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INVERSE GAUSSIAN DISTRIBUTION: A BAYESIAN APPROACH

by

J.A. Achcar, H. Bolfarine
and
J. Rodrigues

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INVERSE GAUSSIAN DISTRIBUTION: A BAYESIAN APPROACH

Jorge Alberto Achcar[1], Heleno Bolfarine[2], and Josemar Rodrigues[2]

[1]Departamento de Estatística, UFSCar
C.P. 676 - 13560 - São Carlos, SP, Brasil

[2]IME - Universidade de São Paulo
C.P. 20570 - 01498 - São Paulo, SP, Brasil

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ABSTRACT: A Bayesian analysis of the inverse Gaussian model is presented considering Jeffreys invariant priors for the parameters of interest. When the posterior densities cannot be obtained explicity, we use the Laplace's method for approximation of integrals. We also consider the problem of comparison of two treatments. At the end, we consider nonparametric Bayesian methods considering the Gauss Newton approach and the Newton Raphson approach.

KEY WORDS: inverse Gaussian distribution, Bayesian methods, Laplace's method, Gauss Newton approach, Newton Raphson approach.

1. INTRODUCTION

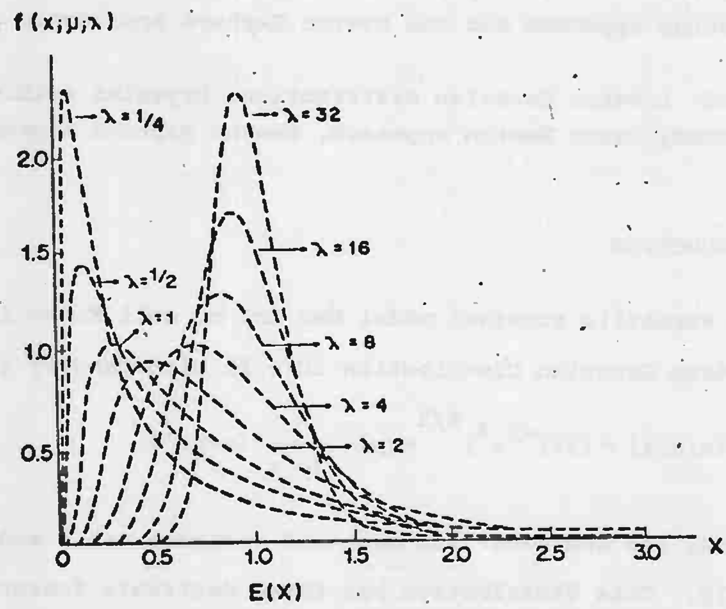
A versatile survival model but not so well known is given by the Inverse Gaussian Distribution IG(μ,λ) with density given by:

$$f(x;\mu,\lambda) = (2\pi\lambda^{-1}x^3)^{1/2}\exp\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\} \quad (1)$$

where $x>0$, $\lambda>0$ and $\mu>0$. Its mean and variance are μ and μ^3/λ, respectively. This distribution has three desirable features: a wide variety of shapes of the probability density curve (see figure 1), analytical tractability of many inferential results, and a motivation from a plausible stochastic setting of the failure process.

The basic properties of this distribution were studied by Tweedie (1957) and many results have been presented in the literature in the last years. A survey of these results is given by Folks and Chhikara (1978).

In this paper, we present Bayesian analyses of the inverse Gaussian model considering one and two sample problems. In section 2, we consider a Bayesian analysis considering a Jeffreys invariant prior for both parameters μ and λ and in section 3, we assume λ known. In section 4, we present two examples considering one and two samples. In section 5 we consider nonparametric Bayesian methods considering the Gauss Newton approach and the Newton Raphson approach.



2. A BAYESIAN APPROACH CONSIDERING A NONINFORMATIVE PRIOR FOR THE PARAMETERS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n where X_i has an inverse Gaussian distribution $IG(\mu,\lambda)$ with density (1). The likelihood function for μ and λ is given by:

$$\ell(\mu,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \left\{\prod_{i=1}^{n} x_i^{-3/2}\right\} . \tag{2}$$

$$. \exp\left\{- \frac{\lambda}{2\mu^2} \sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2} \sum_{i=1}^{n} 1/x_i\right\}$$

The logarithm of the likelihood is given by:

$$L(\mu,\lambda) = \frac{n}{2} \ell n(\lambda) - \frac{n}{2} \ell n(2\pi) - \frac{3}{2} \sum_{i=1}^{n} \ell n(x_i) - \tag{3}$$

$$- \frac{\lambda\mu^{-2}}{2} \sum_{i=1}^{n} x_i + n\lambda\mu^{-1} - \frac{\lambda}{2} \sum_{i=1}^{n} \frac{1}{x_i}$$

2.1. *The Jeffreys Invariant Prior*

The Jeffreys noninformative prior for μ and λ is given by $\pi(\mu,\lambda) \propto \{\det I(\mu,\lambda)\}^{1/2}$, where $I(\mu,\lambda)$ is the Fisher information matrix (see for example Box and Tiao, 1973). Since the second derivatives of the log-likelihood function for μ and λ are given by

$$\partial^2 L/\partial\mu^2 = - 3\lambda\mu^{-4} \sum_{i=1}^{n} x_i + 2n\lambda\mu^{-3},$$

$$\partial^2 L/\partial\lambda^2 = - n\lambda^{-2}/2 \quad \text{and} \quad \partial^2 L/\partial\mu\partial\lambda = \mu^{-3} \sum_{i=1}^{n} x_i - n\mu^{-2},$$

we have:

$$I(\mu,\lambda) = \begin{pmatrix} n\lambda & 0 \\ 0 & n/2\lambda^2 \end{pmatrix} \tag{4}$$

Thus, the Jeffreys invariant prior for μ and λ is proportional to $(\lambda\mu^3)^{-1/2}$ where $\mu>0$ and $\lambda>0$.

2.2. Joint Posterior Density for μ and λ

Using the Jeffreys prior density $\pi(\mu,\lambda) \propto (\lambda\mu^3)^{-1/2}$, we find the posterior density for μ and λ:

$$\pi(\mu,\lambda|\text{data}) = \frac{c\lambda^{(n-1)/2}}{\mu^{3/2}} \exp\{-\frac{\lambda\mu^{-2}}{2}\sum_{i=1}^{n} x_i + n\lambda\mu^{-1} - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}\} \tag{5}$$

where $\mu>0$, $\lambda>0$ and c is given by,

$$c^{-1} = \int_0^\infty\int_0^\infty \frac{\lambda^{(n-1)/2}}{\mu^{3/2}} \exp\{-\frac{\lambda\mu^{-2}}{2}\sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}\}d\lambda d\mu$$

The marginal posterior density for μ is given by:

$$\pi(\mu|\text{data}) = \frac{c}{\mu^{3/2}}\int_0^\infty \lambda^{(n-1)/2} \exp\{-A\lambda\}d\lambda$$

where $A = \frac{\mu^{-2}}{2}\sum_{i=1}^{n} x_i - n\mu^{-1} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}$.

That is,

$$\pi(\mu|\text{data}) \propto \mu^{-3/2}\left(\frac{\mu^{-2}}{2}\sum_{i=1}^{n} x_i - n\mu^{-1} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-(n+1)/2} \tag{6}$$

where $\mu>0$. In the log-scale, we have:

$$\pi(\xi|\text{data}) \propto e^{-\xi/2}\left(\frac{e^{-2\xi}}{2}\sum_{i=1}^{n} x_i - ne^{-\xi} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-(n+1)/2} \quad (7)$$

where $-\infty<\xi=\ell n(\mu)<\infty$.

2.3. *Marginal Posterior for* λ

The marginal posterior density for λ is given by:

$$\pi(\lambda|\text{data}) = c\int_0^\infty \exp\{nL_\lambda(\mu)\}d\mu \quad (8)$$

where $L_\lambda(\mu) = \frac{1}{n}\{(\frac{n-1}{2})\ell n(\lambda) - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i} - \frac{3}{2}\ell n(\mu) -$

$$- \frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu}\}$$

Using the Laplace's method for approximating integrals (see, Tierney and Kadane, 1986), we find the marginal posterior density for λ, given by:

$$\pi(\lambda|\text{data}) \propto \frac{\chi^{*1/2}\,\lambda^{(n-1)/2}\exp\{-\frac{\lambda}{2}\sum_{i=1}^{n}\frac{1}{x_i}\}}{\hat{\mu}^{*3/2}\exp\{\frac{\lambda}{2\hat{\mu}^{*2}}\sum_{i=1}^{n} x_i - \frac{n\lambda}{\hat{\mu}^*}\}} \quad (9)$$

where $\lambda>0$, $\hat{\mu}^* = (-2n\lambda + \sqrt{4n^2\lambda^2 + 24\lambda\sum_{i=1}^{n} x_i})/6$, $\chi^* = -1/\left.\frac{d^2L_\lambda(\mu)}{d\mu^2}\right|_{\mu=\hat{\mu}^*}$

and $d^2L_\lambda(\mu)/d\mu^2 = \frac{1}{n}\{\frac{3}{2\mu^2} - \frac{3\lambda}{\mu^4}\sum_{i=1}^{n} x_i + \frac{2n\lambda}{\mu^3}\}$.

3. A BAYESIAN ANALYSIS WITH λ KNOWN

The likelihood function for μ considering λ known is:

$$\ell(\mu) \propto \exp\{-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu}\} \tag{10}$$

where μ>0. Since $\ln\ell(\mu) \propto -\frac{\lambda}{2\mu^2}\sum_{i=1}^{n} x_i + \frac{n\lambda}{\mu}$, we have $d^2\ln\ell(\mu)/d\mu^2 = -3\lambda\mu^{-4}\sum_{i=1}^{n} x_i + 2n\lambda\mu^{-3}$ and $E\{-d^2\ln\ell(\mu)/d\mu^2\} = n\lambda\mu^{-3}$. Thus, the Jeffreys noninformative prior for μ is proportional to $\mu^{-3/2}$, where μ>0.

3.1. *Posterior Density for μ*

With the Jeffreys noninformative prior for μ we find the posterior density for μ (considering λ known):

$$\pi(\mu|\text{data}) \propto \mu^{-3/2}\exp\{-\frac{\lambda n\bar{x}}{2\mu^2} + \frac{n\lambda}{\mu}\} \tag{11}$$

where μ>0.

In the log-scale, we have:

$$\pi(\xi|\text{data}) \propto e^{-\xi/2}\exp\{-\frac{\lambda n\bar{x}}{2}e^{-2\xi} + n\lambda e^{-\xi}\} \tag{12}$$

where $-\infty<\xi = \ln(\mu)<\infty$.

3.2. *Comparison of Two Treatments Considering λ_1 and λ_2 Known*

Let $X_1, X_2, \ldots, X_{n_1}$ be a random sample of size n_1 a distribution $IG(\mu_1,\lambda_1)$ and let $Y_1, Y_2, \ldots, Y_{n_2}$ be a random sample of size n_2 of a distribution $IG(\mu_2,\lambda_2)$. We assume independence between the two samples.

The Jeffreys invariant prior for μ_1 and μ_2 is proportional to $\mu_1^{-3/2} \cdot \mu_2^{-3/2}$.

The joint posterior density for μ_1 and μ_2 is given by:

$$\pi(\mu_1, \mu_2 | \text{data}) \propto \mu_1^{-3/2} \mu_2^{-3/2} \exp\{- \frac{\lambda_1 n_1 \bar{x}}{2\mu_1^2} - \frac{\lambda_2 n_2 \bar{y}}{2\mu_2^2}\} \cdot \exp\{\frac{n_1 \lambda_1}{\mu_1} + \frac{n_2 \lambda_2}{\mu_2}\} \tag{13}$$

where $\mu_1 > 0$ and $\mu_2 > 0$.

Considering the transformation $\xi_1 = \mu_1/\mu_2$ and $\xi_2 = \mu_2$ (Jacobian $= \xi_2$), the marginal posterior density for ξ_1 is:

$$\pi(\xi_1 | \text{data}) = c \int_0^\infty e^{nL_{\xi_1}(\xi_2)} d\xi_2 \tag{14}$$

where $\xi_1 > 0$, c is normalizing constant, $L_{\xi_1}(\xi_2) = \frac{1}{n} \{-2\ell n \xi_2 - \frac{B_1}{2\xi_2^2} + \frac{B_2}{\xi_2}\}$,

$B_1 = \lambda_1 n_1 \xi_1^{-2} \bar{x} + \lambda_2 n_2 \bar{y}$, $B_2 = n_1 \lambda_1 \xi_1^{-1} + n_2 \lambda_2$ and $n = n_1 + n_2$.

The marginal posterior density for $\xi_1 = \mu_1/\mu_2$ approximated by the Laplace's method is:

$$\pi(\xi_1 | \text{data}) \propto \frac{\Sigma_2^{*1/2}}{\xi_1^{3/2} \hat{\xi}_2^{*2}} \exp\{- \frac{B_1}{2\hat{\xi}_2^{*2}} + \frac{B_2}{\hat{\xi}_2^*}\} \tag{15}$$

where $\xi_1 > 0$, $\hat{\xi}_2^* = (- B_2 + \sqrt{B_2^2 + 8B_1})/4$, $\Sigma_2^* = - 1 / \left. \frac{d^2 L_{\xi_1}(\xi_2)}{d\xi_2^2} \right|_{\xi_2 = \hat{\xi}_2^*}$

and $d^2 L_{\xi_1}(\xi_2)/d\xi_2^2 = \frac{1}{n} \{\frac{2}{\xi_2^2} - \frac{3B_1}{\xi_2^4} + \frac{2B_2}{\xi_2^3}\}$.

4. EXAMPLES

4.1. An Example with One Sample

In table 1, we have simulated data of an Inverse Gaussian distribution with parameters $\mu=1$ and $\lambda=5$.

TABLE 1 - *Generated Data of a IG(μ,λ) with μ=1 and λ=5 (n=30)*

0.26	0.41	0.22	0.69	0.53	0.66
0.56	0.52	0.80	1.00	1.03	0.81
0.79	0.87	0.95	0.97	0.79	0.83
0.83	1.29	1.11	1.43	1.23	1.81
1.73	1.67	1.63	1.23	2.36	2.10

From table 1, we have $n=30$, $\sum_{i=1}^{30} x_i = 31.11$, $\bar{x} = 1.037$, $\sum_{i=1}^{30} x_i^2 = 40.305$, $S^2 = 0.2774$ (sample variance) and $\sum_{i=1}^{30} \frac{1}{x_i} = 39.2645$.

The marginal posterior density for μ in the log-scale (from (7)), is given by:

$$\pi(\xi|\text{data}) \propto e^{-\xi/2}(15.55e^{-2\xi} - 30e^{-\xi} + 19.6322)^{-15.5} \qquad (16)$$

where $-\infty<\xi = \ell n(\mu)<\infty$.

The mode of this posterior density is $\hat{\xi} = 0.0306$ (solving the equation $948.85\, e^{-2\xi} - 900\, e^{-\xi} - 19.63 = 0$). Thus, $\hat{\mu} = e^{\hat{\xi}} = 1.031$. It is important to observe that the maximum likelihood estimator for μ is given by $\hat{\mu} = n^{-1} \sum_{i=1}^{n} x_i = 1.037$.

The marginal posterior density for λ (from (9)) is given by:

$$\pi(\lambda|\text{data}) \propto \frac{\chi^{*1/2}\ \lambda^{14.5}\ \exp\{-19.63\lambda\}}{\hat{\mu}^{*3/2}\ \exp\{\frac{15.55}{\hat{\mu}^{*2}} - \frac{30\lambda}{\hat{\mu}^{*}}\}} \tag{17}$$

where $\lambda>0$, $\hat{\mu}^* = \{-60\lambda + \sqrt{3600\lambda^2 + 746.64\lambda}\}/6$ and $\chi^* = -30\hat{\mu}^{*4}/\{1.5\hat{\mu}^{*2} - 93.33\lambda + 60\lambda\hat{\mu}^*\}$.

In figures 2 and 3 we have the graphs of the posterior densities for $\xi = \ell n(\mu)$ and λ. The mode of the marginal posterior density for λ is $\hat{\lambda} = 2.70$. The maximum likelihood estimator for λ is given by $n/\sum_{i=1}^{n}(\frac{1}{x_i} - \frac{1}{\bar{x}}) = 2.903$.

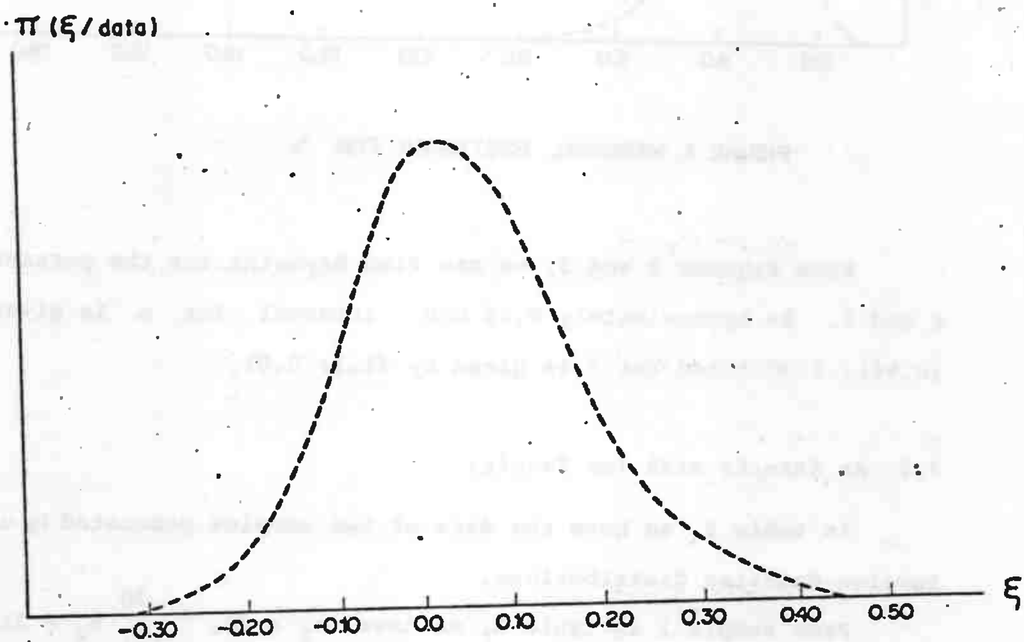


FIGURE 2. MARGINAL POSTERIOR FOR ξ

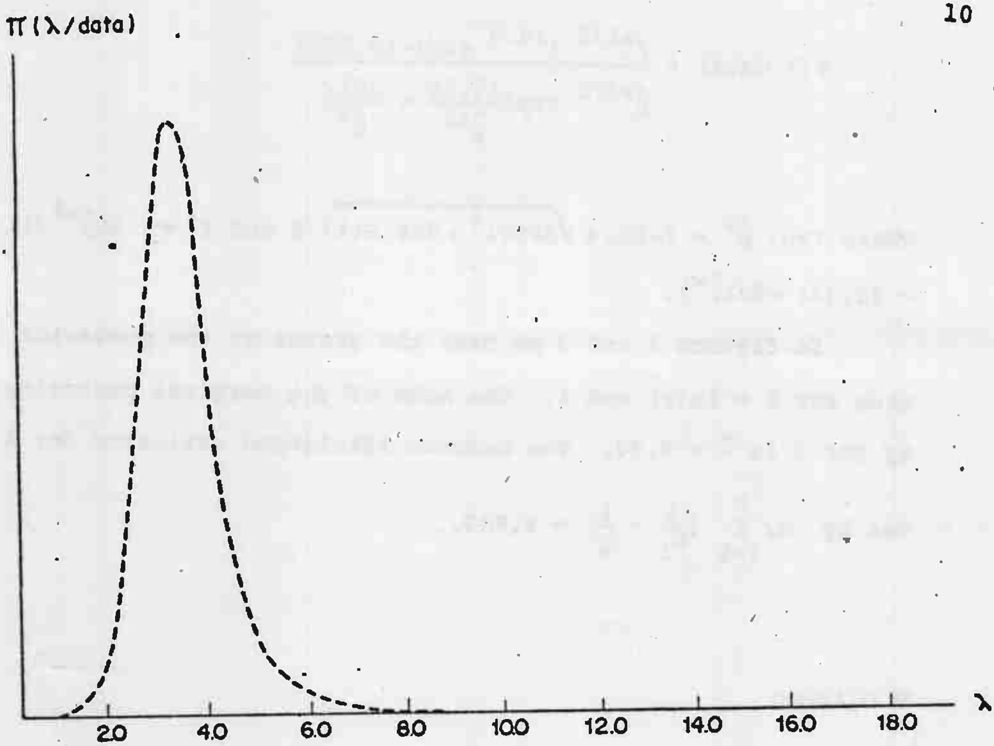


FIGURE 3. MARGINAL POSTERIOR FOR λ

From figures 2 and 3, we can find Bayesint. for the parameters ξ and λ. An approximately 0.95 HPD interval for μ is given by (0.861; 1.[illegible]0) and for λ is given by (1.5; 5.0).

4.2. *An Example with Two Samples*

In table 2, we have the data of two samples generated by using inverse Gaussian distributions.

From sample 1 in table 2, we have $n_1 = 30$, $\sum_{i=1}^{30} x_i = 31.11$, $\bar{x} = 1.037$, $\sum_{i=1}^{30} x_i^2 = 40{,}305$ and $\sum_{i=1}^{30} \frac{1}{x_i} = 39.264$. From sample 2, we have $n_2 = 30$, $\sum_{i=1}^{30} y_i = 59.68$, $\bar{y} = 1.99$, $\sum_{i=1}^{30} y_i^2 = 142.34$ and

TABLE 2 - *Generated Data of $IG(\mu,\lambda)$*

SAMPLE 1 - $\mu_1 = 1$, $\lambda_1 = 5$, $n_1 = 30$					
0.26	0.41	0.22	0.69	0.53	0.66
0.56	0.52	0.80	1.00	1.03	0.81
0.79	0.87	0.95	0.97	0.79	0.83
0.83	1.29	1.11	1.43	1.23	1.81
1.73	1.67	1.63	1.23	2.36	2.10
SAMPLE 2 - $\mu_2 = 2$, $\lambda_2 = 10$, $n_2 = 30$					
0.50	1.20	1.60	2.50	3.30	1.30
2.00	2.58	3.40	0.72	1.30	2.10
2.60	4.20	0.80	1.40	2.20	2.65
0.85	1.55	2.35	2.70	1.00	1.58
2.40	2.80	1.10	1.60	2.40	3.00

$\sum_{i=1}^{30} \frac{1}{y_i} = 19.47$. We also observe that the maximum likelihood estimator for λ_2 is given by $\hat{\lambda}_2 = n_2 / \sum_{i=1}^{30} (\frac{1}{y_i} - \frac{1}{\bar{y}}) = 6.83$.

Considering $\lambda_1 = 5$ and $\lambda_2 = 10$ known, the posterior density for $\xi_1 = \mu_1/\mu_2$ approximated by the Laplace's method is (from (15)) given by:

$$\pi(\xi_1|\text{data}) \propto \frac{\Sigma_2^{*1/2}}{\xi_1^{3/2}\ \hat{\xi}_2^{*2}} \exp\{-\frac{B_1}{2\hat{\xi}_2^{*2}} + \frac{B_2}{\hat{\xi}_2^{*}}\} \tag{18}$$

where $B_1 = 155.55\xi_1^{-2} + 596.79$, $B_2 = 150\xi_1^{-1} + 300$, $\hat{\xi}_2^* = \{-(\frac{150}{\xi_1} + 300) + \frac{1}{\xi_1}\sqrt{23744.4 + 90000\xi_1 + 94774\xi_1^2}\}/4$ and $\Sigma_2^* = n\hat{\xi}_2^{*4}\{3B_1 - 2\hat{\xi}_2^{*2} - 2B_2\hat{\xi}_2^{*}\}^{-1}$.

In figure 4, we have the graph of the posterior density for ξ_1. We observe that the mode of this posterior density is given by $\hat{\xi}_1 = 0.52$. An HPD interval for ξ_1 with probability 0.95 is given (approximately) by (0.40; 0.70).

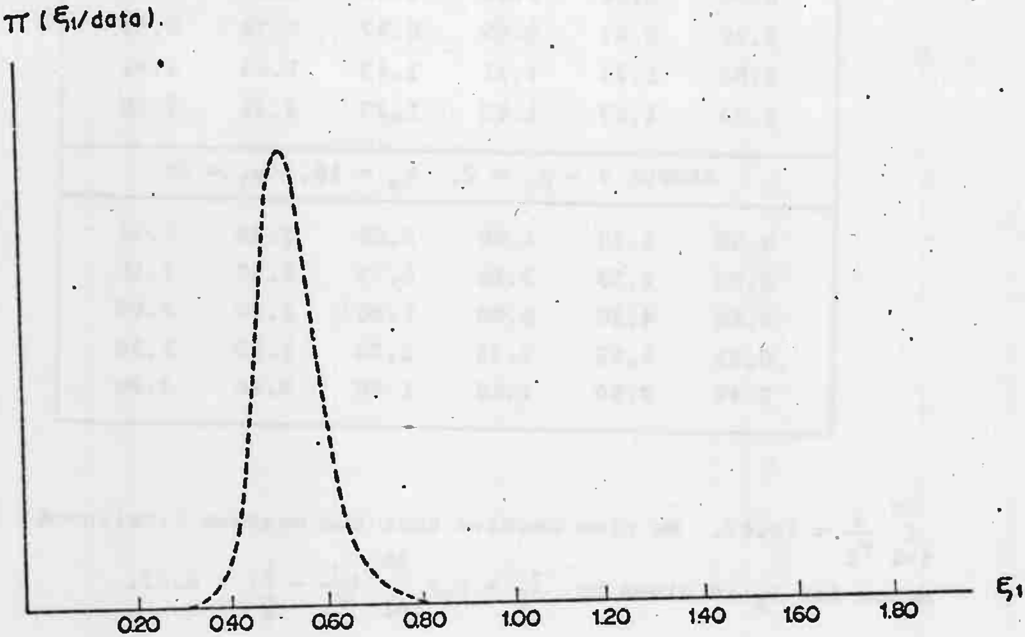


FIGURE 4. MARGINAL POSTERIOR FOR $\xi_1 = \mu_1 / \mu_2$

5. USE OF NONLINEAR LEAST-SQUARES THEORY

In the above sections, we developed Bayesian parametrical methods for the problem of one and two samples. We verified close agreement between the Bayesian estimators and the maximum likelihood estimators, using noninformative priors. For example, in section 4 (example 4.1), we found $\hat{\lambda}_B = 2.70$ (Bayesian estimator) and

$\hat{\lambda}_{MLE} = 2.90$ (maximum likelihood estimator). Also, we found $\hat{\mu}_B = 1.031$ and $\hat{\mu}_{MLE} = 1.037$. We observe that both estimators for λ are not very good in the sense that the data were generated considering $\lambda=5$ and $\mu=1$. We could improve the Bayesian estimators above considering an informative prior structure for the parameters involved.

As an alternative to the strict Bayesian approach, it may be tried the use of nonparametric Bayesian methods. In this direction, we propose a distribution free nonlinear Bayesian approach that generalizes the linear procedure with prior information that was proposed by Duncan and Horn (1972). The approach combines in a natural way, prior and data information and it requires only the first two moments of the distributions of the data and of the parameters involved.

Considering a random sample $X_1, X_2, \ldots, X_n$ of a $IG(\mu,\lambda)$ distribution and the reparametrization $\alpha=\lambda$, $\beta=\rho^2$, where $\rho^2 = \mu/\sigma$ and $\sigma^2 = \mu^3/\lambda$, we have:

$$E(\underset{\sim}{X}|\underset{\sim}{\theta}) = \underset{\sim}{f}(\underset{\sim}{\theta}) = \frac{\alpha}{\beta}\,\underset{\sim}{1}_n$$

$$\text{var}(\underset{\sim}{X}|\underset{\sim}{\theta}) = g(\underset{\sim}{\theta})I_n = \frac{\alpha^2}{\beta^3}\,I_n \tag{19}$$

where $\underset{\sim}{X}' = (X_1, X_2, \ldots, X_n)$, $\underset{\sim}{1}_n' = (1,1,\ldots,1)$, $\underset{\sim}{\theta}' = (\alpha,\beta)$ and I_n is the $(n\times n)$ identity matrix.

We observe that $\underset{\sim}{f}(\underset{\sim}{\theta})$ is an $(n\times 1)$ vector of twice continuously differentiable functions in $\underset{\sim}{\theta}$, and $g(\underset{\sim}{\theta})$ is a known function of $\underset{\sim}{\theta}$.

For the prior structure for $\underset{\sim}{\theta}'=(\alpha,\beta)$, we consider $E(\underset{\sim}{\theta}) = \underset{\sim}{\theta}_o$ (known) and $\text{var}(\underset{\sim}{\theta}) = \Sigma_{NL}$ (known).

5.1. A General Approach

Under model (19), $\hat{\underset{\sim}{\theta}}$ is the nonlinear Bayes least-squares estimator for $\underset{\sim}{\theta}$ if it minimizes

$$Q(\underset{\sim}{\theta},\underset{\sim}{\theta}_o) = \begin{pmatrix} \underset{\sim}{\theta}_o - \underset{\sim}{\theta} \\ \underset{\sim}{x} - \underset{\sim}{f}(\underset{\sim}{\theta}) \end{pmatrix}' \begin{pmatrix} \Sigma_{NL}^{-1} & 0 \\ 0 & E^{-1}[g(\underset{\sim}{\theta})]I \end{pmatrix} \begin{pmatrix} \underset{\sim}{\theta}_o - \underset{\sim}{\theta} \\ \underset{\sim}{x} - \underset{\sim}{f}(\underset{\sim}{\theta}) \end{pmatrix} \quad (20)$$

(see Duncan and Horn, 1972).

We may expand $\underset{\sim}{f}(\underset{\sim}{\theta})$ in a Taylor series about $\underset{\sim}{\theta}_{(i)}$ (the subscript i indicating the i^{th} iteraction), and retain only the first two terms:

$$\underset{\sim}{f}(\underset{\sim}{\theta}) = \underset{\sim}{f}(\underset{\sim}{\theta}_{(i)}) + F(\underset{\sim}{\theta}_{(i)})(\underset{\sim}{\theta} - \underset{\sim}{\theta}_{(i)}) \quad (21)$$

where F is the Jacobian matrix. So, we can rewrite (20) as,

$$Q(\underset{\sim}{\theta},\underset{\sim}{\theta}_o) \cong (\underset{\sim}{X}_{(i)} - Z_{(i)}\underset{\sim}{\theta})' V^{-1} (\underset{\sim}{X}_{(i)} - Z_{(i)}\underset{\sim}{\theta}) \quad (22)$$

where $\underset{\sim}{X}_{(i)} = \begin{pmatrix} \underset{\sim}{\theta}_o \\ \underset{\sim}{X}^*_{(i)} \end{pmatrix}$, $Z_{(i)} = \begin{pmatrix} I_p \\ F(\underset{\sim}{\theta}_{(i)}) \end{pmatrix}$, $V^{-1} = \begin{pmatrix} \Sigma_{NL}^{-1} & 0 \\ 0 & E^{-1}[g(\underset{\sim}{\theta})]I \end{pmatrix}$,

and $\underset{\sim}{X}^*_{(i)} = \underset{\sim}{X} + \underset{\sim}{f}(\underset{\sim}{\theta}_{(i)}) + F(\underset{\sim}{\theta}_{(i)})\,\underset{\sim}{\theta}_{(i)}$.

From Lemma 2.4 in Duncan and Horn, 1972, the quadratic form (22) is minimized by the familiar weighted combination,

$$\hat{\underset{\sim}{\theta}}_{(i+1)} = C_{(i)}\hat{\underset{\sim}{\theta}}_{(i)} + (I - C_{(i)})\,\underset{\sim}{\theta}_o \quad (23)$$

where $\hat{\underset{\sim}{\theta}}_{(i)} = [F'(\underset{\sim}{\theta}_{(i)})F(\underset{\sim}{\theta}_{(i)})]^{-1} F'(\underset{\sim}{\theta}_{(i)})\underset{\sim}{X}^*_{(i)}$ and

$$C_{(i)} = \left[\Sigma_{NL}^{-1} + \frac{F'(\underset{\sim}{\theta}_{(i)})F(\underset{\sim}{\theta}_{(i)})}{E[g(\underset{\sim}{\theta})]}\right]^{-1} \frac{F'(\underset{\sim}{\theta}_{(i)})F(\underset{\sim}{\theta}_{(i)})}{E[g(\underset{\sim}{\theta})]} .$$

Or, we can write,

$$\hat{\underset{\sim}{\theta}}_{(i+1)} = C_{(i)}\underset{\sim}{\theta}_{(i)} + (I - C_{(i)})\underset{\sim}{\theta}_o +$$

$$+ \left(\Sigma_{NL}^{-1} + \frac{F'(\underset{\sim}{\theta}_{(i)})F(\underset{\sim}{\theta}_{(i)})}{E[g(\underset{\sim}{\theta})]}\right)^{-1} \frac{F'(\underset{\sim}{\theta}_{(i)})}{E[g(\underset{\sim}{\theta})]} (\underset{\sim}{x} - \underset{\sim}{f}(\underset{\sim}{\theta}_{(i)})).$$

The above iterative procedure is known in the literature as the Gauss-Newton procedure with prior information. By using Brunk (1980), it is not difficult to show that $\hat{\theta}_{(i+1)}$ has the smallest posterior squared mean in the class of linear estimators on $\underset{\sim}{X}^*_{(i)}$. Starting with an initial value $\underset{\sim}{\theta}_{(o)} = \underset{\sim}{\theta}_o$, the process continues until $||\underset{\sim}{\theta}_{(i+1)} - \underset{\sim}{\theta}_{(i)}|| \leq \varepsilon$. The asymptotic covariance matrix is given by,

$$\text{cov}(\hat{\underset{\sim}{\theta}}) = \left(\Sigma_{NL}^{-1} + \frac{F'(\hat{\underset{\sim}{\theta}})F(\hat{\underset{\sim}{\theta}})}{E[g(\underset{\sim}{\theta})]}\right)^{-1} \tag{24}$$

The first step of (23) gives an approximate solution given by:

$$\hat{\underset{\sim}{\theta}}_{(1)} = \underset{\sim}{\theta}_o + \left(\Sigma_{NL}^{-1} + \frac{F'(\underset{\sim}{\theta}_o)F(\underset{\sim}{\theta}_o)}{E[g(\underset{\sim}{\theta})]}\right)^{-1} \frac{F'(\underset{\sim}{\theta}_o)}{E[g(\underset{\sim}{\theta})]} (\underset{\sim}{x} - \underset{\sim}{f}(\underset{\sim}{\theta}_o)) \tag{25}$$

5.2. *Application to the Inverse Gaussian Distribution*

With model (19), let us consider $E(\underset{\sim}{\theta}) = (\alpha_o, \beta_o)' = \underset{\sim}{\theta}_o$ and $\text{cov}(\underset{\sim}{\theta}) = \text{diag}(\sigma_\alpha^2, \sigma_\beta^2)$ where α_o, β_o, σ_α^2 and σ_β^2 are known. Also, we have in our notation, $\underset{\sim}{f}'(\underset{\sim}{\theta}) = (\alpha/\beta, \ldots, \alpha/\beta)$ and $g(\underset{\sim}{\theta}) = \alpha^2/\beta^3$.

Expanding $g(\underset{\sim}{\theta})$ in a first order Taylor series about $\underset{\sim}{\theta}_o$ and taking its expectation with respect to the prior distribution, we have:

$$E[g(\underset{\sim}{\theta})] \cong \frac{\alpha_o^2}{\beta_o^3} I \tag{26}$$

From (25), we obtain:

$$\hat{\alpha}_{(1)} \doteq \frac{n\beta_o^2\bar{x}}{\Delta_1\alpha_o^2\sigma_\beta^2} + \left(1 - \frac{n\beta_o}{\Delta_1\alpha_o^2\sigma_\beta^2}\right)\alpha_o ,$$

$$\hat{\beta}_{(1)} = \frac{n}{\Delta_1\sigma_\alpha^2} + \left(1 - \frac{n\bar{x}}{\Delta_1\alpha_o\sigma_\alpha^2}\right)\beta_o \tag{27}$$

where $\Delta_1 = \dfrac{1}{\sigma_\alpha^2\sigma_\beta^2} + \dfrac{n}{\sigma_\alpha^2\beta_o} + \dfrac{n\beta_o}{\alpha_o^2\sigma_\beta^2}$

From (24), we have:

$$\text{var}(\hat{\alpha}_{(1)}) \cong \frac{1}{\Delta_1}\left(\frac{1}{\sigma_\beta^2} + \frac{n}{\beta_o}\right)$$

and

$$\text{var}(\hat{\beta}_{(1)}) \cong \frac{1}{\Delta_1}\left(\frac{1}{\sigma_\alpha^2} + \frac{n\beta_o}{\alpha_o^2}\right) \tag{28}$$

Considering $\beta_o = \bar{x}^2/S^2$ and $\alpha_o = (\bar{x}^2/S^2)\bar{x}$ where $\bar{x}$ is the sample mean and S^2 is the sample variance, we find (from (27)), $\hat{\alpha}_{(1)} = (\bar{x}^2/S^2)\bar{x}$ and $\hat{\beta}_{(1)} = \bar{x}^2/S^2$. That is, the solution is given by $\hat{\alpha} = (\bar{x}^2/S^2)\bar{x}$ and $\hat{\beta} = \bar{x}^2/S^2$.

With the data in table 1, we have $\bar{x} = 1.037$ and $S^2 = 0.2774$. Thus, $\hat{\alpha} = 4.02$ and $\hat{\beta} = 3.88$. Since, $\lambda=\alpha$ and $\mu=\alpha/\beta$, we find $\hat{\lambda} = 4.02$ (close to the value $\lambda=5$) and $\hat{\mu} = 1.037$. Thus, we see that this estimator is better than the estimators obtained in section 4. In table 3 we have estimated values of λ for different initial values α_o and β_o.

TABLE 3 - *Estimated Values of λ for the Data in Table 1 Using the Gauss Newton Approach*

$\sigma_\alpha^2 = \sigma_\beta^2 = 1,\ \mu = 1,\ \lambda = 5,\ n = 30,\ \rho^2 = 5$			
α_o	β_o	$\hat{\lambda} = \hat{\alpha}_{(1)}$	$var(\hat{\alpha}_{(1)})$
4.02	3.88	4.02	0.548
	4.00	4.08	0.534
	4.50	4.36	0.478
	5.00	4.68	0.430
	6.00	5.45	0.350

5.3. *The Newton Raphson Approach*

As an alternative to the Gauss Newton approach above, we have the Newton Raphson approach. From (19) and (20), it is not hard to show that the quadratic form to be minimized is given by:

$$Q(\alpha,\beta) = \frac{(\alpha_o-\alpha)^2}{\sigma_\alpha^2} + \frac{(\beta_o-\beta)^2}{\sigma_\beta^2} + \frac{\beta^3}{\alpha^2}\sum_{i=1}^{n}(x_i - \frac{\alpha}{\beta})^2 \qquad (29)$$

Differentiating $Q(\alpha,\beta)$ with respect to α and β, it follows the values of α and β minimizing $Q(\alpha,\beta)$ are given by the solution of:

$$g_1(\alpha,\beta) = \frac{(\alpha_o-\alpha)}{\sigma_\alpha^2} + \frac{\beta^3}{\alpha^3}\sum_{i=1}^{n} x_i^2 - \frac{\beta^2}{\alpha^2}\sum_{i=1}^{n} x_i = 0$$

$$(30)$$

and $$g_2(\alpha,\beta) = \frac{(\beta_o-\beta)}{\sigma_\beta^2} - \frac{3\beta^2}{2\alpha^2}\sum_{i=1}^{n} x_i^2 + \frac{2\beta}{\alpha}\sum_{i=1}^{n} x_i - \frac{n}{2} = 0$$

The Newton-Raphson algorithm can be used to solve the above equations and it is given by:

$$\alpha_{(i+1)} = \alpha_{(i)} - \frac{1}{\Delta_{2(i)}} (A_{11}^{(i)} g_1^{(i)} + A_{12}^{(i)} g_2^{(i)})$$

$$\beta_{(i+1)} = \beta_{(i)} - \frac{1}{\Delta_{2(i)}} (A_{12}^{(i)} g_1^{(i)} + A_{22}^{(i)} g_2^{(i)}) \qquad (31)$$

where $g_1^{(i)}$ and $g_2^{(i)}$ are the functions $g_1(\alpha,\beta)$ and $g_2(\alpha,\beta)$ with α and β replaced by $\alpha_{(i)}$ and $\beta_{(i)}$ respectively, $\Delta_{2(i)} = A_{11}^{(i)} . A_{22}^{(i)} - A_{12}^{(i)} . A_{12}^{(i)}$, where $A_{11}^{(i)}$, $A_{12}^{(i)}$ and $A_{22}^{(i)}$ are equal to

$$A_{11} = \frac{1}{\sigma_\beta^2} + \frac{3\beta}{\alpha^2} \sum_{i=1}^{n} x_i^2 - \frac{2}{\alpha} \sum_{i=1}^{n} x_i$$

$$A_{22} = \frac{1}{\sigma_\alpha^2} + \frac{3\beta^3}{\alpha^4} \sum_{i=1}^{n} x_i^2 - \frac{2\beta^2}{\alpha^3} \sum_{i=1}^{n} x_i \qquad (32)$$

$$A_{12} = \frac{3\beta^2}{\alpha^3} \sum_{i=1}^{n} x_i^2 - \frac{2\beta}{\alpha^2} \sum_{i=1}^{n} x_i$$

with α and β replaced by $\alpha_{(i)}$ and $\beta_{(i)}$, respectively.

The posterior covariance matrix of α and β may be estimated by:

$$\hat{var}(\alpha|\underset{\sim}{x}) = \frac{\hat{A}_{11}}{\hat{\Delta}_2}$$

$$\hat{var}(\beta|\underset{\sim}{x}) = \frac{\hat{A}_{22}}{\hat{\Delta}_2} \qquad (33)$$

and $$\hat{cov}(\alpha,\beta|\underset{\sim}{x}) = \frac{\hat{A}_{12}}{\hat{\Delta}_2}$$

where $\hat{A}_{11}$, $\hat{A}_{12}$, $\hat{A}_{22}$ and $\hat{\Delta}_2$ are given by A_{11}, A_{12}, A_{22} and Δ_2, with α and β replaced by $\hat{\alpha}$ and $\hat{\beta}$ the solution of (31).

In table 4, we have the estimated values of λ for the data in table 1.

TABLE 4 - *Estimated Values of λ for the Data in Table 1 Using the Newton Raphson Approach*

$\sigma_\alpha^2 = \sigma_\beta^2 = 1$, $\mu = 1$, $\lambda = 5$, $n = 30$, $\rho^2 = 5$			
α_o	β_o	$\hat{\lambda} = \hat{\alpha}$	$var(\hat{\alpha})$
4.02	3.88	4.02	0.410
	4.00	4.15	0.380
	4.50	4.48	0.330
	5.00	4.84	0.300
	6.00	5.98	0.250

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