Neutral Functional Differential Equations in L_p-space

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1 Introduction

This work is a part of [7].

We will consider *neutral functional differential equations* (NFDE) of the form

$$\frac{d}{dt}\mathcal{D}(x_t) = F(x_t) \tag{1.1}$$

where $\mathcal{D}$ and F are given continuosly Fréchet-differentiable functionals from the space C with values in a given Banach space X, C being the space of all continuous functions $\varphi : [-r, 0] \to X$ with norm $||\varphi||_\infty = \sup\{|\varphi(\theta)| : -r \leq \theta \leq 0\}$, $r > 0$ a given constant (*the delay*) and x_t the map $x_t : [-r, 0] \to X$ defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

Under appropriate hypotheses on the functionals $\mathcal{D}$ and F, one can prove that, for each φ in C, there exists a unique continuous solution $x : (-\infty, +\infty) \to X$ of equation (1.1) such that $x_0 = \varphi$ and $\mathcal{D}(x_t) = \mathcal{D}(\varphi) + \int_0^t F(x_s)\,ds$ for any real number t.

The *flow* is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators $S(t) : C \to C$ defined by $S(t)\varphi = x_t$, φ in C. The flow is a group of transformations with the following properties, for all t and s in $(-\infty, \infty)$: (i) $S(t)$ is continuously Fréchet-differentiable on C; (ii) $S(t)S(s) = S(t+s)$; (iii) $S(-t) = S(t)^{-1}$; (iv) for each φ in C, the map $t \mapsto S(t)\varphi$ is continuous, that is, the flow is strongly continuous with respect to t.

A point c in X (identified with the constant function ($\varphi(\theta) \equiv c$) is fixed under the flow if and only if $F(c) = 0$. Such a point is said to be *hyperbolic* if the spectrum $\sigma(T(t))$ is disjoint from the unit circle $S^1 = \{u \in \mathbb{C} : |u| = 1\}$, for $t \neq 0$, where $\{T(t)\}_{t\in\mathbb{R}}$ is the flow of the *linearized equations at* c, given by:

$$\frac{d}{dt}Dy_t = Ly_t \tag{1.2}$$

where $D = \mathcal{D}'(c)$, $L = F'(c)$ are the Fréchet derivatives at c.

It is known (cf. [3], [4]) that, when $X = \mathbb{C}^n$, for almost all t in $\mathbb{R}$: $\sigma(T(t)\backslash\{0\} = \overline{e^{t\Lambda}}\backslash\{0\}$, where Λ is the set of all characteristic roots λ in $\mathbb{C}$, that is $\Delta(\lambda) : X \to X$ is not injective, where $\Delta(\lambda)q \triangleq \lambda D[e^{\lambda\cdot} q] - L[e^{\lambda\cdot} q]$, q in X. The set $e^{t\Lambda}$ is in fact the set of all eigenvalues of $T(t)$.

When c is hyperbolic then the phase space can be written as the direct sum $C = C^u \oplus C^s$ of closed invariant subspaces characterized by $C^u = \{\varphi \in C : T(t)\varphi \to 0 \text{ as } t \to -\infty\}$ and $C^s = \{\varphi \in C : T(t)\varphi \to 0 \text{ as } t \to +\infty\}$. In this case, there is a neighbourhood V of c in C such that the sets

$$W^u = \{\varphi \in V : \mathcal{S}(t)\varphi \text{ tends in } V \text{ to c as } t \text{ goes to } -\infty\}$$

$$\text{and} \quad W^s = \{\varphi \in V : \mathcal{S}(t)\varphi \text{ tends in } V \text{ to } c \text{ as } t \text{ goes to } +\infty\}$$

are differentiable manifolds tangent at c to C^u and C^s respectively. W^u is the *unstable manifold* and W^s is the *stable manifold* of c. There exists a homeomorphism $h : C \to C$, $h - I$ bounded, $h(c) = c$, such that for each $\tau > 0$ the above neighbourhood V can be chosen in order to $T(t)h\varphi = h\mathcal{S}(t)\varphi$ for all t in $[-\tau, \tau]$ and all φ in V, that is, h is a *conjugation* between the flows.

It is relevant to search subsets of C which are invariant under the flow. The most important are the Sobolev spaces $W^{1,p} = \{\varphi \in C : \varphi(\theta) \equiv \varphi(0) + \int_0^\theta \dot\varphi(\sigma)\, d\sigma,\ \dot\varphi \text{ in } L_p([-r,0], X)\}$ with the norm $||\varphi||_{W^{1,p}} = (|\varphi(0)|^p + ||\dot\varphi||_p^p)^{1/p}$ if $1 \le p < \infty$ and $||\varphi||_{W^{1,\infty}} = \sup\{|\varphi(0)|, ||\dot\varphi||_\infty\}$.

It is also interesting to argue in the opposite direction: to allow the initial function φ to be discontinuous. Relevant phase-spaces are: (i) the space of functions with bounded variation on $[-r, 0]$; (ii) the space $\mathcal{G}^+$ of regulated right-continuous functions on $[-r, 0[$ with finite left-hand side limits on $]-r, 0]$ with the sup-norm; (iii) the space $L_p = L_p([-r, 0], X)$, $1 \le p \le \infty$.

The above mencioned results are extended in [1] for the phase space $\mathcal{G}^+$.

We will do here the extension for initial functions in L_p; the phase space being $X \times L_p$. Although our functionals $\mathcal{D}$ and F are not defined on L_p, the compositions $t \mapsto \mathcal{D}(x_t)$ and $t \mapsto F(x_t)$ belong to $L_p^{\rm loc}(\mathbb{R}, X)$ for each x in $L_p^{\rm loc}(\mathbb{R}, X)$. We will prove that, for each (ξ, φ) in $X \times L_p$, there exists a unique function x in $L_p^{\rm loc}(\mathbb{R}, X)$ such that $x_0 = \varphi$ and $\mathcal{D}(x_t) = \xi + \int_0^t F(x_s)\, ds$ for almost all t in $\mathbb{R}$. The flow is defined by $\mathcal{S}(t)(\xi, \varphi) = (\xi + \int_0^t F(x_s)\, ds, x_t)$ and $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}$ is a group of strongly continuous Hadamard-differentiable maps of $X \times L_p$, which is strongly-continuous in t if $1 \le p < \infty$. This result and the above results about conjugation of flows are found in [7].

We work here with the equation (2.1) of §2 but the results can be extended to more general equations (see [7]).

The space X in Theorems 2.1 and 2.2 can be any Banach space.

2 The Fundamental Theory

We state the fundamental theorems on the solutions of the NFDE

$$\frac{d}{dt}E(x(t), x(t-r)) = f(x(t), x(t-r)) + \int_{-r}^{0} a(\theta)g(x(t+\theta))\,d\theta \tag{2.1}$$

where E and f are continuously Fréchet-differentiable maps from $X \times X$ into X, X is the n-dimensional Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$, $g : X \to X$ is also continuously Fréchet-differentiable and a is an integrable map from $[-r, 0]$ into the space $\mathcal{L}(X)$ of continuous linear operators from X into itself.

We fix p, $1 \leq p \leq \infty$, and look for solutions of (2.1) in $L_p^{\mathrm{loc}}(\mathbb{R}, X)$.

We assume that:

(H1) The derivatives $DE \equiv \left[\frac{\partial E}{\partial p_0}\ \frac{\partial E}{\partial p_r}\right]$, $Df \equiv \left[\frac{\partial f}{\partial p_0}\ \frac{\partial f}{\partial p_r}\right]$ and Dg are all bounded on their domains.

(H2) There exist injective continuous linear operators L_0 and L_r from X into itself and a real constant c, $0 \leq c < 1$, such that

$$\|I - L_0\frac{\partial E}{\partial p_0}(p_0, p_r)\| \leq c \quad \text{and} \quad \|I - L_r\frac{\partial E}{\partial p_r}(p_0, p_r)\| \leq c$$

for all $(p_0, p_r) \in X \times X$, where I is the identity.

(H3) $\|a\|_{p'} < \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

We observe that if x belongs to $L_p^{\mathrm{loc}}(\mathbb{R}, X)$, then the maps $t \mapsto \mathcal{D}(x_t) \triangleq E(x(t), x(t-r))$ and $t \mapsto F(x_t) \triangleq f(x(t), x(t-r)) + \int_{-r}^{0} a(\theta)g(x(t+\theta))\,d\theta$ also belong to $L_p^{\mathrm{loc}}(\mathbb{R}, X)$.

We say that x in $L_p^{\mathrm{loc}}(\mathbb{R}, X)$ is a solution of (2.1) if there exists a constant vector ξ in X such that the relation

$$\mathcal{D}(x_t) = \xi + \int_0^t F(x_s)\,ds \tag{2.2}$$

holds a.e. for t in $(-\infty, \infty)$; in this case, x is also a solution of the integral equation (2.2).

We denote by L_p the Banach space $L_p([-r, 0], X)$ with the p-norm.

Theorem 2.1 *For each $(\xi, \varphi) \in X \times L_p$ there exists a unique solution x in $L_p^{\mathrm{loc}}(\mathbb{R}, X)$ of equation (2.2) such that $x_0 = \varphi$. Moreover, if $1 \leq p < \infty$, the map $(t, \xi, \varphi) \mapsto x_t \in L_p$ is continuous. This map is not continuous when $p = \infty$ although for each t in $(-\infty, \infty)$, the map $(\xi, \varphi) \mapsto x_t \in L_\infty$ is continuous.*

Proof. Given $(\xi,\varphi) \in X \times L_p$, we consider the map $\mathcal{F}$ from $L_p([0,T],X)$ into itself defined by

$$\mathcal{F}[y](t) = y(t) - L_0[\mathcal{D}(y_t) - \xi - \int_0^t F(y_s)\,ds] \quad \text{a.e. for } t \text{ in } [0,T], \tag{2.3}$$

where we put $y_0 = \varphi$.

For sufficiently small $T > 0$, independent of (ξ,φ), $\mathcal{F}$ is a contraction, uniformly in (ξ,φ). The fixed point x of $\mathcal{F}$ is the solution defined on $[0,T]$.

We use an induction argument to obtain the solution on $[0,\infty)$.

Analogously, the solution on $[-T-r,-r]$ is the fixed point of the map $\tilde{\mathcal{F}}$ defined by

$$\tilde{\mathcal{F}}[y](t-r) = y(t-r) - L_r[\mathcal{D}(y_t) - \xi - \int_0^t F(y_s)\,ds] \quad \text{a.e. for } t \text{ in } [-T,0],$$

where again we put $y_0 = \varphi$.

Again, by induction, we get the solution on $(-\infty,\infty)$.

Since the map $t \mapsto x_t \in L_p$ is continuous when $1 \leq p < \infty$ and not continuous when $p = \infty$ the rest of the proof is a consequence of the continuity of each contraction with respect to (ξ,φ,y) and the independence on (ξ,φ) of the contraction constant.

The flow corresponding to equation (2.2) is the family $\{S(t)\}_{t\in\mathbb{R}}$ of operators $S(t)$ from $X \times L_p$ onto itself defined by

$$S(t)(\xi,\varphi) = (\xi + \int_0^t F(x_s)\,ds, x_t)$$

where $x = x(\xi,\varphi)$ is the solution of equation (2.2) through (ξ,φ).

By Theorem 2.1, the flow is a group of homeomorphisms under composition which is strongly continuous when $1 \leq p < \infty$.

The next theorem is proved in [7].

Theorem 2.2 *Suppose that E, f and g are twice continuously Fréchet-differentiable with*

$$||D^{(2)}E||, ||D^{(2)}f|| \text{ and } ||D^{(2)}g||$$

bounded on their domains. The following statements are true:

(i) For any t in $(-\infty,\infty)$, $S(t)$ is strongly-continuous H-differentiable at any point (ξ,φ) and $DS(t)(\xi,\varphi)(\Delta\xi,\Delta\varphi) = (\Delta\xi + \int_0^t M(s)z_s\,ds, z_t)$ where z is the solution of the linearized equation around the solution $S(t)(\xi,\varphi)$

$$A_0(t)z(t) + A_r(t)z(t-r) = \Delta\xi + \int_0^t M(s)z_s\,ds \quad \text{a.e. for } t \text{ in } (-\infty,\infty) \tag{2.4}$$

through $(\Delta\xi, \Delta\varphi)$, *where* $M(t)z_t = B_0(t)z(t) + B_r(t)z(t-r) + \int_{-r}^{0} B(t,\theta)z(t+\theta)\,d\theta$, $B(t,\theta) = a(\theta)Dg(x(t+\theta))$, $B_i(t) = \frac{\partial f}{\partial p_i}(x(t), x(t-r))$ *and* $A_i(t) = \frac{\partial E}{\partial p_i}(x(t), x(t-r))$ *a.e. for* t *in* $(-\infty, \infty)$, $i = 0, r$.

( ) *Suppose* $2 \leq p \leq \infty$ *and* a *is essentially bounded. If* E *is linear then* $S(t)$ *is continuously-Fréchet-differentiable (class* C^1*) for any* t *in* $\mathbb{R}$*. Conversely, when* p *is not* ∞*, if either the operator* $\mathcal{F}$ *or the operator* $\tilde{\mathcal{F}}$ *given in the proof of Theorem 2.1 is Fréchet-differentiable at some point* (ξ, φ, y) *in its domain, then* E *is linear.*

3 The linear flow

If E, f and g are linear, the equation (2.2) can be written as

$$Dy_t = \xi + \int_0^t Ly_s\,ds \quad \text{a.e. for } t \text{ in } (-\infty,\infty) \tag{3.1}$$

where

$$Dy_t = A_0 y(t) + A_r y(t-r),$$

$$Ly_t = B_0 y(t) + B_r y(t-r) + \int_{-r}^{0} B(\theta) y(s+\theta)\,d\theta\ ,$$

a.e. for $t \in (-\infty, \infty)$. The functions A_0, A_r, B_0, B_r are in $\mathcal{L}(X)$, B is in $L_{p'}([-r,0], \mathcal{L}(X))$ and the hypothesis (H2) becomes: A_0 and A_r are invertible in $\mathcal{L}(X)$. In this case, the flow of equation (3.1), denoted by $\{T(t)\}_{t\in\mathbb{R}}$, is a group of linear isomorphisms on $X \times L_p$.

We will need also the non-homogeneous equation

$$Dy_t = \xi + \int_{t_0}^t Ly_s\,ds + Q(t) \quad \text{a.e. for } t \text{ in } (-\infty,\infty) \tag{3.2}$$

where Q is in $L_p^{\text{loc}}(\mathbb{R}, X)$, $t_0 \in \mathbb{R}$.

The existence and uniqueness of solution y in $L_p^{\text{loc}}(\mathbb{R}, X)$ for equation (3.2) with initial conditions $y_{t_0} = \varphi$ can be proved with the same arguments used in Theorem 2.1. If $1 \leq p < \infty$, the solution depends continuously on ξ in X, φ in L_p, t_0 in $\mathbb{R}$ and $Q_{|[a,b]}$ in $L_p([a,b], X)$, for each compact interval $[a,b]$ containing t_0, and the solution $y(t_0,\xi,\varphi,Q)$ can be decomposed as $y_t(t_0,\xi,\varphi,Q) = \pi_2 T(t-t_0)(\xi,\varphi) + K(t,t_0)Q_{|[t_0,t]}$, where π_2 is the projection $(\xi,\varphi) \mapsto \varphi$ and $K(t,t_0)$ is a continuous linear operator from $L_p([t_0,t], X)$ into L_p (if $t < t_0$ we interchange the order in $[t_0,t]$).

Let $\{T^0(t)\}_{t\in\mathbb{R}}$ be the flow of the difference equation

$$Dy_t = 0 \tag{3.3}$$

The flow $\{T^0(t)\}_{t\in\mathbb{R}}$ is the π_2-projection of the restriction to $\{0\}\times L_p$ of the flow corresponding to equation (3.1) with $L\equiv 0$.

The main purpose of this section is to describe the spectra of $T^0(t)$ and $T(t)$. Define $H(\lambda)\triangleq A_0+A_r e^{-\lambda r}$ and $\Delta(\lambda)\triangleq \lambda H(\lambda)-B_0-B_r e^{-\lambda r}-\int_{-r}^0 B(\theta)e^{\lambda\theta}\,d\theta$, for λ in C.

Theorem 3.1 *(i) For $1\le p<\infty$, the infinitesimal generator of the group $\{T^0(t)\}_{t\in\mathbb{R}}$ is given by $G^0\varphi=\dot\varphi$, $\mathcal{D}(G^0)=\{\varphi\in W^{1,p}: D(\varphi)=0\}$ where $W^{1,p}=\{\varphi\in L_p:\dot\varphi\in L_p\}$.*

(ii) The spectrum of G^0 consists only on eigenvalues and $\lambda\in\sigma(G^0)$ if and only if $\det H(\lambda)=0$.

(iii) If $p=2$, then $\sigma(T^0(t))=\overline{e^{t\sigma(G^0)}}$.

Proof. The proof is due to D. Henry [5].

(i) Let φ be in $\mathcal{D}(G^0)$. Then, putting $x_t=T^0(t)\varphi$ and $y_t=T^0(t)G^0\varphi=G^0T^0(t)\varphi=\frac{d}{dt}T^0(t)\varphi=\frac{d}{dt}x_t$ we have $x_t=\varphi+\int_0^t y_s\,ds$ so that, a.e. for θ in $[-r,0]$, $x(t+\theta)=\varphi(\theta)+\int_\theta^{t+\theta}y(s)\,ds$. Hence, for each θ in $[-r,0]$ for which this last relation holds, the right-hand side is an absolutely continuous function of t in $\mathbb{R}$ and so is x with $\dot x=\varphi$. Then $\varphi=x_0$ is in $W^{1,p}$, $\dot\varphi=y_0=G^0\varphi$ and $D\varphi=0$ because when the solution is continuous, the equation is satisfied for all t in $\mathbb{R}$.

Conversely, if φ is in $W^{1,p}$, $D\varphi=0$, then, putting $x_t=T^0(t)\varphi$, $y_t=T^0(t)\dot\varphi$ and $z(t)=x(t)-\varphi(0)-\int_0^t y(s)\,ds$, we have $z_0=0$ and $D(z_t)=0$ a.e. for t in $\mathbb{R}$. By uniqueness, $z\equiv 0$, so x is absolutely continuous and $\dot x=y$ and then φ is in $\mathcal{D}(G^0)$ and $G^0\varphi=\dot\varphi$.

(ii) For a given ψ in L_p and λ in C, the solution φ of $G^0\varphi-\lambda\varphi=\psi$ is given by $\varphi(\theta)=e^{\lambda\theta}\varphi(0)+\int_{-r}^0 e^{\lambda(\theta-u)}\psi(u)\,du$. This solution φ is in $\mathcal{D}(G^0)$ if and only if $D\varphi=H(\lambda)\varphi(0)+A_r\int_{-r}^0 e^{-\lambda(r+u)}\psi(u)\,du$ is equal to zero. This last condition determines $\varphi(0)$ uniquely if and only if $\det H(\lambda)\neq 0$. From this we conclude that $\sigma(G^0)=\{\lambda\in \mathrm{C}:\det H(\lambda)=0\}$ and contains only eigenvalues.

(iii) It is easy to see that, for λ such that $\operatorname{Re}\lambda\ge\alpha$ and $\det H(\lambda)\neq 0$, there exists a constant c_α such that $\|(\lambda-G^0)^{-1}\|_{\mathcal{L}(L_2)}\le c_\alpha\|H(\lambda)^{-1}\|_{\mathcal{L}(X)}$.

A theorem of Gearhart and Herbst [6] says: $e^{\lambda t}$ is in $\sigma(T^0(t))$ if and only if either there exists an integer k such that $\lambda+\frac{2\pi ik}{t}$ is in $\sigma(G^0)$ or for any integer k, $\lambda+\frac{2\pi ik}{t}$ is not in $\sigma(G^0)$ but $(\lambda+\frac{2\pi ik}{t}-G^0)^{-1}$ is not uniformly bounded as $k\to\pm\infty$.

If we have this last possibility then $\operatorname{dist}(\sigma(G^0),\{\lambda+\frac{2\pi ik}{t}: k \text{ integer}\})=0$. In fact, $\det H(\lambda)$ is an analytic almost periodic function of λ, and this implies that $H(\lambda)^{-1}$ is uniformly bounded in any set $\{\lambda\in\mathrm{C}:\alpha\le\operatorname{Re}\lambda\le\beta,\ \operatorname{dist}(\sigma(G^0),\lambda)\ge\delta\}$ for $\alpha<\beta$, $\delta>0$ (see [3]). Therefore, $e^{\lambda t}\in\overline{e^{t\sigma(G^0)}}$ and the proof is complete.

Remark 3.1 *We have taken $p = 2$ in Theorem 3.1 (iii) because the Theorem of Gearhart and Herbst applies to strongly continuous semigroups defined on Hilbert spaces and no comparable result is known for semigroups on Banach spaces.*

Remark 3.2 *Theorem 3.1 has been proved in [5] when X is any Hilbert space and D has the special form*

$$Dx_t \triangleq x(t) + \sum_{k=1}^{\infty} A_k x(t - r_k) + \int_{-r}^{0} A(\theta)x(t+\theta)\,d\theta$$

where A_k and $A(\theta)$ are compact operators. The possibility $r = +\infty$ was also allowed.

In the lfollowing, we use the identification $T^0(t) : L_p \to L_p \equiv \{0\} \times L_p$.

Lemma 3.2 *For $1 < p \leq \infty$, the map $T(t) - T^0(t)\pi_2$ is compact.*

Proof. The proof follows by observing that $T(t) - T^0(t)\pi_2 = (Q, K^0(t)Q)$, where $K^0(t)$ is the above operator $K(t,0)$ corresponding to equation (3.2) with $L \equiv 0$ and $Q(t) = \pi_1 T(t)(\xi, \varphi)$, π_1 being the projection $(\xi, \varphi) \mapsto \xi$. The map $(\xi, \varphi) \mapsto Q$ is a compact linear map from $X \times L_p$ into the space of continuous functions from $[0, t]$ (or $[t, 0]$ if $t < 0$) to X with the uniform norm, by the Arzelá-Ascoli theorem.

Theorem 3.3 *(i) For $1 \leq p < \infty$, the infinitesimal generator of the group $\{T(t)\}_{t \in \mathbf{R}}$ is given by $G(\xi, \varphi) = (L\varphi, \dot{\varphi})$, $\mathcal{D}(G) = \{(\xi, \varphi) \in X \times W^{1,p} : \xi = D\varphi\}$.*

(ii) The spectrum of G consists only on eigenvalues and $\lambda \in \sigma(G)$ if and only if $\det \Delta(\lambda) = 0$.

(iii) If $p = 2$, then $\sigma(T(t)) = \overline{e^{t\sigma(G)}}$, a. e. for t in $(-\infty, \infty)$.

Proof. The proof of (i) and (ii) follows the same steps of the proof of (i) and (ii) of Theorem 3.1.

(iii) Let $Z = \{\operatorname{Re} \lambda : \lambda \in \sigma(G^0)\}$. let (β, γ) be an open bounded interval disjoint from the closure $\overline{Z}$. Let $U = \{u \in \mathbf{C} : e^{\beta t} < |u| < e^{\gamma t}\}$. Since for all t we have

$$\overline{e^{t\sigma(G^0)}} \subset \{u \in \mathbf{C} : |u| = e^{t\xi}, \xi \in \overline{Z}\} \tag{3.4}$$

using Theorem 3.1 (iii). we conclude that U is disjoint from $\sigma(T^0(t))$. Also $\sigma(T^0(t)\pi_2) = \sigma(T^0(t)) \cup \{0\}$ and U is disjoint from $\sigma(T^0(t)\pi_2)$. By Lemma 3.2 and a result of Gohberg and Krein stated in [3] Lemma 4.2, it follows that $U \subset \tilde{\rho}(T(t))$ or $U \subset e^{t\sigma(G)}$. where $\tilde{\rho}(T(t))$ is the set of all normal points (regular points or isolated eigenvalues with finite multiplicities) of $T(t)$.

Since $e^{t\sigma(G)}$ is countable, U must be disjoint from the essential spectrum $\sigma_e(T(t))$, the complement of $\tilde{\rho}(T(t))$, which implies that $\sigma_e(T(t))$ is contained in the set $\{u \in \mathrm{C} : |u| = e^{t\xi}, \xi \in \overline{Z}\}$.

From [4], the inclusion (3.4) is, in fact, an equality holding for almost all t in $(-\infty, \infty)$, so that $\sigma_e(T(t))$ is contained in $\overline{e^{t\sigma(G^0)}}$ a.e. for t in $(-\infty,\infty)$ and, since $\overline{e^{t\sigma(G^0)}} \subset \overline{e^{t\sigma(G)}}$ we conclude the proof.

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