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Cubic Algebras of Exponent 2: Basic Properties

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ABSTRACT

We introduce a class of baric algebras containing all Bernstein and train algebras of rank 3 satisfying a given train polynomial and establish its basic properties: existence of idempotents, Peirce decomposition, etc.

1 Bernstein and train algebras

Let A be a commutative, not necessarily associative, algebra over an infinite field F with characteristic not 2 and 3. If $\omega : A \to F$ is a nonzero homomorphism, the ordered pair (A, ω) is called a *baric algebra* over F and ω is its *weight* function. For every $a \in A$, $\omega(a)$ is the weight of a. The set N of elements of weight zero is an ideal of codimension 1 in A. The set $H = \{x \in A : \omega(x) = 1\}$ is an affine hyperplane and if $a \in H$, we have a direct sum decomposition $A = Fa \oplus N$. Given $a \in A$, we denote by L_a the linear operator of N defined by $L_a(x) = ax$, for all $x \in N$.

We review two classes of baric algebras which have been extensively studied in the last years. If (A, ω) satisfies the identity

$$(x^2)^2 = \omega(x^2)x^2 \tag{1}$$

it is called a Bernstein algebra. It follows from (1) that ω is the unique nonzero homomorphism from A to the field. The set of idempotent elements of a Bernstein algebra (A, ω) is given by $I(A) = \{x^2 : x \in H\}$ and hence every Bernstein algebra has at least one idempotent. For each idempotent e the linear operator $L_e \colon N \longrightarrow N$ satisfies the equation $2L_e^3 - 3L_e^2 + L_e = 0$ and hence we get the Peirce decomposition $A = Fe \oplus U \oplus Z$ where $U = \{x \in N : 2ex = x\}$ and $Z = \{x \in N : ex = 0\}$. The inclusions $U^2 \subseteq Z$, $UZ \subseteq U$, $Z^2 \subseteq U$, $UZ^2 = (0)$ hold in A. Moreover, we have the identities

$$u^3 = 0 = u(uz) = uz^2 = u^2z^2 = (uz)^2 = (u^2)^2 \tag{2}$$

for all $u \in U$ and $z \in Z$. If $e \in H$ is an idempotent, then every idempotent e' in H is described by $e' = e + \bar{u} + \bar{u}^2$ where $\bar{u} \in U$. Bernstein algebras are closely related to the Bernstein problem stated in [1]. Recent developments in this theory can be found, for example, in [6, 7, 8]. In [15] the reader will find the foundations of the theory.

The second class of baric algebras that we consider consists of train algebras. If the baric algebra (A, ω) satisfies identically some equation of the form

$$x^n + \gamma_1\omega(x)x^{n-1} + \cdots + \gamma_{n-1}\omega(x)^{n-1}x = 0 \tag{3}$$

for some elements $\gamma_1, \ldots, \gamma_{n-1}$ in the field F, it is called a *principal train algebra* (for simplicity *train algebra*). The formal expression $p(x) = x^n + \gamma_1\omega(x)x^{n-1} + \cdots + \gamma_{n-1}\omega(x)^{n-1}x$ is a train polynomial with coefficients $\gamma_1, \ldots, \gamma_{n-1}$ and degree n. In equation (3) the powers x^i are defined by $x^1 = x$ and $x^i = x^{i-1}x$ for $i \geq 2$. When the identity (3) has minimal degree, $p(x)$ is uniquely defined and is called the train equation of A, where the integer n is the *rank* of A. The ideal $N = \{x \in A : \omega(x) = 0\}$ satisfies the monomial identity $x^n = 0$.

We now will assume that (A, ω) is a train algebra of rank ≤ 3, that is, A satisfies the train equation

$$x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x)^2x = 0. \tag{4}$$

If $\gamma \neq \frac{1}{2}$, then the set of idempotent elements is given by

$$I(A) = \left\{\frac{1}{1-2\gamma}(x^2 - 2\gamma x) \,:\, x \in H\right\}. \tag{5}$$

Given an idempotent $e \in A$, the linear operator $L_e : N \to N$ satisfies the equation $2L_e^2 - (1+2\gamma)L_e + \gamma id_N = 0$ and so we obtain the Peirce decomposition $A = Fe \oplus U \oplus V$ where $U = \{x \in N : 2ex = x\}$ and $V = \{x \in N : ex = \gamma x\}$. The following relations hold: $U^2 \subseteq V$, $UV \subseteq U$, $V^2 = (0)$ and

$$u^3 = 0 = u(uv) = v(vu) = v^2 = (uv)^2 \tag{6}$$

for all $u \in U$ and $v \in V$. If $e \in H$ is an idempotent, every other idempotent e' has the form

$$e' = e + \bar{u} + \frac{1}{1-2\gamma}\bar{u}^2. \tag{7}$$

where $\bar{u} \in U$. For more information about train algebras, see [2],[3],[5],[11].

We close this section recalling some well known connections between the theory of train and Bernstein algebras (see [12], [14], [16] and [17]). A Bernstein algebra is Jordan if and only if satisfies the train equation $x^3 - \omega(x)x^2 = 0$. Every Bernstein algebra A such that $A = A^2$ satisfies the equation $x^4 - \frac{3}{2}\omega(x)x^3 + \frac{1}{2}\omega(x)^2x^2 = 0$. It has been proved that if a Bernstein algebra is also train of rank r, it must satisfy $P(a) = 0$ for all $a \in H$ where $P(X) = X^2(X-1)(2X-1)^{r-3} \in F[X]$. In this paper, we establish a new connection between the two theories.

We denote by $F[X_1, \dots, X_m]$ the algebra of all polynomials in m associative commutative indeterminates over the infinite field F and $F\langle X_1, \dots, X_m\rangle$ denotes the algebra of all polynomials in m nonassociative commutative indeterminates. For a polynomial $T(X) \in F\langle X\rangle$ we denote its degree by ∂T.

2 Train algebras of degree n and exponent k

Let $p(x) = x^n + \gamma_1\omega(x)x^{n-1} + \dots + \gamma_{n-1}\omega(x)^{n-1}x$ be a train polynomial of degree n. Consider the class of all baric algebras (A, ω) such that, for some fixed integer $k \geq 1$, the principal powers x^k of every element x in A satisfy $p(x^k) = 0$. Explicitly, we have the identity

$$(x^k)^n + \gamma_1\omega(x^k)(x^k)^{n-1} + \cdots + \gamma_{n-1}\omega(x^k)^{n-1}x^k = 0 \tag{8}$$

for all $x \in A$. Equivalently

$$(x^k)^n + \gamma_1(x^k)^{n-1} + \cdots + \gamma_{n-1}x^k = 0$$

for all $x \in H$. Such algebras will be called *(principal) train algebras of degree n and exponent k*, when n is minimal. For all these algebras the ideal $N = \ker\omega$ satisfies the monomial identity $(x^k)^n = 0$ and this ensures that the weight function ω is the unique nonzero homomorphism from A to F. As an example, baric algebras of degree 2 and exponent 2 satisfy $(x^2)^2 = \omega(x)^2x^2$ so they are exactly the Bernstein algebras. When $k = 1$ we have the usual concept of train algebra.

Let (A, ω) be a baric algebra with an idempotent $e \in H$. Choose an element $x \in N$, a polynomial $T(X) \in F\langle X\rangle$ (or $F[X]$) such that $T(0) = 0$, and consider all the elements of the form $T(e + \lambda x)$, $\lambda \in F$. We collect terms with equal powers of λ and we obtain

$$T(e + \lambda x) = T(1)e + \sum_{i=1}^{\partial T} \lambda^i D_x^i T(e).$$

For simplicity, denote $D_x^1 T(e)$ by $D_x T(e)$. It was proved in [9], as a reformulation of a result in [10], that for every $T(X) \in F[X]$ satisfying $T(0) = T(1) = 0$, the vector $D_x T(e)$ is equal to $\underline{T}(L_e)(x)$ where $\underline{T}(X) := (2X-1)X^{-1}(X-1)^{-1}T(X)$. On the other hand, we can prove, using induction on k, that if $T(X) = X^k$, then $D_x T(e) = (2L_e^{k-1} + L_e^{k-2} + \cdots + L_e^2 + L_e)(x)$. These two facts prove the following

Lemma 1. *If $P(X) \in F[X]$ satisfies $P(0) = P(1) = 0$, and $T(X) = P(X^k)$, then $D_xT(e) = \underline{P}(L_e) \circ (2L_e^{k-1} + L_e^{k-2} + \cdots + L_e^2 + L_e)(x)$.*

Consider the polynomial $P(X) := X^n + \gamma_1 X^{n-1} + \cdots + \gamma_{n-1}X \in F[X]$. In view of the above lemma we have

Theorem 1. *Let (A, ω) be a train algebra satisfying (8). If e is an idempotent, then for all $x \in N$ we have $\underline{P}(L_e) \circ (2L_e^{k-1} + L_e^{k-2} + \cdots + L_e^2 + L_e)(x) = 0$.*

Corollary 1. *Let (A, ω) be a train algebra satisfying (8) for $k = 2$. Then for every idempotent e and every vector $x \in N$ we have that $L_e \circ \underline{P}(L_e)(x) = 0$.*

We will be concerned only with the case $n = 3$ and $k = 2$, that is, we study the class of baric algebras satisfying the equation

$$(x^2)^3 - (1+\gamma)\omega(x^2)(x^2)^2 + \gamma\omega(x^2)^2x^2 = 0 \tag{9}$$

For simplicity, they will be called *cubic of exponent 2*. This class obviously contains all train algebras satisfying (10). Moreover, if we rewrite equation (9) as

$$x^2\left((x^2)^2 - \omega(x)^2x^2\right) = \gamma\omega(x)^2\left((x^2)^2 - \omega(x)^2x^2\right)$$

it becomes clear that all Bernstein algebras also belong to this class, for all γ. Apart from these subclasses, there are many other examples. For instance, according to [4], Theorem 2, the join of a Bernstein and a train algebra satisfying (4) will belong to this class. The following example is neither Bernstein nor train of rank 3. Let A be a 4-dimensional commutative algebra with basis $\{e, u, z, v\}$ and multiplication table given by $e^2 = e$, $eu = \frac{1}{2}u$, $ev = \gamma v$, $z^2 = \delta v$ and other products are equal to zero, where $\delta \in F$. If we define ω by $\omega(e) = 1$, $\omega(u) = \omega(z) = \omega(v) = 0$, then (A, ω) satisfies (9).

Observe that the first linearization of (9) is

$$\begin{aligned}(x^2)^2(xy) + 2(x^2(xy))x^2 - (1+\gamma)[2\omega(x^2)x^2(xy) + \omega(xy)(x^2)^2]\\ +\gamma[\omega(x)^4xy + 2\omega(x^3y)x^2] = 0\end{aligned} \tag{10}$$

and the second linearization is

$$\begin{aligned}&(x^2)^2(yz) + 4(x^2(xz))(xy) + 4(x^2(xy))(xz) + 2(x^2(yz))x^2 + 4((xy)(xz))x^2\\ &= (1+\gamma)[2\omega(x^2)x^2(yz) + 4\omega(x^2)(xy)(xz) + 4\omega(xz)x^2(xy) + 4\omega(xy)x^2(xz)\\ &+\omega(yz)(x^2)^2] - \gamma[\omega(x)^4yz + 4\omega(x^3z)(xy) + 4\omega(x^3y)(xz) + 6\omega(x^2yz)x^2].\end{aligned}$$

In particular, for $x, y, z \in H$ and $y = z$, we obtain

$$\begin{aligned}(x^2)^2y^2 + 8(x^2(xy))(xy) + 2x^2(x^2y^2) + 4x^2(xy)^2\\ -(\gamma+1)[2x^2y^2 + 4(xy)^2 + 8x^2(xy) + (x^2)^2]\\ +\gamma[y^2 + 8xy + 6x^2] = 0\end{aligned} \tag{11}$$

As baric algebras satisfying (1) or (4) with $2\gamma \neq 1$ always have idempotents of weight 1, it is natural to expect that the same holds for equation (9). For this, we need the following lemmas.

Lemma 2. *Let B be a commutative algebra that satisfies the identity $(x^2)^3 = 0$ for all $x \in B$. If $\dim B^2 \leq 2$, then $\dim(B^2)^2 \leq 1$ and $((B^2)^2)^2 = (0)$.*

Proof. First we assume that B satisfies the identity $x^2x^2 = 0$. Linearizing this identity we obtain

$$x^2(xy) = 0, \qquad x^2y^2 + 2(xy)^2 = 0. \tag{12}$$

If $\dim B^2 \leq 1$, then the result is clear. In the other case there exist $a, b \in B$ such that a^2, b^2 is a basis of B^2. Obviously, $(a^2)^2 = (b^2)^2 = 0$. In view of the first identity of (12) $a^2(ab) = 0$ and $b^2(ab) = 0$. Then, because $ab \in \langle a^2, b^2 \rangle$, we get that $(ab)^2 = 0$. Now the second identity of (12) implies that $a^2b^2 = -2(ab)^2 = 0$. Finally, we assume that there exists $a \in B$ such that $a^2a^2 \neq 0$. Then $u := a^2$ and $v := a^2a^2$ are linearly independent and hence form a basis of B^2. Therefore, $u^2 = v$, $uv = 0$ and $v^2 = \alpha u + \beta v$ with $\alpha, \beta \in F$. We now have that $0 = (u^2)^3 = v^3 = v(\alpha u + \beta v) = \beta v^2 = \beta(\alpha u + \beta v)$, so $\beta = 0$. On the other hand, $0 = ((u+v)^2)^3 = (\alpha u + v)^3 = (\alpha u + v)(\alpha u + \alpha^2 v) = \alpha^3 u + \alpha^2 v$, which implies that $\alpha = 0$. ☐

Corollary 2. *Let B be a commutative algebra satisfying the identity $(x^2)^3 = 0$. If $\dim B \leq 2$, then $\dim B^2 \leq 1$ and $(B^2)^2 = (0)$.*

In the following we denote by Q the set $\{x^2 \ : x \in H\}$.

Lemma 3. *If A satisfies (9) with $\gamma \neq 0$, then for all $x \in Q$ we have the identity*

$$(x^2)^2x = -(1+\gamma)(x^2)^2 + (4\gamma^2 + 4\gamma + 2)x^2 - \gamma(4\gamma + 3)x.$$

Proof. Replacing $y = x^2$ in (11), we obtain:
$0 = (x^2)^2x^3 + 2x^2(x^2x^3) - (\gamma+1)[2x^2x^3 + (x^2)^2] + \gamma[x^3 + 2x^2] = (\gamma+1)(x^2)^3 - \gamma(x^2)^2 + 2(\gamma+1)(x^2)^3 - 2\gamma x^2x^3 - 2(\gamma+1)^2(x^2)^2 + 2\gamma(\gamma+1)x^3 - (\gamma+1)(x^2)^2 + \gamma(\gamma+1)x^2 - \gamma^2 x + 2\gamma x^2 = -\gamma(x^2)^2x + 3(\gamma+1)(x^2)^3 - 2\gamma x^2x^3 - (\gamma+1)(2\gamma+3)(x^2)^2 + 2\gamma(\gamma+1)x^3 + \gamma(\gamma+3)x^2 - \gamma^2 x = -\gamma(x^2)^2x + 3(\gamma+1)^2(x^2)^2 - 3\gamma(\gamma+1)x^2 - 2\gamma(\gamma+1)(x^2)^2 + 2\gamma^2x^3 - (\gamma+1)(2\gamma+3)(x^2)^2 + 2\gamma(\gamma+1)^2x^2 - 2\gamma^2(\gamma+1)x + \gamma(\gamma+3)x^2 - \gamma^2 x = -\gamma(x^2)^2x - \gamma(\gamma+1)(x^2)^2 + 2\gamma^2x^3 + 2\gamma(\gamma^2+\gamma+1)x^2 - \gamma^2(2\gamma+3)x = -\gamma(x^2)^2x - \gamma(\gamma+1)(x^2)^2 + 2\gamma^2(\gamma+1)x^2 - 2\gamma^3 x + 2\gamma(\gamma^2 + \gamma + 1)x^2 - \gamma^2(2\gamma+3)x = -\gamma(x^2)^2x - \gamma(\gamma+1)(x^2)^2 + 2\gamma(2\gamma^2 + 2\gamma + 1)x^2 - \gamma^2(4\gamma+3)x$. So $(x^2)^2x = -(\gamma+1)(x^2)^2 + (4\gamma^2 + 4\gamma + 2)x^2 - \gamma(4\gamma+3)x$, as desired. ☐

Lemma 4. *If A satisfies (9), then for every $x \in Q$ we have:*
a) If $\gamma = 0$, then $((x^2)^2)^2 = (x^2)^2$;
b) If $\gamma \neq 0$, $((x^2)^2)^2 = (1 + 2\gamma - 4\gamma^2)(x^2)^2 + 2\gamma(4\gamma^2 + 2\gamma - 1)x^2 - 8\gamma^3 x$.

Proof. If $\gamma = 0$, replacing y by x^2 in (11) we obtain:

$$\begin{aligned}0 &= ((x^2)^2)^2 + 8(x^2x^3)x^3 + 2(x^2)^4 + 4x^2(x^3)^2 - [2(x^2)^3 + 4(x^3)^2 + 8x^2x^3 + (x^2)^2] \\ &= ((x^2)^2)^2 + 8(x^2)^2 + 2(x^2)^2 + 4(x^2)^2 - 2(x^2)^2 - 4(x^2)^2 - 8(x^2)^2 - (x^2)^2 \\ &= ((x^2)^2)^2 - (x^2)^2\end{aligned}$$

Substituting $y = x^2$ in (11) and using Lemma 3 we prove (b). ☐

Now, we assume that $\gamma \neq 0$. For $x \in Q$ we consider the elements $e := x$, $f := x^2 - x$ and $h := (x^2)^2 - x^2$. Then we have:

$$
\begin{aligned}
e^2 &= e+f \\
ef &= x^3-x^2=(\gamma+1)x^2-\gamma x-x^2=\gamma f \\
eh &= (x^2)^2x-x^3=(x^2)^2x-(\gamma+1)x^2+\gamma x-(\gamma+1)(x^2)^2+(4\gamma^2+3\gamma+1)x^2 \\
&\quad -2\gamma(2\gamma+1)x-(\gamma+1)h+(4\gamma^2+2\gamma)f=2\gamma(2\gamma+1)f-(\gamma+1)h \\
f^2 &= (x^2-x)^2=-x(x^2-x)+(x^2)^2-x^3=-\gamma(x^2-x)+(x^2)^2-(\gamma+1)x^2+\gamma x= \\
&\quad (x^2)^2-(2\gamma+1)x^2+2\gamma x=-2\gamma f+h \\
fh &= (x^2-x)((x^2)^2-x^2)=x^2((x^2)^2-x^2)-(x^2)^2x+x^3=\gamma((x^2)^2-x^2) \\
&\quad -(x^2)^2x+(\gamma+1)x^2-\gamma x=(2\gamma+1)(x^2)^2-(4\gamma^2+4\gamma+1)x^2+2\gamma(2\gamma+1)x= \\
&\quad (2\gamma+1)(-2\gamma f+h) \\
h^2 &= ((x^2)^2-x^2)^2=-x^2((x^2)^2-x^2)+((x^2)^2)^2-(x^2)^3=-\gamma((x^2)^2-x^2) \\
&\quad +((x^2)^2)^2-(\gamma+1)(x^2)^2+\gamma x^2=-4\gamma^2(x^2)^2+4\gamma^2[2\gamma+1]x^2-8\gamma^3x= \\
&\quad -4\gamma^2[(x^2)^2-x^2]+8\gamma^3(x^2-x)=-4\gamma^2(h-2\gamma f)
\end{aligned}
$$

These relations show that the subspace B generated by e, f and h is a baric subalgebra with $\dim B \le 3$. Taking $g := h-2\gamma f$, we have the multiplication table:

$$e^2=e+f,\ ef=\gamma f,\ eg=-(\gamma+1)g,\ f^2=g,\ fg=g,\ g^2=-4\gamma(1+2\gamma)g.$$

Therefore, $\{e,f,g\}$ is a generator system of the subalgebra B and we remark that f and g span the subalgebra $\ker(\omega)\cap B$. By Corollary 2, $0=(f^2)^2=-4\gamma(1+2\gamma)g$ and so for $\gamma\neq-\frac{1}{2}$, we have that $g=0$. Finally we assume that $\gamma=-\frac{1}{2}$. Then

$$e^2=e+f,\ ef=-\frac{1}{2}f,\ eg=-\frac{1}{2}g,\ f^2=g,\ fg=g,\ g^2=0$$

Let $a=e+\lambda f$ with $\lambda\in F$. Since B is a cubic algebras of exponent 2, we have that $0=2(a^2)^3-((a^2)^2+a^2)=-2\lambda^3(1-\lambda)g$ and hence $g=0$. Thus, we have proved that $g=0$ for all cubic algebra of exponent 2 with γ different from zero.

One consequence of the above relation and (a) of Lemma 4 is the following relevant fact, which establishes that if A is a cubic algebra of exponent 2, then A is train for the plenary powers.

Theorem 2. *If (A,ω) is cubic of exponent 2 then $((x^2)^2)^2-(1+2\gamma)\omega(x)^4(x^2)^2+2\gamma\omega(x)^6x^2=0$, for all $x\in A$.*

The following Theorem generalizes the form of the idempotents of a Bernstein algebra and of a train algebra of rank ≤ 3.

Theorem 3. *If (A,ω) is a cubic algebra of exponent 2 with $\gamma\neq\frac{1}{2}$, then A has idempotent elements. Moreover, the set of idempotent elements is given by*

$$I(A)=\left\{\frac{1}{1-2\gamma}\Big((x^2)^2-2\gamma x^2\Big)\ :\ x\in H\right\}.$$

Proof. If $\gamma=0$, from (a) of Lemma 4, $(x^2)^2$ is an idempotent. Let $\gamma\neq0$. Now, for $x\in H$, if x^2 is an idempotent then the result is obvious. Otherwise, $e=x^2$ and $f=(x^2)^2-x^2$ are linearly independent and span a subspace B which is a subalgebra with multiplication table:

$$e^2=e+f,\qquad ef=\gamma f,\qquad f^2=0.$$

Finally, one element $a = e + \lambda f$ is an idempotent if and only if $(1-2\gamma)\lambda - 1 = 0$, and so $a = ((x^2)^2 - 2\gamma x^2)/(1-2\gamma)$. □

Example 1. For the case $\gamma = \frac{1}{2}$ we show an example without idempotent elements. Let A be an algebra with basis $\{e, f\}$ and multiplication table

$$e^2 = e + f, \qquad ef = \frac{1}{2}f, \qquad f^2 = 0.$$

For the weight function $\omega(\alpha e + \beta f) = \alpha$, A is a cubic algebra of exponent 2 with $\gamma = \frac{1}{2}$, but A has not idempotent elements, as it is easily verified.

In the next, we will assume $\gamma \neq \frac{1}{2}$ and also that $\gamma \neq 0$.

Proposition 1. *Let (A, ω) be a cubic algebra of exponent 2 satisfying (9) and let e be an idempotent of weight 1 in A. Then for every $x \in N$, we have the following identities:*

(a) $2e(e(ex)) = (1+2\gamma)e(ex) - \gamma ex$

(b) $2e(ex^2 + 2(ex)^2) + 8(ex)(e(ex)) = (1+2\gamma)ex^2 + 4(1+\gamma)(ex)^2 - \gamma x^2$

(c) $e((ex)x^2) + (ex)(ex^2) + 2(ex)^3 + (e(ex))x^2 = (1+\gamma)(ex)x^2$

(d) $e(x^2)^2 + 8(ex)((ex)x^2) + 2(ex^2)x^2 + 4(ex)^2x^2 = (1+\gamma)(x^2)^2$

(e) $(ex)(x^2)^2 + 2((ex)x^2)x^2 = 0$

(f) $(x^2)^3 = 0$.

Conversely if (A, ω) is any baric algebra with an idempotent e of weight 1, decomposed as $A = Fe \oplus N$ such that the above conditions (a), ..., (f) hold for every $x \in N$, then (A, ω) is cubic of exponent 2 satisfying (9).

Proof. (*sketch*) Take $a = \alpha e + x$, $\alpha \in F$ and $x \in N$. When x is replaced in (9) and if we compare the coefficients of $\alpha^0, \alpha^1, \ldots, \alpha^5$, we get equations $(a), \ldots, (f)$. The converse is carried in the same way. □

Let e be an idempotent of the cubic algebra A of expoent 2. In view of Corollary 1 we have the Peirce decomposition $A = Fe \oplus U \oplus Z \oplus V$ where

$$U := \{x \in N : 2ex = x\},\ Z := \{x \in N : ex = 0\},\ V := \{x \in N : ex = \gamma x\}.$$

Theorem 4. *Let (A, ω) be a cubic algebra of exponent 2 satisfying (9). Then the dimensions of U_e, Z_e and V_e are invariant on the algebra, that is, their dimensions are independent of the idempotent e chosen in H.*

Proof. Suppose that $a \in H$ is a nonzero idempotent and $\{a_1, a_2, \ldots, a_{n-1}\}$ is a basis of $N = \ker\omega$. Consider the mapping f of F^{n-1} onto $I(A)$ defined by the following expression. We take a $n-1$-uple $(\lambda_1, \ldots, \lambda_{n-1}) \in F^{n-1}$, define $b = \sum_{k=1}^{n-1} \lambda_k a_k$ and then define f by

$$f(\lambda_1, \ldots, \lambda_{n-1}) = \frac{1}{1-2\gamma}\left(\left((a+b)^2\right)^2 - 2\gamma\,(a+b)^2\right).$$

and the endomorphism of N into itself defined by $L_f(x) := f(\lambda_1, \ldots, \lambda_{n-1})x$ for all $x \in N$. Obviously, the characteristic polynomial of L_f can be written as follows

$$\sum_{t=0}^{n-1} \theta_t(\lambda_1, \ldots, \lambda_{n-1})X^t$$

where $\theta_t(\lambda_1, \dots, \lambda_{n-1}) \in F[\lambda_1, \dots, \lambda_{n-1}]$. On the other hand, by Corollary 1, the minimal polynomial of the left multiplication by an idempotent, restricted to N, divides $X(X-\frac{1}{2})(X-\gamma)$. Therefore, for each $\lambda_1, \dots, \lambda_{n-1} \in F$ the characteristic polynomial of $L_{f(\lambda_1,\dots,\lambda_{n-1})}$ belongs to the finite set $\{(X-\frac{1}{2})^{n(1)}(X-\gamma)^{n(2)}X^{n(3)} : n(1)+n(2)+n(3) = n-1\}$. This implies that $\mathrm{Im}(\theta_t)$ is finite. Then, as the image of a polynomial function is either infinite or an unique point, the polynomials $\theta_t(\lambda_1, \dots, \lambda_{n-1})$ are constant. This proves that the characteristic polynomial of L_e must be independent of e. □

Now we can associate with each cubic algebra of exponent 2 with $\gamma \neq 0, \frac{1}{2}$, $A = Fe \oplus U \oplus Z \oplus V$, the *type* (m, r, s) of the algebra where $m - 1 := \dim U$, $r := \dim Z$ and $s := \dim V$. Clearly the sum of the three integers of the type of A equals the dimension of A.

A generalization of the multiplicative structure of a Bernstein and train algebra of rank ≤ 3, is given by the following result.

Theorem 5. *Let (A, ω) be a cubic algebra of exponent 2 satisfying (9) and let $A = Fe \oplus U \oplus Z \oplus V$ be its Peirce decomposition relative to the idempotent e. The following relations hold:*

$$U^2 \subseteq Z \oplus V, \quad UV \subseteq U \oplus Z, \quad NZ \subseteq U \oplus V, \quad V^2 = (0).$$

Proof. We consider the second linearization of (9). Replacing $x \longrightarrow e$ and $z \longrightarrow x$ and assuming that $x, y \in N$, we have the identity

$$\begin{aligned} e(yx) + 4(e(ex))(ey) + 4(e(ey))(ex) + 2e(e(yx)) + 4e((ey)(ex)) \\ = 2(1+\gamma)[e(yx) + 2(ex)(ey)] - \gamma yx \end{aligned} \tag{13}$$

In particular, if $x \in Z$, this equation reduces to $\gamma(yx) = (1+2\gamma)e(yx) - 2e(e(yx))$, all $y \in N$. Now express yx as $yx = u_0 + z_0 + v_0$, where $u_0 \in U$, $z_0 \in Z$ and $v_0 \in V$. Then $\gamma(u_0 + z_0 + v_0) = (1+2\gamma)e(u_0 + z_0 + v_0) - 2e(e(u_0 + z_0 + v_0)) = (1+2\gamma)(1/2u_0 + \gamma v_0) - 2e(1/2u_0 + \gamma v_0) = (1+2\gamma)(1/2u_0 + \gamma v_0) - 1/2u_0 - 2\gamma^2 v_0 = \gamma(u_0 + v_0)$. By comparison of components we have $z_0 = 0$. Then $yx = u_0 + v_0 \in U \oplus V$ for all $y \in N$. In particular, we have $NZ \subseteq U \oplus V$.

Again from (13), taking $y \in U$ we have $e(yx) + 2y(e(ex)) + y(ey) + 2(e(e(yx)) + 2e(y(ey)) = 2(1+\gamma)[e(yx) + y(ex)] - \gamma yx$. For $x \in U$, this equation becomes $e(e(yx)) = \gamma e(yx)$. Calling $yx = u_0 + z_0 + v_0$ with $u_0 \in U$, $z_0 \in Z$ and $v_0 \in V$, we have, as above, $u_0 = 0$ so that $U^2 \subseteq Z \oplus V$. Again from (13), if $x \in V$, we have $e(yx) = 2e(e(yx))$ and calling $yx = u_0 + z_0 + v_0$ as above, we get $v_0 = 0$ so that $UV \subseteq U \oplus Z$. Finally, replace $y, x \in V$ in (13) to get $\gamma(4\gamma^2 - 4\gamma + 1)x = (1 + 2\gamma - 4\gamma^2)e(yx) - 2e(e(yx))$ and again from $yx = u_0 + z_0 + v_0$ we get $u_0 = z_0 = v_0 = 0$ so that $yx = 0$ and $V^2 = (0)$. □

Replacing x by $u + z + v$ in (c), (d) and (e) of Proposition 1 we obtain, among others, the following identities:

$$eu^3 + (eu^2)u = \gamma u^3 \tag{14}$$

$$2e(u(uz)) + 2(e(uz))u = (2\gamma + 1)u(uz) \tag{15}$$

$$2e(uz^2) + 2(ez^2)u = (2\gamma + 1)uz^2 \tag{16}$$

$$\begin{gathered} 4e(uz)^2 + 2e(u^2z^2) + 2u(uz^2) + 2(eu^2)z^2 + 8(e(uz))(uz) \\ +2(ez^2)u^2 + u^2z^2 = 2(1+\gamma)(2(uz)^2 + u^2z^2) \end{gathered} \tag{17}$$

$$2e(u(uv)) + 2(e(uv))u + 2\gamma e(u^2v) = u(uv) + \gamma u^2 v \tag{18}$$

$$e((uv)v) + (e(uv))v = (1-\gamma)(uv)v \tag{19}$$

$$\begin{gathered} e(uv)^2 + 2(e(uv))(uv) \\ = -2\gamma((uv)v)u - 2\gamma((uv)u)v - 2\gamma^2(u^2v)v + (1-\gamma)(uv)^2 \end{gathered} \tag{20}$$

$$2u(u^2(uv)) + 2u^2(u(uv)) + 2(uv)u^3 + \gamma(2u^2(u^2v) + v(u^2)^2) = 0 \tag{21}$$

$$e(z^2v) + (ez^2)v = z^2 v \tag{22}$$

$$e((zv)v) + (e(zv))v = (zv)v \tag{23}$$

$$((uv)v)v = 0 \tag{24}$$

$$((zv)v)v = 0 \tag{25}$$

$$u(u^2(uz)) + u^2(u(uz)) + (uz)u^3 = 0 \tag{26}$$

$$u(u^2z^2) + u^2(uz^2) + u^3z^2 + 2(u(uz)^2 + 2(uz)(u(uz))) = 0 \tag{27}$$

We can also use equations (e) and (f) to get more complicated identities of degree 5 and 6 in u, v, z but we do not pursue in this direction. Some of the identities will be used now.

At this point a natural question is to ask under which conditions a cubic baric algebra of exponent 2 is Bernstein or train.

Proposition 2. *The cubic baric algebra $A = Fe \oplus U \oplus Z \oplus V$ is Bernstein if and only if $V = (0)$.*

Proof. We assume that $V = (0)$. In view of Theorem 5 we have that $U^2 \subseteq Z$ and $UZ \oplus Z^2 \subseteq U$. We note that A is Bernstein if and only if satisfies the relations (2). From (14), (15) and (16) we obtain $u^3 = 0$, $u(uz) = 0$ and $uz^2 = 0$ respectively. Next, because $x^3 = 0$ for all $x \in U$ and $z^2 \in U$, we have that $u^2z^2 = -2u(uz^2) = 0$ since we proved above that $uz^2 = 0$. Using the identity $u^2z^2 = 0$ and (17) we get $(uz)^2 = 0$. Finally, from (d) of Proposition primpro for $x \longrightarrow u$ we have that $(u^2)^2 = 0$. The converse is trivial. □

Proposition 3. *The cubic baric algebra $A = Fe \oplus U \oplus Z \oplus V$ is train if and only if $Z = (0)$.*

Proof. We now assume that $Z = (0)$. In view of Theorem 5 we have that $U^2 \subseteq V$ and $UV \subseteq U$ and $V^2 = (0)$. We note that A is train if and only if it satisfies the relations (6). By (14) we have that $u^3 = 0$. Next, from (18), (19) and (20) we obtain that $u(uv) = 0$, $v(vu) = 0$ and $(uv)^2 = 0$ respectively. The converse is trivial. □

Proposition 4. *A 3-dimensional baric algebra satisfying (9) with $\gamma \neq 0, \frac{1}{2}$ is Bernstein or train of rank ≤ 3 or isomorphic, to one of the form $A = Fe \oplus Fz \oplus Fv$ from the following list:*

$$\begin{aligned} A(1):&\quad e^2 = e, \quad ez = 0, \quad ev = \gamma v, \\ A(2):&\quad e^2 = e, \quad ez = 0, \quad ev = \gamma v, \quad z^2 = v \\ A(3):&\quad e^2 = e, \quad ez = 0, \quad ev = \gamma v, \quad zv = v, \\ A(4):&\quad e^2 = e, \quad ez = 0, \quad ev = \gamma v, \quad z^2 = v, \quad zv = v, \end{aligned}$$

and the rest of the products are equal to zero.

3 A particular case

We have already remarked that the join of a train algebra $A_1 = Fe_1 \oplus U_1 \oplus V$ satisfying (4) and a Bernstein algebra $A_2 = Fe_2 \oplus U_2 \oplus Z$, is cubic of exponent 2. In this case, $A_1 \times A_2 = Fe \oplus (U_1 \oplus U_2) \oplus Z \oplus V$ and, for instance, $(U_1 \oplus U_2)Z \subseteq U_2, ZV = 0$, etc. In this case, the inclusions in Theorem 5 are simplified.

In this section we present some results for a cubic algebra of exponent 2 when we know the existence of one idempotent element e for which the following multiplicative structure holds:

$$U^2 \subseteq Z \oplus V, \quad UZ + UV + Z^2 \subseteq U, \quad ZV = (0), \quad V^2 = (0). \tag{28}$$

The above relations include all Bernstein and all train algebras of rank ≤ 3. The join of a Bernstein and a train algebra of rank ≤ 3 is also a cubic algebra of exponent 2 with this multiplicative structure.

We know that relations (14) to (27) hold in this case, but we have also:

Proposition 5. *Suppose A is a cubic algebra of exponent 2, e an idempotent for which (28) holds. Then we have the following identities for every $u \in U$, $z \in Z$ and $v \in V$:*

$$(uv)v = u^2v = (uv)^2 = u^4 = (u^2)^2 = 0 \tag{29}$$
$$u(u(u(uv))) = u(u(u(uz))) = u(u(uz^2)) = 0 \tag{30}$$
$$2(eu^2)u + (1-2\gamma)u^3 = 0 \tag{31}$$
$$u(u^2z^2) + u^3z^2 = 0 \tag{32}$$
$$u(uz)^2 + 2(u(uz))(uz) = 0 \tag{33}$$
$$u(uv) \in Z \quad \text{and} \quad u(uz),\ uz^2,\ u(u^2z) \in V. \tag{34}$$

Proof. The identities (29) and (31) are obtained from (19), (28), (20), (??) and (14) together with (28). For example, from (??) and (28) we get that $2u^4 + (1-2\gamma)(u^2)^2 = 0$. Since $u^4 \in Z \oplus V$ and $(u^2)^2 \in U$ we obtain that $u^4 = 0$ and $(u^2)^2 = 0$. The linearizations of $u^4 = 0$ and $(u^2)^2 = 0$ are given by

$$u^3u' + (u^2u')u + 2u(u(uu')) = 0, \quad u^2(uu') = 0. \tag{35}$$

Taking $u' = uv$, we obtain $u^3(uv) + u(u^2(uv)) + 2u(u(u(uv))) = 0$ and looking now to (21) and considering that $u^2v = (u^2)^2 = u^2(u(uv)) = 0$ by (28) and (29), we have $u^3(uv) + u(u^2(uv)) = 0$.

This proves $u(u(u(uv))) = 0$. Next, replacing $u' = uz$ in identities of (35) we obtain $u^3(uz) + u(u^2(uz)) + 2u(u(u(uz))) = 0$ and $u^2(u(uz)) = 0$. Now, from (26) we have $u^3(uz) + u(u^2(uz)) = 0$ since we have proved that $u^2(u(uz)) = 0$. Therefore we have $u(u(u(uz))) = 0$.

We know that $u^2(uz^2) = 0$ because $uz^2 \in U$ and hence from (27) we obtain $u(u^2z^2) + u^3z^2 = -2u(uz)^2 - 4(u(uz))(uz) \in (Z \oplus V) \cap U = (0)$. Considering this fact and taking $u' = z^2$ in the first identity of (35) we prove that $u(u(uz^2)) = 0$. □

Theorem 6. *Let (A, ω) be a cubic algebra of exponent 2 and e an idempotent for which (28) holds. The set of idempotent elements of A is given by:*

$$I(A) = \left\{ e + u + u^2 + \frac{2}{1-2\gamma} eu^2 \ : \ u \in U \right\}. \tag{36}$$

Proof. Let e' be another idempotent element of A. Then its decomposition is given by $e' = e + u + z + v$ with uniquely determined $u \in U$, $z \in Z$ and $v \in V$; furthermore (28) implies $e'^2 = e + (u + 2uz + 2uv + z^2) + (u^2 + 2\gamma v)$ with $u + 2uz + 2uv + z^2 \in U$ and $u^2 + 2\gamma v \in Z \oplus V$. Therefore the idempotent condition $e'^2 = e'$ is equivalent to

$$u + 2uz + 2uv + z^2 = u \quad \text{and} \quad u^2 + 2\gamma v = z + v. \tag{37}$$

The first identity implies $2uz + 2uv + z^2 = 0$ and the second identity implies $eu^2 + 2\gamma^2 v = \gamma v$, that is, $eu^2 = \gamma(1 - 2\gamma)v$. Therefore $z = u^2 - (1 - 2\gamma)v = u^2 - \gamma^{-1}eu^2$.

For $z = u^2 - \gamma^{-1}eu^2$ and $v = \gamma^{-1}(1 - 2\gamma)^{-1}eu^2$, equations (28), (29) and (31) say that $2uz + 2uv + z^2 = 0$ and hence (37) is satisfied. Therefore the set of idempotent elements is given by (36). □

If e' is another idempotent of A, we denote as $A = Fe' \oplus U_{e'} \oplus Z_{e'} \oplus V_{e'}$ the Peirce decomposition of A relative to e'. The following Proposition gives the relation between the two decompositions.

Proposition 6. *Suppose A is a cubic algebra of exponent 2 and e an idempotent for which (28) holds. If $e' = e + \bar{u} + \bar{u}^2 + 2(1-2\gamma)^{-1}e\bar{u}^2$, $\bar{u} \in U$, is another idempotent of A, then:*

$$U_{e'} = \left\{ u + 2\bar{u}u + \frac{4}{1-2\gamma} e(\bar{u}u) : u \in U \right\}$$
$$Z_{e'} = \left\{ z - 2\bar{u}z - 2\bar{u}^2 z - \frac{4}{(1-2\gamma)\gamma} \bar{u}\left[\bar{u}(\bar{u}z + \bar{u}^2 z)\right] + \frac{2}{\gamma}\bar{u}(\bar{u}z + \bar{u}^2 z) : z \in Z \right\}$$
$$V_{e'} = \left\{ v - \frac{2}{(1-2\gamma)\gamma} \bar{u}\left[\gamma v + \bar{u}v - 2\bar{u}(\bar{u}v)\right] : v \in V \right\}$$

Proof. Let $x = u + z + v$ be an element in $N = \ker \omega$. Then (28) implies that $e'x$ is equal to

$$\left(\frac{u}{2} + \bar{u}z + \bar{u}v + \bar{u}^2 u + \bar{u}^2 z + \frac{2}{1-2\gamma}(e\bar{u}^2)u \right) \oplus \left(\bar{u}u + \gamma v \right) \tag{38}$$

where the first summand is in U and the second belongs to $Z \oplus V$.

I. The condition $x \in U_{e'}$, that is, $2e'x = x$ is equivalent to

$$\bar{u}z + \bar{u}v + \bar{u}^2u + \bar{u}^2z + \frac{2}{1-2\gamma}(e\bar{u}^2)u = 0, \quad z + (1-2\gamma)v = 2\bar{u}u. \tag{39}$$

Multiplying the last expression by e, we obtain $v = \frac{2\gamma}{1-2\gamma}e(\bar{u}u)$. Now, replacing v in $z+(1-2\gamma)v = 2\bar{u}u$ we get that $z = 2\bar{u}u - \frac{2}{\gamma}e(\bar{u}u)$. Conversely, if $x = u + 2\bar{u}u + \frac{4}{1-2\gamma}e(\bar{u}u)$, then (39) is satisfied since by the first linearization of (31) we have

$$2(e\bar{u}^2)u + 4(e(\bar{u}u))\bar{u} + (1-2\gamma)\bar{u}^2u + 2(1-2\gamma)\bar{u}(\bar{u}u) = 0. \tag{40}$$

II. In view of (38) the condition $x \in Z_{e'}$, that is, $e'x = 0$ is equivalent to

$$u + 2\Big[\bar{u}z + \bar{u}v + \bar{u}^2u + \bar{u}^2z + \frac{2}{1-2\gamma}(e\bar{u}^2)u\Big] = 0, \quad \bar{u}u + \gamma v = 0. \tag{41}$$

Therefore, we obtain $v = -\gamma^{-1}\bar{u}u$. Using this identity and (40) we have that

$$\frac{2}{1-2\gamma}(e\bar{u}^2)u = \frac{2\gamma}{1-2\gamma}\bar{u}v - \bar{u}^2u$$

and replacing this expression in the first identity of (41) we have

$$u + 2\bar{u}z + 2\bar{u}^2z + \frac{2}{1-2\gamma}\bar{u}v = 0. \tag{42}$$

Multiplying by $\bar{u}$ we obtain $\bar{u}u + 2\bar{u}(\bar{u}z) + 2\bar{u}(\bar{u}^2z) + 2(1-2\gamma)^{-1}\bar{u}(\bar{u}v) = 0$. But $\bar{u}u, \bar{u}(\bar{u}z), \bar{u}(\bar{u}^2z) \in V$ and $\bar{u}(\bar{u}v) \in Z$, so $\bar{u}u = -2[\bar{u}(\bar{u}z) + \bar{u}(\bar{u}^2z)] = -\gamma v$ and $v = \frac{2}{\gamma}[\bar{u}(\bar{u}z) + \bar{u}(\bar{u}^2z)]$. Replacing v in (42) we get $u = -2\bar{u}z - 2\bar{u}^2z - 4(1-2\gamma)^{-1}\gamma^{-1}[\bar{u}(\bar{u}(\bar{u}z)) + \bar{u}(\bar{u}(\bar{u}^2z))]$. Conversely, if $x = [-2\bar{u}z - 2\bar{u}^2z - 4(1-2\gamma)^{-1}\gamma^{-1}[\bar{u}(\bar{u}(\bar{u}z)) + \bar{u}(\bar{u}(\bar{u}^2z))]] + z + \frac{2}{\gamma}[\bar{u}(\bar{u}z) + \bar{u}(\bar{u}^2z)]$ then the identities of (30) show that $\bar{u}u + \gamma v = 0$. Now it is easily proved that $u + 2\big[\bar{u}z + \bar{u}v + \bar{u}^2u + \bar{u}^2z + \frac{2}{1-2\gamma}(e\bar{u}^2)u\big] = 0$ using (40).

III. Finally, in view of (38), the condition $x \in V_{e'}$ is equivalent to

$$\frac{1-2\gamma}{2}u + \bar{u}z + \bar{u}v + \bar{u}^2u + \bar{u}^2z + \frac{2}{1-2\gamma}(e\bar{u}^2)u = 0, \quad \bar{u}u - \gamma z = 0. \tag{43}$$

From (40) and considering that $\bar{u}u \in Z$, we obtain

$$\frac{1}{1-2\gamma}(e\bar{u})u = -\bar{u}^2u - 2\bar{u}(\bar{u}u) = -\bar{u}^2u - 2\gamma\bar{u}z.$$

Using this fact together with $\bar{u}^2z = \bar{u}^2(\bar{u}u) = 0$ we obtain from the first identity of (43) that

$$\frac{1-2\gamma}{2}u + (1-2\gamma)\bar{u}z + \bar{u}v = 0.$$

Then $\gamma z = \bar{u}u = -2\bar{u}[\bar{u}z + (1-2\gamma)^{-1}\bar{u}v] = -2\bar{u}(\bar{u}z) - 2(1-2\gamma)^{-1}\bar{u}(\bar{u}v)$. On the other hand, $\bar{u}u$, $\bar{u}(\bar{u}v) \in Z$ and $\bar{u}(\bar{u}z) \in V$. This implies that

$$z = -\frac{2}{(1-2\gamma)\gamma}\bar{u}(\bar{u}v), \quad u = \frac{4}{(1-2\gamma)\gamma}\bar{u}(\bar{u}(\bar{u}v)) - \frac{2}{1-2\gamma}\bar{u}v.$$

Conversely, if u and z are as above, then $\bar{u}u = 4(1-2\gamma)^{-1}\gamma^{-1}\bar{u}(\bar{u}(\bar{u}(\bar{u}v))) - 2(1-2\gamma)^{-1}\bar{u}(\bar{u}v) = -2(1-2\gamma)^{-1}\bar{u}(\bar{u}v) = \gamma z$. Now it is easy to prove the first relation of (43). □

We remark that if A is Bernstein in Proposition 6, $e(\bar{u}u) = 0$ because $\bar{u}u \in Z$, and so $U_{e'} = \{u + 2\bar{u}u \ : u \in U\}$, which coincides with the expression of $U_{e'}$ in a Bernstein algebra. On the other hand, if A is a train algebra of rank 3, $e(\bar{u}u) = \gamma\bar{u}u$ and $U_{e'} = \{u + 2(1-2\gamma)^{-1}\bar{u}u \ : u \in U\}$, which agrees with $U_{e'}$ in A. Also we may observe that if A is a train algebra of rank 3 in Proposition 6, $\bar{u}(\bar{u}v) = 0$, and so $V_{e'} = \{\ v - 2(1-2\gamma)^{-1}\bar{u}v : v \in V\}$, an expression which coincides with $V_{e'}$ in this algebra. Now, if A is Bernstein, then $\bar{u}(\bar{u}z) = \bar{u}(\bar{u}z) = 0$ and so $Z_{e'} = \{z - 2\bar{u}z - 2\bar{u}z : z \in Z\}$.

Corollary 3. $\dim U^2$ *is invariant (under change of idempotents in A).*

Proof. Let e be an idempotent such that its Peirce decomposition satisfies (28). Suppose that e' is another idempotent. Consider $Fe \oplus U \oplus Z \oplus V$ and $Fe' \oplus U_{e'} \oplus Z_{e'} \oplus V_{e'}$ the Peirce decompositions of A relative to the idempotent elements e and $e' = e + \bar{u} + \bar{u}^2 + 2(1-2\gamma)^{-1}e\bar{u}^2$, with $\bar{u} \in U$. Now, let u_1', u_2' be two elements of $U_{e'}$, with $u_1' = u_1 + 2\bar{u}u_1 + 4(1-2\gamma)^{-1}e(\bar{u}u_1)$ and $u_2' = u_2 + 2\bar{u}u_2 + 4(1-2\gamma)^{-1}e(\bar{u}u_2)$. Using (28) we have

$$\begin{aligned} u_1'u_2' &= u_1u_2 + 2u_1(\bar{u}u_2) + \frac{4}{1-2\gamma}u_1[e(\bar{u}u_2)] + 2u_2(\bar{u}u_1) + 4(\bar{u}u_1)(\bar{u}u_2) \\ &+ \frac{4}{1-2\gamma}[e(\bar{u}u_2)](\bar{u}u_1) + \frac{4}{1-2\gamma}u_2[e(\bar{u}u_1)] + \frac{4}{1-2\gamma}[e(\bar{u}u_1)](\bar{u}u_2). \end{aligned}$$

Then, because $U^2 \subseteq Z \oplus V$, $[(Fe)(Z \oplus V)](Z \oplus V) = (0)$ and from the second linearization of (31) we have

$$u_1'u_2' = u_1u_2 - 2\bar{u}(u_1u_2) - \frac{4}{1-2\gamma}\bar{u}[e(u_1u_2)] - 2\bar{u}^2(u_1u_2)$$

an element of $U_{e'}^2$. Now the linear mapping $\varphi : U^2 \to U_{e'}^2$ defined by $\varphi(x) = x - 2\bar{u}x - 2\bar{u}^2x - 4(1-2\gamma)^{-1}\bar{u}(ex)$ is an isomorphism since $x \in U^2 \subseteq Z \oplus V$ and $2(\bar{u}x + \bar{u}^2x + 2(1-2\gamma)^{-1}\bar{u}(ex)) \in U$. □

Since $N = U \oplus Z \oplus V$ and $N^2 = (Z^2 + UZ + UV) \oplus U^2$ we have

Corollary 4. *The dimension of $Z^2 + UZ + UV$ is an invariant of A.*

References

[1] Bernstein, S.: Principe de stationarité et généralisation de la loi de Mendel, *C.R. Acad. Sci. Paris* **177**, 528-531, 1923.

[2] Costa, R.: Principal train algebras of rank 3 and dimension ≤ 5, *Proc. Edinb. Math. Soc.* **33**, 61-70, 1990.

[3] Costa, R.: On train algebras of rank 3, *Linear Alg. Appl.* **149**, 1-12, 1991.

[4] Costa, R. and Guzzo Jr. H.: Indecomposable baric algebras II, *Linear Alg. Appl.* **196**, 233-242, 1994.

[5] Etherington, I.M.H.: Commutative train algebras of ranks 2 and 3, *J. London Math. Soc.* **15**, 136-149, 1940. Corrigendum, ibid 20, 238, 1945.

[6] Gonzalez S., Gutierrez J. Carlos, Martinez C.: Classification of Bernstein algebras of type $(3, n-3)$, *Comm. Alg.* **23**(1), 201-213, 1995.

[7] Gutierrez, J. Carlos: The Bernstein problem in dimension 6, *J. of Algebra* **185**, 420-439, 1996.

[8] Gutierrez, J. Carlos: The Bernstein problem for the type $(3, n-3)$, *Linear Alg. Appl.* **269**, 17-32, 1998.

[9] Gutierrez, J. Carlos: Principal and plenary train algebras, submitted.

[10] Guzzo Jr, H.: The Peirce decomposition for commutative train algebras, *Comm. Alg.* **22**(14), 5745-5757, 1994.

[11] Guzzo Jr. H. and Vicente P.: Some properties of commutative train algebras of rank 3. *Comm. Alg.* (to appear).

[12] Guzzo Jr. H. and Vicente P.: Train algebras of rank n which are Bernstein or power associative algebras. *Nova Journal of Mathematics, Game Theory and Algebra* (to appear).

[13] Lelis M.L.: Formas p-invariantes em álgebras de Bernstein, PhD thesis, Instituto de Matemática e Estatística da Universidade de São Paulo, 1996.

[14] Lyubich, Yu.I.: Bernstein algebras, *Uspekhi Mat. Nauk* **32**(6), 261-263, 1977.

[15] Lyubich, Yu.I.: Mathematical Structures in Population Genetics, Biomathematics **22**, Springer, Berlin-Heidelberg-New York, 1992.

[16] Ouattara, M.: Sur les algèbres de Bernstein qui sont des T-algèbres, *Linear Alg. Appl.*,**148**, 171-178, 1991.

[17] Walcher, S.: On Bernstein algebras which are train algebras, *Proc. Edinb. Math. Soc.*, **35** *(2), 159-166, 1992.*

[18] Walcher, S.: Algebras with satisfy a train equation for the first three plenary powers, Arch. Math. (Basel) **56**, 547-551, 1991.

[19] Worz, A.: Algebras in Genetics, Lecture Notes in Biomathematics, Vol. **36**, Springer, 1980.