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EXISTENCE OF QUASI STATIONARY DISTRIBUTIONS. A RENEWAL DYNAMICAL APPROACH.

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Existence of quasi stationary distributions. A renewal dynamical approach.

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Summary. Quasi stationary distributions (qsd) are described as fix points of a transformation T in the space of probability measures. For a given probability measure μ we give sufficient conditions for the existence of $\lim_{n\to\infty} T^n\mu$ and for this limit to be a qsd. For the birth and death chain, calling $R(\delta_1)$ the absorbing time of the chain with starting measure δ_1, that concentrates mass on the state 1, we show that the existence of qsd is equivalent to $Ee^{\theta R(\delta_1)} < \infty$ for some positive θ. Moreover we prove that $T^n\delta_1$ converges to the minimal qsd. The method is based on the study of the renewal process with interarrival times distributed as the absorbing time of the chain with initial measure μ. The key tool is the fact that the residual time of that renewal process has the same distribution as the absorbing time of $T\mu$.

Keywords. Quasi stationary distributions. Renewal processes. Residual time. Birth and Death process.

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1. Introduction.

Let $X(t)$ be a continuous time irreducible Markov chain in $\{0\} \cup S$, where S is a denumerable set. Let Q be the corresponding transition rates matrix: $q(x,y)$ is the rate of jumping from x to $y \neq x$, $q(x,x) = -\sum_{y\neq x} q(x,y)$ and it is finite. The state 0 is absorbing: $q(0,x) = 0$, for $x \in S$ and all states connect. We assume $ER^x < \infty$ for all $x \in S$, where R^x is the time of absorption at 0 for $X^x(t)$, the chain starting at $x \in S$.

A quasi stationary distribution (qsd) μ is a probability measure on S with the property that, starting with μ and conditioning that at time t the chain has not been absorbed, the distribution is still μ, *i.e.*

$$\frac{\sum_{y\in S}\mu(y)P(X^y(t)=x)}{\sum_{y\in S}\mu(y)P(X^y(t)\neq 0)}=\mu(x). \tag{1.1}$$

This implies that if $\sum_{y\in S} q(y,x) < \infty$, then any qsd μ satisfies the forward equation

$$\sum_{y\in S}\mu(y)\left(q(y,x)+q(y,0)\mu(x)\right)=0, \quad \text{for all } x\in S. \tag{1.2}$$

Reciprocally, if μ satisfies (1.2) then μ is qsd (See Lemma 2.3 below). The existence of qsd for chains with denumerable state space has been established for the branching process in a pioneer work by Yaglom (1947), for the asymmetric random walk by Seneta (1966) and for the birth and death chain by Good (1968), Kijima (1990) and Ferrari, Martinez and Picco (1991). For discrete time this has been studied by Seneta and Vere-Jones (1966) and for the finite state space by Darroch and Seneta (1965). Yaglom and Seneta Vere-Jones compute the limit as time $t\to\infty$, of the transition probability functions given that the chain is not absorbed at time t. This has been called Yaglom Limit. Kijima (1990) computed the left eigenvector of the transition matrix for the birth and death process obtaining that a sufficient condition for the existence of a qsd is that the birth and death rates at state i converge as $i\to\infty$ to constants $\lambda<\mu$. Ferrari, Martinez and Picco (1991) improved that condition by computing explicitely the solutions of (1.2) that were previously described formally by Cavender (1978). They found that the necessary and sufficient condition for the existence of a qsd in the birth and death chain is (1.3) below.

The main novelty in this paper is the characterization of qsd's as another limit. Equation (1.2) can be interpreted by saying that μ is the invariant measure for the process on S with transition matrix Q_μ given by

$$q_\mu(x,y)=q(x,y)+q(x,0)\mu(y), \quad x,y\in S,$$

so that a probability measure μ is qsd if and only if $\mu Q_\mu = 0$. But this suggest the introduction of the transformation $T:\mu\mapsto\nu$, where ν is the unique solution of $\nu Q_\mu=0$. Under suitable conditions, if we call $\mu_n = T^n\mu$, then $\lim_{n\to\infty}\mu_n$ is qsd. A natural object to study in this context is the absorbing time $R(\mu)$, the first time that the chain hits 0 when initially distributed according to μ. One would expect that if $R(\mu_n)$ converges to a non degenerate random variable, then μ_n will also converge. We show that $R(\mu_{n+1})$ is the residual time of the renewal process induced by $R(\mu_n)$ and this allows us to obtain that a sufficient condition for $R(\mu_n)$ to converge is that $ER(\mu_n)$ is a non decreasing sequence.

We were not able to complete this program in general, but it works for the birth and death process. We use the method to show in Theorem 1 of Section 4 that a necessary and sufficient condition for the existence of a qsd in the birth and death process is that

$$\text{there exists } \theta > 0 \text{ such that } Ee^{\theta R^1} < \infty, \tag{1.3}$$

where R^1 is the absorbing time at 0 of the chain starting at 1. We notice that (1.3) is equivalent to exponential decay of the tail of the distribution of R^1:

$$\text{there exist positive constants } C_1, C_2 \text{ such that } P(R^1 > t) \le C_1 e^{-C_2 t}. \tag{1.4}$$

This is implied, in the birth and death case, by

$$\limsup_{i\to\infty} \frac{q_{i,i+1}}{q_{i,i-1}} < 1, \quad i \in \mathbb{N},$$

for instance, where $q_{i,j}$ is the rate of jump from i to j. But better sufficient conditions -with infinitely many $q_{i,i+1}/q_{i,i-1} > 1$ for example- can be easily constructed. This result was proven by Ferrari, Martinez and Picco (1991) using the discrete chain.

In contrast with ergodic Markov chains, for which the stationary measure is unique, there are in general infinitely many qsd's. This has been pointed out by Cavender (1978). However one of those qsd can be characterized as the "minimal" one: the one that has minimal expected absorbing time at 0. In the case of the random walk the minimal qsd is the Yaglom limit of δ_x, for any x in S, where δ_x is the measure concentrating on x. Indeed, the Yaglom limit computed by Seneta (1966) turns out to be the minimal qsd exibited by Cavender (1978). For the birth and death chain we prove, under condition (1.3), that if $\mu_0 = \delta_1$, then $\lim_{n\to\infty} \mu_n$ is exactly the minimal qsd. This is done in Theorem 2 of Section 4, where we deal with the birth and death case. In sections 2 and 3 we define the tranformation T and introduce the family of renewal processes associated with this transformation.

2. Dynamical system.

For any probability measure μ on S define a Markov chain $Y_\mu(t)$ on S with transition rates matrix Q_μ with entries

$$q_\mu(x,y) = q(x,y) + q(x,0)\mu(y), \quad x,y \in S. \tag{2.1}$$

It is convenient to have a construction of $Y_\mu(t)$ related to $X^\mu(t)$, the absorbing chain with initial distribution μ. Let $\{X_k(t) : k = 1,2,\ldots\}$ be a sequence of independent copies of $X^\mu(t)$ with absorbing times $t_k = \inf\{t : X_k(t) = 0\}$. Define $s_0 = 0$, $s_k = \sum_{i=1}^k t_i$ for $k \geq 1$ and

$$Y_\mu(t) = \sum_{k=1}^\infty X_k(t - s_{k-1})1\{t \in [s_{k-1}, s_k)\}. \tag{2.2}$$

In words, each time that a copy of the chain $X^\mu(t)$ is absorbed, it is substituted immediatly by a new copy. It is easy to see that the chain constructed in this way has transitions rates given by (2.1). Observe that s_k are the ocurrence times of a renewal process with interarrival intervals distributed as $R(\mu)$, the absorbing time of the chain with initial measure μ.

If Q_μ is ergodic and positive recurrent, then there exists an unique invariant measure, *i.e.* an unique probability measure ν satisfying $\nu Q_\mu = 0$. Let T be the transformation $T : \mu \mapsto \nu$. We prove now that the quasi stationary distributions are the fix points of T.

Lemma 2.3. *A probability measure μ is qsd if and only if $\mu Q_\mu = 0$.*

Proof. If μ is qsd, μ satisfies (1.1) which implies immediatly that $\mu Q_\mu = 0$. Reciprocally, if $\mu Q_\mu = 0$ then

$$\mu(x) = \sum_{y \in S} P(Y_\mu^y(t) = x)\mu(y). \tag{2.4}$$

To prove that the solutions of this equation are solutions of (1.1) it suffices to show that $P(Y_\mu^y(t) = x) = P(X^y(t) = x) + P(X^y(t) = 0)\mu(x)$. Now

$$P(Y_\mu^y(t) = x) = P(Y_\mu^y(t) = x, R^y > t) + P(Y_\mu^y(t) = x | R^y \leq t)P(R^y \leq t) \tag{2.5}$$

By (2.2) we take care of the first term in the right hand side of (2.5): $\{Y_\mu^y(t) = x, R^y > t\} = \{X^y(t) = x\}$. For the second term, observe that the position of the chain $Y_\mu(t)$ at the time of absorption of $X(t)$ has distribution μ. Then the strong Markov property gives

$$\begin{aligned} P(Y_\mu^y(t) = x | R^y \leq t) &= \sum_{z \in S} P(Y_\mu^y(R^y) = z) P(Y_\mu^z(t - R^y) = x) \\ &= \sum_{z \in S} \mu(z) P(Y_\mu^z(t - R^y) = x) = \mu(x), \end{aligned} \tag{2.6}$$

by (2.4). To conclude observe that $P(R^y \leq t) = P(X^y(t) = 0)$.♣

Now we show that the limit of $T^n \mu$ is qsd.

Lemma 2.7. *Assume that $\sum_y q(y,x) < \infty$ for all x. If $\mu_\infty = \lim_{n\to\infty} T^n \mu$ exists for some μ and μ_∞ is a probability measure, then μ_∞ is quasi stationary.*

Proof. Call $\mu_n = T^n \mu$. Since $\mu_n Q_{\mu_{n-1}} = 0$, we have

$$\lim_{n\to\infty} \left[\sum_{y \in S} \mu_n(y) (q(y,x) + q(y,0)\mu_{n-1}(x)) \right] = 0 \quad \text{for all } x \in S.$$

Since $\mu_n(y) \leq 1$, by dominated convergence we can interchange the limit and the sum to obtain that $\mu_\infty = \lim_{n\to\infty} \mu_n$ satisfies the qsd equation (1.2).♣

3. The associate renewal processes.

Let τ_i be the interarrival times of a renewal process. That is τ_i are independent and identically distributed as a given variable $\tau \geq 0$. Let $N(t)$ be the number of renewals up to time t and $\sigma_i = \sum_{k=1}^{i} \tau_k$ be the time of the i-th renewal. The distribution of the residual time $\mathrm{Res}(\tau)$ is given by

$$P(\mathrm{Res}(\tau) > s) = \lim_{t\to\infty} P(\sigma_{N(t)+1} - t > s).$$

The residual of τ is a non degenerate random variable if $E\tau < \infty$ which we assume.

Let $R(\mu) = \inf\{t : X^\mu(t) = 0\}$, be the absorbing time at 0 of the chain $X(t)$ when the starting measure is μ. In the next lemma we show that under reasonable conditions the residual time of $R(\mu)$ has the same distribution as $R(T\mu)$ where T is the transformation introduced in Section 2.

Lemma 3.1. *If $ER(\mu) < \infty$, then $R(T\mu) = \mathrm{Res}(R(\mu))$ in distribution.*

Proof. By (2.2), the distribution of the residual time $\mathrm{Res}(R(\mu))$ is given by

$$P(\mathrm{Res}(R(\mu)) < s) = \lim_{t\to\infty} \sum_{x\in S} P(Y_\mu(t) = x)P(R^x < s). \tag{3.2}$$

From $ER(\mu) < \infty$ we get that $Y_\mu(t)$ is positive recurrent so $T\mu$ is a probability measure and, as $t \to \infty$, $Y_\mu(t)$ converges in distribution to $T\mu$. We would be done if we were ready to assume that for all $s > 0$ it holds $\sum_{x\in S} P(R^x < s) < \infty$. In that case, by dominated convergence we would have that the right hand side of (3.2) equals

$$\sum_{x\in S}(T\mu)(x)P(R^x < s) = P(R(T\mu) < s).$$

To prove the lemma without the extra assumption let $\bar{Y}_\mu(t)$ be a copy of the chain $Y_\mu(t)$ with initial (stationary) distribution $T\mu$: $P(\bar{Y}_\mu(0) = y) = (T\mu)(y)$. Hence $P(\bar{Y}_\mu(t) = y) = (T\mu)(y)$ for all $t \geq 0$ and

$$P(R(T\mu) < s) = \lim_{t\to\infty} \sum_{x\in S} P(\bar{Y}_\mu(t) = x)P(R^x < s). \tag{3.3}$$

From (3.2) and (3.3) we can write

$$\begin{aligned}
&|P(\mathrm{Res}(R(\mu)) < s) - P(R(T\mu) < s)| \\
&\qquad \leq \lim_{t\to\infty} \sum_{x\in S} |P(Y_\mu(t) = x) - P(\bar{Y}_\mu(t) = x)|P(R^x < s) \\
&\qquad \leq \lim_{t\to\infty} E \sum_{x\in S} |1\{Y_\mu(t) = x\} - 1\{\bar{Y}_\mu(t) = x\}|P(R^x < s) \\
&\qquad \leq \lim_{t\to\infty} 2P(Y_\mu(w) \neq \bar{Y}_\mu(w), \text{ for all } w \leq t)
\end{aligned} \tag{3.4}$$

To show the third inequality observe first that the sum in the third line of (3.4) has at most two terms. Then the inequality holds by considering any joint realization (coupling) of $Y_\mu(t)$ and $\bar{Y}_\mu(t)$ such that they will continue together after meeting. To prove that the last line in (3.4) vanishes we consider the following coupling: the two chains are independent up to the first moment that they meet. Since the chain is positive recurrent, this meeting occurs in a random time W_1 that is finite with probability one. The lemma is proven.♣

Let μ be a probability measure such that $E[R(\mu)]^n < \infty$ for all n. Define $\mu_0 = \mu$ and $\mu_n = T^n\mu$. Calling $R_n = R(\mu_n)$ we have under the conditions of Lemma 3.1 that $R_{n+1} = \text{Res}(R_n)$ so that by elementary renewal theory, the distributions of R_n and R_{n+1} are linked by the following

$$P(R_{n+1} > s) = \int_s^\infty \frac{P(R_n > w)}{ER_n} dw. \tag{3.5}$$

The expected value of R_n is given in function of the moments of R_0 by

$$ER_n = \frac{ER_0^{n+1}/(n+1)!}{ER_0^n/n!}. \tag{3.6}$$

By induction this implies that

$$\frac{ER_m^n}{n!} = \prod_{k=m}^{n+m-1} ER_k, \tag{3.7}$$

for $n, m \geq 0$, where we have used the convention $\prod_{k=m}^{m-1} ER_k = 1$.

Lemma 3.8. *If there exists $\theta > 0$ such that $Ee^{\theta R_0} < \infty$ and ER_n is non decreasing in n, then ER_n are uniformly bounded and, as $n \to \infty$, R_n converges in distribution to R_∞, an exponentially distributed random variable with mean $ER_\infty = \lim_{n\to\infty} ER_n$.*

Proof. We have

$$Ee^{\theta R_0} = \sum_{n=0}^{\infty} \theta^n \frac{ER_0^n}{n!}. \tag{3.9}$$

Call $b_n = ER_n$ and $a_n = \theta^n ER_0^n/n!$. By (3.6) we have that $b_n = \frac{a_{n+1}}{\theta a_n}$ and, since b_n is non decreasing, $a_{n+1}/a_n \geq a_n/a_{n-1}$ for all n. Hence, if for some n_0, $a_{n_0} \geq a_{n_0-1}$, then the sequence is non decreasing for $n \geq n_0$. But this contradicts the hypothesis $\sum a_n < \infty$. This implies that a_n must decrease. So

$$ER_n = b_n = \frac{a_{n+1}}{\theta a_n} \leq \frac{1}{\theta}$$

implies that ER_n converges to a finite number that we call ER_∞. Using (3.7),

$$Ee^{\theta R_m} = \sum_{n=0}^{\infty} \theta^n \frac{ER_m^n}{n!} = \sum_{n=0}^{\infty} \theta^n \prod_{k=m}^{m+n-1} ER_k. \tag{3.10}$$

The convergence radii of the series $Ee^{\theta R_m}$ is $1/ER_\infty$. In fact, as $n \to \infty$ the quantities $(\prod_{k=0}^{n-1} ER_k)^{1/n}$ converge to ER_∞. For any $\theta < 1/ER_\infty$, (3.10) and monotone convergence imply that

$$\lim_{m\to\infty} Ee^{\theta R_m} = \sum_{n=0}^{\infty} \theta^n (ER_\infty)^n = \frac{1}{1-\theta ER_\infty},$$

which is the moment generating function of an exponential random variable with mean ER_∞. From this R_n converges to R_∞, in distribution. Remark that also the series $Ee^{\theta R_\infty}$ has convergence radious $1/ER_\infty$.♣

In the next result we give a condition for the non decreasing property of the ER_n.

Lemma 3.11. *Assume that $ER_0^n < \infty$ for all n. Then a sufficient condition for ER_n to be non decreasing in n is that*

$$E\left[R_1^n \bar{R}_0^n (R_1 - \bar{R}_0)\right] \geq 0, \tag{3.12}$$

where $\bar{R}_0$ is a copy of R_0 independent of R_1.

Proof. Using (3.5) as was used in (3.6) one gets $ER_{n+1} = \dfrac{ER_1^{n+1}}{(n+1)ER_1^n}$, from where

$$ER_{n+1} - ER_n = \frac{ER_1^{n+1}ER_0^n - ER_0^{n+1}ER_1^n}{(n+1)ER_1^nER_0^n} = \frac{E\left[R_1^n\bar{R}_0^n(R_1 - \bar{R}_0)\right]}{(n+1)ER_1^nER_0^n}.♣$$

4. The birth and death case.

In this Section we treat the birth and death process. The state space is $\{0\} \cup S$, where $S = \{1, 2, \ldots\}$. The transition matrix has components $q_{0,j} = 0$, for all j, $q(i,i) = q_{ii} = -1$, $q(i,j) = q_{i,j}$, for $i \neq 0$ and $q_{i,j} = 0$ if $|i - j| > 1$. The process evolves according to the (backward) equation

$$p'_{i,j}(t) = \sum_{k=0}^{\infty} q_{i,k} p_{k,j}(t), \tag{4.1}$$

where $p_{i,j}(t)$ is the probability that the chain starting at i be at j by time t. Recall that $R^i = R(\delta_i)$ is the absorbing time at 0 of the chain starting at state i. The main result of this section is the following.

Theorem 1. *A necessary and sufficient condition for the existence of qsd for the birth and death process is that there exists $\theta > 0$ such that $Ee^{\theta R^1} < \infty$.*

The condition is necessary. In fact, if there exists a qsd μ, then $R(\mu)$ verifies $P(R(\mu) > s + t) = P(R(\mu) > s)P(R(\mu) > t)$. So $R(\mu)$ is exponentially distributed. Then for $\theta < 1/ER(\mu)$ we have $Ee^{\theta R(\mu)} < \infty$. Since $R^1 = R(\delta_1)$ is stochastically dominated by $R(\mu)$ we deduce $Ee^{\theta R^1} < \infty$.

In order to prove the sufficient condition we assume throughout that $Ee^{\theta R^1} < \infty$ for some $\theta > 0$. We prove in the next Lemma that the distribution of the absorbing time determines the initial measure. Let μ be a probability measure on S and $X^\mu(t)$ be a birth and death process with initial measure μ: $P(X^\mu(0) = k) = \mu(k)$.

Lemma 4.2. *The function $R : \mu \mapsto R(\mu)$ is one to one: if $\mu \neq \nu$ then $R(\mu) \neq R(\nu)$.*

Proof. We have

$$p_{i,j}^{(n)}(t) = \sum_{k=0}^{\infty} q_{i,k}^{(n)} p_{k,j}(t),$$

where $p_{i,j}^{(n)}(t)$ is the n-th derivative of $p_{i,j}(t)$ and $q_{i,k}^{(n)}$ is the (i,k)-term of the matrix Q^n. So

$$p_{i,0}^{(n)}(0) = q_{i,j}^{(n)} = \sum_{k_1,\dots,k_{n-2}} q_{i,k_1} q_{k_1,k_2} \cdots q_{k_{n-2},1} q_{1,0}.$$

Then $p_{n,0}^{(n)}(0) = q_{n,n-1} q_{n-1,n-2} \cdots q_{1,0}$ and $p_{k,0}^{(n)}(0) = 0$ if $k > n$.

The fact that the $q_{i,j}$ are bounded and that for any i only a finite number of k verify that $q_{k,i} \neq 0$, imply that the n-derivative $F^{(n)}(s) = \frac{d^n}{dt^n} P(R(\mu) \leq t)|_{t=s}$ exists and is given by

$$F^{(n)}(s) = \sum_{k=0}^{\infty} \mu(k) p_{k,0}^{(n)}(s).$$

Since $p_{k,0}^{(n)}(0) = 0$ if $k > n$ the above sum evaluated at $s = 0$ only takes into account $k = 1, \dots, n$. From $p_{k,0}^{(n)}(0) = q_{n,n-1} \cdots q_{1,0} > 0$ we deduce that $\mu(n)$ is a linear function of $F^{(1)}(0), \dots, F^{(n)}(0)$.♣

Corollary 4.3. *A probability measure μ is a qsd if and only if the absorbing time $R(\mu)$ is exponentially distributed.*

Proof. We must only show that the condition is sufficient. Let $\mu_t(x) = P_\mu(X(t) = x)/P_\mu(X(t) \neq 0)$. If $R(\mu)$ is exponentially distributed, then $P(R(\mu_t) > s) = P(R(\mu) > t + s | R(\mu) > t) = P(R(\mu) > s)$. Hence, by Lemma 4.2 $\mu_t = \mu$.♣

Let $\mu_0 = \delta_1$, $\mu_n = T^n \delta_1$ and $R_n = R(\mu_n)$. In the next Lemma and Corollary we show that the sequence R_n converges. Notice that the assumption $Ee^{\theta R^1} < \infty$ and (3.6) imply that $ER_n^m < \infty$ for all $m, n \geq 0$.

Lemma 4.4. *The sequence ER_n is non decreasing in n.*

Proof. First notice that the birth and death chain satisfies the conditions of Lemma 3.1, so $R_{n+1} = \mathrm{Res}(R_n)$. Hence, it suffices to prove that (3.12) holds. Start two independent Markov chains with matrix Q. One called $X_0(t)$ with initial distribution $\mu_0 = \delta_1$ and the other called $X_1(t)$ with initial distribution μ_1. Let W be the first time that $X_1(t) = X_0(t)$, and Z be the place of this meeting. Notice that $W < \infty$ with probability one because the absorbing times of the two chains have finite expectation. Then the left hand side of (3.12) reads:

$$E(R_1^n \bar{R}_0^n (R_1 - \bar{R}_0) | Z = 0) P(Z = 0) + E(R_1^n \bar{R}_0^n (R_1 - \bar{R}_0) | Z > 0) P(Z > 0). \tag{4.5}$$

In $\{Z = 0\}$ we have $X_1(t) > X_0(t)$ for all $t < W$, so $R_1 > R_0$ and the expectation in the first term of (4.5) is strictly positive. By the strong Markov property, the expectation in the second term of (4.5) can be written

$$E((W + R^Z)^n (W + \bar{R}^Z)^n (W + R^Z - W - \bar{R}^Z) | Z > 0), \tag{4.6}$$

where R^Z and $\bar{R}^Z$ are two independent copies of the absorbing time of $X(t)$ both starting at Z. The Lemma follows because (4.6) is zero by symmetry.♣

Corollary 4.7. *The sequence R_n converges in distribution to a non degenerate random variable called R_∞. Furthermore, calling $F_n^{(m)}(s) = \frac{d^m}{dt^m} P(R_n \leq t)|_{t=s}$, for all $m \geq 1$,*

$$\lim_{n \to \infty} F_n^{(m)}(t) = F_\infty^{(m)}(t).$$

Proof. The first part follows from Lemma 3.8 and the fact that for the birth and death process ER_n is non decreasing by Lemma 4.4. For the convergence of the derivatives observe that

$$F_n^{(m)}(t) = (-1)^{m+1} \frac{1 - F_{n-m}(t)}{ER_{n-1} \dots ER_{n-m}},$$

so that, by Lemma 3.8 the derivatives also converge.♣

Lemma 4.8. *The limit $\mu_\infty = \lim_{n\to\infty} T^n \delta_1$ exists and it is a probability measure. Furthermore, $R(\mu_\infty) = R_\infty$.*

Proof. From Lemma 4.2 we have that the distribution of the absorbing time when starting by μ determines univoquely μ througth the derivatives of the distribution function of the absorbing time. Since the derivatives are in fact functions of the moments of R and all the moments converge, this proves that the measures μ_n converge to a measure μ_∞. To prove that μ_∞ is a probability, we argue by contradiction. Assume that there exists $\varepsilon > 0$ such that for any $N > 0$, $\sum_{i=1}^{N} \mu_\infty(i) < 1 - \varepsilon$. Then for any N and $n \geq n(N)$ sufficiently large we have $\sum_{i=1}^{N} \mu_n(i) < 1 - \frac{1}{2}\varepsilon$. But this implies that $ER_n = ER(\mu_n) > \varepsilon N/2$. Since N is arbitrarily large, this is in contradiction with $ER_n \leq ER_\infty < \infty$. So μ_∞ is a probability measure. Finally, since $\sum_{x\geq 1} P(R^x < s) < \infty$, we get $R_\infty = R(\mu_\infty)$.♣

Proof of Theorem 1. The necessary condition was proven after the statement of the theorem. The reciprocal follows from the convergence of μ_n to a probability measure proved in Lemma 4.8 and the fact that this limit is a qsd, property given by Lemma 2.7. An alternative proof is that R_∞ is exponentially distributed and with the same distribution as $R(\mu_\infty)$. Hence, by Corollay 4.3, μ_∞ must be qsd.♣

Now we show that μ_∞ is the minimal qsd.

Theorem 2. *The limiting measure $\mu_\infty = \lim_{n\to\infty} T^n \delta_1$ is the minimal qsd.*

Proof. Recall that the set of qsd $\{\mu\}$ is ordered by the parameter $\gamma_\mu = E(R(\mu))$. Moreover

$$Ee^{\theta R(\mu)} = \frac{1}{1 - \theta\gamma_\mu},$$

which converges for any $\theta < 1/\gamma_\mu$. Let us show that if μ' is a non minimal qsd (*i.e.* there exists a qsd μ satisfying $\gamma_\mu < \gamma_{\mu'}$) then $\mu_\infty \neq \mu'$. Take $\theta \in (\frac{1}{\gamma_{\mu'}}, \frac{1}{\gamma_\mu})$ then $Ee^{\theta R(\mu')} = \infty$ and

$Ee^{\theta R(\mu)} < \infty$. So $Ee^{\theta R(\delta_1)} \le Ee^{\theta R(\mu)} < \infty$. By the proof of Lemma 3.8, $Ee^{\theta R(\mu_\infty)} < \infty$, so $\mu_\infty \neq \mu'$. Hence μ_∞ is the minimal qsd *i.e.* $\gamma_{\mu_\infty} = \inf\{\gamma_\mu : \mu \text{ is qsd }\}$. Remark that, since $\mu(1) = 1/(\gamma_\mu q_{1,0})$, this implies that μ_∞ is characterized as the qsd with the biggest $\mu(1)$. ♣

5. Final remarks.

One would like to use the approach to show the existence of qsd for chains with rates other than nearest neighbor unidimensional. The condition $\sum_y q(y,x) < \infty$ is necessary for the statement of the forward equation (1.2) and to prove that $\lim_{n\to\infty} T^n\mu$ must be qsd in Lemma 2.7. The nearest neighbor one dimensional property of the birth and death chain is used in two key places: (a) in the proof of the monotonicity of the ER_n in equation (3.12). This property may very well be quite general, but in order to use the symmetry as in (4.6) we need that two independent copies of the chain necessary cross; (b) in the reconstruction of μ_n from R_n in Lemma 4.2, but we believe that this is just a technical constraint.

We conclude by posing a couple of (natural) open problems. Assume that R_n is non decreasing and converges to R_∞. Then, is it true that

$$\frac{P(R_0 > t+s)}{P(R_0 > t)} \to P(R_\infty > s)? \tag{5.1}$$

In other words, the existence of the limit of the R_n is sufficient to guarantee the existence of the Yaglom limit of the chain, when starting at a single state? This should be true in view of the fact that for the random walk it holds and also because both sides of (5.1) depend on the tail of the distribution of R^1.

From the results of Seneta (1966) and Cavender (1978) mentioned at the end of the introduction and Lemma 4.10, we get that for the random walk

$$\lim_{t\to\infty} P(X^{\delta_1}(t) = x | X^{\delta_1}(t) \neq 0) = \lim_{n\to\infty} (T^n\delta_1)(x).$$

Namely, the Yaglom limit and the limit defined in Section 2, both starting from a measure concentrating on a single state, are the same and equal to the minimal measure. Under which conditions this is true for Markov chains satisfying (1.3)?

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