of Proposition 4.6 imply that $B_t Y_{t^{-1}} = Y_1$ and $X_t = X_1 B_t$ (recall that each Y_r, X_r, are unital D_r^B and

$D^B_{r^{-1}}$ modules respectively). Hence $D^A_t = X_t Y_{t^{-1}} = X_1 B_t Y_{t^{-1}} = X_1 Y_1 = A_1$. Then $A_t = D^A_t A_t = A_1 A_t = A_t$, $\forall t \in G$, so A is strongly-graded, which ends the proof. $\quad\square$

5. Product partial actions

Let us begin by recalling the usual definition of a partial group action on an algebra.

Definition 5.1. A partial action α of a group G on an algebra A consists of a family of two-sided ideals D_t in A ($t \in G$) and algebra isomorphisms $\alpha_t : D_{t^{-1}} \to D_t$, such that for all $s, t \in G$ the following properties are verified:

(1) α_1 is the identity isomorphism $A \to A$,
(2) $\alpha_s(D_{s^{-1}} \cap D_t) = D_s \cap D_{st}$,
(3) $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$, for any $x \in D_{t^{-1}} \cap D_{(st)^{-1}}$.

We say that a partial action α is *idempotent* if each domain D_t is an idempotent ideal. In this work we will prefer to replace in the above definition intersections by products as follows.

Definition 5.2. A product partial action α of a group G on an algebra A consists of a family of two-sided ideals D_t in A ($t \in G$) and algebra isomorphisms $\alpha_t : D_{t^{-1}} \to D_t$, such that for all $s, t \in G$ the following properties are verified:

(1) α_1 is the identity isomorphism $A \to A$,
(2) $D^2_t = D_t$, and $D_s D_t = D_t D_s$,
(3) $\alpha_s(D_{s^{-1}} D_t) = D_s D_{st}$,
(4) $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$, for any $x \in D_{t^{-1}} D_{(st)^{-1}}$.

For instance, a partial action whose domains are all unital is a product partial action. Other examples of product partial actions are the partial actions on C*-algebras, since in this case the product of two closed ideals is equal to their intersection.

Notice that an idempotent ideal I in an algebra A is a unital A-bimodule. Hence, given a product partial action $\alpha = \{\alpha_t : D_{t^{-1}} \to D_t, t \in G\}$ of G on A, each ideal D_t, $t \in G$, is a unital A-bimodule.

The following result shows that, under certain circumstances, product partial actions can be obtained from global actions by restriction. We will see in Theorem 8.2 that in fact any product partial action can be obtained, up to Morita equivalence, in such a way.

Proposition 5.3. *Let A be an idempotent ideal in the algebra B, and suppose that $\beta : G \times B \to B$ is an action such that $A\beta_t(A) = \beta_t(A)A$, $\forall t \in G$. Define $D_t := A\beta_t(A)$. Then $\beta_t(D_{t^{-1}}) = D_t$, $\forall t \in G$, and $\alpha := (\{\alpha_t\}, \{D_t\}_{t \in G})$ is a product partial action, where $\alpha_t(x) = \beta_t(x)$, $\forall t \in G$ and $x \in D_{t^{-1}}$.*

Proof. It is clear that $\beta_t(D_{t^{-1}}) = D_t$, and also that each $\beta_t(A)$ is idempotent because so is A. Moreover $A\beta_t(A) = \beta_t(A)A$, so we have $D_t^2 = A^2\beta_t(A)^2 = D_t$, and

$$D_s D_t = A\beta_s(A)A\beta_t(A) = A\beta_t(\beta_{t^{-1}s}(A)A)A = A\beta_t(A\beta_{t^{-1}s}(A))A$$
$$= A\beta_t(A)\beta_s(A)A = D_t D_s.$$

On the other hand:

$$\alpha_s(D_{s^{-1}}D_t) = \beta_s(A\beta_{s^{-1}}(A)A\beta_t(A)) = \beta_s(A)A\beta_s(A)\beta_{st}(A) = D_s D_{st}.$$

Finally, condition (4) of Definition 5.2 is obviously satisfied. □

The product partial action α obtained in Proposition 5.3 will be called the *restriction* of the global action β. The action β is called a *globalization* of α (as well as any global action isomorphic to β). A globalization β of a product partial action α is called *minimal* if $B = \sum_{t \in G} \beta_t(A)$, in which case we have $B^2 = B$. Note that each $\beta_t(A)$ is a unital B-bimodule, due to the fact that A is an idempotent ideal in B.

In [6], a partial action $\alpha := (\{\alpha_t\}, \{D_t\}_{t \in G})$ was called regular if

$$D_{t_1} \cap D_{t_2} \cap \ldots \cap D_{t_n} = D_{t_1} D_{t_2} \ldots D_{t_n}, \quad \forall t_1, t_2, \ldots, t_n \in G.$$

Evidently any regular partial action is a product partial action. As it was mentioned in [6, p. 4961], any C^*-algebraic partial action is regular as well as any partial action on a von Neumann regular ring. Notice that in a von Neumann regular ring the ideals are idempotent and the intersection of any two of them coincides with their product. The same is true for the ideals of the Jacobian algebra $\mathbb{A}_n$ over a field of characteristic zero introduced by V. Bavula with respect to the Jacobian Conjecture (see [9, Theorem 3.1, Corollary 3.10]). It follows that any partial action on $\mathbb{A}_n$ is regular. Moreover, the same can be said about the algebra $\mathbb{I}_n$ of integro-differential operators on a polynomial algebra thanks to the ideal equivalence with $\mathbb{A}_n$ established in [11, Theorem 3.1]. Furthermore, [10, Corollary 7.4] implies that any idempotent partial action on the algebra of one-sided inverses of a polynomial algebra is regular.

Given a partial action or a product partial action α of G on A, we define the skew group algebra $A \rtimes_\alpha G$ of A by α as the direct sum $\oplus_{t \in G} D_t \delta_t$, with the product determined by the rule

$$(a\delta_r)(b\delta_s) = \alpha_r(\alpha_r^{-1}(a)b)\delta_{rs}, \tag{4}$$

where $r, s \in G, a \in D_r, b \in D_s$. It was established in [17, Corollary 3.2] that $A \rtimes_\alpha G$ is associative if α is an idempotent partial action of a group G on a unital algebra A. Notice that the proof does not use the fact that A is unital and, moreover, it works for any product partial action α.

Let $\alpha = \{\alpha_t : D_{t^{-1}} \to D_t, t \in G\}$ and $\alpha' = \{\alpha'_t : D'_{t^{-1}} \to D'_t, t \in G\}$ be product partial actions of G on A and A', respectively. By a *morphism* $\varphi : \alpha \to \alpha'$ we mean an algebra homomorphism $\varphi : A \to A'$ such that $\varphi(D_t) \subseteq D'_t$ and $\varphi(\alpha_t(a)) = \alpha'_t(\varphi(a))$ for all $t \in G$, $a \in D_{t^{-1}}$. A morphism $\varphi : \alpha \to \alpha'$ induces an algebra homomorphism $\varphi^{\rtimes} : A \rtimes_\alpha G \to A \rtimes_{\alpha'} G$. In fact the correspondence $(\alpha \xrightarrow{\varphi} \alpha') \mapsto (A \rtimes_\alpha G \xrightarrow{\varphi^{\rtimes}} A \rtimes_{\alpha'} G)$ is a functor.

Remark 5.4. Every partial action $\alpha := (\{\alpha_t\}, \{D_t\}_{t \in G})$, such that each D_t is unital, can be seen as a restriction of a global action, which is essentially unique if a minimality condition is required (see [17]). We refer to this global action as the enveloping action of α. Note that the enveloping algebra is idempotent [20, Theorem 3.1], which implies that so is the corresponding skew group algebra of the enveloping action.

Proposition 5.5. *The skew group algebra of a product partial action is partially-strongly-graded, and it is strongly-graded if and only if the partial action is global.*

Proof. Let $\alpha := (\{\alpha_t\}, \{D_t\}_{t \in G})$ be a product partial action on the algebra A. Let $B := A \rtimes_\alpha G$. Then $B = \oplus_{t \in G} B_t$, where $B_t = D_t \delta_t$, and the product is given by $d_s \delta_s \, d_t \delta_t = \alpha_s(\alpha_s^{-1}(d_s) d_t) \delta_{st}$. Thus $B_s B_t = \alpha_s(\alpha_s^{-1}(D_s) D_t) \delta_{st} = \alpha_s(D_{s^{-1}} D_t) \delta_{st} = D_s D_{st} \delta_{st}$. In particular $B_1 B_t = B_t = B_t B_1$, $\forall t \in G$. Moreover we have

$$B_t B_{t^{-1}} B_t = (D_t D_1 \delta_1)(D_t \delta_t) = D_t \delta_t = B_t.$$

As for the last statement, it is clear that the skew group algebra of a global action is strongly-graded. To prove the converse note that, according to (3) of Definition 5.2 and equality (4) above, we have

$$(D_r \delta_r)(D_s \delta_s) = D_r D_{rs} \delta_{rs}, \quad \forall r, s \in G,$$

which implies $D_{rs} \subseteq D_r$, $\forall r, s \in G$, hence $D_r = A$, $\forall r$. $\square$

Moreover, we have the next.

Proposition 5.6. *Let α be an idempotent partial action of a group G on an algebra A. Then $A \rtimes_\alpha G$ is partially-strongly-graded if and only if α is a product partial action.*

Proof. The 'if' part follows from Proposition 5.5. For the 'only if' part suppose that $B = A \rtimes_\alpha G$ satisfies the partial representation property. If we write $B = \oplus_{t \in G} B_t$, where $B_t = D_t \delta_t$, then we are assuming that

$$B_t B_{t^{-1}} B_t = B_t,$$

for all $t \in G$. Since this is equivalent to the two equalitites in (2), we obtain, on one hand,

$$B_{r^{-1}}B_r B_s = D_{r^{-1}}D_s\delta_s = B_{r^{-1}}B_{rs} = \alpha_{r^{-1}}(D_r D_{rs})\delta_s,$$

which gives

$$\alpha_r(D_{r^{-1}}D_s) = D_r D_{rs} \tag{5}$$

for all $r, s \in G$. On the other hand,

$$B_r B_s B_{s^{-1}} = \alpha_r(D_{r^{-1}}D_s)\delta_r = B_{rs}B_{s^{-1}} = \alpha_{rs}(D_{s^{-1}r^{-1}}D_{s^{-1}})\delta_r,$$

that is

$$\alpha_r(D_{r^{-1}}D_s) = \alpha_{rs}(D_{s^{-1}r^{-1}}D_{s^{-1}}),$$

for all $r, s \in G$. Thanks to (5) the latter results in

$$D_r D_{rs} = D_{rs}D_r,$$

for all $r, s \in G$, showing that the domains D_t commute with each other. In combination with (5) this shows that α is a product partial action. $\quad\square$

Theorem 5.7. *Let α be a product partial action on A with a minimal globalization β acting on B. Then $A \rtimes_\alpha G$ and $B \rtimes_\beta G$ are graded-equivalent.*

Proof. Taking the subsets $X := \oplus_{t\in G}\beta_t(A)\delta_t$ and $Y := \oplus_{t\in G}A\delta_t$ in $B \rtimes_\beta G$, we show that $\big(B \rtimes_\beta G, A \rtimes_\alpha G, X, Y\big)$ is a graded-equivalence. We only need to show that it is a Morita equivalence, for all the objects of the context are graded under the grading of $B \rtimes_\beta G$. Using the fact that each $\beta_t(A)$ is a unital B-bimodule, we have:

$$(B \rtimes_\beta G)X = \sum_{s,t}(B\delta_s)(\beta_t(A)\delta_t) = \sum_{s,t}B\beta_{st}(A)\delta_{st} = \sum_{s,t}\beta_{st}(A)\delta_{st} = X.$$

$$Y(B \rtimes_\beta G) = \sum_{s,t}(A\delta_s)(B\delta_t) = \sum_{s,t}A\beta_s(B)\delta_{st} = \sum_t A\delta_t = Y.$$

$$XY = \sum_{s,t}\beta_s(A)^2\delta_{st} = \sum_r\sum_t \beta_{rt^{-1}}(A)\delta_r = \sum_r B\delta_r = B \rtimes_\beta G.$$

$$YX = \sum_{s,t}(A\beta_{st}(A)\delta_{st} = \sum_{s,t}D_{st}\delta_{st} = A \rtimes_\alpha G.$$

Finally:

$$X(A \rtimes_\alpha G) = X(YX) = (XY)X = (B \rtimes_\beta G)X = X.$$

$$(A \rtimes_\alpha G)Y = (YX)Y = Y(XY) = Y(B \rtimes_\beta G) = Y. \quad\square$$

6. Partial smash products

Let $B = \oplus_{t \in G} B_t$ be a graded algebra. It is readily checked that the set $I^B :=$ $\oplus_{r,s \in G} B_{r^{-1}} B_s e_{r,s}$ is a two-sided ideal of $B \# G$.

Definition 6.1. We will say that $I^B := \oplus_{r,s \in G} B_{r^{-1}} B_s e_{r,s}$ is the partial smash product of the G-graded algebra B. We may occasionally denote it also by $B \#_p G$.

Observe that $I^B = B \# G$ if and only if B is strongly-graded. The easy proof of the next fact is left to the reader:

Proposition 6.2. Let $B = \oplus_{t \in G} B_t$ be a graded algebra, such that the B_1-bimodule B_t is unital for each $t \in G$. Then the algebras B, B_1 and I^B are idempotent.

We proceed with the following fact.

Proposition 6.3. Let $B = \oplus_{t \in G} B_t$ be a partially-strongly-graded algebra and $I := I^B$, its partial smash product. Then

(1) The linear β^B-orbit of I is all of $B \# G$.
(2) For every $t \in G$ we have

$$I \beta_t^B(I) = \beta_t^B(I) I = \oplus_{r,s} B_{r^{-1}} B_t B_{t^{-1}} B_s e_{r,s},$$

so that the restriction $\gamma^B := \beta^B|_I$ of β^B to I is a product partial action, and β^B is a minimal globalization of γ^B.

Proof. Since B is partially-strongly-graded, each B_t is a unital B_1-module and by Proposition 6.2 the ideal I is idempotent. Therefore

$$B \# G = \oplus_{r,s \in G} \beta_s^B(B_{r^{-1}s} B_1 e_{s^{-1}r,1}) \subseteq \sum_{s \in G} \beta_s^B(I) \subseteq B \# G,$$

which proves our first statement.

Let us see next that γ is a product partial action. On one hand:

$$\beta_t^B(I) = \oplus_{r,s} B_{r^{-1}} B_s e_{tr,ts} = \oplus_{u,v} B_{u^{-1}t} B_{t^{-1}v} e_{u,v}.$$

Then, since $B_1 = \sum_{u \in G} B_{u^{-1}} B_u$, we have:

$$I \beta_t^B(I) = \sum_{r,s,u,v} B_{r^{-1}} B_s B_{u^{-1}t} B_{t^{-1}v} e_{r,s} e_{u,v} = \sum_{r,u,v} B_{r^{-1}} B_u B_{u^{-1}t} B_{t^{-1}v} e_{r,v}$$

$$= \sum_{r,u,v} B_{r^{-1}} B_u B_{u^{-1}} B_t B_{t^{-1}} B_v e_{r,v} = \oplus_{r,s} B_{r^{-1}} B_t B_{t^{-1}} B_s e_{r,s}$$

On the other hand:

$$\beta_t^B(I)I = \sum_{r,s,u,v} B_{r^{-1}t} B_{t^{-1}s} B_{u^{-1}} B_v e_{r,s} e_{u,v} = \sum_{r,u,v} B_{r^{-1}t} B_{t^{-1}u} B_{u^{-1}} B_v e_{r,v}$$

$$= \sum_{r,u,v} B_{r^{-1}} B_t B_{t^{-1}} B_u B_{u^{-1}} B_v e_{r,v} = \oplus_{r,s} B_{r^{-1}} B_t B_{t^{-1}} B_s e_{r,s}.$$

Thus $I\beta_t^B(I) = \beta_t^B(I)I$, so γ^B is a product partial action by Proposition 5.3, and β^B is a minimal globalization of γ^B by (1). $\square$

Note that in the proof of (1) of Proposition 6.3 the only restriction on the grading of B we used is that each B_t is a unital B_1-bimodule. Thus we have:

Corollary 6.4. *Let $B = \oplus_{t\in G}B_t$ be a graded algebra such that every B_t is a unital B_1-bimodule. Then $B\#_p G$ is β^B-invariant if and only if $B\#_p G = B\#G$, that is, if and only if B is strongly-graded.*

Theorem 6.5. *If $B = \oplus_{t\in G}B_t$ is a partially-strongly-graded algebra, then $(B\#G) \rtimes_{\beta^B} G$ and $I^B \rtimes_{\gamma^B} G$ are graded-equivalent.*

Proof. This is a direct consequence of Proposition 6.3 and Theorem 5.7. $\square$

We shall call γ^B from Proposition 6.3 the *canonical partial action* of G on the partial smash product I^B.

If $\phi : A \to B$ is a homomorphism of G-graded algebras and $\phi^\# : A\#G \to B\#G$ is the corresponding homomorphism between the smash products, it is clear that $\phi^\#(I^A) \subseteq I^B$. Thus $\phi^\#$ induces a homomorphism $\phi^\natural : I^A \to I^B$. Besides, since $\beta_t^B \phi^\# = \phi^\# \beta_t^A$, we have that $\gamma_t^B \phi^\natural = \phi^\natural \gamma_t^A \ \forall t \in G$, so that $\phi^\natural : \gamma^A \to \gamma^B$ is a homomorphism of partial actions. Therefore $\phi^\natural$ induces a homomorphism $\phi^\rtimes : I^A \rtimes G \to I^B \rtimes G$. It turns out that the maps $(A \xrightarrow{\phi} B) \mapsto (\gamma^A \xrightarrow{\phi^\natural} \gamma^B)$ and $(A \xrightarrow{\phi} B) \mapsto (I^A \rtimes G \xrightarrow{\phi^\rtimes} I^B \rtimes G)$ are functors from the category of graded algebras to the category of partial actions on algebras and to that of partially-strongly-graded algebras, respectively.

Proposition 6.6. *Let $B = \oplus_{t\in G}B_t$ be a partially-strongly-G-graded algebra. Then $I^B \rtimes_{\gamma^B} G$ is naturally isomorphic, as a graded algebra, to the graded subalgebra $\oplus_{r,s,t} B_{r^{-1}} B_t B_s e_{r,s}$ of $\mathrm{FMat}_G(B)$.*

Proof. Let $\psi_B : (B\#G) \rtimes_{\beta^B} G \to \mathrm{FMat}_G(B)$ be the natural isomorphism defined in Theorem 2.1. Recall that ψ_B was determined by $be_{r,s}\delta_t \xrightarrow{\psi_B} be_{r,s}\Delta_t = be_{r,t^{-1}s}$, $\forall r,s,t \in G$, $b \in B_{r^{-1}s}$. Since $I^B \rtimes_{\gamma^B} G = \oplus_{t\in G} I_t \delta_t$, where $I_t = I\beta_t^B(I) = \beta_t^B(I)I = \oplus_{r,s} B_{r^{-1}} B_t B_{t^{-1}} B_s e_{r,s}$, it follows that $\psi_B(I^B \rtimes_{\gamma^B} G) = \oplus_{t\in G} \psi_B(I_t \delta_t) = \oplus_{t\in G} I_t \Delta_t$. Now:

$$I_t \Delta_t = \oplus_{r,s \in G} B_{r^{-1}} B_t B_{t^{-1}} B_s e_{r,t^{-1}s} = \oplus_{r,u} B_{r^{-1}} B_t B_{t^{-1}} B_{tu} e_{r,u}$$

$$= \oplus_{r,s} B_{r^{-1}} B_t B_{t^{-1}} B_t B_s e_{r,s} = \oplus_{r,s} B_{r^{-1}} B_t B_s e_{r,s}.$$

To verify naturality, just note that if $\phi : A \to B$ is a homomorphism of partially-strongly-graded algebras, the natural map $\tilde{\phi} : (A \# G) \rtimes_{\beta^A} G \to (B \# G) \rtimes_{\beta^B} G$ sends $I^A \rtimes_{\gamma^A} G$ into $I^B \rtimes_{\gamma^B} G$, so the following diagram commutes:

$$
\begin{array}{ccccc}
(A \#_p G) \rtimes_{\gamma^A} G & \xrightarrow{\ inc\ } & (A \# G) \rtimes_{\beta^A} G & \xrightarrow[\cong]{\psi_A} & \mathrm{FMat}_G(A) \quad \square \\
\downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi^{\mathrm{fin}}} \\
(B \#_p G) \rtimes_{\gamma^B} G & \xrightarrow[inc]{} & (B \# G) \rtimes_{\beta^B} G & \xrightarrow[\psi_B]{\cong} & \mathrm{FMat}_G(B)
\end{array}
$$

7. Equivalences of product partial actions

7.1. Morita equivalence

If $\theta = \{\bar{D}_{t^{-1}} \xrightarrow{\theta_t} \bar{D}_t\}_{t \in G}$ is a product partial action of the group G on the algebra C, we say that a subset S of C is θ-invariant if $\theta_t(S \cap \bar{D}_{t^{-1}}) = S \cap \bar{D}_t$, $\forall t \in G$. Suppose that $\mathsf{M} = (A, A', X, Y, \tau_A, \tau_{A'})$ is a Morita context. If $X' \subseteq X$ and $Y' \subseteq Y$, in general we will write $X'Y'$ instead of $\tau_A(X' \otimes_{A'} Y')$ and $Y'X'$ instead of $\tau_{A'}(Y' \otimes_A X')$.

Definition 7.1. Let $\alpha = \{D_{t^{-1}} \xrightarrow{\alpha_t} D_t\}_{t \in G}$ and $\alpha' = \{D'_{t^{-1}} \xrightarrow{\alpha'_t} D'_t\}_{t \in G}$ be product partial actions of G on algebras A and A', respectively. We say that α is Morita equivalent to α' if there exists a Morita equivalence $\mathsf{M} = (A, A', X, Y, \tau_A, \tau_{A'})$ between A and A', and a product partial action $\theta = \{\bar{D}_{t^{-1}} \xrightarrow{\theta_t} \bar{D}_t\}$ on the context algebra C of M such that:

(i) $Y D_t X = D'_t$.
(ii) $\theta|_A = \alpha$ and $\theta|_{A'} = \alpha'$.

The pair (M, θ) will be called a Morita equivalence between α and α'. We simbolize this relation by writing $\alpha \overset{M}{\sim} \alpha'$

Remark 7.2. Note that the above definition implies that each ideal D_t corresponds to D'_t under this equivalence. In fact, condition (i) above implies

$$D_t = (XY)D_t(XY) = X(Y D_t X)Y = X D'_t Y.$$

Moreover,

$$D_t X = D_t X D'_t = X D'_t$$

and

$$D'_t Y = D'_t Y D_t = Y D_t, \quad \forall t \in G.$$

Define $X_t = D_t X D'_t$ and $Y_t = D'_t Y D_t$. Then

$$X_t Y_t = D_t \qquad \text{and} \qquad Y_t X_t = D'_t, \quad \forall t \in G. \tag{6}$$

$$D_t X = X_t = X D'_t \qquad \text{and} \qquad D'_t Y = Y_t = Y D_t, \quad \forall t \in G. \tag{7}$$

Remark 7.3. It follows from the previous remark that if α and α' are Morita equivalent product partial actions, then α is a global action if and only if so is α'.

Remark 7.4. It is easy to see, using Remark 7.2, that in Definition 7.1 condition (i) is equivalent to the following one:

(i') There exist families $\{X_t\}_{t \in G}$ and $\{Y_t\}_{t \in G}$ of subbimodules of X and Y respectively such that (6) and (7) hold.

Remark 7.5. Observe that the proof of [6, Proposition 2.11] works for product partial actions. It follows, in particular, that $\bar{D}_t = \begin{pmatrix} D_t & X_t \\ Y_t & D'_t \end{pmatrix}$, $\forall t \in G$, and all of the subsets $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$, $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$ of the context algebra C are θ-invariant. We will identify the subsets above with A, A', X and Y respectively. Notice that with this identification and slightly abusing the notation we may write

$$\theta_t(ax) = \alpha_t(a)\,\theta_t(x), \theta_t(xa') = \theta_t(x)\,\alpha'_t(a'), \tag{8}$$

for all $a \in D_t^A, x \in X, a' \in D_t^{A'}$, and similar equalities can be written for the bimodule Y. Observe also that if both of α and α' are global actions, then θ must be a global action as well.

Let (M, θ) be a Morita equivalence between the product partial actions α and α', where $\mathsf{M} = (A, A', X, Y, \tau_A, \tau_{A'})$. In what follows we use the notation of Remark 7.2. Every X_t is a unital (D_t, D'_t)-bimodule, and every Y_t is a unital (D'_t, D_t)-bimodule. Therefore we have natural maps $j_t : X_t \otimes_{D'_t} Y_t \to X \otimes_{A'} Y$ and $j'_t : Y_t \otimes_{D_t} X_t \to Y \otimes_A X$. We define $\tau_t := \tau_A j_t : X_t \otimes_{D'_t} Y_t \to D_t$ and $\tau'_t := \tau_{A'} j'_t : Y_t \otimes_{D_t} X_t \to D'_t$. Then τ_t is a D_t-bimodule map, and τ'_t is a D'_t-bimodule map. Let us see that τ_t is surjective. Since τ_A is surjective, given $a \in A$ there exist $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ such that $\tau_A(\sum_{i=1}^n x_i \otimes y_i) = a$. Now, given $d_1, d_2 \in D_t$, we have $d_1 x_1, \ldots, d_1 x_n \in X_t$, $y_1 d_2, \ldots, y_n d_2 \in Y_t$, and $\tau_t(\sum_{i=1}^n d_1 x_i \otimes y_i d_2) = d_1 a d_2$. Thus $\tau_t(X_t \otimes_{D'_t} Y_t) \supseteq D_t A D_t = D_t$, because D_t is idempotent and unital over A. Similarly, every τ'_t is surjective. Then we have:

Proposition 7.6. *The context* $\mathsf{M}_t := (D_t, D'_t, X_t, Y_t, \tau_t, \tau'_t)$ *is a Morita equivalence between* D_t *and* D'_t.

Remark 7.7. Observe that the alternative definition of the Morita equivalence of regular partial actions given in [6, Proposition 2.11] also holds for product partial actions without any change in the proof.

Proposition 7.8. *Morita equivalence of product partial actions is an equivalence relation.*

Proof. Just follow, *mutatis mutandis*, the proof in [6, Proposition 2.12] of the same result for regular partial actions, taking into account Remark 7.7. □

Proposition 7.9. *If α and α' are Morita equivalent product partial actions on A and A' respectively, then $A \rtimes_\alpha G$ and $A' \rtimes_{\alpha'} G$ are strongly-graded-equivalent (the converse is also true: see Theorem 8.3).*

Proof. We use the notation of Definition 7.1. Let $\bar{A} := A \rtimes_\alpha G$, $\bar{B} := A' \rtimes_{\alpha'} G =$, and $\bar{C} := C \rtimes_\theta G$. Thus $\bar{A} = \oplus_{t \in G} D_t \delta_t$, $\bar{B} = \oplus_{t \in G} D'_t \delta_t$, and $\bar{C} = \oplus_{t \in G} \bar{D}_t \delta_t$. Consider $e := (L, R) \in \mathfrak{M}_1(\bar{C})$, where $L\left(\begin{pmatrix} a & x \\ y & b \end{pmatrix} \delta_t\right) = \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \delta_t$ and $R\left(\begin{pmatrix} a & x \\ y & b \end{pmatrix} \delta_t\right) = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix} \delta_t$.
To see that $\bar{A}$ and $\bar{B}$ are strongly-graded-equivalent, it is enough to see that the pair $(\bar{C}, e)$ satisfies (2) of Proposition 4.7. First note that

$$
\bar{C}_r e \bar{C}_s = \begin{pmatrix} D_r & X_r \\ Y_r & D'_r \end{pmatrix} \delta_r e \begin{pmatrix} D_s & X_s \\ Y_s & D'_s \end{pmatrix} \delta_s = \begin{pmatrix} D_r & X_r \\ Y_r & D'_r \end{pmatrix} \delta_r \begin{pmatrix} D_s & X_s \\ 0 & 0 \end{pmatrix} \delta_s
$$

$$
= \theta_r \left(\begin{pmatrix} D_{r^{-1}} & X_{r^{-1}} \\ Y_{r^{-1}} & D'_{r^{-1}} \end{pmatrix} \begin{pmatrix} D_s & X_s \\ 0 & 0 \end{pmatrix} \right) \delta_{rs} = \theta_r \left(\begin{pmatrix} D_{r^{-1}} D_s & D_{r^{-1}} X_s \\ Y_{r^{-1}} D_s & Y_{r^{-1}} X_s \end{pmatrix} \right) \delta_{rs}
$$

$$
= \theta_r \left(\begin{pmatrix} D_{r^{-1}} D_s & D_{r^{-1}} D_s X \\ Y D_{r^{-1}} D_s & Y D_{r^{-1}} D_s X \end{pmatrix} \right) \delta_{rs} = \begin{pmatrix} D_r D_{rs} & D_r D_{rs} X \\ Y D_r D_{rs} & Y D_r D_{rs} X \end{pmatrix} \delta_{rs}.
$$

By (6), (7) and the computations above we obtain, for $r = t$, $s = 1$:

$$
\bar{C}_t e \bar{C}_1 = \begin{pmatrix} D_t D_t & D_t D_t X \\ Y D_t D_t & Y D_t D_t X \end{pmatrix} \delta_t = \begin{pmatrix} D_t & X_t \\ Y_t & D'_t \end{pmatrix} \delta_t = \bar{C}_t.
$$

Then it follows that $\bar{C} e \bar{C} = \bar{C}$. Similar computations show that also $\bar{C}(1 - e)\bar{C} = \bar{C}$. On the other hand:

$$
e \begin{pmatrix} a & x \\ y & b \end{pmatrix} \delta_t e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta_t
$$

$$
= \begin{pmatrix} a & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta_t = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \delta_t.
$$

Similarly, $(1-e)\begin{pmatrix} a & x \\ y & b \end{pmatrix}\delta_t(1-e) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\delta_t$. Then $e\bar{C}e \cong \bar{A}$ and $(1-e)\bar{C}(1-e) \cong \bar{B}$. Now, since $D_t^{\bar{C}} = \bar{D}_t\delta_1$, we have $eD_t^{\bar{C}}e = \begin{pmatrix} D_t & 0 \\ 0 & 0 \end{pmatrix}\delta_1 \cong D_t\delta_1 = D_t^{\bar{A}}$, and also $(1-e)D_t^{\bar{C}}(1-e) \cong D_t^{\bar{B}}$. Finally, using again the identities (6), (7) we get:

$$D_t^{\bar{C}}eD_t^{\bar{C}} = \bar{D}_t\delta_1 e\bar{D}_t\delta_1 = \begin{pmatrix} D_t & X_t \\ Y_t & D_t' \end{pmatrix}\begin{pmatrix} D_t & X_t \\ 0 & 0 \end{pmatrix}\delta_1 = \begin{pmatrix} D_t & X_t \\ Y_t & D_t' \end{pmatrix}\delta_1 = D_t^{\bar{C}},$$

and similarly $D_t^{\bar{C}}eD_t^{\bar{C}} = D_t^{\bar{C}}$, which ends the proof. $\square$

Theorem 7.10. *Let $B = \oplus_{t \in G}B_t$ be a partially-strongly-graded algebra and I^B its partial smash product. Then B is strongly-graded-equivalent to the partial skew group algebra $I^B \rtimes_{\gamma^B} G$.*

Proof. Let $X_t = \oplus_{r \in G}X_t(r)$, $Y_t = \oplus_{r \in G}Y_t(r)$ and $B' := \oplus_{t \in G}B_t e_{1,t}\delta_t$ be the subsets of $B\#G$ defined in the proof of Theorem 3.3, so $X_t(r) = B_{r^{-1}t}e_{r,t}$, and $Y_t(r) = B_r e_{1,r}$ $\forall r,t \in G$. As it is shown in Theorem 3.3, if $X = \oplus_{t \in G}X_t\delta_t$ and $Y = \oplus_{t \in G}Y_t\delta_t$, then $((B\#G)\rtimes G, B', X, Y)$ is a graded-equivalence. We define

$$X' = (I^B \rtimes_{\gamma^B} G)X = \oplus_{t \in G}X_t'\delta_t \quad \text{and} \quad Y' = Y(I^B \rtimes_{\gamma^B} G) = \oplus_{t \in G}Y_t'\delta_t.$$

We are going to show that $(I^B \rtimes_{\gamma^B} G, B', X', Y')$ gives a strong-graded-equivalence.

We start by recalling that the partial action $\gamma := \gamma^B = \{\gamma_t : I_{t^{-1}} \to I_t\}$ of G on $I = I^B$ is the restriction of β with

$$I_t = \oplus_{r,s}B_{r^{-1}}B_tB_{t^{-1}}B_s e_{r,s}.$$

We compute first X' as follows. Notice that

$$(B_{r^{-1}}D_u^B B_s e_{r,s}\delta_u)(B_{v^{-1}u^{-1}t}e_{v,u^{-1}t}\delta_{u^{-1}t}) = B_{r^{-1}}D_u^B B_s B_{v^{-1}u^{-1}t}e_{r,s}e_{uv,t}\delta_t$$
$$= [s = uv]B_{r^{-1}}D_u^B B_s B_{s^{-1}t}e_{r,t}\delta_t.$$

Consequently, since $X_t'\delta_t = \sum_{u,v}(I_u\delta_u)(X_{u^{-1}t}(v)\delta_{u^{-1}t})$, we obtain

$$X_t' = \sum_{r,s,v}\sum_u [s = uv]B_{r^{-1}}(D_u^B B_s)B_{s^{-1}t}e_{r,t}$$
$$= \sum_r B_{r^{-1}}\sum_s B_s B_{s^{-1}t}e_{r,t} = \sum_r B_{r^{-1}}B_t e_{r,t},$$

using Proposition 3.2. Then $X' = \oplus_{t,r}B_{r^{-1}}B_t e_{r,t}\delta_t$. Computing next Y', we have

$$(B_r e_{1,r}\delta_u)(B_{v^{-1}}D_{u^{-1}t}^B B_w e_{v,w}\delta_{u^{-1}t}) = B_r e_{1,r}\beta_u(B_{v^{-1}}D_{u^{-1}t}^B B_w e_{v,w})\delta_t$$
$$= [r = uv]B_r B_{r^{-1}u}D_{u^{-1}t}^B B_w e_{1,uw}\delta_t.$$

Setting $s = uw$ we obtain

$$Y_t' = \sum_{r,s,u,v} [r = uv] B_r B_{r^{-1}u} D_{u^{-1}t}^B B_{u^{-1}s} e_{1,s}$$

$$= \sum_{u,s} (\sum_r B_r B_{r^{-1}u}) D_{u^{-1}t}^B B_{u^{-1}s} e_{1,s} = \sum_{u,s} B_u D_{u^{-1}t}^B B_{u^{-1}s} e_{1,s}$$

$$= D_t^B \sum_s (\sum_u B_u B_{u^{-1}s}) e_{1,s} = D_t^B \sum_s B_s e_{1,s},$$

using (3). Thus $Y' = \oplus_{s,t} D_t^B B_s e_{1,s} \delta_t$.

Let us see that $X'Y' = I \rtimes_\gamma G$. We have

$$X'Y' = (I \rtimes_\gamma G)(XY)(I \rtimes_\gamma G) = (I \rtimes_\gamma G)((B\#G) \rtimes_\beta G)(I \rtimes_\gamma G).$$

Then

$$X'Y' = \oplus_t \sum_{uvw=t} (I_u \delta_u)((B\#G)\delta_v)(I_w \delta_w)$$

$$= \oplus_t \sum_{uvw=t} (I_u \delta_u)((B\#G)\delta_v)(I_w \delta_w) = \oplus_t \sum_{uvw=t} (I\beta_u(I)(B\#G)\delta_{uv})(\beta_w(I)I\delta_w)$$

$$= \oplus_t \sum_{uvw=t} I\beta_u(I)\beta_t(I)\beta_{uv}(I)\delta_t = \oplus_t \sum_{uvw=t} I_u I_{uv} I_t \delta_t = \oplus_t I_t \delta_t = I \rtimes_\gamma G.$$

To prove that $Y'X' = B'$, we compute, for each $t, u, r, s \in G$:

$$(D_u^B B_s e_{1,s} \delta_u)(B_{r^{-1}} B_{u^{-1}t} e_{r,u^{-1}t} \delta_{u^{-1}t}) = [s = ur] D_u^B B_{ur} B_{r^{-1}} B_{u^{-1}t} e_{1,t} \delta_t$$

Then, since $D_u^B B_{ur} B_{r^{-1}} = B_u D_r^B$, we have:

$$Y'X' = \sum_{t,u,r,s} [s = ur] B_u D_r^B B_{u^{-1}t} e_{1,t} \delta_t = \sum_{t,u,r} B_u D_r^B B_{u^{-1}t} e_{1,t} \delta_t$$

$$= \sum_{t,u} B_u B_{u^{-1}t} e_{1,t} \delta_t = \oplus_t B_t e_{1,t} \delta_t = B'.$$

Note that X' and Y' are unital over $I \rtimes_\gamma G$ and B'. In fact:

$$X'B' = X'Y'X' = (I \rtimes_\gamma G)X' = (I \rtimes_\gamma G)^2 X = (I \rtimes_\gamma G)X = X'.$$

$$B'Y' = Y'X'Y' = Y'(I \rtimes_\gamma G) = Y(I \rtimes_\gamma G)^2 = Y(I \rtimes_\gamma G) = Y'.$$

We have proved so far that $(I \rtimes_\gamma G, B', X', Y')$ is a graded-equivalence. Next we check that it is also a strong-graded-equivalence. On one hand:

$$(X'_t\delta_t)(Y'_{t^{-1}}\delta_{t^{-1}}) = X'_t\beta_t(Y'_{t^{-1}})\delta_1 = \sum_{r,s} B_{r^{-1}}B_t D^B_{t^{-1}}B_s e_{r,t}e_{t,ts}\delta_1$$

$$= \sum_{r,s} B_{r^{-1}}B_t B_s e_{r,ts}\delta_1 = \sum_{r,s'} B_{r^{-1}}B_t B_{t^{-1}s'}e_{r,s'}\delta_1 = \sum_{r,s} B_{r^{-1}}D^B_t B_s e_{r,s}\delta_1$$

$$= I_t\delta_1 = I_t\beta_t(I_{t^{-1}})\delta_1 = (I_t\delta_t)(I_{t^{-1}}\delta_{t^{-1}}).$$

On the other hand:

$$(Y'_t\delta_t)(X'_{t^{-1}}\delta_{t^{-1}}) = Y'_t\beta_t(X'_{t^{-1}})\delta_1 = \sum_{r,s} D^B_t B_r B_{s^{-1}}B_{t^{-1}}e_{1,r}e_{ts,1}\delta_1$$

$$= D^B_t\sum_{r,s}[r = ts]B_r B_{r^{-1}t}B_{t^{-1}}e_{1,1}\delta_1 = D^B_t\sum_r D^B_r e_{1,1}\delta_1$$

$$= D^B_t e_{1,1}\delta_1 = (B_t e_{1,t}\delta_t)(B_{t^{-1}}e_{1,t^{-1}}\delta_{t^{-1}}). \quad \square$$

Corollary 7.11. *If $B = \oplus_{t\in G}B_t$ is a strongly-graded algebra, then B is strongly-graded-equivalent to $(B\#G)\rtimes_{\beta^B}G$.*

By [19, (15)] the crossed product by a twisted partial group action is partially-strongly-graded. Thus we have:

Corollary 7.12. *The crossed product by any twisted partial group action is strongly-graded-equivalent to the skew group algebra of a product partial action.*

Theorem 7.13. *Let $\mathsf{M} = (A, B, X, Y, \tau_A, \tau_B)$ be a graded Morita context between G-graded algebras A and B. Let $C = \oplus_{t\in G}C_t$ be the corresponding graded context algebra, so $C_t = \begin{pmatrix} A_t & X_t \\ Y_t & B_t \end{pmatrix}$ Then:*

(1) *The algebras $C\#G$ and $\begin{pmatrix} A\#G & X\#G \\ Y\#G & B\#G \end{pmatrix}$ are isomorphic, and the restrictions of β^C to $A\#G$ and $B\#G$ are β^A and β^B respectively, and $X\#G$, $Y\#G$, are β^C-invariant.*
(2) *$\mathsf{M}\#G := (A\#G, B\#G, X\#G, Y\#G)$ is a Morita context, where the traces are given by the product in $C\#G$. If M is a Morita equivalence so is $\mathsf{M}\#G$.*
(3) *If M is a graded-equivalence, then β^A and β^B are Morita equivalent actions, so $(A\#G)\rtimes_{\beta^A}G$ and $(B\#G)\rtimes_{\beta^B}G$ are strongly-graded-equivalent.*

Proof. Since $C_{r^{-1}s} = \begin{pmatrix} A_{r^{-1}s} & X_{r^{-1}s} \\ Y_{r^{-1}s} & B_{r^{-1}s} \end{pmatrix}$, it is clear that the linear map $C\#G \rightarrow \begin{pmatrix} A\#G & X\#G \\ Y\#G & B\#G \end{pmatrix}$ determined by

$$ce_{r,s} = \begin{pmatrix} a & x \\ y & b \end{pmatrix}e_{r,s} \longmapsto \begin{pmatrix} ae_{r,s} & xe_{r,s} \\ ye_{r,s} & be_{r,s} \end{pmatrix},$$

$c \in C_{r^{-1}s}$, is an isomorphism. This proves the first claim of (1). The second assertion of (1) is clear. Since (3) follows directly from (1), (2) and Proposition 7.9, we concentrate on the proof of (2). We will suppose that M is a Morita equivalence and prove that then so is $\mathsf{M}\#G$. Note that, since $AX = X$ and $YA = Y$, for each $t \in G$ we have $X_t = \sum_u A_u X_{u^{-1}t}$ and $Y_t = \sum_u Y_u A_{u^{-1}t}$, due to the fact that X and Y are graded modules. Hence

$$(A\#G)(X\#G) = \sum_{r,s,u,v} A_{r^{-1}s} e_{r,s} X_{u^{-1}v} e_{u,v} = \sum_{r,s,u,v} [s = u] A_{r^{-1}s} X_{s^{-1}v} e_{r,v}$$

$$= \sum_{r,s,v} A_{r^{-1}s} X_{s^{-1}v} e_{r,v} = \sum_{r,u,v} A_u X_{u^{-1}r^{-1}v} e_{r,v} = \oplus_{r,v} X_{r^{-1}v} e_{r,v} = X\#G,$$

which shows that $X\#G$ is a unital left $A\#G$-module. Similarly we obtain that $Y\#G$ is a unital right $B\#G$-module:

$$(Y\#G)(A\#G) = \sum_{r,u,v} Y_u A_{u^{-1}r^{-1}v} e_{r,v} = \oplus_{r,v} Y_{r^{-1}v} e_{r,v} = Y\#G.$$

Let us see that the traces are surjective. Recall that τ_A and τ_B are surjective and respect the gradings, so that $\tau_A(\sum_u (X_u \otimes_B Y_{u^{-1}t})) = A_t$ and $\tau_B(\sum_u (Y_u \otimes_A X_{u^{-1}t})) = B_t$, $\forall t \in G$. Then:

$$(X\#G)(Y\#G) = \sum_{r,s,u,v} X_{r^{-1}s} e_{r,s} Y_{u^{-1}v} e_{u,v}$$

$$= \sum_{r,s,u,v} [s = u] \tau_A(X_{r^{-1}s} \otimes_B Y_{s^{-1}v}) e_{r,v} = \oplus_{r,v} A_{r^{-1}v} e_{r,v} = A\#G.$$

Similar computations show that $(Y\#G)(X\#G) = B\#G$.

Finally:

$$(B\#G)(Y\#G) = (Y\#G)(X\#G)(Y\#G) = (Y\#G)(A\#G) = (Y\#G).$$
$$(X\#G)(B\#G) = (X\#G)(Y\#G)(X\#G) = (A\#G)(X\#G) = (X\#G). \quad \square$$

Proposition 7.14. *Let* (M, γ) *be a Morita equivalence between the (global) actions* β *and* β', *where* $\mathsf{M} = (B, B', X, Y, \tau_B, \tau_{B'})$. *Let* A *be an idempotent ideal of* B *such that* $A\beta_t(A) = \beta_t(A)A$, $\forall t \in G$, *and let* $\alpha := \beta|_A$ *be the product partial action obtained by the restriction of* β *to* A. *Let* $A' := YAX$ *be the ideal that corresponds to* A *via the equivalence* M. *Then*

(1) $\beta'_s(A')\beta'_t(A') = Y\beta_s(A)\beta_t(A)X$, $\forall s, t \in G$.
(2) *Let* $\mathsf{N} = (A, A', X_1, Y_1, \tau_A, \tau_{A'})$, *where* $X_1 := AX$, $Y_1 = YA$, τ_A *is the composition of* τ_B *with the natural map* $X_1 \otimes_{A'} Y_1 \to X \otimes_{B'} Y$, *and* $\tau_{A'}$ *is defined analogously. Then* N *is a Morita equivalence between* A *and* A', *and the corresponding context algebra*

C_{N} *is an idempotent ideal of the context algebra* C_{M} *of* M*, such that* $\gamma_s(C_{\mathsf{N}})\gamma_t(C_{\mathsf{N}}) = \gamma_t(C_{\mathsf{N}})\gamma_s(C_{\mathsf{N}})$*,* $\forall s, t \in G$*.*

(3) *Let* $\alpha' := \beta'|_{A'}$ *and* $\theta := \gamma|_{C_{\mathsf{N}}}$ *be the restrictions of* β' *and* γ *to* A' *and* C_{N} *respectively. Then* (N, θ) *is a Morita equivalence between* α *and* α'*.*

Proof. Since X and Y are γ-invariant we have:

$$\beta'_t(A') = \gamma_t(YAX) = \gamma_t(Y)\gamma_t(A)\gamma_t(X) = Y\beta_t(A)X.$$

Next observe that every $\beta_t(A)$ is idempotent because so is A. Then $B\beta_t(A) = \beta_t(A)$ and, since $XY = B$:

$$Y\beta_s(A)\beta_t(A)X = Y\beta_s(A)(XY\beta_t(A))X = (Y\beta_s(A)X)(Y\beta_t(A)X)$$
$$= \beta'_s(A')\beta'_t(A').$$

We see next that N is a Morita equivalence. Since A and A' are idempotent, then X_1 and Y_1 respectively a left unital A-module and a right unital A'-module. On the other hand:

$$X_1 A' = AX(YAX) = ABAX = AX = X_1, \text{ and similarly } A'Y_1 = Y_1.$$

Reasoning exactly as in the proof of Proposition 7.6, we have that τ_A and $\tau_{A'}$ are surjective. Let us show that C_{N} is an ideal in C_{M}:

$$\begin{pmatrix} A & AXA' \\ A'YA & A' \end{pmatrix} \begin{pmatrix} B & X \\ Y & B' \end{pmatrix} = \begin{pmatrix} AB + AXA'Y & AX + AXA'B' \\ A'YAB + A'Y & A'YAX + A'B' \end{pmatrix}$$
$$= \begin{pmatrix} A + AXY & AX + AXA' \\ A'YA + A'Y & A'A' + A' \end{pmatrix} = \begin{pmatrix} A & AXA' \\ A'YA & A' \end{pmatrix} = C_{\mathsf{N}}.$$

A similar computation shows that also $C_{\mathsf{M}}C_{\mathsf{N}} = C_{\mathsf{N}}$, so C_{N} is an ideal of C_{M}. Next we compute $\gamma_s(C_{\mathsf{N}})\gamma_t(C_{\mathsf{N}})$. Note that (1), with $s = t$, implies that $\beta_t(A)X = X\beta'_t(A')$ and $\beta'_t(A')Y = Y\beta_t(A)$, $\forall t \in G$. Recall also from the proof of Proposition 5.3 that, since $A\beta_t(A) = \beta_t(A)A$ $\forall t$, then $\beta_s(A)\beta_t(A) = \beta_t(A)\beta_s(A)$ $\forall s, t$. Then also $\beta'_s(A')\beta'_t(A') = \beta'_t(A')\beta'_s(A')$ $\forall s, t$, by (1). Then using (8) we have:

$$\gamma_s(C_{\mathsf{N}})\gamma_t(C_{\mathsf{N}}) = \begin{pmatrix} \beta_s(A) & \beta_s(A)X \\ \beta'_s(A')Y & \beta'_s(A') \end{pmatrix} \begin{pmatrix} \beta_t(A) & \beta_t(A)X \\ \beta'_t(A')Y & \beta'_t(A') \end{pmatrix}$$
$$= \begin{pmatrix} \beta_s(A)\beta_t(A) + \beta_s(A)X\beta'_t(A')Y & \beta_s(A)\beta_t(A)X + \beta_s(A)X\beta'_t(A') \\ \beta'_s(A')Y\beta_t(A) + \beta'_s(A')\beta'_t(A')Y & \beta'_s(A')Y\beta_t(A)X + \beta'_s(A')\beta'_t(A') \end{pmatrix}$$
$$= \begin{pmatrix} \beta_s(A)\beta_t(A) & \beta_s(A)\beta_t(A)X \\ \beta'_s(A')\beta'_t(A')Y & \beta'_s(A')\beta'_t(A') \end{pmatrix} = \gamma_t(C_{\mathsf{N}})\gamma_s(C_{\mathsf{N}}).$$

Taking $s = 1 = t$ in the above equalities we get $C_{\mathsf{N}}^2 = C_{\mathsf{N}}$.

To prove (3) note first that Proposition 5.3, together with (1) and (2), show that α, α' and θ are product partial actions. The domains of α, α' and θ are respectively $D_t = A\beta_t(A)$, $D_t' = A'\beta_t'(A')$, and $\bar{D}_t = C_{\mathsf{N}}\gamma_t(C_{\mathsf{N}}) = \begin{pmatrix} A\beta_t(A) & A\beta_t(A)X \\ A'\beta_t'(A')Y & A'\beta_t'(A') \end{pmatrix}$, $\forall t \in G$. So it is clear that $\alpha = \theta|_A$ and $\alpha' = \theta|_{A'}$. Finally, by (1):

$$Y_1 D_t X_1 = Y A\beta_t(A)AX = A'\beta_t'(A') = D_t', \quad \forall t \in G,$$

which ends the proof. □

Theorem 7.15. *Let $\mathsf{M} = (A, B, X, Y, \tau_A, \tau_B)$ be a strong-graded-equivalence between partially-strongly-G-graded algebras A and B, and let γ^A and γ^B be the canonical partial actions of G on I^A and I^B respectively. Then γ^A and γ^B are Morita equivalent partial actions.*

Proof. Let $C = \oplus_{t \in G} C_t$ be the graded Morita ring of M. By Theorem 7.13 we know that $(\mathsf{M}\#G, \beta^C)$ is a Morita equivalence between β^A and β^B. The ideal I^A of $A\#G$ is idempotent and $I^A \beta_t^A(I^A) = \beta_t^A(I^A)I^A$, $\forall t \in G$ (recall Proposition 6.3). Thus all we need to do is to show that $(Y\#G)I^A(X\#G) = I^B$, and then use Proposition 7.14. By Proposition 4.2 and Corollary 4.3, we have for each $r, s \in G$:

$$\sum_{u,v \in G} (Y_{r^{-1}u} e_{r,u})(A_{u^{-1}} A_v e_{u,v})(X_{v^{-1}s} e_{v,s}) = \sum_{u,v \in G} Y_{r^{-1}u} A_{u^{-1}} A_v X_{v^{-1}s} e_{r,s}$$

$$= \sum_{v \in G} Y_{r^{-1}} A_v X_{v^{-1}s} e_{r,s} = Y_{r^{-1}} X_s e_{r,s} = B_{r^{-1}} B_s.$$

Then $(Y\#G)I^A(X\#G) = I^B$. □

Corollary 7.16. *Let A and B be partially-strongly-G-graded algebras, and let γ^A and γ^B be their corresponding canonical partial actions. Then A and B are strongly-graded-equivalent if and only if γ^A and γ^B are Morita equivalent partial actions.*

Proof. Just combine Propositions 4.8, 7.9 with Theorems 7.10, 7.15. □

7.2. Weak equivalence

Proposition 7.9 suggests the following notion:

Definition 7.17. Let α and α' be product partial actions on the algebras A and A' respectively. We say that α and α' are weakly equivalent whenever $A \rtimes_\alpha G$ and $A' \rtimes_{\alpha'} G$ are graded-equivalent.

Of course Morita equivalence implies weak equivalence. From Proposition 3.6 we immediately obtain:

Proposition 7.18. *Weak equivalence of product partial actions is an equivalence relation.*

Proposition 7.19. *Let B and B' be strongly-graded algebras. Then B and B' are strongly-graded-equivalent if and only if they are graded-equivalent.*

Proof. We only need to prove the converse. Since B and B' are strongly-graded algebras we have $I^B = B\#G$ and $I^{B'} = B'\#G$. Hence, by Theorem 7.10, B is strongly-graded-equivalent to $(B\#G) \rtimes_{\beta^B} G$ and B' is strongly-graded-equivalent to $(B'\#G) \rtimes_{\beta^{B'}} G$. Furthermore, thanks to (3) of Theorem 7.13 $(B\#G) \rtimes_{\beta^B} G$ and $(B'\#G) \rtimes_{\beta^{B'}} G$ are strongly-graded-equivalent. Then, in view of Proposition 4.8, we conclude that B and B' are strongly-graded-equivalent. $\square$

8. Globalization of product partial actions

As we show in the present section, much of the work done for regular partial actions in [6, Sections 4 and 5] can be extended to the case of product partial actions.

Definition 8.1. Let α be a product partial action, and β a global action. We say that β is a Morita enveloping action of α if β is a minimal globalization of a product partial action α' which is Morita equivalent with α.

Theorem 8.2. *Let $\alpha = \{D_{t^{-1}} \xrightarrow{\alpha_t} D_t\}_{t \in G}$ be a product partial action of a group G on an algebra A. Then α has a Morita enveloping action. More precisely, if $B = A \rtimes_\alpha G$, and $\alpha' := \gamma^B$ is the canonical partial action of G on I^B, and $\beta' := \beta^B$ is the canonical action on $B\#G$, then α and α' are Morita equivalent product partial actions, and β' is a Morita enveloping action for α.*

Proof. Note that β' is a minimal globalization of $\alpha' = \beta'|_{I^B}$ by Proposition 6.3. Now it is very easy to adapt the proof of [6, Theorem 4.1], keeping in mind Remark 7.7. Indeed, obviously, there is no need to prove property (2) from [6] for the ideals $D'_t = I^B \beta_t^B(I^B)$. Nevertheless, equality (12) from [6] should be used in the proof. However, the latter is an immediate consequence of equality (9) from [6], which in our case is given by Proposition 6.3. The rest of the proof goes without any change. $\square$

In what follows, given a product partial action α on an algebra A, we will denote by β^α the canonical action of G on $(A \rtimes_\alpha G)\#G$, and by $\gamma^\alpha = \beta^\alpha|_{I^{(A \rtimes_\alpha G)}}$ the canonical partial action of G on $I^{(A \rtimes_\alpha G)}$. Thus, according to Theorem 8.2, $\alpha \overset{M}{\sim} \gamma^\alpha$, and β^α is a Morita enveloping action for α. We refer to β^α as the *canonical Morita enveloping action of α*. Note that, in virtue of the comments preceding Remark 5.4 and Proposition 6.6, the correspondences that send α to γ^α and to β^α, as well as to the corresponding skew group algebras, determine functors.

As announced before, the converse of Proposition 7.9 holds:

Theorem 8.3. *Let $\alpha = \{D_{t^{-1}} \overset{\alpha_t}{\to} D_t\}_{t \in G}$ and $\alpha' = \{D'_{t^{-1}} \overset{\alpha'_t}{\to} D'_t\}_{t \in G}$ be product partial actions of G on algebras A and A', respectively. Then the skew group algebras $A \rtimes_\alpha G$ and $A' \rtimes_{\alpha'} G$ are strongly-graded-equivalent if and only if α and α' are Morita equivalent.*

Proof. For the 'only if' part, we have $\alpha \overset{M}{\sim} \gamma^\alpha$ and $\alpha' \overset{M}{\sim} \gamma^{\alpha'}$ by Theorem 8.2 and $\gamma^\alpha \overset{M}{\sim} \gamma^{\alpha'}$ by Theorem 7.15. Then $\alpha \overset{M}{\sim} \alpha'$ by Proposition 7.8. The 'if' part is Proposition 7.9.   □

Corollary 8.4. *Two global actions on idempotent algebras are weakly equivalent if and only if they are Morita equivalent.*

Proof. The claim follows at once from Proposition 7.19 and Theorem 8.3 above.   □

The next two results will show that any Morita enveloping action of a product partial action α is Morita equivalent to the canonical Morita enveloping action β^α of α.

Theorem 8.5. *Let $\alpha = \{\alpha_t : D_{t^{-1}} \to D_t\}_{t \in G}$ and $\alpha' = \{\alpha'_t : D'_{t^{-1}} \to D'_t\}_{t \in G}$ be Morita equivalent product partial actions of G on algebras A and A', respectively. Then $\beta^\alpha \overset{M}{\sim} \beta^{\alpha'}$.*

Proof. By Proposition 7.9 the skew group algebras $A \rtimes_\alpha G$ and $A' \rtimes_{\alpha'} G$ are strongly-graded-equivalent. Then our claim follows from (3) of Theorem 7.13.   □

Proposition 8.6. *Let $\alpha = \{\alpha_t : D_{t^{-1}} \to D_t\}_{t \in G}$ be a product partial action of G on A. If $\beta : G \times B \to B$ is a minimal globalization of α, then $\beta \overset{M}{\sim} \beta^\alpha$.*

Proof. The skew group algebras $A \rtimes_\alpha G$ and $B \rtimes_\beta G$ are graded-equivalent by Theorem 5.7. Then β^α and β^β are Morita equivalent by (3) of Theorem 7.13. Since $B \rtimes_\beta G$ is strongly-graded, and $I^{B \rtimes_\beta G} = (B \rtimes_\beta G)\#G$, then $B \rtimes_\beta G$ is in fact strongly-graded-equivalent to $\big((B \rtimes_\beta G)\#G\big) \rtimes_{\beta^\beta} G$ by Corollary 7.11. Thus β and β^β are Morita equivalent by Theorem 8.3. Since Morita equivalence of actions is an equivalence relation (recall Proposition 7.8), it follows that β^α and β are Morita equivalent.   □

Corollary 8.7. *If β and β' are minimal globalizations of the Morita equivalent product partial actions α and α' respectively, then β and β' are Morita equivalent actions.*

Proof. This follows at once from Theorem 8.5 and Proposition 8.6.   □

Our previous results can be summarized as follows:

Theorem 8.8. *Let $\alpha = \{\alpha_t : D_{t^{-1}} \to D_t\}_{t \in G}$ be a product partial action of G on A. Then α has a Morita enveloping action, which is unique up to Morita equivalence. Moreover, for every Morita enveloping action $\beta : G \times B \to B$ of α, the skew group algebras $A \rtimes_\alpha G$ and $B \rtimes_\beta G$ are graded-equivalent.*

Proof. Theorem 8.2 ensures the existence of a Morita enveloping action for α, and its uniqueness up to Morita equivalence follows from Proposition 7.8 and Corollary 8.7. The last claim is a consequence of Theorem 5.7, Proposition 7.9 and Proposition 3.6. □

To conclude the section, we summarize several characterizations of Morita and weak equivalences of product partial actions.

Proposition 8.9. *Let α and α' be product partial actions of G on the algebras A and A' respectively. Then the following are equivalent:*

(1) α *and* α' *are Morita equivalent.*
(2) $A \rtimes_\alpha G$ *and* $A' \rtimes_{\alpha'} G$ *are strongly-graded-equivalent.*
(3) γ^α *and* $\gamma^{\alpha'}$ *are Morita equivalent.*
(4) $I^A \rtimes_{\gamma^\alpha} G$ *and* $I^{A'} \rtimes_{\gamma^{\alpha'}} G$ *are strongly-graded-equivalent.*

Proof. Theorem 8.3 implies that (1) and (2) are equivalent, as well as (3) and (4). Finally, by Corollary 7.16, (2) and (3) are equivalent. □

Proposition 8.10. *Let α and α' be product partial actions of G on the algebras A and A' respectively, with corresponding Morita enveloping actions β and β', acting on B and B' respectively. Then the following are equivalent:*

(1) α *and* α' *are weakly equivalent.*
(2) β *and* β' *are weakly equivalent.*
(3) γ^α *and* $\gamma^{\alpha'}$ *are weakly equivalent.*
(4) β^α *and* $\beta^{\alpha'}$ *are weakly equivalent.*
(5) α *and* β' *are weakly equivalent.*
(6) $A \rtimes_\alpha G$ *and* $A' \rtimes_{\alpha'} G$ *are graded-equivalent.*
(7) $A \rtimes_\beta G$ *and* $A' \rtimes_{\beta'} G$ *are graded-equivalent.*
(8) $I^A \rtimes_{\gamma^\alpha} G$ *and* $I^{A'} \rtimes_{\gamma^{\alpha'}} G$ *are graded-equivalent.*
(9) $((A \rtimes_\alpha G)\#G) \rtimes_{\beta^\alpha} G$ *and* $((A' \rtimes_{\alpha'} G)\#G) \rtimes_{\beta^{\alpha'}} G$ *are graded-equivalent.*
(10) $A \rtimes_\alpha G$ *and* $B' \rtimes_{\beta'} G$ *are graded-equivalent.*

Moreover, by Corollary 8.4, (2) and (4) above can be replaced by (2') and (4') below:

(2') β *and* β' *are Morita equivalent.*
(4') β^α *and* $\beta^{\alpha'}$ *are Morita equivalent.*

Proof. Combining Theorems 5.7, 8.2, 8.8 and Proposition 7.18, and recalling that Morita equivalence implies weak equivalence, we see that α, β, γ^α and β^α are weak equivalent, as well as α', β', $\gamma^{\alpha'}$, $\beta^{\alpha'}$. Hence the first five assertions are equivalent to each other.

Finally, according to the definition of weak equivalence, the last five sentences are just rephrasings of the first five ones, all the ten sentences are equivalent. □

9. Stabilization of graded algebras

Following [19] we shall say that a (non-necessarily graded) algebra A possesses *orthogonal local units* if there exists a set of (non-necessarily central) pairwise orthogonal idempotents E in A such that

$$A = \bigoplus_{e \in E} Ae = \bigoplus_{e \in E} eA. \tag{9}$$

Algebras A with (9) are also called algebras with enough idempotents (see [24]). Note that algebras with orthogonal local units generalize algebras with a countable set of local units (see [19, p. 3300]).

It is proved in [19] that if A and B are Morita equivalent algebras with orthogonal local units, then there is an isomorphism of algebras

$$\mathrm{FMat}_{\mathcal{X}}(A) \cong \mathrm{FMat}_{\mathcal{X}}(B), \tag{10}$$

where $\mathcal{X}$ is an appropriately chosen infinite set. Furthermore, it is shown in [6] that if A and B are skew group algebras of Morita equivalent regular partial actions of a group G on algebras with orthogonal local units, then the isomorphism in (10) is graded. In the latter case, we now know by Theorem 8.3 that A and B are strongly-graded-equivalent.

Given a G-graded algebra A and a set of indexes $\mathcal{X}$, define a G-grading on $\mathrm{FMat}_{\mathcal{X}}(A)$ by taking the g-homogeneous component of $\mathrm{FMat}_{\mathcal{X}}(A)$ to be $\mathrm{FMat}_{\mathcal{X}}(A_g)$, $g \in G$. Note that this grading is different from that one considered in Theorem 2.1. We give the next:

Theorem 9.1. *Let*

$$A = \bigoplus_{e \in E} Ae = \bigoplus_{e \in E} eA,$$

and

$$B = \bigoplus_{f \in F} Bf = \bigoplus_{f \in F} fB,$$

be partially-strongly-G-graded algebras with orthogonal local units. Suppose that A and B are strongly-graded-equivalent. Then for any infinite set of indexes $\mathcal{X}$, whose cardinality is bigger than or equal to those of E and F, there exists a graded isomorphism of algebras (10).

Proof. Let $(A, B, {}_A X_B, {}_B Y_A, \tau_A, \tau_B)$ be a strong-graded-equivalence between A and B. In particular, A and B are Morita equivalent as non-necessarily graded algebras, so that

[19, Corollary 8.4] implies the existence of an isomorphism of algebras (10), and we need to check that it is a graded isomorphism. If A and B are skew group algebras of regular partial actions of G on algebras with orthogonal local units, this was verified in [6, Theorem 6.1] by showing that the maps involved in the construction of (10) are all graded. It turns out that the arguments given in the proof of [6, Theorem 6.1] can be easily adapted to our more general case. The adaptation has to be done at the starting point of the process and the subsequent steps follow the same way. Since we have a strong-graded-equivalence between A and B, and these algebras are partially-strongly-graded, it follows by Corollary 4.3 that $(A_1, B_1, X_1, Y_1, \tau_A^1, \tau_B^1)$ is a Morita equivalence. Obviously, $E \subseteq A_1$ and $F \subseteq B_1$. Consequently, for any $f \in F$, using the trace map τ_B^1, we may write $f = \sum_{i=1}^{n_f} y_i x_i$, where $x_i = x_i^{(f)} \in X_1$, and $y_i = y_i^{(f)} \in Y_1$, $x_i = x_i f$, and $f y_i = y_i$ for all i. Then it is readily seen that the map

$$\pi_f : A^{n_f} \ni (r_1, r_2, \dots, r_{n_f}) \mapsto \sum r_i x_i \in Xf$$

is a graded epimorphism of left A-modules, as well as its splitting map

$$\rho_f : Xf \ni yf \mapsto (yf y_1, yf y_2, \dots, yf y_{n_f}) \in A^{n_f}.$$

Taking $K_f = \mathrm{Ker}\, \pi_f$ and denoting by μ_f the embedding of K_f in A^{n_f}, we see that all maps in the exact sequences

$$0 \longrightarrow K_f \xrightarrow{\mu_f} A^{n_f} \xrightarrow{\pi_f} Xf \longrightarrow 0,$$

and

$$0 \longleftarrow K_f \xleftarrow{\tau_f} A^{n_f} \xleftarrow{\rho_f} Xf \longleftarrow 0,$$

preserve the G-gradings. Similar maps are constructed replacing A by B and X by Y. The subsequent steps involve the use of the functor $X_B \otimes_-$, application of the above maps to direct summands, rearrangements of direct summands, each time resulting in graded maps, and leading to a graded isomorphism $A^{(\mathcal{X})} \to X^{(\mathcal{X})}$ of left A-modules, which is finitely determined in the sense of [19, Definition 7.3], and whose inverse is also finitely determined. Here $A^{(\mathcal{X})}$ (respectively, $X^{(\mathcal{X})}$) stands for the direct sum of copies of A (respectively, X), indexed by the elements of $\mathcal{X}$. An important point is to interpret the finitely determined isomorphism $A^{(\mathcal{X})} \to X^{(\mathcal{X})}$ as a so-called row and column summable $\mathcal{X} \times \mathcal{X}$-matrix $[\psi]$ over $\mathrm{RCFMat}_{(e,f) \in E \times F}(eXf)$:

$$[\psi] \in \mathrm{RCSumMat}_X(\mathrm{RCFMat}_{(e,f) \in E \times F}(eXf)).$$

Then $[\psi]$ is used to define the maps

$$\Psi : \mathrm{FMat}_{\mathcal{X}}(A) \to \mathrm{FMat}_{\mathcal{X}}(X), \quad \text{and} \quad \Psi' : \mathrm{FMat}_{\mathcal{X}}(B) \to \mathrm{FMat}_{\mathcal{X}}(X),$$

and, with the help of some preliminary results from [19] the isomorphism (10) is obtained as the composition $(\Psi')^{-1} \circ \Psi$. The fact that $A^{(\mathcal{X})} \to X^{(\mathcal{X})}$ is graded implies that each entry of $[\psi]$ belongs to the 1-homogeneous component $eX_1 f$ of eXf. The latter yields that (10) is graded. The details can be seen in the proofs of [19, Theorem 8.2, Corollary 8.4] and in the comments to them given to justify [6, Theorem 6.1]. $\quad\square$

Data availability

No data was used for the research described in the article.

Acknowledgments

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