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Hydrodynamic Vortex on Surfaces

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Abstract The equations of motion for a system of point vortices on an oriented Riemannian surface of finite topological type are presented. The equations are obtained from a Green's function on the surface. The uniqueness of the Green's function is established under hydrodynamic conditions at the surface's boundaries and ends. The hydrodynamic force on a point vortex is computed using a new weak formulation of Euler's equation adapted to the point vortex context. An analogy between the hydrodynamic force on a massive point vortex and the electromagnetic force on a massive electric charge is presented as well as the equations of motion for massive vortices. Any noncompact Riemann surface admits a unique Riemannian metric such that a single vortex in the surface does not move ("Steady Vortex Metric"). Some examples of surfaces with steady vortex metric isometrically embedded in $\mathbb{R}^3$ are presented.

Keywords Point vortex dynamics · Special metrics · Robin function · Euler equation weak solution · Steady vortex metric

Mathematics Subject Classification 70F99 · 76B47 · 53C07 · 31A15

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1 Introduction

The motion of point vortices on a planar domain is a classical subject in fluid mechanics that goes back to Helmholtz, Kelvin, and Kirchhoff (see the survey Aref 1983 or the book Newton 2001). Point vortices on the plane can be introduced in the following way.

The motion of an incompressible inviscid fluid on the plane is governed by the usual Euler's equations:

$$\partial_t v + \nabla_v v = -\nabla p, \quad \operatorname{div} v = 0 \tag{1.1}$$

where the fluid density was taken equal to one, v and p are the fluid velocity and pressure, respectively, and $\nabla_v = v_1 \partial_{x_1} + v_2 \partial_{x_2}$. This equation implies that the vorticity function $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$ is carried by the flow of the fluid, namely $\partial_t \omega + \nabla_v \omega = 0$. This remarkable fact due to Helmholtz and Kelvin implies that if the initial fluid velocity field is irrotational, then it will remain irrotational for all time. Irrotational flows can be easily understood. Beyond initially irrotational flows, the simplest possible situation occurs when the vorticity is initially concentrated at a finite set of points. Each point is called a vortex. The circulation around a vortex is called the vortex strength. In this case, several different physical and mathematical arguments using Euler's equations (see, for instance, Friedrichs 1966; Marchioro and Pulvirenti 1994) lead to the conclusion that the vorticity remains concentrated at the vortices, their strength remain constant, and the vortex dynamics is well defined by a set of ordinary differential equations as far as vortex collision do not occur. These are the Helmholtz–Kirchhoff point vortex equations on the plane.

Our main goal in these notes is to obtain equations for the motion of point vortices on a curved surface. According to Borisov et al. (2010), the first to consider the motion of point vortices on a curved surface, the sphere embedded in $\mathbb{R}^3$, was Zermello in 1902 then Lamb (1916) (paragraph 160), Gromeka 1952, and more recently Bogomolov (1977) and Hally (1979, 1980). Subsequently, this question was considered by Kimura and Okamoto (1987), Kimura (1999), Boatto and Koiller (2008), Turner et al. (2010), Boatto and Koiller (2013), Dritschel and Boatto (2015), and many others. The paper by Borisov et al. (2010) has an interesting historical review on the early research on hydrodynamic vortices on surfaces, and the papers by Boatto and Koiller (2013) and Dritschel and Boatto (2015) have lists of interesting references. The equations for the motion of vortices, including massive vortices, on general compact surfaces are given in Boatto and Koiller (2013).

Our main contributions in this paper are: to review from a mathematical perspective, namely providing proofs, several statements already present in the literature, particularly in Boatto and Koiller (2008) and Boatto and Koiller (2013); to settle on conditions at the surface's ends to determine the equations for the motion of vortices on general noncompact surfaces of finite topological type; and to extend to general noncompact Riemann surfaces a previous result of Gustafsson (1979) saying that any planar domain with prescribed circulations at its ends has a Riemannian metric where a single vortex does not move, we called this metric "Steady Vortex Metric."

This paper is divided as follows. Section 2 contains most of the background material and definitions used in the paper, including the definition of surfaces of finite topological type. In Sect. 3, we establish the hydrodynamic conditions at the surface's ends and then prove that there exists a unique Green's function, up to an additive constant, that satisfies these conditions. The difference between boundary conditions on hyperbolic and on parabolic ends is new. The proof of existence of the Green's function uses standard theorems from potential theory. Section 4 contains the definition of background flow, a harmonic flow that satisfies some conditions at the surface's ends, and the definition of systems of vortices. This section is an extension of the work of Lin (1943) on vortices in planar domains to vortices in surfaces of finite topological type. Section 5 contains a weak formulation of Euler's equation adapted to point vortices that allows for the computation of the force on a vortex. This weak formulation of Euler's equation is new. It can be useful in the study of solutions to Euler's equations that contain both point vortices and nonuniform background vorticity. The expression for the force on a vortex is used in Sect. 6 to establish an analogy between the hydrodynamic force on a vortex and the electromagnetic force on an electric charge under constant magnetic field. The Hamiltonian nature of the equations of motion of the system of vortices, with and without mass, follows easily. As a corollary, we obtain that the equation for the motion of massless vortices is nothing else but the guiding-center approximation for the motion of electric charges in strong magnetic fields. Sections 5 and 6 contain extensions of the results in Grotta Ragazzo et al. (1994) on massive vortices in planar domains to massive vortices in surfaces of finite topological type. In the last section, we generalize to noncompact Riemann surfaces a result due to Gustafsson (1979) saying that any planar domain with prescribed circulations at its ends has a unique Riemannian metric compatible with its conformal structure where a single vortex does not move. We called this metric "Steady Vortex Metric." It is a generalization of the "capacity metric" of classical potential theory to the hydrodynamic context. We finally present some examples of surfaces with a steady vortex metric isometrically embedded, some only partially embedded, in $\mathbb{R}^3$. We remark that our contributions in this last section are: to realize the property of steadiness of a single vortex in the special class of metrics given in Gustafsson (1979), to generalize the results in Gustafsson (1979) from surfaces of genus zero to surfaces of larger genus as well as to surfaces with more than one parabolic end, and to provide examples of surfaces with steady vortex metrics embedded in $\mathbb{R}^3$.

2 Surfaces and Hypotheses

In this paper, a surface S is a connected and oriented two-dimensional manifold. All mathematical structures on S are supposed to be C^∞ unless otherwise stated.

Hydrodynamics on a surface requires the surface to be endowed with a Riemannian metric g. The pair (S, g) is called a Riemannian surface. In the following, we recall Cartan's method of moving frames Spivak (1999) that will be used to describe the local geometry of the surface. On a given chart on the surface, let $\{V_1, V_2\}$ be a positively oriented pair of smooth orthonormal vector fields (a moving frame) and $\{\theta_i, \theta_j\}$ be the dual frame of differential one-forms, namely $\langle V_i, V_j\rangle = \delta_{ij}$, $\langle\theta_i, \theta_j\rangle = \delta_{ij}$ and $\theta_j = \langle V_j, \cdot\rangle$, $i, j = 1, 2$. On this chart, the Riemannian volume form μ is given by $\mu = \theta_1 \wedge \theta_2$. The connection one-forms ω_{ij} are uniquely defined by: $d\theta_i = -\sum_j \omega_{ij} \wedge \theta_j$ and $\omega_{ij} = -\omega_{ji}$, for $i, j = 1, 2$. For surfaces, the only nontrivial connection forms are $\omega_{12} = -\omega_{21}$. The covariant derivative can be written in terms of the connection one-forms as $\nabla_{V_k} V_j = \sum_i \omega_{ij}(V_k) V_i$, for $i, j, k = 1, 2$. The exterior derivative of ω_{12} gives the curvature form $d\omega_{12} = K\mu$ where K is the Gaussian curvature of the surface. The Hodge-star operator "$*$" associated with the Riemannian metric is a linear map onto differential forms which is determined by its action on the elements of the basis $\{1, \theta_1, \theta_2, \theta_1 \wedge \theta_2\}$:

$$*1 = \theta_1 \wedge \theta_2 = \mu, \qquad *\theta_1 = \theta_2, \qquad *\theta_2 = -\theta_1, \qquad *\mu = 1.$$

The divergence of a j-form σ is defined as the $(j-1)$-form given by $\operatorname{div} \sigma = *d*\sigma$ (in several Riemannian geometry texts the divergence is defined with the opposite sign). Notice that our definition coincides with the usual definition of divergence when the surface is the Euclidean plane. Finally, the Laplacian of a j-form σ is defined as the j-form given by $\Delta\sigma = (d \operatorname{div} + \operatorname{div} d)\sigma$. If ϕ is a function (a 0-form), then $\Delta\phi = *d*\mathrm{d}\phi$ which coincides with the usual Laplacian when the Riemannian surface is the Euclidean plane.

By duality, the Hodge-star operator also acts on vectors with $*V_1 = V_2$ and $*V_2 = -V_1$. Notice that $*$ rotates vectors by $\pi/2$ and, therefore, defines a complex structure $J : TS \to TS$ on S. A surface S endowed with a complex structure J is called a Riemann surface (S, J). Two different Riemannian metrics g_1 and g_2 on S are said conformally equivalent if $g_1 = \lambda^2 g_2$, where the conformal factor $\lambda : S \to \mathbb{R}$ is strictly positive. Notice that two conformally equivalent metrics generate the same Riemann surface (S, J). Two Riemannian surfaces (S_1, g_1) and (S_2, g_2) are said conformally equivalent if there exists a diffeomorphism $h : S_1 \to S_2$ that preserves angles, namely $h^* g_2 = \lambda^2 g_1$ with $\lambda > 0$. Two Riemann surfaces (S_1, J_1) and (S_2, J_2) are conformally equivalent if there is a diffeomorphism $h : S_1 \to S_2$ that is biholomorphic, namely $h^* J_2 = J_1$. So, conformally equivalent Riemannian surfaces generate conformally equivalent Riemann surfaces.

Any Riemannian surface is locally conformally equivalent to an open set of the Euclidean plane. In the following, we introduce a particular coordinate system, "isothermic coordinates," that establishes the equivalence. These coordinates are very convenient for doing computations.

Centered at each point p of S, there exists a disk U and a function $x : U \to \mathbb{R}$ such that: $\Delta x = 0$ (x is harmonic)[1], $x(p) = 0$, and $\mathrm{d}x \neq 0$ in U. Since x is harmonic, $*dx$ is closed and a function $y : U \to \mathbb{R}$ exists such that $\mathrm{d}y = *dx$ and [2] $y(p) = 0$. Since $**\,\mathrm{d}x = -\mathrm{d}x$, y is also harmonic and $\mathrm{d}y \neq 0$ in U. Therefore, the functions $(x, y) : U \to \mathbb{R}^2$ define a coordinate system on a neighborhood of p (sometimes called isothermic coordinates or local uniformizer). Since the $*$ operator rotates one-forms by $+\pi/2$, $\mathrm{d}x$ is everywhere orthogonal to $\mathrm{d}y$ and both have the same norm $|\mathrm{d}x| = |\mathrm{d}y| = 1/\lambda$. Therefore, $\theta_1 = \lambda \mathrm{d}x$, $\theta_2 = \lambda \mathrm{d}y$ define a moving frame on U. In

[1] In an arbitrary local coordinate system (ξ^1, ξ^2), the Riemannian Laplacian is given by $\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi^j} g^{jk} \sqrt{|g|} \frac{\partial}{\partial \xi^k}$ where the sum over repeated indices is assumed, the Riemannian metric is given by $g_{jk} d\xi^j \otimes d\xi^k$, g^{jk} is the inverse of matrix g_{jk}, and $|g|$ is the absolute value of the determinant of the matrix g_{jk}.

[2] In the coordinates of footnote 1, $\mathrm{d}y = *\mathrm{d}x = -\frac{1}{\sqrt{|g|}} g_{ki} \epsilon^{il} \frac{\partial x}{\partial \xi^l} d\xi^k$, where $\epsilon^{il} = -\epsilon^{li}$ and $\epsilon^{12} = 1$.

the coordinates (x, y): $g_{11} = g_{22} = \lambda^2, g_{12} = g_{21} = 0$; $g^{11} = g^{22} = \lambda^{-2}, g^{12} = g^{21} = 0$; $\mu = \lambda^2 \mathrm{d}x \wedge \mathrm{d}y$; if $e_1 = \partial_x, e_2 = \partial_y$, and $\nabla_{e_k} e_l = \sum_i \Gamma^i_{kl} e_i$, then

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{\partial \log \lambda}{\partial x},$$

$$\Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \frac{\partial \log \lambda}{\partial y}; \tag{2.1}$$

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right); \tag{2.2}$$

and

$$K = -\frac{1}{\lambda^2}\left(\frac{\partial^2 \log \lambda}{\partial^2 x} + \frac{\partial^2 \log \lambda}{\partial^2 y}\right). \tag{2.3}$$

Any Riemann surface S has a compact exhaustion Beardon (1984); namely, there are compact subsets $K_1 \subset K_2 \subset ...$ such that $S = \cup K_i$. An end of S with respect to a given K_i is an unbounded (it is not contained in any compact subset of S) connected component of S minus K_i (see Richards 1963 and Li 2000 for definitions and properties of ends of noncompact surfaces). The genus of an end of S with respect to K_i is zero if all simple closed curves separate the end. The surface S is said to have finite genus if there exists a j such that all ends of S with respect to K_j have genus zero. The number of ends of S with respect to K_i is a nondecreasing function of i. So, either the number of ends of S with respect to K_i increases unboundedly as $i \to \infty$ or it stabilizes at some number N. In this last case, S is said to have a finite number N of ends.

The universal covering of any Riemann surface is conformally equivalent to either the two sphere $\mathbb{S}^2$ with its natural Riemannian metric of curvature 1, or to the Euclidean plane $\mathbb{R}^2$, or to the Poincaré disk $\mathbb{H}^2$ with its Riemannian metric of curvature -1. If S is compact, then its universal covering is either $\mathbb{S}^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$, depending on the genus of S being 0, 1, or greater than 1, respectively (see Beardon 1984 for these results). A circle domain in a Riemann surface is a domain such that each connected component of its complement is either a point or a closed geometric disk. Here a geometric disk means a topological disk that lifts to the universal covering (either $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$) as a disk of constant radius.

Theorem 2.1 (Uniformization He and Schramm 1993) *Let S be a Riemann surface with finite genus and at most countably many ends. Then, there is a compact Riemann surface R such that S is conformally homeomorphic to a circle domain S_C in R. Moreover, the pair (R, S_C) is unique up to conformal homeomorphism.*

This theorem is due to He and Schramm (1993). In the case of finitely many ends, the history of this theorem goes back to Koebe (1908) and several other authors. The genus of a noncompact surface S is the genus of the compact surface obtained by capping off all its ends. That is the same as the genus of the surface R in Theorem 2.1.

Hypothesis on the topology of S. If S is noncompact, then S has finite genus and finitely many ends (S is of finite topolgy).

This hypothesis implies that there exists a compact subset K of S such that each one of the N ends of S with respect to K is homeomorphic to an open annulus. An end can be conformally classified as hyperbolic or parabolic. An end is *hyperbolic* if it is conformally equivalent to the annulus $\{r < |z| < 1\}$, $r > 0$, the end corresponding to $|z| = 1$. An end is *parabolic* if it is conformally equivalent to the punctured disk $\{0 < |z| < 1\}$, the end corresponding to $|z| = 0$. In short, we may say that a hyperbolic end admits a uniformization to $\{r < |z| < 1\}$, $r > 0$, while a parabolic end admits a uniformization to $\{0 < |z| < 1\}$. In Theorem 2.1, each disk of R minus S_C corresponds to a hyperbolic end of S and each isolated point in R minus S_C corresponds to a parabolic end.

At this moment, it is convenient to analyze some examples. The simplest example of a surface with one parabolic end is $\mathbb{R}^2$ with the usual complex structure induced by the Euclidean metric. If $\mathbb{R}^2$ is identified with the complex plane $z = x + iy$, and $\mathbb{S}^2$ is embedded in $\mathbb{R}^3$ as $\mathbb{S}^2 = \{(x, y, t) : x^2 + y^2 + t^2 = 1\}$, then a simple computation shows that

$$f_1(x, y, t) = \frac{x + iy}{1 - t} \tag{2.4}$$

is a biholomorphism of $\mathbb{S}^2 - \{(0, 0, 1)\}$ (which is a circle domain) onto $\mathbb{R}^2$: "the stereographic projection." The end of $\mathbb{R}^2$ is mapped to the north pole of $\mathbb{S}^2$. This shows that the end of $\mathbb{R}^2$ is parabolic.

The simplest example of a surface with one hyperbolic end is $\{|z| < 1\}$ with the usual complex structure. The inverse of the function defined in (2.4) restricted to $\{|z| < 1\}$ is a one-to-one conformal map from $\{|z| < 1\}$ to the south hemisphere of $\mathbb{S}^2$, which is a circle domain in $\mathbb{S}^2$. Notice that the usual complex structure in $\{|z| < 1\}$ is compatible with both the Euclidean metric, $ds = |dz|$, and the Poincaré metric, $ds = 2|dz|/(1 - |z|^2)$. In the first case, $\{|z| = 1\}$ is just the boundary of $\{|z| < 1\}$ in the sense that any point in the surface has a finite distance to $\{|z| = 1\}$, while in the second $\{|z| = 1\}$ is at an infinite distance from any point in $\{|z| < 1\}$.

The upper half-plane $\{z = x + iy : y > 0\}$ with the usual complex structure also has a single hyperbolic end since

$$\{w \in \mathbb{C} : |w| < 1\} \to \mathbb{C}, \qquad z = \frac{w + i}{iw + 1},$$

maps conformally the Poincaré disk to the upper half-plane. Again the boundary $y = 0$ gives rise to a hyperbolic end.

Another example of a "surface with boundary" is the slit plane given by the Euclidean plane $\{z = x + yi \in \mathbb{C}\}$ minus the segment $\{-1 \leq x \leq 1, y = 0\}$. The Joukowsky map

$$\{w \in \mathbb{C} : |w| > 1\} \to \mathbb{C}, \qquad z = \frac{1}{2}\left(w + \frac{1}{w}\right),$$

is a one-to-one conformal map onto the slit plane. The inverse of this map composed with the inverse of f_1 in Eq. (2.4) gives a holomorphic one-to-one map from the slit plane onto a circle domain in $\mathbb{S}^2$. The slit plane has two ends: ∞ that is parabolic and $\{-1 \leq x \leq 1, y = 0\}$ that is hyperbolic. Notice that, under the Joukowsky map, to each point at the "boundary" $\{-1 \le x \le 1, y = 0\}$ of the slit plane there corresponds two points of $\{|w| = 1|\}$.

Surfaces of revolution can be constructed rotating the graph of a positive function $x \to f(x)$, $x \in (a, b)$, about the x-axis one complete revolution. Let θ be the angle of rotation about the x-axis. The azimuthal projection of the surface onto the Euclidean annulus $\mathcal{A} = \{z = re^{i\theta} \in \mathbb{C} : \tilde{a} < |z| < \tilde{b}\}$ maps the meridian θ of the surface of revolution to the ray with the same argument in $\mathcal{A}$ and maps the parallel x of the surface of revolution to the circle of radius $r(x)$ in $\mathcal{A}$. The transformation is conformal Dritschel and Boatto (2015) if $r(x)$ is a solution to

$$\lambda = \frac{f}{r} = \pm\frac{\sqrt{1 + f'^2}}{r'}. \tag{2.5}$$

where λ is the conformal parameter such that $|\cdot|_S = \lambda|\cdot|_{\mathcal{A}}$ and the sign $\pm$ depends on $r'(x)$ be either positive or negative, respectively. For the catenoid $f(x) = \cosh(x)$, $x \in \mathbb{R}$, that implies $r(x) = e^x$. So the catenoid is conformally mapped to the punctured plane, and therefore, it has two parabolic ends. Another example is the pseudosphere that is generated by the *tractrix*, a function that satisfies the differential equation $f'(x) = -f(x)(1 - f^2(x))^{-1/2}$, for $x > 0$, with $f(0) = 1$ and $f(\infty) = 0$. In this case, $r(x) = \exp[1 - 1/f(x)]$. So, the boundary $x = 0$ is mapped to $r = 1$ and the end $x = \infty$ is mapped to $r = 0$. Therefore, the pseudosphere has one hyperbolic end and one parabolic end.

From these examples, it becomes clear that most surfaces with boundaries that appear in the usual problems of fluid mechanics can be handled within the framework of surfaces of ends:
Each boundary component is identified with a hyperbolic end.

3 Euler's Equation and Vortex Definition

The generalization of Euler's equation (1.1) from the Euclidean plane to a Riemannian surface is direct, and the covariant derivative, gradient, and divergence in Eq. (1.1) must be interpreted as those associated with the Riemannian metric. Using that the Lie derivative L_v of the volume form μ satisfies $L_v\mu = di_v\mu = (\operatorname{div} v)\mu$, we can write Euler's equation (1.1) in terms of the velocity form u (the dual form, $u = \langle v, \cdot\rangle$, of the velocity vector v) as

$$\partial_t u + \nabla_v u = -dp \qquad di_v\mu = 0 \tag{3.1}$$

where $\nabla_v u$ is the dual of $\nabla_v v$. Using the identity

$$\langle\nabla_v v, \cdot\rangle = du(v, \cdot) + d\left(\frac{|v|^2}{2}\right) = \underbrace{i_v du + du(v)}_{L_v u} - d\left(\frac{|u|^2}{2}\right) \tag{3.2}$$

Euler's equation can be rewritten as

$$\partial_t u + L_v u = -d\left(p - \frac{|u|^2}{2}\right) \qquad L_v\mu = 0 \tag{3.3}$$

The vorticity form is defined as the differential two-form $\mathrm{d}u = \omega\mu$, while the vorticity is defined as the function ω. Taking the exterior derivative of Eq. (3.3), we get

$$\partial_t \omega + L_v \omega = 0$$

that is the Kelvin–Helmholtz identity which shows that the vorticity is carried by the flow of the fluid.

Notice that if u is the velocity form at a point p, then $*u$ is a one-form that determines the infinitesimal flux of fluid across any direction at p. The same quantity is given by the one-form $i_v\mu = \mu(v, \cdot)$, so $*u = i_v\mu$. The fluid incompressibility, $\mathrm{div}\, u = *d*u = 0$, implies that $*u$ is closed. Suppose that $*u$ is exact. This implies that there is a function ψ, the stream function, such that $\mathrm{d}\psi = *u$ or, equivalently,

$$u = - * \mathrm{d}\psi \tag{3.4}$$

If C is a curve that joins two points p and q in S, then the integral

$$\int_C *u = \int_C \mathrm{d}\psi = \psi(p) - \psi(q)$$

gives the volume (area) of fluid streaming across the curve C per unit time. Finally, ψ and ω are related by the equation $\mathrm{d}u = -d * d\psi = \omega\mu$ that implies $-\Delta\psi = \omega$.

The next natural step is to define a vortex at a point p as the velocity field generated by a δ-concentrated vorticity at p. The associated stream function satisfies $-\Delta\psi = \Gamma\delta_p$ as in the case of the Euclidean plane. Some caution must be taken with this step, though. For instance, if the equation $-\Delta\psi = \Gamma\delta_p$ is integrated over a compact boundaryless surface, then using Stokes theorem

$$\Gamma \int \delta_p \mu = \Gamma = - \int *d * \mathrm{d}\psi\, \mu = - \int d * \mathrm{d}\psi = 0$$

that gives $\Gamma = 0$. So, the definition of a point vortex in a compact boundaryless surface must be modified in the following way.

A Green's function of a Riemannian surface S is a solution in distribution sense to either

$$-\Delta_q G(q, p) = \delta_p(q) - V^{-1}, \qquad \text{if } S \text{ is compact,} \tag{3.5}$$

where $V = \int_S \mu$ is the volume of S, or

$$-\Delta_q G(q, p) = \delta_p(q) \qquad \text{if } S \text{ is noncompact.} \tag{3.6}$$

Notice that in the compact case the extra term $-V^{-1}$ corresponds to a constant background vorticity that compensates for the vorticity of the vortex. Equations (3.5) and (3.6) must be interpreted in the following way. The function G is a solution to either Eq. (3.5) or (3.6) if for all functions ϕ on $C_c^\infty(S)$ ($\phi \in C^\infty$ and has compact support) either

$$\phi(p) = \frac{1}{V} \int_S \phi\mu - \int_S G(q, p)\Delta\phi(q)\mu(q). \tag{3.7}$$

or

$$\phi(p) = -\int_S G(q, p)\Delta\phi(q)\mu(q), \tag{3.8}$$

respectively. Using a partition of unity, it is possible to show that in Eqs. (3.7) and (3.8) it is enough to consider functions ϕ with support in a single isothermic coordinate domain. For a function ϕ with support in a single isothermic domain

$$\int_S G(q, p)\Delta\phi(q)\mu(q) = \int_S G(\cdot, p)\left(\frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 y}\right)\mathrm{d}x \wedge \mathrm{d}y$$

$$\int_S \phi\mu = \int_S \phi\lambda^2 \mathrm{d}x \wedge \mathrm{d}y$$

Notice that the first of these equations does not depend on the conformal factor λ, while the second does depend. This shows that the solutions G to Eq. (3.6) are conformal invariants; namely, they are the same for all Riemannian metrics that give rise to the same Riemann surface, while the solutions to Eq. (3.5) are not conformal invariants.

Definition In a *compact surface* S, a *point vortex* of intensity Γ at a point p is the velocity field associated with the stream function $\Gamma G(\cdot, p)$, where G is the solution to Eq. (3.5), unique up to the addition of a constant (the constant depends neither on q nor on p)[3] , that has the following properties: $G(q, p)$ is C^∞ on $S \times S$ minus the diagonal, G is symmetric $G(q, p) = G(p, q)$, and G is bounded from below (see Aubin 1998, Theorem 4.13, for the existence of G).

The definition of a vortex when S is noncompact is more complicated since it requires boundary conditions at the ends of S. At first, we define the circulation associated with a stream function $q \to \psi(q)$ around a particular end. Let γ be a simple closed curve that separate the topologically annular end from the surface. Orient γ such that the end remains at the right- hand side of the curve. Then, the circulation around this end is

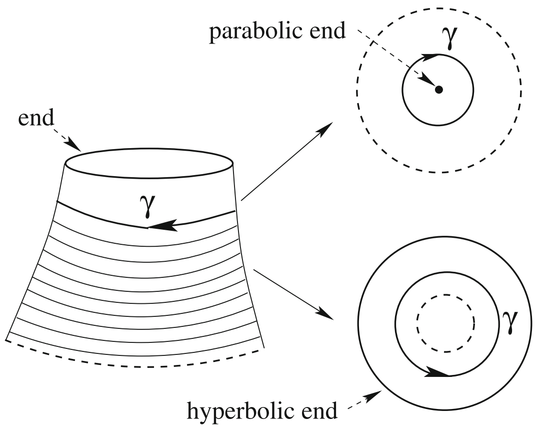
$$-\oint_\gamma *\mathrm{d}\psi = \oint_\gamma u, \tag{3.9}$$

where $u = - * \mathrm{d}\psi$ is the velocity form (see Fig. 1).

The analysis of the usual boundary condition at the parabolic end of the Euclidean plane $\{z = x + iy \in \mathbb{C}\}$ will give the clue for a general statement on the boundary condition at a parabolic end. Let $f_1 : \{\mathbb{S}^2 - \{(0, 0, 1)\} \to \mathbb{C}$ be the stereographic projection given in Eq. (2.4), where $\mathbb{S}^2 = \{(x, y, t) : x^2 + y^2 + t^2 = 1\}$. The end of $\mathbb{C}$ is mapped to the north pole of $\mathbb{S}^2$, denoted as P. If a holomorphic function u is defined on a punctured neighborhood U of P, then either $|u|$ is bounded in U and u can be extended as a holomorphic function to P or u is unbounded on U and P is a pole or an essential singularity of u. Notice that the usual Green's function $G(z, w) = -\mathrm{Re}(1/2\pi)\log(z - w)$ on $\mathbb{R}^2$ lifts to $\mathbb{S}^2$ under the map f_1 as a function

[3] The constant may be determined by the normalization $\int_S G(q, p)\mu(q) = 0$.

Fig. 1 Orientation of the curve γ used to compute the circulation around an end



with a logarithmic singularity at the north pole. The easiest way to see this is to use a second analytic chart $f_2 : \mathbb{S}^2 - \{(0, 0, -1)\} \to \mathbb{C}$

$$f_2(x, y, t) = \frac{x - iy}{1 + t} = \tilde{z} \tag{3.10}$$

that covers the north pole of $\mathbb{S}^2$. Notice that $f_1 f_2 = 1$ which implies that the transition map is $\tilde{z} = 1/z$ on $\mathbb{C} - \{0\}$. Therefore, in the chart on $\mathbb{S}^2$ that misses the south pole:

$$-\frac{1}{2\pi} \log \left| \frac{1}{\tilde{z}} - w \right| = \frac{1}{2\pi} \left[+ \log |\tilde{z}| - \log |w| - \log \left| \tilde{z} - \frac{1}{w} \right| \right]$$

which shows the existence of a vortex of intensity -1 at the north pole of $\mathbb{S}^2$ independently on the position of the vortex of intensity 1. On $\mathbb{C}$, there are infinitely many different harmonic polynomials. Under conformal mapping, these harmonic polynomials lift to the sphere as meromorphic functions with one pole at P. Clearly, for any h holomorphic on $\mathbb{C}$ the function $G(z, w) = -\mathrm{Re}[h(z) + (1/2\pi) \log(z - w)]$ solves Eq. (3.6). So, boundary conditions must be imposed at a parabolic end to select a solution to Eq. (3.6). Motivated by all this considerations, we state the following.

Definition A stream function ψ is said to be *simple at a parabolic end* if on a uniformizing punctured disk $\{0 < |z| < 1\}$,

$$\lim_{|z| \to 0} \psi(z) - \frac{\Gamma}{2\pi} \log |z| = \text{constant}$$

where Γ is the circulation of ψ (possibly zero) around a parabolic end. Notice that if $\psi(z)$ is harmonic on $\{0 < |z| < 1\}$, then $\psi(z) - \frac{\Gamma}{2\pi} \log |z|$ extends to a harmonic function at $z = 0$.

As mentioned above, a hyperbolic end may appear as a boundary or an infinity of a given surface. The simplest example is $\{|z| < 1\}$ where under the Euclidean metric

$|z| = 1$ is interpreted as a boundary and under the Poincaré metric as an infinity. There is no doubt that the correct boundary condition in the Euclidean case is that the fluid cannot penetrate the boundary. If the fluid flow is associated with a stream function ψ, then $\psi(z)$ must be constant for $|z| = 1$. Is the same boundary condition appropriate for the Poincaré metric? Notice that there are many solutions to Eq. (3.6) on $\{|z| < 1\}$. Indeed, any function of the form

$$G(z, w) = -\frac{1}{2\pi} \log |z - w| + \operatorname{Re} u(z, w),$$

where $u(\cdot, w)$ is holomorphic on $\{|z| < 1\}$ is a solution to Eq. (3.6). The symmetry condition $G(z, w) = G(w, z)$ only constrains u to be symmetric. So, a natural question is: what is the "physical" principle that constraints the behavior of the Green's function on the Poincaré disk $\mathbb{H}^2$ at $|z| = 1$? The answer comes from the symmetries of the Poincaré metric. The group of isometries of $\mathbb{H}^2$ is the group of Möbius transformations

$$z \to \frac{az + \overline{c}}{cz + \overline{a}} \quad \text{where} \quad (a, c) \in \mathbb{C}^2, \quad |a|^2 - |c|^2 = 1 \tag{3.11}$$

The invariance of $G(\cdot, w)$ under isometries and the action of these transformations on the circle $|z| = 1$ requires $G(z, w)$ to be constant for $|z| = 1$. This plus the symmetry condition implies that there is a unique solution to Eq. (3.6) up to a constant:

$$G(z, w) = -\frac{1}{2\pi} \log \frac{|z - w|}{|z\overline{w} - 1|} \tag{3.12}$$

In Kimura (1999), the same hydrodynamical Green's function was first obtained through comparison with the Green's function on the sphere. These considerations lead to the following definition.

Definition A stream function ψ is said to be *constant at a hyperbolic end* if on a uniformizing annulus $\{r < |z| < 1\}, r > 0$, f extends continuously to $|z| = 1$ as a constant function or, equivalently, if $\lim_{|z|\to 1} f(z)$ =constant.

The following definition of vortex in a noncompact surface is a generalization of the definition of "hydrodynamic Green's function" due to Lin (1941, 1943). There are other definitions, as for instance, that in Gustafsson (1979), Flucher and Gustafsson (1997). Lin constructs a Green's function for multiply connected domains in the plane imposing that the circulation around each "internal" boundary component is null and the circulation around a particular boundary component, called the "external" boundary, is 2π. Gustafsson defines the hydrodynamic Green's function with prescribed circulation around each boundary component. In Lin's approach, the circulation around each boundary component is prescribed a posteriori and is part of the definition of a system of vortices.

Definition *Riemann surface with external end.* Let S be a Riemann surface with $N_P \geq 0$ parabolic ends and $N_H \geq 0$ hyperbolic ends, such that $N_P + N_H > 0$ (S is noncompact). Choose one end and call it "*the external end*" (if $N_P + N_H = 1$ there is no choice). The surface S with such a distinguished end will be called a Riemann surface with external end.

Theorem 3.1 (Hydrodynamic Green's function) *There exists a Green's function G solution to Eq. (3.6), unique up to the addition of a constant (the constant depends neither on q nor on p), that satisfies the following properties: $G(q, p)$ is C^∞ on $S \times S$ minus the diagonal, G is symmetric $G(q, p) = G(p, q)$, the circulation (3.9) associated with $G(\cdot, p)$ around all ends are null except for that around the external end that is equal to 1, $G(\cdot, p)$ is simple at each parabolic end, and $G(\cdot, p)$ is constant at each hyperbolic end.*

We remark that G is the same for all Riemannian metrics compatible with the conformal structure.

Proof The solutions to Eq. (3.6) and the boundary conditions are invariant under conformal equivalence. Therefore, Theorem 2.1 implies that we can look for a solution to Eq. (3.6) on a circular domain S_C in a compact surface R. Let $P_1, \dots, P_{N_P}$ be the points in R that correspond to parabolic ends of S and $\gamma_1, \dots, \gamma_{N_H}$ be circles in R that correspond to hyperbolic ends of S. The circles γ_i are oriented as the boundary of the domain

$$\hat{S} = S_C \cup P_1 \cup \ldots \cup P_{N_P}, \tag{3.13}$$

which corresponds to the orientation in Fig. 1.

Case where $N_H = 0$. The external end has to be a parabolic end, for instance, that corresponding to P_1. In this case, we have to construct a Green's function $g(q, p, P_1)$ that satisfies the equation $-\Delta_q g(q, p, P_1) = \delta_p(q) - \delta_{P_1}(q)$ on R. This sets the circulation associated with $g(\cdot, p, P_1)$ around the end P_1 equals to 1. Let $\hat{G}(q, p)$ be the Green's function on R that satisfies $-\Delta_q \hat{G}(q, p) = \delta_p(q) - V^{-1}$. Then,

$$g(q, p, P_1) = \hat{G}(q, p) - \hat{G}(q, P_1) - \hat{G}(p, P_1) \tag{3.14}$$

satisfies $g(q, p, P_1) = g(p, q, P_1)$ and $-\Delta_q g(q, p, P_1) = \delta_p(q) - \delta_{P_1}(q)$. So, $g(\cdot, \cdot, P_1)$ is the required hydrodynamical Green's function. Notice that if $\hat{g}$ is any other function with the same properties, then $\Delta_q(g - \hat{g}) = 0$ on S_C. Since R is compact and $g - \hat{g}$ is simple at each parabolic end of S_C, then $g(q, p, P_1) = \hat{g}(q, p, P_1) + c(p, P_1)$. The symmetry of $\hat{g}$ and g implies that $c(p, P_1) = c(P_1)$ as required.

Case where $N_P = 0$. In this case, $N_H = N$ and the circular domain S_C has the circular boundary components $\gamma_1, \dots, \gamma_N$. We will use the following result.

Theorem 3.2 (Aubin 1998 Theorem 4.17) *There exists a Green's function F on S_C, solution to the equation $-\Delta F(q, p) = \delta_p(q)$, which has the following properties: $F(q, p)$ is C^∞ on $S_C \times S_C$ (including the boundary points in ∂S_C) minus the diagonal $\{p = q\}$; $F(q, p) = 0$ for $p \in S_C - \{q\}$ and $q \in \partial S_C$; $F(q, p) > 0$ for q and p in the interior of S_C; and F is symmetric $F(q, p) = F(p, q)$.*

If $N = 1$, then the function F is the required hydrodynamic Green's function. So, in the following we suppose that $N > 1$ and that the external end is γ_N. The following argument is classical and can be found in Gustafsson (1979). From Theorem 3.2 follows the existence of the functions (the harmonic measures on S_C)

$$\omega_i(q) = -\int_{\gamma_i} *d_\xi F(\xi, q) \qquad i = 1, \dots, N \tag{3.15}$$

which are harmonic on S_C, satisfy $\omega_i(q) = 0$ if $q \in \gamma_j$ with $j \neq i$, and $\omega_i(q) = 1$ if $q \in \gamma_i$.

The so-called period of the conjugate function associated with ω_j around γ_i, which is minus the circulation of ω_j around γ_i, is given by

$$p_{ij} = \int_{\gamma_i} *\mathrm{d}\omega_j$$

Let P be the $(N-1) \times (N-1)$ matrix with entries p_{ij} (the period matrix). The matrix P is symmetric because

$$p_{ij} = \int_{\gamma_i} *\mathrm{d}\omega_j = \int_{\partial S_C} \omega_i * \mathrm{d}\omega_j = \int_{S_C} \mathrm{d}\omega_i \wedge *\mathrm{d}\omega_j = \int_{S_C} \mathrm{d}\omega_j \wedge *\mathrm{d}\omega_i.$$

The matrix P is positive because, for any $a_1, \dots a_{N-1}$,

$$\sum_i \sum_j a_i a_j p_{ij} = \int_{S_C} d\,[\underbrace{\sum_i a_i \omega_i}_{\phi}] \wedge * \, d[\underbrace{\sum_j a_j \omega_j}_{\phi}] = \int_{S_C} |\mathrm{d}\phi|^2 \mu.$$

Since $\phi(q) = a_i$ for $q \in \gamma_i$ and $i = 1, \dots, N-1$, $\phi(q) = 0$ for $q \in \gamma_N$, and ϕ is harmonic on W, if at least one $a_i \neq 0$, then ϕ is nonconstant and $\sum_i \sum_j a_i a_j p_{ij} > 0$. Therefore, P has an inverse A with entries a_{ij} which is also symmetric.

For future use, we state the following.

Proposition 3.1 *There exist harmonic functions $h_1, \dots, h_{N-1}$ defined on S_C such that:*

(a) $h_i(q) = const$ for $q \in \gamma_j$, $j = 1, \dots N$, the constant may depend on i and j;
(b) The circulation associated with h_i is: 1 around γ_i, -1 around γ_N, and 0 around γ_j, for all $j = 1, \dots, N-1$ with $j \neq i$.

The functions h_i, $i = 1, \dots, N-1$ in the proposition are given by

$$h_i(q) = -\sum_j a_{ij}\omega_j(q). \tag{3.16}$$

Then, their associated circulations satisfy

$$-\int_{\gamma_k} *dh_i = \sum_j a_{ij} p_{jk} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

and $h_i(q)$ has a constant value on each γ_j, $j = 1, \dots, N$.

Now, as given in Eq. (3.15), the circulation associated with $F(\cdot, p)$ around γ_i is $\omega_i(p)$. So, let

$$G_N(q, p) = F(q, p) - \sum_{i=1}^{N-1} \omega_i(p)h_i(q) = F(q, p) + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \omega_i(p)a_{ij}\omega_j(q) \tag{3.17}$$

This function is symmetric, satisfies $-\Delta G_N = \delta_p$, has a constant value on each boundary component, and has an associated null circulation around each boundary γ_i, for $i = 1, \dots, N-1$. Notice that the circulation associated with G_N around γ_N has to be 1 to compensate for the vortex at p. So, G_N is the required hydrodynamic Green's function.

Finally, let $\hat{G}_N$ be any other function with the same properties as G_N. Then, $u = G_N - \hat{G}_N$ satisfies: $\Delta_q u = 0$ on S_C, $u(\cdot, p)$ is constant on each boundary component of S_C, and the circulation associated with $u(\cdot, p)$ around each γ_i is null. Therefore, $\int_{S_C} d_q u \wedge * d_q u = \int_W |d_q u|^2 \mu = 0$ that implies $u = u(p)$. The symmetry of $G_N(q, p)$ and $\hat{G}_N(q, p)$ with respect to the exchange of q and p implies that $u(p) =$ constant. So, the required Green's function $G_N(q, p)$ is unique up to a constant. This finishes the proof of the case $N_P = 0$.

Case where $N_H > 0$, $N_P > 0$, and the external end is hyperbolic.

Let $\hat{S}$ be the surface in R defined in Eq. (3.13). Notice that the boundary components of $\hat{S}$ are the circles $\gamma_1, \dots, \gamma_{N_H}$. Let $\hat{G}_N(q, p)$ be the Green's function of the case $N_P = 0$, now defined on $\hat{S}$. If $G_N(q, p)$ is the restriction of $\hat{G}_N$ to S_C, then $G_N(\cdot, p)$ is simple at $P_1, \dots, P_{N_P}$ and has null circulation around these points. Therefore, G_N is the required hydrodynamic Green's function.

Case where $N_H > 0$, $N_P > 0$, and the external end is parabolic.

Suppose that the external end is at P_1. Again, let $\hat{S}$ be the surface in R defined in Eq. (3.13). Let $G_1(q, p), \dots, G_{N_H}(q, p)$ be the Green's function on $\hat{S}$ obtained in the case $N_P = 0$. Consider the function

$$g(q, p, P_1) = G_i(q, p) - G_i(q, P_1) - G_i(p, P_1), \tag{3.18}$$

for some $i \in \{1, \dots, N_H\}$. For a fixed P_1, this function comply with all the requirements in the definition of a vortex when P_1 is the external border. Notice that $g(\cdot, p, P_1)$ is harmonic on a neighborhood of each P_i, $i = 2, \dots, P_{N_P}$, that implies the circulation associated with $g(\cdot, p, P_1)$ around these ends is zero. The same argument used in the proof of the uniqueness, up to a constant, of G_N in the case $N_P = 0$ can be also used in this case.

This finishes the proof of the existence of the hydrodynamical Green's function in all cases. □

Definition In a *noncompact surface S* with an external end, a *point vortex* of intensity Γ at a point p is the velocity field associated with the stream function $\Gamma G(\cdot, p)$, where G is the Green's function of Theorem 3.1.

In Sect. 5, it will be important to know the behavior of $G(q, p)$ for q close to p. In order to study this question, let $\ell(q, p)$ denote the Riemannian distance between q and p. For a given p in S, consider the function $q \to A(q, p) = G(q, p) - (-1/2\pi)\log \ell(q, p)$ defined for q in a ball B_p centered at p of sufficiently small radius. In order to do computations with this function, it is convenient to introduce geodesic polar coordinates (ρ, ζ) centered at p. These are polar coordinates (ρ, ζ) on T_pS which parametrize B_p through the exponential map (see Spivak 1999 chapter 3 B). In these coordinates, $\ell(q, p) = \rho$ and

$$\theta_1 = d\rho, \quad \theta_2 = \sqrt{h}d\zeta, \quad \mu = \theta_1 \wedge \theta_2 = \sqrt{h}d\rho \wedge d\zeta$$
$$\sqrt{h} = \rho - \frac{K(p)}{6}\rho^3 + \mathcal{O}(\rho^4), \quad \Delta f = \frac{1}{\sqrt{h}}\left[\partial_\rho(\sqrt{h}\partial_\rho f) + \partial_\zeta\left(\frac{\partial_\zeta f}{\sqrt{h}}\right)\right] \tag{3.19}$$

where $K(p)$ is the Gaussian curvature of S at p that corresponds to $\rho = 0$. The function $q \to \log[\ell(q, p)]$ defines a distribution which when applied to a C^∞ function ϕ with compact support in B_p gives

$$\phi \to -\frac{1}{2\pi}\int_U \log[\ell(q, p)]\phi\mu = -\frac{1}{2\pi}\int_U (\log\rho)\phi\sqrt{h}\mathrm{d}\rho \wedge d\zeta$$

Using the polar coordinates, the Laplacian of this distribution can be computed (Aubin 1998 p. 107; the Laplacian used in Aubin 1998 has a different sign than that used here):

$$-\Delta_q \log[\ell(q, p)] = \delta_p + H(q, p), \quad \text{with} \quad H(q, p) = -\frac{K(p)}{6\pi} + \mathcal{R}(q, p), \tag{3.20}$$

where $\mathcal{R}(q, p)$ is C^∞ for $q \neq p$ and there exists a constant c such that $|\mathcal{R}(q, p)| < c\ell(p, q)$. So, $\mathcal{R}$ is a continuous Lipschitz function and

$$-\Delta_q A(q, p) = -\frac{K(p)}{6\pi} + \text{const} + \mathcal{R}(q, p) \tag{3.21}$$

where const $\neq 0$ is the inverse V^{-1} of the volume when S is a compact surface and const $= 0$ when S is a noncompact surface. So, we obtain the following.

Proposition 3.2 *Given p in S, there is a constant $c(p) > 0$ such that for $\ell(q, p) < c(p)$ the hydrodynamic Green's function can be written as*

$$G(q, p) = -\frac{1}{2\pi}\log\ell(q, p) + A(q, p)$$

where $A(q, p) = A(p, q)$ is defined on a neighborhood of the diagonal in $S \times S$, A is C^2, A is C^∞ for $q \neq p$, and A satisfies Eq. (3.21).

4 Definition of a System of Vortices

Our definition of a system of vortices has two ingredients: the flow induced by the vortices themselves and a "background flow." By a background flow, we mean a harmonic flow on S, which exists in the absence of vortices. On the plane, any harmonic function is the potential of a harmonic flow. A vortex interacting with most of these harmonic flows may go to infinity in finite time. We avoid these undesirable (unphysical) situations imposing that the background flow must satisfy the same boundary conditions at the ends of S as those satisfied by the vortices.

We remark that compact surfaces do not have nonconstant harmonic functions but may have a finite-dimensional space of harmonic flows. These harmonic flows are nonexact in the sense that they neither come from the gradient of a potential nor admit a stream function. The dimension of the space of nonexact harmonic flows is equal to the dimension of the first de Rham cohomology of S. For instance, for the torus $\mathbb{R}^2/\mathbb{Z}^2$ this space is two-dimensional and it is generated by the vector fields (1, 0) and (0, 1). A noncompact surface with genus greater than zero may also admit nonexact harmonic flows. In the definition of background flow we give below, we avoid including nonexact harmonic flows. If they would be included, then the set of equations for the motion of the vortices would not be strictly Hamiltonian, in the sense of being generated by a global Hamiltonian function.

Definition *Background flow.* Let S be a Riemann surface with external end. Suppose that $N = N_P + N_H > 1$ (if $N_P + N_H = 1$ there is no background flow). Then for any given numbers $c_1, \ldots, c_{N-1}$, there exists a harmonic function ψ_0 on S, "the background stream function," with the following properties:

(a) ψ_0 is simple at each parabolic end;
(b) ψ_0 is constant at each hyperbolic end;
(c) If the $N - 1$ elements of the set of ends minus the external end are labeled by the integers $i = 1, \ldots, N - 1$, then the circulation associated with ψ_0 around each element i is equal to c_i and the circulation around the external end is $1 - (c_1 + \cdots + c_{N-1})$.

The function ψ_0 is unique up to the addition of a constant. Finally, the background flow in S is the velocity field associated with the stream function ψ_0.

Proof of the existence the background flow. We follow the notation of the proof of the existence of the hydrodynamic Green's function. We labeled the ends as $\gamma_1, \ldots \gamma_{N_H}, P_{N_H+1}, \ldots, P_N$, where $N = N_H + N_P \geq 2$, $N_H \geq 0$, and $N_P \geq 0$.

Case where the external end is hyperbolic. Let the external end be γ_{N_H} with $N_H > 0$. Let $h_1, \ldots, h_{N_H-1}$ be the functions given in Proposition 3.1 and G_{N_H} be the hydrodynamic Green's function in Eq. (3.17). Then,

$$\psi_0(q) = \sum_{i=1}^{N_H-1} c_i h_i(q) - \sum_{i=N_H+1}^{N} c_i G_{N_H}(q, P_i) \tag{4.1}$$

Case where the external end is parabolic. Let the external end be P_N with $N > 1$. Let $G_i(q, p)$ be the hydrodynamic Green's function with nontrivial circulation around γ_i and $g(q, P_i, P_N)$ be the hydrodynamic Green's function in equation (3.14). Then,

$$\psi_0(q) = \sum_{i=1}^{N_H} c_i G_i(q, P_N) - \sum_{i=N_H+1}^{N-1} c_i g(q, P_i, P_N) \tag{4.2}$$

The stated properties of all these functions ψ_0 follow from those of g, G_i and h_i. □

Definition A it system of point vortices on a surface S is the velocity field generated by:

(a) A collection of $n \geq 1$ point vortices of intensities $\Gamma_1, \ldots, \Gamma_n$ at distinct points $q_1, \ldots, q_n$ in S and
(b) A background flow associated with a stream function ψ_0.

The stream function of the system is given by

$$\psi(p) = \psi_0(p) + \sum_{i=1}^{n} \Gamma_i G(p, q_i) \tag{4.3}$$

where $G(p, q)$ is the hydrodynamic Green's function and ψ_0 is the stream function of the background flow. If S is compact, then the background flow is null.

Gustafsson (1979) introduces the hydrodynamic Green's function in a slightly different way than Lin (1941, 1943) and us. In some situations, as in Theorem 7.1 below, Gustafsson's definition is more convenient.

Definition *(Gustafsson's Green's function)* Suppose that a Riemann surface has N ends. Let $c_1, \ldots, c_N$ be a given collection of numbers such that $c_1 + \ldots + c_N = 1$. The Gustafsson's Green's function G_c has the same properties as the hydrodynamic Green's function G in Theorem 3.1 except for the circulations at the ends. The circulation of G_c around the end i is c_i.

If N is the label of the external end of S, then the circulations of $G_c - G$ are $c_1, \ldots, c_{N-1}, c_N - 1$. Therefore, there exists a background flow ψ_0 with these circulations such that

$$G_c(q, p) = G(q, p) + \psi_0(q) + \psi_0(p). \tag{4.4}$$

5 The Force Upon a Point Vortex and Its Motion

In a system of point vortices on the plane, each vortex moves according to the velocity induced by the other vortices. This can be understood as a loose consequence of the Kelvin–Helmholtz theorem. On a surface, the same is expected.

On the plane, there a several different derivations of the equations for the motion of a system of vortices, the Helmholtz–Kirchhoff equations. The derivation we found most inspiring uses a momentum balance. It can be found in the book by Friedrichs (1966), in Gustafsson (1979), Grotta Ragazzo et al. (1994), and in other places. This derivation not only gives the Helmholtz–Kirchhoff equations but also the force on a material point at the vortex location.

On a curved surface, the momentum balance is delicate. On the Euclidean plane, vectors can be parallel translated freely and this allows for the integration of vector fields over a curve obtaining a vector as a result. On a curved Riemannian surface, vectors can be parallel translated along curves but due to the curvature the result depends on the curve and not only on the endpoints. Nevertheless, the balance of momentum argument can be adapted to surfaces in different ways. One of them is in Viglioni (2013). Another, which is presented below, uses a balance of momentum in its weak form. The idea of a balance of momentum in weak form was used in Llewellyn Smith (2011) to study the motion of dipoles in the Euclidean plane. We remark that the question of vortices with mass on a curved surface was first considered by Boatto and Koiller (2008). They proposed an expression for the force on a vortex in a surface that is a natural generalization of that for a vortex in the plane. Here we demonstrate that their expression is correct. The computation of forces on a rigid body in the sphere is given in Borisov et al. (2010).

There are several possibilities of writing Euler's equation in weak form (see, for instance Majda and Bertozzi 2002). The idea behind a weak formulation of Euler's equation is the following. Let $C_c^\infty(S)$ denote the space of C^∞ functions on S with compact support. If Euler's equation (3.3) is satisfied for some smooth velocity form u, then for any $\phi \in C_c^\infty(S)$ the following identity holds

$$\underbrace{\int_S (\partial_t u) \wedge \mathrm{d}\phi}_{=\partial_t \int_S u \wedge \mathrm{d}\phi} + \int_S L_v u \wedge \mathrm{d}\phi = -\int_S \underbrace{\mathrm{d}\left(p - \frac{|u|^2}{2}\right) \wedge \mathrm{d}\phi = 0.}_{\int_{\partial S}\left(p - \frac{|u|^2}{2}\right)\mathrm{d}\phi} \tag{5.1}$$

A smooth one-form u that satisfies Eq. (5.1) for all test functions $\phi \in C_c^\infty(S)$ is called a weak solution to Eq. (5.1). Equation (5.1) is said to be a weak formulation of Euler's equation if all weak solutions of (5.1) that are smooth are also solutions of the differential Euler's equation (3.3). Here we are only interested in very particular solutions so we will not pay attention to this general question. The weak formulation has two very interesting properties: the pressure disappears from the problem, and it is possible to consider solutions u that are not differentiable; the only requirement is that the integrations in the left-hand side of the equation are well defined. Unfortunately, for the vortex problem this is not the case: there is no hope that the second integral in the left-hand side of (5.1) makes sense in this form. So, in order to avoid the vortex singularity we take the principal value of the integral. Summarizing, we introduce the following.

Definition *Point vortex weak solution of Euler's equation.* We say that Euler's equation has a weak solution containing n point vortices at $q_1, \ldots, q_n$ if for every $\phi \in C_c^\infty(S)$ the following identity holds

$$\partial_t \int_S u \wedge \mathrm{d}\phi + \sum_{i=1}^{n} \lim_{\epsilon \to 0} \int_{S \setminus \{B_\epsilon(q_1) \cup \ldots \cup B_\epsilon(q_n)\}} L_v u \wedge \mathrm{d}\phi = 0 \tag{5.2}$$

where $B_\epsilon(q_i)$ is the ball of geodesic radius ϵ about q_i. This definition can be applied even for a system of vortices embedded in a flow with vorticity.

Now, we turn to the problem of the determination of a force on a vortex. Consider a system of n point vortices in S such that the position $q_i(t)$ of each vortex is a prescribed smooth function of time. In particular, the vortex may stay at rest. In order to keep this motion, an external force must be exerted upon each vortex and through the vortex upon the fluid. So, besides the pressure, that is an internal force caused upon a fluid particle by the surrounding particles, an external force one-form $-F$ must be added to the right-hand side of Euler's equation (3.3):

$$\partial_t u + L_v u = -d\left(p - \frac{|u|^2}{2}\right) - F \qquad L_v \mu = 0 \tag{5.3}$$

The origin of the minus sign in front of F is that at the end we are interested on the force that the fluid acts upon the vortices. The force F must be supported at the vortices positions, so $F = F^1 + \ldots + F^n$ where F^i has support at $q_i(t)$. Clearly, a differential one-form in the usual sense cannot be supported at a single point. But one can think on a δ-sequence of differential one-forms F^i_ϵ such that as $\epsilon \to 0$ their supports tend to a point, while their strengths grow in such a way that their limit under integration is well defined. By this, we mean that for any given smooth one-form λ on S the map $\lambda \to \int_S F^i \wedge \lambda = \lim_{\epsilon \to 0} \int_S F^i_\epsilon \wedge \lambda \in \mathbb{R}$ is well defined, it is linear, and it depends only on the value of λ at q_i. Rather than a force form, F^i should be called a force current because, indeed, it is a 1-current in the sense of de Rham (1984). If x is a coordinate system in a neighborhood of q_i such that: $x(q_i) = (0, 0)$, $\lambda_{q_i} = \lambda_1 \mathrm{d}x_1 + \lambda_2 \mathrm{d}x_2$, and $F^i = F^i_1 \delta_0 \mathrm{d}x_1 + F^i_2 \delta_0 \mathrm{d}x_2$, where δ_0 is the Dirac δ-function at the origin, then

$$\int_S F^i \wedge \lambda = F^i_1 \lambda_2 - F^i_2 \lambda_1 = - * \lambda_{q_i}(\vec{F}_i)$$

where $\vec{F}_i$ is a vector at $T_{q_i} S$ with coordinates $\vec{F}_i = F^i_1 \partial_{x_1} + F^i_2 \partial_{x_2}$. Therefore, the introduction of the force form $-F = -F^1 - \ldots - F^n$ at the right-hand side of Euler's equation (3.3) leads to the following.

Definition *Point vortex weak solution of Euler's equation under external forces upon the vortices*. We say that Euler's equation with prescribed forces on the vortices (5.3) has a weak solution containing n point vortices at $q_1, \ldots, q_n$ if for every $\phi \in C^\infty_c(S)$ the following identity holds

$$\partial_t \int_S u \wedge \mathrm{d}\phi + \sum_{i=1}^{n} \lim_{\epsilon \to 0} \int_{S \setminus \{B_\epsilon(q_1) \cup \ldots \cup B_\epsilon(q_n)\}} L_v u \wedge \mathrm{d}\phi = \sum_{i=1}^{n} * \mathrm{d}\phi_{q_i}(\vec{F}_i) \tag{5.4}$$

where $B_\epsilon(q_i)$ is the ball of geodesic radius ϵ about q_i.

On the other hand, if the velocities of the n vortices are prescribed and the u in Eq. (5.3) is that of a system of point vortices, then a force $F_i(t) = \langle \vec{F}_i, \cdot \rangle$ upon the

i^{th} vortex, with $i = 1, \ldots, n$, is well defined if, and only if, for every $\phi \in C_c^\infty(S)$ Eq. (5.4) holds.

In order to simplify the analysis, let U_i be a neighborhood of q_i such that $\{q_j\} \cap U_i = \emptyset$ for all $j \neq i$. Restricting the test functions to $\phi \in C_c^\infty(U_i)$, only the term $*\mathrm{d}\phi_{q_i}(\vec{F}_i)$ appears on the write hand side of Eq. (5.4). In this way, Eq. (5.4) becomes localized in a neighborhood of q_i and the index i can be omitted in q_i, $\vec{F}_i$, U_i, etc, with no risk of confusion.

Using Proposition 3.2, the velocity form $u = - * d\psi$ can be split into two parts: a singular one due to the vortex at q and a remainder C^1 form. Using geodesic polar coordinates (ρ, ζ) centered at q, Eq. (3.19), and Proposition 3.2, we obtain

$$u = - * \mathrm{d}\psi = \frac{\Gamma}{2\pi}\mathrm{d}\zeta + u_r \tag{5.5}$$

where ψ is the stream function of the vortex system, Eq. (4.3), and u_r is a C^1 differential form in U. Using these expressions, the integrals in Eq. (5.4) can be computed. If B_ϵ is the ball of radius $\epsilon > 0$ centered at q, then

$$\begin{aligned} \int_S u \wedge \mathrm{d}\phi &= \underbrace{\int_{B_\epsilon} u \wedge \mathrm{d}\phi}_{\mathcal{O}(\epsilon)} + \int_{U\setminus B_\epsilon} \underbrace{u \wedge \mathrm{d}\phi}_{-\mathrm{d}[\phi u]+\phi \mathrm{d}u} \\ &= \int_{\partial B_\epsilon} \underbrace{\phi u}_{\phi(q)\frac{\Gamma}{2\pi}d\zeta+\mathcal{O}(\rho)} + \int_{U\setminus B_\epsilon} \phi \mathrm{d}u + \mathcal{O}(\epsilon) \end{aligned}$$

where ∂B_ϵ is oriented as the boundary of B_ϵ. In $U \setminus B_\epsilon$, either S is compact and $\mathrm{d}u = c\mu$ where c is some real constant (possibly zero) or S is noncompact and $\mathrm{d}u$ is null. Therefore, the limit as $\epsilon \to 0$ in the equation above gives

$$\int_S u \wedge \mathrm{d}\phi = \Gamma\phi(q) + c\int_U \phi\mu$$

and since the last integral does not depend on time

$$\partial_t \int_S u \wedge \mathrm{d}\phi = \Gamma \mathrm{d}\phi_q(\dot{q}) \tag{5.6}$$

For the other integral in Eq. (5.4), using $L_v u = i_v \mathrm{d}u + \mathrm{d}i_v u$, we get

$$\int_{S\setminus B_\epsilon} L_v u \wedge \mathrm{d}\phi = \int_{S\setminus B_\epsilon} (i_v du) \wedge \mathrm{d}\phi - \int_{\partial B_\epsilon} |u|^2 \mathrm{d}\phi$$

In $S \setminus B_\epsilon$, as above, $du = c\mu$ where c is possibly zero. Thus, using $L_v\mu = 0$,

$$(i_v du) \wedge \mathrm{d}\phi = -c\mathrm{d}\phi(v)\mu = -cL_v(\phi\mu) = -cd(\phi i_v \mu).$$

Using this identity

$$\int_{S\setminus B_\epsilon}(i_v \mathrm{d}u)\wedge \mathrm{d}\phi = c\int_{\partial B_\epsilon}\phi i_v\mu = c\int_{\partial B_\epsilon}\phi(*u)$$
$$= c\int_{\partial B_\epsilon}\phi\left(-\frac{\Gamma}{2\pi}\frac{d\rho}{\rho}+*u_r\right) = +c\int_{\partial B_\epsilon}\phi * u_r = \mathcal{O}(\epsilon)$$

It remains to compute the integral $\int_{\partial B_\epsilon}|u|^2\mathrm{d}\phi$. Using the orthonormal moving frame in Eq. (3.19), $\theta_1 = d\rho, \theta_2 = (\rho + \mathcal{O}(\rho^3))d\zeta$, and $u = \frac{\Gamma}{2\pi}d\zeta + u_r$ we obtain

$$\int_{\partial B_\epsilon}|u|^2\mathrm{d}\phi = \left(\frac{\Gamma}{2\pi\epsilon}\right)^2\underbrace{\int_{\partial B_\epsilon}\mathrm{d}\phi}_{=0}$$
$$+\frac{\Gamma}{\pi\epsilon}\int_{\partial B_\epsilon}\langle\theta_2, u_r\rangle\mathrm{d}\phi + \underbrace{\int_{\partial B_\epsilon}|u_r|^2\mathrm{d}\phi}_{=\mathcal{O}(\epsilon)} + \mathcal{O}(\epsilon)$$

Using Cartesian coordinates on T_qS, $x_1 = \rho\cos\zeta$, $x_2 = \rho\sin\zeta$, the expression for u_r becomes

$$u_r = u_{r1}^0\mathrm{d}x_1 + u_{r2}^0\mathrm{d}x_2 + \kappa_1\mathrm{d}x_1 + \kappa_2\mathrm{d}x_2$$
$$= (u_{r1}^0\cos\zeta + u_{r2}^0\sin\zeta)\mathrm{d}\rho + (-u_{r1}^0\sin\zeta + u_{r2}^0\cos\zeta)\rho\mathrm{d}\zeta + \kappa_1\mathrm{d}x_1 + \kappa_2\mathrm{d}x_2$$

where $u_r(0) = u_r^0 = u_{r1}^0\mathrm{d}x_1 + u_{r2}^0\mathrm{d}x_2$ is a form with constant coefficients and κ_1, κ_2 are C^1 functions that vanish at the origin. Similarly,

$$\mathrm{d}\phi = \phi_1\mathrm{d}x_1 + \phi_2\mathrm{d}x_2 + \mathcal{O}(|x|)\mathrm{d}x_1 + \mathcal{O}(|x|)\mathrm{d}x_2$$
$$= (\phi_1\cos\zeta + \phi_2\sin\zeta)\mathrm{d}\rho + (-\phi_1\sin\zeta + \phi_2\cos\zeta)\rho d\zeta + \mathcal{O}(|x|)\mathrm{d}x_1 + \mathcal{O}(|x|)\mathrm{d}x_2$$

With this

$$\int_{\partial B_\epsilon}\langle\theta_2, u_r\rangle\mathrm{d}\phi = \epsilon\int_{\partial B_\epsilon}[\phi_1 u_{r1}^0\sin^2\zeta + \phi_2 u_{r2}^0\cos^2\zeta$$
$$-(\phi_2 u_{r1}^0 + \phi_1 u_{r2}^0)\cos\zeta\sin\zeta]d\zeta + \mathcal{O}(\epsilon^2)$$
$$= \epsilon\pi\,\mathrm{d}\phi(v_r^0) + \mathcal{O}(\epsilon^2)$$

where v_r^0 is the dual of u_r^0. Therefore,

$$\lim_{\epsilon\to 0}\int_{S\setminus B_\epsilon(q_i)}L_v u\wedge\mathrm{d}\phi = -\lim_{\epsilon\to 0}\frac{\Gamma}{\pi\epsilon}\int_{\partial B_\epsilon}\langle\theta_2, u_r\rangle\mathrm{d}\phi = -\Gamma\mathrm{d}\phi_q[v_r(q)]$$

This result plus that in Eq. (5.6) gives for the force $\vec{F}$ in Eq. (5.4)

$$\Gamma\mathrm{d}\phi_q[\dot{q} - v_r(q)] = *\mathrm{d}\phi_q(\vec{F})$$

So, the force vector that acts upon the point vortex at q_i is

$$\vec{F}_i = *[\dot{q}_i - v_r(q_i)]\Gamma_i \tag{5.7}$$

where $*$ is the operation that rotates counterclockwise a vector by $\pi/2$.

Notice that the force upon a material vortex depends on the difference between the vortex velocity and the regularized fluid velocity at the vortex position. If the vortex is at the point q_i, then using Proposition 3.2 and the stream function of a system of vortices (4.3) we obtain

$$\begin{aligned} u_r(q_i) &= \lim_{\ell(q,q_i)\to 0} u(q) - *d_q \frac{\Gamma_i}{2\pi} \log \ell(q, q_i) \\ &= - * \lim_{\ell(q,q_i)\to 0} d_q \left[\psi(q) + \frac{\Gamma_i}{2\pi} \log \ell(q, q_i) \right] \\ &= - * d_{q_i} \left[\psi_0(q_i) + \sum_{j\neq i}^{n} \Gamma_j G(q_i, q_j) \right] \\ &\quad - * \lim_{\ell(q,q_i)\to 0} d_q \underbrace{\left[G(q, q_i) + \frac{1}{2\pi} \log \ell(q, q_i) \right]}_{=A(q,q_i)} \Gamma_i \end{aligned}$$

The function $q \to A(q, q_i)$ is C^2, Proposition 3.2, so the limit above commutes with the differential. Moreover, $A(q, q_i) = A(q_i, q)$ that implies $d_{q_i} A(q_i, q_i) = 2 d_q A(q, q_i)|_{q=q_i}$. Therefore, the equation above can be written as:

$$u_r(q_i) = - * d_{q_i} \left[\psi_0(q_i) + \sum_{j\neq i}^{n} \Gamma_j G(q_i, q_j) + \frac{\Gamma_i}{2} R(q_i) \right] \tag{5.8}$$

where the function R is the Robin function from potential theory that is given by

$$R(p) = \lim_{\ell(q,p)\to 0} \left[G(q, p) + \frac{1}{2\pi} \log \ell(q, p) \right] \tag{5.9}$$

Since G is unique up to an additive constant, the same happens to R. Notice that if S is noncompact, then G is the same for all Riemannian metrics in the same conformal class. On the contrary, the Robin function does depend on the Riemannian metric through the Riemannian distance function.

Theorem 5.1 *The Robin function is C^∞ on S. Let g_0 and g_1 be two different Riemannian metrics on S in the same conformal class, $|\cdot|_1 = \lambda(p)|\cdot|_0$. Let R_0 and R_1 be their Robin function, respectively. If S is noncompact, then the hydrodynamic Green's function for both Riemannian surfaces can be taken as the same and*

$$R_1(p) = R_0(p) + \frac{1}{2\pi} \log \lambda(p) \tag{5.10}$$

(*If the Green's function were different by a constant, then this constant would have to be added to the right-hand side of Eq. (5.10)*).

Proof Let p and q be sufficiently close to be in the domain U of a local uniformizer z. Suppose that U is small enough such that any two points in U are connected by a single geodesic in U. In this coordinates, the length elements of the metrics g_0 and g_1 are $\lambda_0|\mathrm{d}z|$ and $\lambda_1|\mathrm{d}z|$, respectively. Notice that $\lambda_1 = \lambda\lambda_0$. To simplify the notation, we write $z(q) = w$ and $z(p) = z$. In these coordinates,

$$G_j(q, p) = -\frac{1}{2\pi}\log|z - w| + f_j(z, w)$$

where $f_j(z, w) = f_j(w, z)$. The function f_j is C^∞ because for any Riemannian metric $\Delta_z \log|z - w| = 2\pi\delta_w$ and $G_j(q, p)$ is a Green's function that satisfies either Eq. (3.5) or (3.6).

Let $\ell(p, q)$ be the length of the unique geodesic connecting q to p. In U, the geodesics satisfy an equation of the form $\dot{z}_t = v_t, \dot{v}_t = F(z_t, v_t)$. So, $z_t = z_0 + tv_0 + \mathcal{O}(t^2)$. Let ℓ_0 and ℓ_1 denote the distance function for the metrics g_0 and g_1, respectively. If the g_j-Riemannian norm of v_0 at the point z_0 is equal to one, $|v_0|_j = 1$, then $t = \ell_j(z_0, z_t)$, as far as $z_t \in U$. Therefore, for $z_0 = z$ and $z_t = w$, we obtain

$$|w - z| = t|v_0|[1 + \mathcal{O}(t)] = t\frac{|v_0|_j}{\lambda_j(z)}[1 + \mathcal{O}(t)] = t\frac{1}{\lambda_j(z)}[1 + \mathcal{O}(t)]$$

that implies

$$\log\ell_j(z, w) = \log|w - z| + \log\lambda_j(z) + \log[1 + \mathcal{O}(\ell_j(z, w))]$$

Therefore

$$G_j(q, p) + \frac{1}{2\pi}\log\ell_j(q, p) = f_j(z, w) + \frac{1}{2\pi}\log\lambda_j(z) + \frac{1}{2\pi}\log[1 + \mathcal{O}(\ell_j(z, w))]$$

Taking the limit as $|z - w| \to 0$, we obtain

$$R_j(z) = f_j(z, z) + \frac{1}{2\pi}\log\lambda_j(z)$$

that is a C^∞ function.

If S is noncompact, Eq. (3.6) does not depend on the metric and the Green's functions G_0 and G_1 can be taken as the same that implies $G_0 = G_1 = G$ and $f_0 = f_1 = f$. So,

$$R_1(z) - R_0(z) = \frac{1}{2\pi}\log\lambda_1(z) - \frac{1}{2\pi}\log\lambda_0(z)$$

that implies Eq. (5.10). □

The Robin's function R_c associated with the Gustafsson's Green's function G_c is given by

$$R_c(p) = \lim_{\ell(q,p)\to 0}\left[G_c(q,p) + \frac{1}{2\pi}\log \ell(q,p)\right] \tag{5.11}$$

The relation $G_c(q, p) = G(q, p) + \psi_0(q) + \psi_0(p)$ in Eq. (4.4) implies that

$$R_c(p) = R(p) + 2\psi(p).$$

Therefore, under the same hypotheses in Theorem 5.1 the Robin functions R_{c0} and R_{c1} associated with the conformal metrics g_0 and g_1, respectively, and to the same Gustafsson's Green's function on S also satisfy Eq. (5.10), namely

$$R_{c1}(p) = R_{c0}(p) + \frac{1}{2\pi}\log\lambda(p). \tag{5.12}$$

6 Equations of Motion and the Electromagnetic Analogy

A field of forces F on a surface S is a map $F : TS \to TS^*$ (see, for instance, Oliva 2002). The motion of a particle of mass m at a point q and with velocity $\dot{q}$ is determined by the following generalization of Newton's equation

$$\left\langle m\frac{D\dot{q}}{dt}, \cdot\right\rangle = F(q, \dot{q})(\cdot)$$

The force upon a vortex at q_j is obtained from Eqs. (5.7) and (5.8):

$$\begin{aligned} F_j &= \langle *[\dot{q}_j - v_r(q_j)]\Gamma_j, \cdot\rangle = \Gamma_j i_{\dot{q}_j}\mu \\ &\quad - \underbrace{d_{q_j}\left[\Gamma_j\psi_0(q_j) + \Gamma_j\sum_{k\neq j}^{n}\Gamma_k G(q_j, q_k) + \frac{\Gamma_j^2}{2}R(q_j)\right]}_{=d_{q_j}W} \end{aligned}$$

where

$$W(q_1,\ldots,q_n) = \sum_{i=1}^{n}\left[\Gamma_i\psi_0(q_i) + \frac{\Gamma_i^2}{2}R(q_i) + \frac{1}{2}\sum_{k\neq i}^{n}\Gamma_i\Gamma_k G(q_i, q_k)\right] \tag{6.1}$$

This force consists of two terms. The first, $\langle *\dot{q}_j\Gamma_j, \cdot\rangle$, is a force perpendicular to the vortex velocity, and it is analogous to a magnetic force upon an electric charge. In order to explain this analogy, consider the case where $S = \mathbb{R}^2$ is the Euclidean horizontal plane embedded in $\mathbb{R}^3$ Grotta Ragazzo et al. (1994). Let e_3 be the unit vertical vector and B be the magnetic field equal to $-e_3$. The Lorentz force of B upon a particle with electric charge Q and velocity v is $Qv \times B$. This force can be written as a map $T\mathbb{R}^2 \to T^*\mathbb{R}^2$ as

$$\langle \cdot, Qv \times (-\mathrm{e}_3)\rangle = \langle Qv \times \cdot, \mathrm{e}_3\rangle = Q dq_1 \wedge dq_2(v, \cdot) = Q\mu(v, \cdot)$$

So B can be identified with the area form of $\mathbb{R}^2$. This identification can be carried over any surface S. So, $*\dot{q}_j \Gamma_j \in T_{q_j} S$ or $\Gamma_j i_{\dot{q}_j} \mu \in T^*_{q_j} S$ is the Lorentz force upon a particle with electric charge Γ_j due to a uniform (with respect to the area form) magnetic field on the surface.

The second term of F_j, $-d_{q_j} W \in T^*_{q_j} S$ or $-\nabla_{q_j} W \in T_{q_j} S$, is analogous to the electric force upon a particle with charge Γ_j due to an electric potential W. This W is the electrostatic energy due to: the pairwise interaction among the particles $\sum_{i=1}^n \sum_{k\neq i}^n \Gamma_i \Gamma_k G(q_i, q_k)/2$, the interaction of the particles with external agents $\sum_{i=1}^n \Gamma_i \psi_0(q_i)$, and the interaction of the particle with boundaries and with the underlying geometry (representing a space-variable dielectric tensor Jackson 1999) $\frac{1}{2}\sum_{i=1}^n \Gamma_i^2 R(q_i)$.

It is well known that Newton's equation can be written in Hamiltonian form when F is an electromagnetic force. The same is valid for a system of vortices on a surface S. Let $M = S \times \cdots \times S$ be the Cartesian product of n copies of S and $K : TM \to \mathbb{R}$ be the kinetic energy function:

$$K = \frac{1}{2}\sum_{i=1}^n m_i \langle v_i, v_i\rangle$$

Notice that K defines a Riemannian metric on M. The momentum $p_j \in T^*_{q_j} S$ associated with $v_j \in T_{q_j} S$ is given by

$$v_j \to m_j \underbrace{\langle \cdot, v_j\rangle}_{=u_j} = p_j.$$

The cotangent bundle of M has a canonical symplectic structure

$$\tilde{\Omega} = \sum_{i=1}^n \mathrm{d}q_{i1} \wedge \mathrm{d}p_{i1} + \mathrm{d}q_{i2} \wedge \mathrm{d}p_{i2} = \sum_{i=1}^n m_i[\mathrm{d}q_{i1} \wedge \mathrm{d}u_{i1} + \mathrm{d}q_{i2} \wedge \mathrm{d}u_{i2}]$$

Let Ω be the pull back of $\tilde{\Omega}$ to TM by the map $v \to p$. The geodesic vector field X associated with K can be written in Hamiltonian form as (see, for instance, Paternain 1999 Proposition 1.21, or do the computations using a moving frame)

$$dK = i_X \Omega \longrightarrow \begin{cases} \dot{q}_j = v_j \\ m_j \frac{Dv_j}{dt} = 0 \end{cases}$$

Finally, the Newtonian vector field X on TM for the motion of the system of electric charges is:

$$\dot{q}_j = v_j, \qquad m_j \left\langle \frac{Dv_j}{dt}, \cdot \right\rangle = \Gamma_j i_{v_j} \mu - d_{q_j} W, \tag{6.2}$$

that can be written in Hamiltonian form as $dH = i_X \hat{\Omega}$ with

$$H = \frac{1}{2}\sum_{i=1}^{n} m_i \langle v_i, v_i\rangle + W(q_1, \ldots, q_n)$$
$$\hat{\Omega} = \sum_{i=1}^{n} m_i [\mathrm{d}q_{i1} \wedge \mathrm{d}u_{i1} + \mathrm{d}q_{i2} \wedge \mathrm{d}u_{i2}] + \Gamma_i \mu_i \tag{6.3}$$

where $u_i = \langle v_i, \cdot\rangle$ and μ_i is the area form of the i^{th} factor of $S \times \cdots \times S = M$. These equations are also those for the motion of a system of vortex with mass. Notice that $\hat{\Omega}$ is closed, since μ_j is closed, and $\hat{\Omega}$ is nondegenerate, since $\hat{\Omega} \wedge \cdots \wedge \hat{\Omega} = \hat{\Omega}^{2n} = \Omega^{2n}$.

The motion of a system of n vortices with mass depends on two sets of n parameters: the masses $m_1, \ldots, m_n$ and the vorticities $\Gamma_1, \ldots, \Gamma_n$. Let $m = \max\{m_1, \ldots, m_n\}$ and $\Gamma = \max\{|\Gamma_1|, \ldots, |\Gamma_n|\}$. If $\Gamma = 0$, then Eqs. (6.2) and (6.3) become

$$\dot{q}_j = v_j, \qquad m_j \left\langle \frac{Dv_j}{dt}, \cdot \right\rangle = 0, \quad \text{or} \quad dH = i_X \hat{\Omega} \quad \text{with}$$
$$H = \frac{1}{2}\sum_{i=1}^{n} m_i \langle v_i, v_i\rangle \quad \text{and} \quad \hat{\Omega} = \sum_{i=1}^{n} m_i [dq_{i1} \wedge \mathrm{d}u_{i1} + dq_{i2} \wedge \mathrm{d}u_{i2}]$$

that is the equation for the geodesic motion of n noninteracting particles. If $m = 0$, then Eqs. (6.2) and (6.3) become:

$$dW = i_X \hat{\Omega} \quad \text{with} \quad \hat{\Omega} = \sum_{i=1}^{n} \Gamma_i \mu_i \quad \text{and}$$
$$W(q_1, \ldots, q_n) = \sum_{i=1}^{n} \left[\Gamma_i \psi_0(q_i) + \frac{\Gamma_i^2}{2} R(q_i) + \frac{1}{2} \sum_{k \neq i}^{n} \Gamma_i \Gamma_k G(q_i, q_k) \right] \tag{6.4}$$

that is the equation for the motion of (massless) hydrodynamic vortices. So, as m/Γ varies from zero to infinity the dynamics changes from that of hydrodynamic vortices to that of purely inertial massive particles.

Notice that $m \to 0$ is a singular limit in the sense that the phase-space dimension of system (6.2) decreases from $4n$, for $m > 0$, to $2n$, for $m = 0$, with the remarkable property of being a Hamiltonian system even for $m = 0$. This interesting limit is analogous to that of the "guiding-center" approximation for the motion of a charge in a electromagnetic field Arnold et al. (2006), Littlejohn (1980). Consider, for instance, the motion of a particle of mass m and electric charge Q in the plane (x_1, x_2) under a constant magnetic field $-B\mathrm{e}_3$ and a variable electric potential $QW(x_1, x_2)$. This motion is determined by the equation

$$m\ddot{x} = -QB(\dot{x} \times \mathrm{e}_3) - Q\nabla W(x)$$

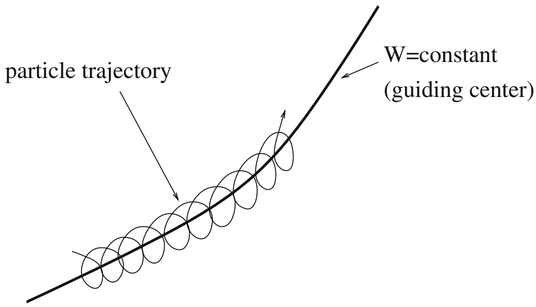


Fig. 2 Guiding-center motion illustration. The magnetic field points downward through the figure

If $m = 0$, we obtain the two-dimensional Hamiltonian system $B(\dot{x} \times \mathrm{e}_3) = -\nabla W(x)$ the integral curves of which are the guiding center. For $m > 0$ small, rescaling time as $t = m\tau$, we obtain

$$\frac{d^2x}{d^2\tau} = -QB\left(\frac{\mathrm{d}x}{d\tau} \times \mathrm{e}_3\right) - mQ\nabla W(x)$$

For $m = 0$, the integral curves of this equation are circles. A solution traces one circle with constant angular τ-velocity QB. From this analysis, we obtain that for $m > 0$ small the approximated motion of a particle is composed by a fast circular motion with constant angular t-velocity QB/m and constant radius $m|\dot{x}(0)|$ plus a slow drift of the circle center along the guiding-center curve (see Fig. 2). This same analysis holds for vortex systems (see Grotta Ragazzo et al. 1994 for systems of vortices on the Euclidean plane). Several results for the dynamics of systems of electric charges can be immediately applied to systems of massive vortex (see, for instance, Bolotin and Negrini 1995 for interesting stability analysis of equilibria).

The dynamics of a single massless vortex of intensity Γ is fully determined by the Robin function and the background flow. In this case, in isothermic coordinates $z = x + iy$, Eq. (6.4) becomes

$$\lambda^2\dot{x} = \partial_y H \quad \lambda^2\dot{y} = -\partial_x H \quad \text{with} \quad H = \psi_0(z) + \frac{\Gamma}{2}R(z). \tag{6.5}$$

Notice that the orbits of the system are contained in the level sets of the function H.

7 Examples and the “Steady Vortex Metric”

The motion of vortex in surfaces has been studied by several authors. See, for instance, Newton (2014) for many interesting references, Aref et al. (2003) for equilibrium configurations of point vortices, Turner et al. (2010) for vortices in minimal surfaces, Dritschel and Boatto (2015) for vortices in surfaces of revolution, Sakajo and Shimizu (2016) for vortices on a toroidal surface, and Boatto and Koiller (2013) for several examples and references. Here we just consider some very simple examples that will lead to a special type of metric that we call Steady Vortex Metric.

At first, consider the disk $\{|z| < 1\}$ with the Euclidean metric $|dz|$, denoted as $\mathbb{D}^2$, and with the Poincaré metric $2|dz|/(1 - |z|^2)$, denoted as $\mathbb{H}^2$. Since the two metrics are in the same conformal class, their hydrodynamical Green's function is the same and is given in Eq. (3.12). This implies that the Robin function (5.9) for $\mathbb{D}^2$ is

$$R_{\mathbb{D}^2}(z) = \lim_{w \to z} \left\{ -\frac{1}{2\pi} \log \frac{|z-w|}{|z\overline{w}-1|} + \frac{1}{2\pi} \log |z-w| \right\} = \frac{1}{2\pi} \log(1 - |z|^2)$$

So, from Eq. (6.5), we obtain that a single vortex of intensity one in $\mathbb{D}^2$ rotates counterclockwise in a circular orbit, the origin is a stable equilibrium, and the angular velocity blows up as $r \to 1$. From Theorem 5.1, we obtain that the Robin function of $\mathbb{H}^2$ is constant. So, a single vortex in the unit disk does not move under the Poincaré metric, independently on its position Kimura (1999). A single vortex does not move also in the Euclidean plane, in the round sphere, and in the flat torus. A metric for which a single vortex does not move regardless its position will be called a *steady vortex metric*.

Theorem 7.1 (Steady vortex metric: noncompact S) *Let S be a noncompact Riemann surface with N ends endowed with a collection of numbers $c_1, \ldots, c_N$, the "Gustafsson's Green's function data." Then, there is a Riemannian metric g compatible with the conformal structure of S such that its associated Gustafsson's Robin function is constant. The metric g is unique up to a multiplicative constant.*

Proof Let g_0 be a Riemannian metric compatible with the conformal structure of S and R_{c0} be its Robin function. If $g = g_1$ is conformal to g_0, $|\cdot|_1 = \lambda |\cdot|_0$, and R_{c1} is imposed to be equal to zero, then Eq. (5.12) implies

$$\lambda(p) = \exp[-2\pi R_{c0}(p)]. \tag{7.1}$$

Any other metric with constant Robin function must be proportional to g_1. □

The steady vortex metric first appeared in the work of Gustafsson (1979) with no special name. For bounded planar domains, Gustafsson compared this special metric with the previously defined "capacity metric" that is obtained in the same as way as the steady vortex metric after replacing the hydrodynamic Green's function by the standard Green's function (it has zero value on all ends of the domain). Gustafsson showed that in nonsimply connected domains the steady vortex metric has negative but nonconstant curvature. In simply connected bounded domains, the curvature is negative and constant (see Aboudi 2005 for further results on the capacity metric on nonsimply connected domains). Our Theorem 7.1 generalizes that of Gustafsson to surfaces of strictly positive genus. Before we talk more about steady vortex metrics, we present the equations of motion for a vortex in a surface of revolution.

The Eq. (2.5) determines the conformal factor for the azimuthal projection of a surface of revolution S onto the Euclidean annulus $\mathcal{A} = \{z = re^{i\theta} \in \mathbb{C} : \tilde{a} < |z| < \tilde{b}\}$. In this case, Eq. (6.5), with the Gustafsson's Robin function instead of the background flow and $\Gamma = 1$, implies

$$\dot{r} = 0 \qquad \dot{\theta} = -\frac{1}{2\lambda^2 r}\partial_r R_c(r)$$

As an example, consider the catenoid presented in the paragraph that contains Eq. (2.5). In this case, $\mathcal{A}$ is the punctured plane. Let c and $1-c$ be the circulations around the ends $-\infty$ and $+\infty$, respectively. If $c = 0$, then the Gustafsson's Robin function $R_{\mathcal{A}c}$ of $\mathcal{A}$ coincides with that of the Euclidean plane that is null. Then, $R_c = R_{\mathcal{A}c} + \frac{1}{2\pi}\log\lambda$ (Eq. 5.12), $\lambda = f/r$ (Eq. 2.5), and the expressions $f(x) = \cosh(x)$ and $r(x) = \mathrm{e}^x$, $x \in \mathbb{R}$, imply

$$R_c(r) = \frac{1}{2\pi}\log\frac{\cosh(\log r)}{r}.$$

Then, a simple computation gives

$$\dot{\theta} = \frac{1}{4\pi\cosh^3(x)}(\cosh x - \sinh x) > 0$$

Notice that the rate of rotation is not symmetric with respect to $x = 0$, which is a plane of symmetry of the catenoid. This is related to the unbalanced circulations at the ends. If we make $c = 1/2$ such that the circulations at both ends are $1/2$, then the Robin function becomes

$$R_c = \frac{1}{2\pi}\log\cosh(\log r)$$

that implies

$$\dot{\theta} = -\frac{1}{4\pi}\frac{\sinh x}{\cosh^3(x)}.$$

This symmetric solution reflects the choice of symmetric boundary conditions at the ends.

At last, we investigate the possibility of embedding isometrically in $\mathbb{R}^3$ annular regions with steady vortex metric. Let R_c be the Robin function of the Euclidean annulus $\mathcal{A} = \{z = r\mathrm{e}^{i\theta} \in \mathbb{C} : \tilde{a} < |z| < \tilde{b}\}$ with circulations c and $1-c$ at the ends $\tilde{a}$ and $\tilde{b}$, respectively. The steady vortex metric in $\mathcal{A}$ is given by $\exp(-2\pi R_c)|\mathrm{d}z| = \lambda|\mathrm{d}z|$. Suppose that $\mathcal{A}$ with the steady vortex metric can be embedded isometrically in $\mathbb{R}^3$ as a surface of revolution. Then, Eq. (2.5) implies, after some computations,

$$\mathrm{d}x = \pm 2\exp(-2\pi R_c)\sqrt{\pi r\partial_r R_c(1 - \pi r\partial_r R_c)}\mathrm{d}r. \tag{7.2}$$

This equation can be integrated to obtain the map $x(r)$ if

$$\pi r\partial_r R_c(1 - \pi r\partial_r R_c) > 0. \tag{7.3}$$

Then, $x(r)$ can be inverted and used to obtain $f(x) = r(x)\exp[-2\pi R_c(r(x))]$, which is the function that generates the required surface of revolution. In this way, if Condition

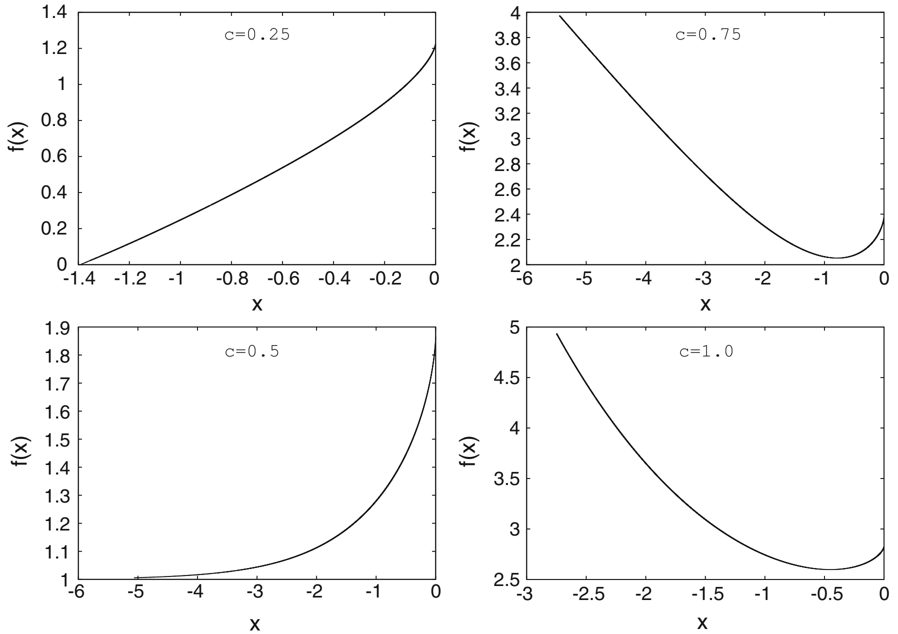


Fig. 3 Generating functions f of the surfaces of revolution with steady vortex metric for several values of circulation at the ends (c is the circulation around the parabolic end and $1-c$ around the hyperbolic end). The hyperbolic ends are never represented in the figure since their neighborhoods cannot be isometrically embedded in $\mathbb{R}^3$. For $c = 0.25$, the parabolic end is at a finite value of x (≈ -1.39). For $c = 0.5$, $\lim_{x\to-\infty} f(x) = 1$ and the parabolic end has a cylindrical shape. For $c = 0.75$, $f(x)$ grows linearly with x as $x \to -\infty$ and the parabolic end has a conical shape. For $c = 1$, $f(x)$ grows exponentially with x as $x \to -\infty$ and the parabolic end has a catenoidal shape

(7.3) is verified at least in an interval (α, β) with $\tilde{a} \le \alpha < \beta \le \tilde{\beta}$, then it is possible to embed at least a part of $\mathcal{A}$ with a steady vortex metric as a surface of revolution.

Let $\mathcal{A}$ be the punctured plane ($\tilde{a} = 0, \tilde{b} = \infty$) with circulations c and $1-c$ around the origin and the infinity, respectively. The two ends can be exchanged using the conformal map $1/z$. So, we can restrict the values of c to the interval $[0, 1/2]$. For $c = 0$, the Robin function is trivial and the embedding in $\mathbb{R}^3$ is also trivial. For $c \in (0, 1/2)$, the Robin function of $\mathcal{A}$ is $R_c = (c/\pi)\log r$ and the computation described in the previous paragraph gives $f = x(1-2c)/\sqrt{4c(1-c)}$, $x \in \mathbb{R}$. This construction suggests, and it can be easily proved, that the set of cones generated by $f(x) = \alpha x$, $x > 0$, $\alpha > 0$, gives rise to a family of surfaces of revolution with steady vortex metric provided that the circulations at the origin and at infinity are c and $1-c$, respectively, with $c = (1 - \alpha/\sqrt{1+\alpha^2})/2$. As $\alpha \to \infty$, the surface approaches a punctured disk with $c = 0$. In the limit $\alpha = 0$, the cone degenerates to the positive x-axis and $c = 1/2$. It can be easily verified that for $c = 1/2$ the cylinder in $\mathbb{R}^3$, generated by $f(x) = 1$, $x \in \mathbb{R}$, is a surface with a steady vortex metric. Notice that all these surfaces, which are conformally equivalent to the punctured Euclidean plane, have steady vortex metric of Gaussian curvature equal to zero.

Now, let $\mathcal{A}$ be the punctured Euclidean disk $\{0 < |z| < 1\}$ with circulations c and $1-c$ around the origin and $|z|=1$, respectively. In this case: $R_c(r) = \frac{1}{2\pi}\log(1-r^2) + \frac{c}{\pi}\log r$, the steady vortex metric is $\lambda|\mathrm{d}z|$ with $\lambda = 1/(1-r^2) + 1/r^{2c}$, and $\pi r \partial_r R_c = (c(1-r^2) - r^2)/(1-r^2)$. For $c=0$, $\lambda|\mathrm{d}z|$ is the Poincaré metric and $\partial_r R_c(r) < 0$ for all $r \in (0,1)$. So, Condition (7.3) is not verified, and no part of the Poincaré punctured disk can be embedded as a surface of revolution in $\mathbb{R}^3$. If $0 < c \leq 1$, then $\pi r \partial_r R_c(r) > 0$ for $0 < r < \beta = \sqrt{c/(1+c)}$. In this case, imposing the condition $x(\beta) = 0$, Eq. (7.2) can be integrated to obtain $r \in (0,\beta) \to x(r)$. This plus the function $r \to f(r) = r\lambda(r)$ is enough to plot the graph of the function $f(x)$ that generates a surface of revolution $S \subset \mathbb{R}^3$ with a vortex steady metric. Notice that this surface represents only a subset $\{0 < |z| < \beta\}$ of the punctured unit disk. The graph of f computed numerically is shown in Fig. 3 for several values of c. Notice that these surfaces have negative nonconstant Gaussian curvature.

This same type of analysis can be repeated for the annulus $\{0 < \tilde{a} < |z| < 1$ using the hydrodynamical Green's function given in Crowdy and Marshall (2005) (in this reference, the hydrodynamic Green's functions on multiply connected domains in $\mathbb{R}^2$ is extensively discussed).

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References

Aboudi, N.: Geodesics for the capacity metric in doubly connected domains. Complex Var. **50**, 7–22 (2005)

Aref, H.: Integrable, chaotic, and turbulent vortex motion in two-dimensional flows. Ann. Rev. Fluid Mech. **15**, 345–389 (1983)

Aref, H., Newton, P.K., Stremler, M.A., Tokieda, T., Vainchtein, D.L.: Vortex Crystals. Adv. Appl. Mech. **39**, 1–79 (2003)

Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics, 3rd edn. Springer, Berlin (2006)

Aubin, T.: Some Nonlinear Problems in Riemannian Geometry, 3rd edn. Springer, Berlin (1998)

Beardon, A.F.: A Primer on Riemann Surfaces. London Math. Soc. Lect. Notes Ser. 78, Cambridge University Press (1984)

Boatto, S., Koiller, J.: Vortices on closed surfaces, arXiv:0802.4313 (2008)

Boatto, S., Koiller, J.: Vortices on closed surfaces. In: Chang, D.E., Holm, D.D., Patrick, G., Ratiu, T. (eds.) Geometry, Mechanics, and Dynamics The Legacy of Jerry Marsden, pp. 185–237. Springer, Berlin (2013)

Bogomolov, V.A.: Dynamics of vorticity at a sphere. Fluid Dyn. **12**, 863–870 (1977)

Bolotin, S., Negrini, P.: Asymptotic solutions of Lagrangian systems with gyroscopic forces. Nonlinear Diff. Eq. Appl. NoDEA **2**, 417–444 (1995)

Borisov, A.V., Mamaev, I.S., Ramodanov, S.M.: Coupled motion of a rigid body and point vortices on a two-dimensional spherical surface. Regul. Chaotic Dyn. **15**, 440461 (2010)

Crowdy, D., Marshall, J.: Analytical formulae for the Kirchhoff–Routh path function in multiply connected domains. Proc. R. Soc. A **461**, 24772501 (2005)

de Rham, G.: Differentiable Manifolds. Springer, Berlin (1984)

Dritschel, D.G., Boatto, S.: The motion of point vortices on closed surfaces. Proc. R. Soc. A **471**, 20140890 (2015)

Flucher, M., Gustafsson, B.: Vortex motion in two-dimensional hydromechanics (Technical report, http://www.math.kth.se/~gbjorn/) (1997)

Friedrichs, K.O.: Special Topics in Fluid Dynamics. Gordon and Breach, New York (1966)

Grotta Ragazzo, C., Koiller, J., Oliva, W.: On the motion of two-dimensional vortices with mass. J. Nonlinear Sci. **4**, 375–418 (1994)
Gustafsson, B.: On the motion of a vortex in two-dimensional flow of an ideal fluid in simply and multiply connected domains, (Technical Report, http://www.math.kth.se/~gbjorn/) (1979)
Hally, D.: Motion of Vortex in Thin Films, PhD Thesis, The University of British Columbia, Canada (1979)
Hally, D.: Stability of streets of vortices on surfaces of revolution with a reflection symmetry. J. Math. Phys. **21**, 211–217 (1980)
Jackson, J.D.: Classical Electrodynamics. Wiley, New York (1999)
He, Z.-X., Schramm, O.: Fixed points. Koebe uniformization and circle packings. Ann. Math. **137**, 369–406 (1993)
Kimura, Y., Okamoto, H.: Vortex motion on a sphere. J. Phys. Soc. Jpn **56**, 4203–4206 (1987)
Kimura, Y.: Vortex motion on surfaces with constant curvature. Proc. R. Soc. Lond. A **455**, 245259 (1999)
Lamb, H.: Hydrodynamics, 4th edn. Cambridge University Press, Cambridge (1916)
Llewellyn Smith, S.G.: How do singularities move in potential flow? Physica D **240**, 16441651 (2011)
Li, P.: Curvature and Function Theory on Riemannian Manifolds, Survey in Differential Geometry in Honor of Atiyah, Bott, Hirzebruch and Singer, VII, International Press, Cambridge, 71–111. (2000)
Lin, C.C.: On the motion of vortices in two dimensions. I. Existence of the Kirchhoff-Routh function. Proc. Nat. Acad. Sci. USA **27**, 570575 (1941). (see also the second part of the paper in the same volume p. 575-577)
Lin, C.C.: On the Motion of Vortices in Two Dimensions, University of Toronto Studies, Applied Mathematics Series. University of Toronto Press, (1943)
Littlejohn, R.G.: "Hamiltonian Theory of Guiding Center Motion", PhD Thesis, Lawrence Berkeley Lab (eletronically available at: http://escholarship.org/uc/item/6b31q0xd) (1980)
Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge (2002)
Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids. Springer, New York (1994)
Newton, P.: The N-vortex Problem: Analytical Techniques. Springer, New York (2001)
Newton, P.: Point vortex dynamics in the post-Aref era, Fluid Dyn. Res. **46** 031401 (11pp) (2014)
Oliva, W.M.: "Geometric Mechanics", Lect. Notes Math. **1798**, Springer, Heidelberg (2002)
Paternain, G.P.: Geodesic Flows. Birkhaŝer, Boston (1999)
Richards, I.: On the classification of noncompact surfaces. Trans. AMS **106**, 259–269 (1963)
Sakajo, T., Shimizu, Y.: Point vortex interactions on a toroidal surface. Proc. R. Soc. A **472**, 20160271 (2016)
Spivak, M.: A Comprehensive Introduction to Differential Geometry, vol. 2, 3rd edn. Publish or Perish, Houston (1999)
Turner, A.M., Vitelli, V., Nelson, D.R.: Vortices on curved surfaces. Rev. Modern Phys. **82**, 1301–1348 (2010)
Viglioni, H.H.B.: Vortex Dynamics on Surfaces and applications to the problem of two vortices on the torus, Doctorate thesis (in Portuguese), Universidade de São Paulo (2013)