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ON THE AVERAGED DYNAMICS OF THE RANDOM FIELD CURIE-WEISS MODEL

by

L.R. Fontes, P. Mathieu
and
P.Picco

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On the Averaged Dynamics of the Random Field Curie-Weiss model

L.R. Fontes [1], P. Mathieu [2], P. Picco [3].

Abstract: *We describe an 'averaged over disorder' dynamics for the magnetization of the random field Curie-Weiss model. Our approach is based on spectral asymptotics and includes results on the random fluctuations of eigenvalues and eigenvectors.*

Key Words: *Metastability, Random magnetic field, Random Spin Systems, Glauber Dynamics, Curie Weiss Model.*

AMS Classification Numbers: 60K35, 82B44, 82D30, 82C44.

Abbreviated tittle: Random Curie-Weiss model.

[1] IME.USP. Caixa Postal 66218, 05315-970, São Paulo, SP, Brazil lrenato@ime.usp.br

[2] CMI, Université de Provence, 39 Rue F. Joliot Curie, 13453 Marseille Cedex 13, France. PMathieu@gyptis.univ-mrs.fr

[3] CPT.CNRS Luminy, case 907, 13288 Marseille Cedex 9, France. Picco@cpt.univ-mrs.fr

I. Introduction

In spite of its lack of physical significance, the Curie-Weiss model is often considered a useful toy model for testing ideas in statistical mechanics. In particular the rigourous formulation of the notion of 'metastability' - the so called 'pathwise approach'- was first introduced to describe the evolution of the magnetization of the Curie-Weiss model in [3]. Metastability was later proved for various systems in statistical mechanics and it became one of the most powerful tools to describe the evolution towards equilibrium of a Markov process.

Following this line, we initiated the study of the random field Curie-Weiss model in [8] as an attempt to get some insight in the dynamical properties of disordered systems. Indeed, although the statics of disordered mean field models has been the object of various papers, see [9] for instance, the investigation of dynamical properties is at its beginning. (See [6] for the R.E.M. process). We refer to [2] for a general overview of the field. In [8] we focused on the description of the almost sure behaviour of the magnetisation for the Glauber dynamics. In this context, 'almost sure' means for a given realization of the external field. We obtained a kind of metastability property for the magnetization via spectral methods. The aim of this paper is to extend the spectral approach to describe the 'averaged over the disorder' dynamics.

Our approach is based on an approximation of the law of the magnetization by the first terms in the spectral decomposition. In [8] we derived the metastability statement from the asymptotics of the eigenvalues of the generator on the exponential scale. This scale corresponds to the law of large numbers for the free energy or, equivalently, for the random field. It allows one to describe the approach to equilibrium on a disorder dependent time scale. For instance the law of the first time the magnetization changes sign is proved to be close to an exponential law with a disorder dependent parameter. The estimates we had on the random parameter were actually quite poor. For instance computing the probability - w.r.t. the disorder of the external field - that the sign change in the magnetization has taken place before some given time is not possible with the only results of [8]. To answer this type of questions, one has to go one step further and prove a central limit theorem for the partition functions (See Proposition 2.1) and for the eigenvalues (See Theorem 2.7). We therefore obtain the description of the 'averaged over the disorder' dynamics (See Theorem 2.4) which allows one to compute the probability that the system is in a given state at any (deterministic) time. Note that the averaged and almost sure behaviours are different: the limiting dynamics for the averaged problem turns out to be non Markovian i.e. the process is not self-averaging. We also prove that the law of the first time the process shifts from a positive magnetization to a negative one is close to a log-normal law.

Actually we shall obtain estimates of the eigenvalues that are much more precise than a mere central limit theorem. As a consequence of our sharp lower bound on the spectral gap of the dynamics, we derive a precise estimate of the rate of convergence of the magnetization towards equilibrium in Proposition 2.3. It shows in particular that the time needed for the magnetization to reach equilibrium is shorter when the fluctuations of the random field are bigger: fluctuations help the process to converge. This result is extended to the full spin dynamics in the next paper.

We use geometrical tools to bound eigenvalues. As explained in [10], one can estimate the spectral gap in terms of the variations of the free energy along paths in the state space. Instead of applying directly the results of [10], we use a combination of geometrical and probabilistic arguments in order to estimate both eigenvalues and eigenvectors. We believe this trick yields shorter proofs. (See Lemma 4.1 in part IV).

II. The model and main results

In this section we describe our model, recall the results from [8] that we shall need and state our results. The main results are Theorems 2.4 and 2.7.

- **The random field Curie-Weiss model:** let $h = (h_i)_{i\in\mathbb{N}}$ be a sequence of independent symmetric Bernoulli random variables defined on some probability space, say $(\Omega, \mathcal{A}, Q)$. That is $Q[h_i = 1] = Q[h_i = -1] = 1/2$, for any i. Define $S_N = \sum_{i=1}^N h_i$.

Let $\beta > 0$ and $\theta \in \mathbb{R}$. Later we shall assume that $\beta > cosh^2(\beta\theta)$.

Most of the quantities that we are going to define depend on h. Usually we shall drop this dependence in the computations. In the sequel, we denote by C a constant which depends on β and θ only. Its value may change from line to line. N_0 is an integer that depends also on β and θ only. Its value may change from line to line. In particular C and N_0 do not depend on h.

- **Equilibrium:** let $\mathcal{S}_N = \{-1,+1\}^N$. We denote by $\sigma = (\sigma_i)_{i=1..N}$ some element in $\mathcal{S}_N$. We define on $\mathcal{S}_N$ the following Hamiltonian:

$$H_N(\sigma) \equiv H_N^h(\sigma) = -\frac{N}{2}\Big(\frac{1}{N}\sum_{i=1}^N \sigma_i\Big)^2 - \theta\sum_{i=1}^N h_i\sigma_i \tag{2.1}$$

Let $\mu_N \equiv \mu_N^h$ be the Gibbs measure on $\mathcal{S}_N$ defined by:

$$\mu_N(\sigma) = \frac{\exp(-\beta H_N(\sigma))}{K_N}$$

where

$$K_N \equiv K_N^h = \sum_{\sigma\in\mathcal{S}_N} e^{-\beta H_N(\sigma)} \tag{2.2}$$

is a normalizing constant.

- **Magnetization:** for any $\sigma \in \mathcal{S}_N$, let $\overline{m}_N(\sigma) = \frac{1}{N}\sum_{i=1}^N \sigma_i$ be the empirical mean or magnetization. We also define

$$m_N^+(\sigma) \equiv m_N^{h,+}(\sigma) = \frac{1}{N}\sum_{i;h_i=+1} \sigma_i$$

and

$$m_N^-(\sigma) \equiv m_N^{h,-}(\sigma) = \frac{1}{N}\sum_{i;h_i=-1} \sigma_i$$

and $m_N(\sigma) = (m_N^+(\sigma), m_N^-(\sigma))$. Therefore $\overline{m}_N(\sigma) = m_N^+(\sigma) + m_N^-(\sigma)$.

m_N clearly takes its values in $[-1,+1]^2$. We denote by $\mathcal{M}_N$ the image of $\mathcal{S}_N$ by m_N. Denoting $N^+ \equiv N^{h,+} = \#\{i : h_i = +1\}$ and $N^- \equiv N^{h,-} = \#\{i : h_i = -1\}$, we have

$$\mathcal{M}_N = \{-\frac{N^+}{N}, \frac{-N^+ + 2}{N}, ..., \frac{N^+}{N}\} \times \{-\frac{N^-}{N}, \frac{-N^- + 2}{N}, ..., \frac{N^-}{N}\}$$

The Hamiltonian can be written in terms of m_N:

$$H_N(\sigma) = -N\left(\frac{1}{2}(m_N^+(\sigma) + m_N^-(\sigma))^2 + \theta(m_N^+(\sigma) - m_N^-(\sigma))\right) \tag{2.3}$$

With a little abuse of notation, we shall also denote by H_N the function defined on $\mathcal{M}_N$ by

$$H_N(m) = -N\left(\frac{1}{2}(m^+ + m^-)^2 + \theta(m^+ - m^-)\right) \tag{2.4}$$

We are interested in the behavior of the magnetization under the law μ_N: let $\mathcal{G}_N \equiv \mathcal{G}_N^h$ be the image of μ_N by m_N. $\mathcal{G}_N$ is a probability measure on $\mathcal{M}_N$. We have, for $m = (m^+, m^-) \in \mathcal{M}_N$,

$$\mathcal{G}_N(m) = \frac{\exp(-\beta N \mathcal{F}_N(m))}{Z_N} \tag{2.5}$$

where

$$Z_N \equiv Z_N^h = \sum_{m \in \mathcal{M}_N} e^{-\beta N \mathcal{F}_N(m)} \tag{2.6}$$

is a normalizing constant and

$$\begin{aligned}\mathcal{F}_N(m) \equiv \mathcal{F}_N^h(m) = &-\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-) \\ &- \frac{1}{\beta N}\log\left(\frac{N^+}{\frac{N^+}{2} + m^+\frac{N}{2}}\right)\left(\frac{N^-}{\frac{N^-}{2} + m^-\frac{N}{2}}\right)\end{aligned} \tag{2.7}$$

satisfies

$$e^{-\beta N \mathcal{F}_N(m)} = \sum_{\sigma; m_N(\sigma) = m} e^{-\beta H_N(\sigma)} \tag{2.8}$$

- Free energy: it is not difficult to see that, as a consequence of the law of large numbers, $\mathcal{F}_N$ converges Q-almost surely, as $N \to +\infty$, to the function

$$\mathcal{F}(m) = -\frac{1}{2}(m^+ + m^-)^2 - \theta(m^+ - m^-) + \frac{1}{2\beta}\left(I(2m^+) + I(2m^-)\right) \tag{2.9}$$

Here, for $x \in [-1,+1]$, $I(x) = \frac{1+x}{2}\log\frac{1+x}{2} + \frac{1-x}{2}\log\frac{1-x}{2}$, and for $|x| \geq 1$, $I(x) = 0$, is the entropy of Bernoulli random variables.
The function $\mathcal{F}$ is symmetric w.r.t. the diagonal. It has three critical points when $\beta > \cosh^2(\beta\theta)$: let m_* be the unique positive solution of the equation:

$$m_* = \frac{1}{2}\left[\tanh(\beta m_* + \beta\theta) + \tanh(\beta m_* - \beta\theta)\right] \tag{2.10}$$

The critical points of $\mathcal{F}$ are

$$\begin{aligned} m_0 &= \left(\frac{1}{2}\tanh(\beta\theta), -\frac{1}{2}\tanh(\beta\theta)\right) \\ m_1 &= \left(\frac{1}{2}\tanh(\beta m_* + \beta\theta), \frac{1}{2}\tanh(\beta m_* - \beta\theta)\right) \\ m_2 &= \left(\frac{1}{2}\tanh(-\beta m_* + \beta\theta), -\frac{1}{2}\tanh(\beta m_* + \beta\theta)\right) \end{aligned} \tag{2.11}$$

m_0 is a saddle point. m_1 and m_2 are two minima.
Let

$$\begin{aligned} T_1^N &= \mathcal{M}_N \cap \{m^+ + m^- > 0\} \\ \overline{T}_1^N &= \mathcal{M}_N \cap \{m^+ + m^- \geq -\tfrac{3}{N}\} \\ \partial T_1^N &= \mathcal{M}_N \cap \{0 \geq m^+ + m^- \geq -\tfrac{3}{N}\} \end{aligned}$$

and define T_2^N and $\overline{T}_2^N$ analogously.
Let $\mathcal{G}_N^1 \equiv \mathcal{G}_N^{h,1}$ be the restriction of $\mathcal{G}_N$ to T_1^N, i.e.

$$\mathcal{G}_N^1(m) = \frac{Z_N}{Z_N^1}\mathcal{G}_N(m)\mathbb{I}(m \in \overline{T}_1^N)$$

$$Z_N^1 = \sum_{m \in \overline{T}_1^N} e^{-\beta N \mathcal{F}_N(m)} \tag{2.12}$$

Define $\mathcal{G}_N^2$ analogously and

$$Z_N^2 = \sum_{m \in \overline{T}_2^N} e^{-\beta N \mathcal{F}_N(m)} \tag{2.13}$$

Define also

$$z_N^1 = \sum_{m \in \partial T_1^N} e^{-\beta N \mathcal{F}_N(m)} \tag{2.14}$$

Clearly, as N tends to $+\infty$, under $\mathcal{G}_N$, the magnetization m_N gets close to one of the two values m_1 or m_2. The asymptotic support of the law of m_N is therefore deterministic. We have the

Proposition 2.1 (Static asymptotics). *Define*

$$a = a(\beta,\theta) = \frac{1}{2\beta}\log\frac{\cosh(\beta m_* + \beta\theta)}{\cosh(\beta m_* - \beta\theta)} \tag{2.15}$$

Then, for $N \geq N_0$, on the set $|S_N| \leq 2\sqrt{N\log N}$, we have:

$$|\log Z_N + \beta N\mathcal{F}(m_1) - \beta a|S_N|| \leq C\log N \tag{2.16}$$

$$|\log Z_N^1 + \beta N\mathcal{F}(m_1) - \beta a S_N| \leq C\log N \tag{2.17}$$

$$|\log z_N^1 + \beta N\mathcal{F}(m_0)| \leq C\log N \tag{2.18}$$

for some positive constant C.
Besides, for Q-almost all realizations of h, for any continuous function ϕ defined on $[-1,+1]^2$, we have

$$\mathcal{G}_N(\phi) - (\alpha_N\phi(m_1) + (1-\alpha_N)\phi(m_2)) \to 0 \tag{2.19}$$

where

$$\alpha_N \equiv \alpha_N^h = \frac{e^{\beta a S_N}}{e^{\beta a S_N} + e^{-\beta a S_N}} \tag{2.20}$$

(2.19) is actually proved in [8], Lemma 4.1.
Note that $Q[|S_N| \geq 2\sqrt{N\log N}] \leq 2\exp(-2\log N)$. Therefore the Borel-Cantelli lemma implies that the difference $\mathcal{G}_N - (\alpha_N\delta_{m_1} + (1-\alpha_N)\delta_{m_2})$ weakly converges to 0 for almost all realizations of h. Note however that α_N itself does not converge almost surely but only in distribution. The equilibrium measure keeps oscillating as N tends to $+\infty$. A more detailed description of these oscillations is given in [7].
- **Dynamics**: for a given realization of h, we define the dynamics, first on the spins σ, then on the magnetization m_N. For $1 \leq i \leq N$, let T^i be the map from $\mathcal{S}_N$ to $\mathcal{S}_N$ defined by $T^i(\sigma)_j = \sigma_j$ for $j \neq i$, $T^i(\sigma)_i = -\sigma_i$. Consider the following operator acting on real valued functions ϕ on $\mathcal{S}_N$:

$$L_N\phi(\sigma) \equiv L_N^h\phi(\sigma) = \frac{1}{N}\sum_{i=1}^{N}(\phi(T^i(\sigma)) - \phi(\sigma))\, e^{-\frac{\beta}{2}[H_N(T^i(\sigma)) - H_N(\sigma)]} \tag{2.21}$$

L_N is the generator of a Markov process which we denote by $\sigma_N(t) \equiv \sigma_N^h(t)$. μ_N is the unique invariant probability measure for σ_N. It is also reversible.

Now let $m_N(t) \equiv m_N(\sigma_N(t))$. It turns out that m_N is also a Markov process with invariant probability measure $\mathcal{G}_N$. Let $\mathcal{L}_N \equiv \mathcal{L}_N^h$ be the generator of m_N. According to formula (2.24) of [8] we have

$$\mathcal{L}_N\phi(m) = \sum_{\substack{\tilde{m}\in\mathcal{M}_N \\ \tilde{m}\sim m}} [\phi(\tilde{m}) - \phi(m)]\,\mathcal{N}_N(\tilde{m}, m)e^{-\frac{\beta}{2}[H_N(\tilde{m})-H_N(m)]} \tag{2.22}$$

where $\tilde{m} \sim m$ means that $\tilde{m}$ and m are neighbors in $\mathcal{M}_N$ and $\mathcal{N}_N(\tilde{m}, m)$ is some correction factor which is between $2/N$ and 1 (See [8]). Call $P_t^N \equiv P_t^{h,N} = e^{t\mathcal{L}_N}$ the semi-group generated by m_N. We shall use the notation $P_m \equiv P_m^h$ to denote the law of the process m_N when $m_N(0) = m$ and E_m to denote the expectation w.r.t. P_m. We insist on the fact that, since h is kept fixed, the measure P_m is Markovian.

We aim at describing the way the law of $m_N(t)$ approaches $\mathcal{G}_N$. Due to the concentration of the measure $\mathcal{G}_N$ on $\{m_1, m_2\}$, we expect the process m_N to spend most of its time around these two points. It suggests that the dynamics can be approximated by a Markovian process with state space $\{m_1, m_2\}$. Let $\tau_N = \inf\{t > 0 : m_N(t) \in \partial T_1^N\}$ be the hitting time of ∂T_1^N.

The main result of [8] is the

Proposition 2.2 (Almost sure metastability). *For almost all realizations of h, there exist two sequences $\Lambda_1^N \equiv \Lambda_1^{h,N}$ and $L_1^N \equiv L_1^{h,N}$ such that, for all sequences $m^N \in T_1^N$ with* $\limsup \mathcal{F}(m^N) < \mathcal{F}(m_0)$, *we have:*
for all $t > 0$,

$$P_{m^N}[L_1^N\tau_N > t] \to e^{-t} \tag{2.23}$$

and, for any continuous function ϕ,

$$\begin{aligned} &E_{m^N}[\phi(m_N(t/\Lambda_1^N))] \\ &\quad - (e^{-t}\phi(m_1) + (1 - e^{-t})(\alpha_N\phi(m_1) + (1 - \alpha_N)\phi(m_2))) \to 0 \end{aligned} \tag{2.24}$$

We define the activation energy: $\Delta\mathcal{F} = \mathcal{F}(m_0) - \mathcal{F}(m_1) = \mathcal{F}(m_0) - \mathcal{F}(m_1)$. It follows from Propositions 2.5 and 2.6 below that $\frac{1}{N}\log L_1^N$ and $\frac{1}{N}\log\Lambda_1^N$ both tend to $-\beta\Delta\mathcal{F}$ as $N \to +\infty$. Nevertheless the two normalizing constants Λ_1^N and L_1^N do depend on h.

(2.23) implies that τ_N is roughly speaking of order $1/L_1^N$ i.e. of order $\exp(\beta N\Delta\mathcal{F})$, provided the process starts inside T_1^N. Furthermore the law of τ_N is close to an exponential law of parameter L_1^N. Roughly speaking, the evolution of the magnetization is: until time τ_N, m_N stays close to m_1, and after time τ_N, it reaches its equilibrium: $\mathcal{G}_N$. Since $\frac{1}{N}\log\Lambda_1^N \to -\beta\Delta\mathcal{F}$, we deduce from (2.24) that, for any $\alpha > \beta\Delta\mathcal{F}$, the law of m_N at time $\exp(\alpha N)$ is close to the equilibrium. We shall obtain a more precise result:

Proposition 2.3 *There exist deterministic constants K and K' such that, for any $N \geq N_0$, for almost all realizations of h, for any continuous function ϕ bounded by 1, for any $m \in \mathcal{M}_N$, if we let $t_N = N^K \exp(\beta N \Delta\mathcal{F} - \beta a |S_N|)$, where the constant a is defined by equation (2.15), then, on the set $|S_N| \leq 2\sqrt{N \log N}$, we have*

$$|E_m[\phi(m_N(t_N))] - \mathcal{G}_N(\phi)| \leq \exp(-N^{K'}) \tag{2.25}$$

Proposition 2.3 implies in particular that a time of order $\exp(\beta N \Delta\mathcal{F})$ is more than sufficient to reach equilibrium, provided we restrict ourselves to a sub sequence N_n s.t. $\beta|S_{N_n}| \geq K \log N_n$. In some sense the fluctuations of the external field h in the free energy help the process to reach equilibrium.

We now turn to the description of the law of m_N averaged over the realizations of h. Let us define the measure $\mathbb{P}_m = P_m^h \times Q$. $\mathbb{E}_m$ stands for the expectation w.r.t. $\mathbb{P}_m$. By definition, if Φ is a measurable function on the paths space, $\mathbb{E}_m[\Phi] = \int P_m^h[\Phi] Q(dh)$. Under $\mathbb{P}_m$, m_N is not a Markov process anymore. We prove the

Theorem 2.4 **(Asymptotics for the averaged dynamics).** *Let $\mathcal{N}$ be a normalized Gaussian random variable. Let a be the constant defined in formula (2.15). For all sequences $m^N \in T_1^N$ with $\limsup \mathcal{F}(m^N) < \mathcal{F}(m_0)$, and for all $\alpha \in \mathbb{R}$, we have:*

$$\mathbb{P}_{m^N}[N^{-1/2}(\log \tau_N - \beta N \Delta\mathcal{F}) \geq \alpha] \to P[\beta a \mathcal{N} \geq \alpha] \tag{2.26}$$

and, for any continuous function ϕ,

$$\begin{aligned} &\mathbb{E}_{m^N}[\phi(m_N(e^{\beta N \Delta\mathcal{F} + \alpha\sqrt{N}}))] \\ &\to (\frac{1}{2} + P[0 \geq \beta a \mathcal{N} \geq \alpha])\phi(m_1) + (\frac{1}{2} - P[0 \geq \beta a \mathcal{N} \geq \alpha])\phi(m_2) \end{aligned} \tag{2.27}$$

Note that the expression (2.27) implies that, after renormalization, the magnetization only takes the two values m_1 and m_2. But (2.27) does not define a Markovian evolution i.e., even in the large N limit, the dynamics remains non Markovian. It reveals the non self-averaging behavior of the process.

Spectral decomposition: the proof of Proposition 2.2 is based on estimates of the eigenvalues of $\mathcal{L}_N$. Since the operator $\mathcal{L}_N$ is symmetric in $L_2(\mathcal{M}_N, \mathcal{G}_N)$, we can consider a spectral decomposition: let $(\Lambda_i^N \equiv \Lambda_i^{h,N})_{i=0\ldots}$ denote the eigenvalues of $-\mathcal{L}_N$ in increasing order, with $\Lambda_0^N = 0$. Let $\psi_i^N \equiv \psi_i^{h,N}$ be the corresponding eigenvectors. We have $\psi_0^N \equiv 1$. We assume that the ψ_i^N form an orthonormal basis of $L_2(\mathcal{M}_N, \mathcal{G}_N)$. We can now express the law of m_N at time t on this basis:

$$E_m[\phi(m_N(t))] = \sum_i \psi_i^N(m) \mathcal{G}_N(\phi \psi_i^N) e^{-\Lambda_i^N t} \tag{2.28}$$

Similarly let $\mathcal{L}_N^K$ be the generator of the process m_N killed at time τ_N. In other word $\mathcal{L}_N^K$ is the restriction of $\mathcal{L}_N$ to functions $\phi \in L_2(\overline{T}_1^N, \mathcal{G}_N^1)$ with $\phi(m) = 0$ for $m \in \partial T_1^N$. $-\mathcal{L}_N^K$ is a symmetric operator on $L_2(\overline{T}_1^N, \mathcal{G}_N^1)$. We denote by L_i^N its eigenvalues, and ϕ_i^N the corresponding normalized eigenfunctions. We then have

$$P_m[\tau_N > t] = \sum_i \phi_i^N(m)\mathcal{G}_N^1(\phi_i^N)e^{-L_i^N t} \tag{2.29}$$

Remark: it follows from the results of [8] that the constants in Proposition 2.2 are indeed the eigenvalues we have just defined.

¿From [8] we know that

Proposition 2.5 (Estimates of eigenvalues). *For almost all realizations of h,*

$$\begin{aligned} \frac{1}{N}\log L_1^N &\to -\beta\Delta\mathcal{F} \\ \frac{1}{N}\log \Lambda_1^N &\to -\beta\Delta\mathcal{F} \end{aligned} \tag{2.30}$$

Estimating the eigenvalues Λ_2^N and L_2^N, one checks that only the first terms really contribute in (2.28) and (2.29). The next result is a consequence of the computation of [8], part 3. (See part IV for the details).

Proposition 2.6 (Spectral approximation). *There exists a deterministic constant K such that, for $t > 0$, if we define $T = t\exp(-K\sqrt{N}\log N) - K\log N$, then, for any realization of h and any $N \geq N_0$ s.t. $|S_N| \leq 2\sqrt{N\log N}$, for any $m \in T_1^N$, we have*

$$|P_m[\tau_N > t] - \phi_1^N(m)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t}| \leq e^{-T} \tag{2.31}$$

Besides, for any continuous function ϕ bounded by 1, and for $m \in \mathcal{M}_N$, we have

$$|E_m[\phi(m_N(t))] - \Big(\mathcal{G}_N(\phi) + \psi_1^N(m)\mathcal{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\Big)| \leq e^{-T} \tag{2.32}$$

Proposition 2.2 is a consequence of the Propositions 2.5 and 2.6. Note that the expressions (2.31) (resp. (2.32)) do look like (2.23) (resp. (2.24)). The only additional information we would need is some estimate of the eigenfunctions (See [8] for the proof). We shall deduce Theorem 2.4 from the fluctuations of the eigenvalues:

Theorem 2.7 (Fluctuations of the eigenvalues). *Let a be the constant defined in (2.15). For any $N \geq N_0$, for almost all realizations of h, on the set $|S_N| \leq 2\sqrt{N\log N}$, we have:*

$$|\log L_1^N + \beta N\Delta\mathcal{F} + \beta a S_N| \leq C\log N \tag{2.33}$$

and

$$|\log \Lambda_1^N + \beta N \Delta \mathcal{F} - \beta a |S_N|| \leq C \log N \tag{2.34}$$

As a consequence, if $\mathcal{N}$ is a normalized Gaussian random variable, then the following convergences hold in law w.r.t. Q:

$$N^{-1/2}(\log L_1^N + \beta N \Delta \mathcal{F}) \to -\beta a \mathcal{N} \tag{2.35}$$

$$N^{-1/2}(\log \Lambda_1^N + \beta N \Delta \mathcal{F}) \to \beta a |\mathcal{N}| \tag{2.36}$$

- **Prerequisites:** Here we recall some of the estimates proved in [8] that we shall need in the sequel. These are rough bounds on the exponential scale.
Let us choose $N \geq N_0$ and a realization of h s.t. $|S_N| \leq 2\sqrt{N \log N}$. Using Stirling's formula as in [8] part 4, it is not difficult to see that, for any $m \in [-1,+1]^2$,

$$|\mathcal{F}_N(m) - \mathcal{F}(m)| \leq C \frac{\log N}{\sqrt{N}} \tag{2.37}$$

It is proved in [8] that, for any i, the following convergences hold almost surely:

$$\frac{1}{N} \log L_i^N \to -c_i$$
$$\frac{1}{N} \log \Lambda_i^N \to -c_i$$

where $c_1 = \beta \Delta \mathcal{F}$ and $c_i = 0$ for $i \geq 2$. Taking into account (2.37), it is immediate to prove that in fact

$$\begin{aligned} |\log L_i^N + N c_i| &\leq C\sqrt{N} \log N \\ |\log \Lambda_i^N + N c_i| &\leq C\sqrt{N} \log N \end{aligned} \tag{2.38}$$

Finally we also have some estimates of ϕ_1^N:

$$1 - \mathcal{G}_N^1(\phi_1^N) \leq e^{-CN} \tag{2.39}$$

and, for any deterministic compact set A s.t. $\sup_{x \in A} \mathcal{F}(x) < \mathcal{F}(m_0)$, Q.a.s.

$$\sup_{m \in A \cap T_1^N} |1 - \phi_1^N(m)| \leq e^{-C'N} \tag{2.40}$$

where C' is a deterministic constant that depends on A. (2.39) and (2.40) can be proved as in [8] part 3.3. with the help of (2.37).

III. Static estimates

Partition functions.

Proof of Proposition 2.1: the proof of Proposition 2.1 is inspired by the arguments of [4]. Let us first note that, by symmetry and because the spins are exchangeable, we may assume without loss of generality that $S_N \geq 0$ and that $h_i = +1$, for $i = 1, ..., (N + S_N)/2$ and $h_i = -1$ for $i = (N + S_N)/2 + 1, ..., N$. Let $M = [N/2]$, where [.] denotes the integer part. Following [4], we introduce a different parametrization of the magnetization: for a configuration $\sigma \in \mathcal{S}_N$, define:

$$\tilde{m}_N^+(\sigma) = \frac{1}{N} \sum_{i=1}^{M} \sigma_i$$

$$\tilde{m}_N^-(\sigma) = \frac{1}{N} \sum_{i=M+1}^{N} \sigma_i$$

We use the notation $\tilde{m}_N(\sigma) = (\tilde{m}_N^+(\sigma), \tilde{m}_N^-(\sigma))$, and we denote by $\tilde{\mathcal{M}}_N$ the image of $\mathcal{S}_N$ by the application $\tilde{m}_N$. $\tilde{\mathcal{M}}_N$ is therefore a deterministic subset of $[-1/2, 1/2]^2$. We have $\overline{m}_N(\sigma) = \tilde{m}_N^+(\sigma) + \tilde{m}_N^-(\sigma)$. Let $D = \{i \geq M + 1 : h_i = 1\}$. Note that the cardinality of D satisfies $|D| = (N + S_N)/2 - M \leq (1 + S_N)/2$. The Hamiltonian H_N can be expressed in these new coordinates as:

$$H_N(\sigma) = -\frac{N}{2}(\overline{m}_N(\sigma))^2 - N\theta(\tilde{m}_N^+(\sigma) - \tilde{m}_N^-(\sigma)) - 2\theta \sum_{i \in D} \sigma_i$$

and

$$Z_N^1 = \sum_{\sigma \in \mathcal{S}_N} \mathbb{1}_{\{\overline{m}_N(\sigma) \geq -3/N\}} e^{-\beta H_N(\sigma)}$$

i.e.

$$Z_N^1 = \sum_{\tilde{m} \in \tilde{\mathcal{M}}_N} \mathbb{1}_{\{\overline{m} \geq -3/N\}} \#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\} e^{\beta N(\frac{1}{2}\overline{m}^2 + \theta(\tilde{m}^+ - \tilde{m}^-))}$$
$$\times \frac{\sum_{\sigma; \tilde{m}_N(\sigma) = \tilde{m}} e^{2\beta\theta \sum_{i \in D} \sigma_i}}{\#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}} \tag{3.1}$$

In this last expression, the only term that depends on h is the set D.
Let $\tilde{\mathcal{F}}_N(\tilde{m}) = -\frac{1}{2}\overline{m}^2 - \theta(\tilde{m}^+ - \tilde{m}^-) - \frac{1}{\beta N} \log \#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}$ and note that

$$|\tilde{\mathcal{F}}_N(\tilde{m}^N) - \mathcal{F}(\tilde{m})| \leq C(\frac{\log N}{N} + \|\tilde{m}^N - \tilde{m}\|) \tag{3.2}$$

for any $\tilde{m}^N \in \tilde{\mathcal{M}}_N, \tilde{m} \in [-1/2, 1/2]^2$.

The minimum of $\mathcal{F}$ in the set $\overline{m} \geq 0$ is achieved at the only point m_1. Since we have assumed that $|S_N| \leq 2\sqrt{N \log N}$, we have $|\sum_{i \in D} \sigma_i| \leq \sqrt{N \log N}$. Taking into account the estimate (3.2), one deduces that there exists a small enough ball, B, centered at point m_1 such that, if we define

$$\tilde{Z}_N^1 = \sum_{\tilde{m} \in \tilde{\mathcal{M}}_N \cap B} e^{-\tilde{\mathcal{F}}_N(\tilde{m})} \frac{\sum_{\sigma; \tilde{m}_N(\sigma)=\tilde{m}} e^{2\beta\theta \sum_{i \in D} \sigma_i}}{\#\{\sigma; \tilde{m}_N(\sigma) = \tilde{m}\}}$$

then, for some deterministic constant C depending on the choice of B, we have $\log \tilde{Z}_N^1 \leq \log Z_N^1 \leq \log \tilde{Z}_N^1 - \log(1 - e^{-CN})$. It is therefore enough to prove the Proposition 2.1 for $\tilde{Z}_N^1$ instead of Z_N^1.

For $\nu \in \mathbb{R}$, define the probability measure

$$E_\nu[f] = \frac{\sum_{\sigma \in S_N} f(\sigma) e^{\nu \sum_{i=M+1}^N \sigma_i}}{\sum_{\sigma \in S_N} e^{\nu \sum_{i=M+1}^N \sigma_i}} \tag{3.3}$$

Note that

$$\sum_{\sigma \in S_N} e^{\nu \sum_{i=M+1}^N \sigma_i} = 2^N \cosh(\nu)^{N-M}$$

For any choice of ν_2 and ν_1, $\tilde{Z}_N^1$ can be rewritten as

$$\begin{aligned}
\tilde{Z}_N^1 = &\sum_{\tilde{m} \in \tilde{\mathcal{M}}_N \cap B} e^{-\beta N \tilde{\mathcal{F}}_N(\tilde{m})} (\frac{\cosh \nu_2}{\cosh \nu_1})^{N-M} e^{-N\tilde{m}^-(\nu_2 - \nu_1)} \\
&\times \frac{E_{\nu_2}[e^{2\beta\theta \sum_{i \in D} \sigma_i} \mathbb{1}_{\tilde{m}_N^-(\sigma) = \tilde{m}^-}]}{E_{\nu_1}[\mathbb{1}_{\tilde{m}_N^-(\sigma) = \tilde{m}^-}]} \\
= &\sum_{\tilde{m} \in \tilde{\mathcal{M}}_N \cap B} e^{-\beta N \tilde{\mathcal{F}}_N(\tilde{m})} (\frac{\cosh \nu_2}{\cosh \nu_1})^{N-M} e^{-N\tilde{m}^-(\nu_2 - \nu_1)} \\
&\times (\frac{\cosh(\nu_2 + 2\beta\theta)}{\cosh \nu_2})^{|D|} \frac{\Psi(\nu_2, \theta)(\tilde{m}^-)}{\Psi(\nu_1, 0)(\tilde{m}^-)}
\end{aligned} \tag{3.4}$$

where

$$\Psi(\nu, \theta)(\tilde{m}^-) = \frac{E_\nu[e^{2\beta\theta \sum_{i \in D} \sigma_i} \mathbb{1}_{\tilde{m}_N^-(\sigma) = \tilde{m}^-}]}{E_\nu[e^{2\beta\theta \sum_{i \in D} \sigma_i}]}$$

Let $\alpha = S_N/(N - M)$. We now choose for ν_1 and ν_2 the solutions of the equations:

$$\begin{aligned}
&\tanh(\nu_1) = 2\tilde{m}^- \\
&\alpha \tanh(\nu_2 + 2\beta\theta) + (1 - \alpha) \tanh(\nu_2) = 2\tilde{m}^-
\end{aligned} \tag{3.5}$$

Since we are only interested in estimates for $\tilde{m} \in B$, and since $|\alpha| \leq 2\sqrt{\log N/N}$, then ν_1 and ν_2 are uniformly bounded as S_N and $\tilde{m}$ vary. Besides we deduce from equation (3.5) that

$$|\alpha(\tanh(\nu_1 + 2\beta\theta) - \tanh(\nu_1)) + (\nu_2 - \nu_1)(\frac{\alpha}{\cosh^2(\nu_1 + 2\beta\theta)} + \frac{1-\alpha}{\cosh^2(\nu_1)})| \leq C(\nu_2 - \nu_1)^2$$

Therefore

$$|(\nu_2 - \nu_1) - \alpha \cosh^2(\nu_1)(\tanh(\nu_1) - \tanh(\nu_1 + 2\beta\theta))| \leq C\frac{\log N}{N}$$

and

$$\Big(- N\tilde{m}^-(\nu_2 - \nu_1) + (N-M)\log\frac{\cosh\nu_2}{\cosh\nu_1} + |D|\log\frac{\cosh(\nu_2 + 2\beta\theta)}{\cosh\nu_2}\Big)$$
$$-\frac{S_N}{2}\log\frac{\cosh(\nu_1 + 2\beta\theta)}{\cosh\nu_1}| \leq C\log N$$

and

$$|\log\tilde{Z}_N^1 - \log\sum_{\tilde{m}\in\tilde{\mathcal{M}}_N\cap B} e^{-\beta N\tilde{\mathcal{F}}_N(\tilde{m})}(\frac{\cosh(\nu_1 + 2\beta\theta)}{\cosh\nu_1})^{S_N/2}\frac{\Psi(\nu_2,\theta)(\tilde{m}^-)}{\Psi(\nu_1, 0)(\tilde{m}^-)}| \leq C\log N \quad (3.6)$$

It now only remains to estimate Ψ. This can be done through a local central limit theorem just as in [4]. Repeating the arguments of Proposition 3.2 of [4], we get that

$$\Psi(\nu,\theta)(\tilde{m}^-) = \frac{1}{2\pi}\int_0^{2\pi} dke^{-ikN\tilde{m}^-}(\frac{\cosh(2\beta\theta + \nu + ik)}{\cosh(2\beta\theta + \nu)})^{|D|}(\frac{\cosh(\nu + ik)}{\cosh(\nu)})^{N-M-|D|}$$

¿From this last expression, following the estimates (3.36) to (3.44) in [4], one deduces that $C/\sqrt{N} \leq \Psi(\nu,\theta)(\tilde{m}^-) \leq 1$ provided that $2\tilde{m}^- = \alpha\tanh(\nu + 2\beta\theta) + (1-\alpha)\tanh(\nu)$. The constant C is chosen deterministic and independent of $\tilde{m} \in B$. Therefore

$$|\log\tilde{Z}_N^1 - \log\sum_{\tilde{m}\in\tilde{\mathcal{M}}_N\cap B} e^{-\beta N\tilde{\mathcal{F}}_N(\tilde{m})}(\frac{\cosh(\nu_1 + 2\beta\theta)}{\cosh\nu_1})^{S_N/2}| \leq C\log N \quad (3.7)$$

¿From the estimate (3.2), it is easy to deduce that one can replace $\mathcal{F}_N$ by $\mathcal{F}$ in this expression i.e.

$$|\log\tilde{Z}_N^1 - \log\sum_{\tilde{m}\in\tilde{\mathcal{M}}_N\cap B} e^{-\beta N\tilde{\mathcal{F}}(\tilde{m})}(\frac{\cosh(\nu_1 + 2\beta\theta)}{\cosh\nu_1})^{S_N/2}| \leq C\log N \quad (3.8)$$

Let us denote by ν_1^* the solution of equation (3.5) for the value $\tilde{m}^- = m_1^- = \frac{1}{2}\tanh(\beta m_* - \beta\theta)$ i.e. $\nu_1^* = \beta m_* - \beta\theta$. By standard Laplace arguments, we deduce from (3.7) that

$$|\log \tilde{Z}_N^1 + \beta N \mathcal{F}(m_1) - \frac{S_N}{2}\log\frac{\cosh(\nu_1^* + 2\beta\theta)}{\cosh\nu_1^*}| \leq C\log N \tag{3.9}$$

(2.17) is proved with $a = \frac{1}{2\beta}\log\frac{\cosh(\beta m_* + \beta\theta)}{\cosh(\beta m_* - \beta\theta)}$.
By symmetry, we also have

$$|\log Z_N^2 + \beta N\mathcal{F}(m_1) + \beta a S_N| \leq C\log N$$

Since $Z_N = Z_N^1 + Z_N^2$, we clearly have $Z_N^1 \vee Z_N^2 \leq Z_N \leq 2(Z_N^1 \vee Z_N^2)$. It yields (2.16).
Let us prove (2.18). As before we can assume that $S_N \geq 0$. As in the proof of (3.9), one gets that

$$|\log z_N^1 + \beta N\mathcal{F}(m_0) - \frac{S_N}{2}\log\frac{\cosh(\nu_1^* + 2\beta\theta)}{\cosh\nu_1^*}| \leq C\log N \tag{3.10}$$

where ν_1^* is now solution of the equation $\tanh(\nu_1^*) = 2m_0^-$. ¿From (2.11), we therefore have $\nu_1^* = -\beta\theta$ and $\cosh(\nu_1^* + 2\beta\theta) = \cosh(\nu_1^*)$.
This entails (2.18). ■

Let us conclude this section by the following corollary:
Lemma 3.1 . *On the set* $|S_N| \leq 2\sqrt{N\log N}$, *we have*

$$|\inf_{m\in \tilde{T}_1^N} \mathcal{F}_N(m) - \mathcal{F}(m_1) + \frac{a}{N}S_N| \leq C\frac{\log N}{N} \tag{3.11}$$

$$|\inf_{m\in\mathcal{M}_N} \mathcal{F}_N(m) - \mathcal{F}(m_1) + \frac{a}{N}|S_N|| \leq C\frac{\log N}{N} \tag{3.12}$$

$$|\inf_{m\in\partial T_1^N} \mathcal{F}_N(m) - \mathcal{F}(m_0)| \leq C\frac{\log N}{N} \tag{3.13}$$

Proof: : The number of points in $\mathcal{M}_N$ being bounded by $(N+1)^2$, we have

$$e^{-\beta N \inf_{m\in \tilde{T}_1^N}\mathcal{F}_N(m)} \leq Z_N^1 \leq (N+1)^2 e^{-\beta N\inf_{m\in\tilde{T}_1^N}\mathcal{F}_N(m)}$$

Combining this inequality with (2.17) yields (3.12). The proof of the (3.11) and (3.13) is identical. ■

Paths.

We now derive some a priori estimates on the landscape of the graph of $\mathcal{F}_N$ that will be used in the proof of Theorem 2.7. Let A be a subset of $\mathcal{M}_N$. By definition, a *path*, γ in A is a sequence $(x_0, x_1, ..., x_k)$ of points belonging to A such that x_i and x_{i+1} are neighbors and $x_i \neq x_j$ for $i \neq j$. The length of a path is therefore always bounded by N^2.
Since m_1 is an absolute minimum of $\mathcal{F}$ and m_0 is the unique saddle point, we know that there exists a continuous function $\gamma : [0,1] \to [-1/2, 1/2]^2$ s.t. $\gamma(0) = m_1$, $\gamma(1) = m_0$ and the function $t \to \mathcal{F}(\gamma(t))$ is increasing. We further assume that the curve $\gamma([0,1])$ lies in $]-1/2, 1/2[^2$. Let m_1^N (resp. m_0^N) be a point in $\mathcal{M}_N$ s.t. the distance $\|m_1^N - m_1\|$ (resp. $\|m_0^N - m_0\|$) is minimal. There exists a path in $\bar{T}_1^N$, say $\gamma_1^N = (x_0, ..., x_k)$, such that $x_0 = m_1^N$, $x_k = m_0^N$ and the distance between x_i and the curve $\gamma([0,1])$ is less than $\sqrt{2}/N$. Furthermore we have the:

Lemma 3.2 . *For $N \geq N_0$, on the set $|S_N| \leq 2\sqrt{N \log N}$, we have*

$$\sup_{x \in \gamma_1^N} \mathcal{F}_N(x) \leq \mathcal{F}(m_0) + C \frac{\log N}{N} \tag{3.14}$$

Proof: : Let K be a compact subset of $]-1/2, 1/2[^2$ that contains the paths γ_1^N for all $N \geq N_0$ and all realizations of h. m_0 is a critical point of $\mathcal{F}$. Therefore

$$|\mathcal{F}(m) - \mathcal{F}(m_0)| \leq C\|m - m_0\|^2 \tag{3.15}$$

Using Taylor expansions and Stirling formula, one immediately gets that, for $m \in \mathcal{M}_N \cap K$,

$$|\mathcal{F}_N(m) - \mathcal{F}(m) - \frac{1}{\beta} \log \frac{(\frac{1}{2} + m^+)(\frac{1}{2} - m^+)}{(\frac{1}{2} + m^-)(\frac{1}{2} - m^-)} \frac{S_N}{N}| \leq C \frac{\log N}{N} \tag{3.16}$$

Let $A > 0$. Let $x \in \gamma_1^N$. First assume that $\|x - m_0\| \leq A\sqrt{\log N/N}$. Since $m_0^+ = -m_0^-$, (3.16) implies that

$$\begin{aligned} |\mathcal{F}_N(x) - \mathcal{F}(x)| \leq & C\|x - m_0\| \frac{|S_N|}{N} + C \frac{\log N}{N} \\ \leq & C(1 + A) \frac{\log N}{N} \end{aligned}$$

on the set $|S_N| \leq 2\sqrt{N \log N}$.
Then, from (3.15), we deduce that

$$\begin{aligned} \mathcal{F}_N(x) &= \mathcal{F}(m_0) + \mathcal{F}_N(x) - \mathcal{F}(x) + \mathcal{F}(x) - \mathcal{F}(m_0) \\ &\leq \mathcal{F}(m_0) + C(1 + A) \frac{\log N}{N} + C\left(\frac{\sqrt{\log N}}{\sqrt{N}}\right)^2 \\ &\leq \mathcal{F}(m_0) + C(2 + A) \frac{\log N}{N} \end{aligned}$$

Assume now that $\|x-m_0\| \geq A\sqrt{\log N/N}$. Using (3.16) and the fact that since m_0 is non degenerate, there exists a constant $C'>0$ such that

$$\begin{aligned}
\mathcal{F}_N(x) &= \mathcal{F}(m_0) + \mathcal{F}_N(x) - \mathcal{F}(x) + \mathcal{F}(x) - \mathcal{F}(m_0) \\
&\leq \mathcal{F}(m_0) + C\|x-m_0\|\sqrt{\frac{\log N}{N}} + C\frac{\log N}{N} - C'\|x-m_0\|^2 \\
&\leq \mathcal{F}(m_0) + \|x-m_0\|(C-AC')\sqrt{\frac{\log N}{N}} + C\frac{\log N}{N} \\
&\leq \mathcal{F}(m_0) + C\frac{\log N}{N}
\end{aligned}$$

provided that we choose $A > C/C'$. ■

IV. Spectral estimates

Proof of Proposition 2.6: we first prove (2.32). For any continuous function ϕ bounded by 1, we have

$$\sqrt{\mathcal{G}_N\left(|E_{.}[\phi(m_N(t)] - \{\mathcal{G}_N(\phi) + \psi_1^N(.)\mathcal{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\}|^2\right)} \leq e^{-\Lambda_2^N t}\sqrt{\mathcal{G}_N(\phi^2)} \leq e^{-t\Lambda_2^N} \tag{4.1}$$

Since, for any $m \in \mathcal{M}_N$, $|\mathcal{F}_N(m)| \leq C$, we have, for any function ψ, $\psi(m)^2 \leq e^{CN}\mathcal{G}_N(\psi^2)$. From (2.38), we deduce that

$$\Lambda_2^N \geq e^{-C\sqrt{N}\log N} \tag{4.2}$$

Therefore (4.1) implies that

$$|E_m[\phi(m_N(t)] - \{\mathcal{G}_N(\phi) + \psi_1^N(m)\mathcal{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t}\}| \leq N^C e^{-te^{-C\sqrt{N}\log N}} = e^{-T} \tag{4.3}$$

for the correct choice of the constant K in Proposition 2.6. The proof of (2.31) is identical: one has to consider the spectral decomposition of the process m^N killed at time τ_N (See (2.29)). ∎

Proof of Theorem 2.7: (2.35) and (2.36) clearly follow from (2.33) and (2.34). Following [8], let us introduce the following Dirichlet forms: for any function ϕ defined on $\mathcal{M}_N$, we denote by $\mathcal{E}_N$ the Dirichlet form of the operator $\mathcal{L}_N$ w.r.t. $\mathcal{G}_N$ i.e.:

$$\mathcal{E}_N(\phi) \equiv -\mathcal{G}_N(\phi[\mathcal{L}_N\phi])$$

According to formula (2.25) in [8], $\mathcal{E}_N$ can also be written:

$$\mathcal{E}_N(\phi) = \frac{1}{2Z_N}\sum_{\substack{m,\tilde{m}\in\mathcal{M}_N\\ \tilde{m}\sim m}} (\phi(\tilde{m}) - \phi(m))^2(\tilde{\mathcal{N}}_N(\tilde{m},m))^{1/2}e^{-\frac{\beta N}{2}[\mathcal{F}_N(\tilde{m})+\mathcal{F}_N(m)]} \tag{4.4}$$

where $\tilde{\mathcal{N}}_N$ is a correction factor bounded from below by $2/N$ and bounded from above by 1. Similarly let $\mathcal{E}_N^1$ be the Dirichlet form of the process m^N killed when reaching ∂T_1^N: the domain of $\mathcal{E}_N^1$ is the set of functions ϕ defined on $\bar{T}_1^N$ vanishing on ∂T_1^N, and we have

$$\begin{aligned}\mathcal{E}_N^1(\phi) &\equiv -\,\mathcal{G}_N^1(\phi[\mathcal{L}_N(\phi)])\\ &= \frac{1}{2Z_N^1}\sum_{\substack{m,\tilde{m}\in T_1^N\\ \tilde{m}\sim m}} (\phi(\tilde{m}) - \phi(m))^2(\tilde{\mathcal{N}}_N(\tilde{m},m))^{1/2}e^{-\frac{\beta N}{2}[\mathcal{F}_N(\tilde{m})+\mathcal{F}_N(m)]}\end{aligned} \tag{4.5}$$

Upper bound for L_1^N: L_1^N is given by the variational principle:

$$L_1^N = \inf \frac{\mathcal{E}_N^1(\phi)}{\mathcal{G}_N^1(\phi^2)} \tag{4.6}$$

where the inf is taken on the domain of $\mathcal{E}_N^1$. Choose as a trial function $\phi(m) = \mathbb{I}(m \in T_1^N)$ in (4.6). The only non zero terms in (4.5) come from neighboring points $(m, \tilde{m})$ such that $m \in T_1^N$ and $\tilde{m} \in \partial T_1^N$. For such points we have $3/N \geq \overline{m} \geq -3/N$. Therefore

$$\mathcal{E}_N^1(\phi) \leq \frac{N^2}{Z_N^1} e^{-\beta N \inf \mathcal{F}_N(m)}$$

where the $\inf \mathcal{F}_N(m)$ is computed for points $m \in \mathcal{M}_N$ s.t. $3/N \geq \overline{m} \geq -3/N$. From (3.13), we know that

$$\inf_{m \in \partial T_1^N} \mathcal{F}_N(m) \geq \mathcal{F}(m_0) - C\frac{\log N}{N}$$

(The same holds true for ∂T_2^N) and therefore

$$\mathcal{E}_N^1(\phi) \leq \frac{N^2}{Z_N^1} e^{-\beta N \mathcal{F}(m_0)} N^C$$

Using (2.17), we get that

$$\begin{aligned} \mathcal{E}_N^1(\phi) \leq & N^C e^{\beta N \mathcal{F}(m_1) - \beta a S_N} e^{-\beta N \mathcal{F}(m_0)} \\ = & N^C e^{-\beta N \Delta \mathcal{F} - \beta a S_N} \end{aligned} \tag{4.7}$$

We also have $\mathcal{G}_N^1(\phi^2) = 1 - z_N^1/Z_N^1 \geq 1/2$ provided that N_0 is chosen big enough (See Proposition 2.1.

Therefore

$$L_1^N \leq N^C e^{-\beta N \Delta \mathcal{F} - \beta a S_N}$$

Lower bound for L_1^N: by definition of the eigenfunction ϕ_1^N, we have

$$L_1^N = \mathcal{E}_N^1(\phi_1^N)$$

Let $\gamma_1^N = (x_0, ..., x_k)$ be the path defined before Lemma 3.2. We have

$$\begin{aligned} |\phi_1^N(m_1^N)|^2 &= |\sum_i \phi_1^N(x_i) - \phi_1^N(x_{i+1})|^2 \\ &\leq \sum_i |\phi_1^N(x_i) - \phi_1^N(x_{i+1})|^2 (\tilde{\mathcal{N}}_N(x_i, x_{i+1}))^{1/2} e^{-\frac{\beta N}{2}[\mathcal{F}_N(x_i) + \mathcal{F}_N(x_{i+1})]} \\ &\times \sum_i (\tilde{\mathcal{N}}_N(x_i, x_{i+1}))^{-1/2} e^{\frac{\beta N}{2}[\mathcal{F}_N(x_i) + \mathcal{F}_N(x_{i+1})]} \\ &\leq 2 Z_N^1 \mathcal{E}_N^1(\phi_1^N) \sqrt{\frac{N}{2}} N^2 e^{\beta N \sup_{x \in \gamma_1^N} \mathcal{F}_N(x)} \\ &\leq N^C L_1^N e^{\beta N \mathcal{F}(m_0) - \beta N \mathcal{F}(m_1) + \beta a S_N} \end{aligned} \tag{4.8}$$

We used the results of Lemma 3.2 and Proposition 2.1. It only remains to note that, provided that N_0 is chosen big enough, we have $\phi_1^N(m_1^N) \geq 1/2$. This follows from (2.40). *Upper bound for* Λ_1^N*:* Λ_1^N is given by the variational principle:

$$\Lambda_1^N = \inf \frac{\mathcal{E}_N(\phi)}{\mathcal{G}_N[(\phi - \mathcal{G}_N(\phi))^2]} \tag{4.9}$$

Choose as a trial function $\phi(m) = \sqrt{Z_N^2/Z_N^1}\mathbb{I}(m \in \bar{T}_1^N) - \sqrt{Z_N^1/Z_N^2}\mathbb{I}(m \in \bar{T}_2^N)$. Then $\mathcal{G}_N(\phi) = 0$ and

$$\begin{aligned}\mathcal{G}_N(\phi^2) =& \frac{Z_N^1 + Z_N^2}{Z_N} - 2\mathcal{G}_N(\bar{T}_N^1 \cap \bar{T}_N^2) \\ \geq& \frac{Z_N^1 - z_N^1 + Z_N^2 - z_N^2}{Z^N} \geq 1 - e^{-CN}\end{aligned}$$

according to Proposition 2.1.
Using formula (4.4), we get that

$$\begin{aligned}\mathcal{E}_N(\phi) \leq& \frac{1}{Z_N}(\sqrt{\frac{Z_N^1}{Z_N^2}} + \sqrt{\frac{Z_N^2}{Z_N^1}})^2 N^C e^{-\frac{\beta N}{2}\inf_{m \in \mathcal{T}_N^1, \tilde{m} \in \mathcal{T}_N^2, m \sim \tilde{m}}[\mathcal{F}_N(m) + \mathcal{F}_N(\tilde{m})]} \\ \leq& N^C \frac{Z_N}{Z_N^1 Z_N^2} e^{-\frac{\beta N}{2}[\inf_{m \in \partial \mathcal{T}_1^N} \mathcal{F}_N(m) + \inf_{m \in \partial \mathcal{T}_2^N} \mathcal{F}_N(m)]}\end{aligned}$$

Using the results of Proposition 2.1 and Lemma 3.1, we therefore get that

$$\Lambda_1^N \leq N^C e^{-\beta N \Delta \mathcal{F} + \beta a |S_N|}$$

Lower bound for Λ_1^N*:* to use the same strategy to bound Λ_1^N as we did for L_1^N, we need some estimates on the eigenfunction ψ_1^N. This is the content of the next lemma: we choose for ψ_1^N the normalized eigenfunction corresponding to Λ_1^N such that $\mathcal{G}_N(\mathbb{I}_{\bar{T}_1^N}\psi_1^N) > 0$. This last condition uniquely determines ψ_1^N. (When N is big enough, Λ_1^N has multiplicity 1 as follows from our estimates of Λ_1^N and Λ_2^N.)

Lemma 4.1 *On the set* $|S_N| \leq 2\sqrt{N \log N}$*, we have*

$$\frac{1}{N} \log |\psi_1^N(m_1^N) - \sqrt{\frac{Z_N^2}{Z_N^1}}| \leq -C \tag{4.10}$$

and

$$\frac{1}{N} \log |\psi_1^N(m_2^N) + \sqrt{\frac{Z_N^1}{Z_N^2}}| \leq -C \tag{4.11}$$

This lemma will be proved later. We first proceed to the end of the proof of Theorem 2.7. Let γ_1^N be the path defined in Lemma 3.2. Define similarly a path γ_2^N in $\bar{T}_2^N$ from m_2^N to m_0^N such that $\sup_{x\in\gamma_2^N} \mathcal{F}_N(x) \leq \mathcal{F}(m_0) + C\log N/N$. Let γ^N be the path from m_1^N to m_2^N obtained by gluing together γ_1^N and γ_2^N. Say $\gamma^N = (x_0, ..., x_k)$. We therefore have

$$\sup_{x\in\gamma^N} \mathcal{F}_N(x) \leq \mathcal{F}(m_0) + C\frac{\log N}{N} \tag{4.12}$$

As in (4.8), we have

$$\begin{aligned}
&|\psi_1^N(m_1^N) - \psi_1^N(m_2^N)|^2 \\
&= |\sum_i \psi_1^N(x_i) - \psi_1^N(x_{i+1})|^2 \\
&\leq \sum_i |\psi_1^N(x_i) - \psi_1^N(x_{i+1})|^2 (\tilde{\mathcal{N}}_N(x_i, x_{i+1}))^{1/2} e^{-\frac{\beta N}{2}[\mathcal{F}_N(x_i)+\mathcal{F}_N(x_{i+1})]} ... \\
&... \sum_i (\tilde{\mathcal{N}}_N(x_i, x_{i+1}))^{-1/2} e^{\frac{\beta N}{2}[\mathcal{F}_N(x_i)+\mathcal{F}_N(x_{i+1})]} \qquad (4.13) \\
&\leq 2Z_N \mathcal{E}_N(\psi_1^N)\sqrt{\frac{N}{2}} N^2 e^{\beta N \sup_{x\in\gamma^N} \mathcal{F}_N(x)} \\
&\leq N^C \Lambda_1^N e^{\beta N \mathcal{F}(m_0) - \beta N \mathcal{F}(m_1) + \beta a |S_N|} \\
&= N^C \Lambda_1^N e^{\beta N \Delta\mathcal{F} + \beta a|S_N|}
\end{aligned}$$

¿From Lemma 4.1 and Proposition 2.1, we have

$$\begin{aligned}
|\psi_1^N(m_1^N) - \psi_1^N(m_2^N)|^2 &\geq \frac{(Z_N)^2}{Z_N^1 Z_N^2}(1 - \frac{\sqrt{Z_N^1 Z_N^2}}{Z_N} e^{-2CN})^2 \\
&\geq N^{-C} e^{2\beta a|S_N|}
\end{aligned}$$

Therefore

$$\Lambda_1^N \geq N^{-C} e^{-\beta N\Delta\mathcal{F} + \beta a|S_N|}$$

Proof of Lemma 4.1: : the proof relies on Proposition 2.6 and the fact that ϕ_1^N converges to 1. Let $0 < \alpha < \alpha' < \Delta\mathcal{F}$ and $t = \exp(\alpha'\beta N)$. Define T as in Proposition 2.6. Clearly $T \geq e^{\alpha\beta N}$ for N big enough. From (2.32) applied with the function $\phi(m) = \mathbb{I}(m \in T_1^N)$, we get that

$$|P_m[m^N(t) \in T_1^N] - \mathcal{G}_N(T_1^N) - \psi_1^N(m)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)e^{-\Lambda_1^N t}| \leq e^{-T} \tag{4.14}$$

In particular

$$\begin{aligned}\psi_1^N(m)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N) &\le e^{\Lambda_1^N t}(e^{-T}+P_m[m^N(t)\in T_1^N]-\mathcal{G}_N(T_1^N))\\ &\le e^{\Lambda_1^N t}(e^{-T}+1-\mathcal{G}_N(T_1^N))\\ &\le e^{\Lambda_1^N t}(e^{-T}+\mathcal{G}_N(\bar{T}_2^N))\\ &\le e^{\Lambda_1^N t}(e^{-T}+\frac{Z_N^2}{Z_N})\end{aligned}$$

Taking into account that $T\ge e^{\alpha\beta N}$, $\alpha'<\Delta\mathcal{F}$ and the estimates (2.38) for Λ_1^N and Proposition 2.1, we obtain

$$\psi_1^N(m)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)\le\frac{Z_N^2}{Z_N}(1+e^{-CN})$$

In particular, integrating over T_1^N,

$$\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)\le\frac{\sqrt{Z_N^1Z_N^2}}{Z_N}(1+e^{-CN})\tag{4.15}$$

and

$$\psi_1^N(m_1^N)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)\le\frac{Z_N^2}{Z_N}(1+e^{-CN})\tag{4.16}$$

¿From (4.14), we also get that, for $m\in T_1^N$,

$$\begin{aligned}\psi_1^N(m)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N) &\ge e^{\Lambda_1^N t}(-e^{-T}+P_m[m^N(t)\in T_1^N]-\mathcal{G}_N(T_1^N))\\ &\ge -e^{-T}+P_m[\tau_N>t]-\mathcal{G}_N(T_1^N)\\ &\ge -2e^{-T}+\phi_1^N(m)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t}-\mathcal{G}_N(T_1^N)\\ &= -2e^{-T}+\phi_1^N(m)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t}-1+\mathcal{G}_N(\bar{T}_2^N)\end{aligned}$$

where we used (2.31). In particular

$$\psi_1^N(m_1^N)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)\ge -2e^{-T}+\phi_1^N(m_1^N)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t}-1+\mathcal{G}_N(\bar{T}_2^N)\tag{4.17}$$

and, integrating over T_1^N,

$$\begin{aligned}\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)^2 \ge&\mathcal{G}_N(T_1^N)(-2e^{-T}+\frac{\mathcal{G}_N(\mathbb{I}_{T_1^N}\phi_1^N)}{\mathcal{G}_N(T_1^N)}\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t}-1+\mathcal{G}_N(\bar{T}_2^N))\\ =&\mathcal{G}_N(T_1^N)(-2e^{-T}+\frac{Z_N^1}{Z_N\mathcal{G}(T_1^N)}(\mathcal{G}_N^1(\phi_1^N))^2e^{-L_1^N t}-1+\mathcal{G}_N(\bar{T}_2^N))\end{aligned}\tag{4.18}$$

Use now (2.38) and (2.40), to deduce from (4.17) that

$$\psi_1^N(m_1^N)\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N) \geq \frac{Z_N^2}{Z_N}(1-e^{-CN}) \tag{4.19}$$

Use (2.38) and (2.39), to deduce from (4.18) that

$$\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N)^2 \geq \mathcal{G}_N(T_1^N)\frac{Z_N^2}{Z_N}(1-e^{-CN})$$

But

$$\mathcal{G}_N(T_1^N) = \frac{Z_1^N - z_1^N}{Z_N} \geq \frac{Z_N^1}{Z_N}(1-e^{-CN})$$

by Proposition 2.1. Therefore

$$\mathcal{G}_N(\mathbb{I}_{T_1^N}\psi_1^N) \geq \frac{\sqrt{Z_N^1 Z_N^2}}{Z_N}(1-e^{-CN}) \tag{4.20}$$

One can now solve equations (4.15) (4.16) (4.19) and (4.20) to conclude the proof of (4.10). The proof for (4.11) is identical. ■

Proof of Proposition 2.3: : for any continuous function ϕ bounded by 1, we have

$$\sqrt{\mathcal{G}_N\big(|E_.[\phi(m_N(t_N))] - \mathcal{G}_N(\phi)|^2\big)} \leq e^{-\Lambda_1^N t_N}\sqrt{\mathcal{G}_N(\phi^2)} \leq e^{-t_N\Lambda_1^N} \tag{4.21}$$

Since, for any $m \in \mathcal{M}_N$, $|\mathcal{F}_N(m)| \leq C$, we have, for any function ψ, $\psi(m)^2 \leq e^{CN}\mathcal{G}_N(\psi^2)$. From Theorem 2.7, we deduce that

$$\Lambda_1^N \geq e^{-\beta N\Delta\mathcal{F}+\beta a|S_N|}N^{-C}$$

hence

$$\Lambda_1^N t_N \geq N^{K-C}$$

Therefore (4.21) implies that

$$|E_m[\phi(m_N(t_N))] - \mathcal{G}_N(\phi)| \leq e^{-N^{K-C}}$$

(2.25) follows by choosing $K > C$. ■

V. The averaged dynamics

We now turn to the proof of Theorem 2.4.

Proof of (2.26): : Let $\alpha \in I\!R$ and $t_N = \exp(\beta N \Delta\mathcal{F} + \alpha\sqrt{N})$. We have

$$
\begin{aligned}
&P_{m^N}[N^{-1/2}(\log \tau_N - \beta N \Delta \mathcal{F}) \geq \alpha] \\
=&P_{m^N}[\tau_N \geq t_N] \\
=&P_{m^N}[\tau_N \geq t_N] - \phi_1^N(m^N)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t_N} + (\phi_1^N(m^N)\mathcal{G}_N^1(\phi_1^N) - 1)e^{-L_1^N t_N} + e^{-L_1^N t_N}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&I\!P_{m^N}[\tau_N \geq t_N] \\
=&I\!P_{m^N}[\tau_N \geq t_N; |S_N| \geq 2\sqrt{N \log N}] \\
&+ Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}(P_{m^N}[\tau_N \geq t_N] - \phi_1^N(m^N)\mathcal{G}_N^1(\phi_1^N)e^{-L_1^N t_N})] \\
&+ Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}(\phi_1^N(m^N)\mathcal{G}_N^1(\phi_1^N) - 1)e^{-L_1^N t_N}] + Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}]
\end{aligned}
\tag{5.1}
$$

In (5.1), the first term tends to 0 since $Q[|S_N| \geq 2\sqrt{N \log N}] \leq 2/N^2$. By Proposition 2.6, the second term is bounded by $\exp(-T)$ with $T = t_N \exp(-K\sqrt{N}\log N) - K \log N \to +\infty$. Therefore the second term also tends to 0. ¿From (2.39) and (2.40), it follows that the third term is bounded by $\exp(-CN)$ and therefore tends to 0. Thus

$$
I\!P_{m^N}[\tau_N \geq t_N] - Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}] \to 0
$$

For any $\epsilon > 0$, write

$$
\begin{aligned}
&Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}] \\
=&Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}1\!\!1_{\alpha\sqrt{N} - \beta a S_N \leq -\epsilon\sqrt{N}}] \\
+&Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}1\!\!1_{\alpha\sqrt{N} - \beta a S_N \geq \epsilon\sqrt{N}}] \\
+&Q[1\!\!1_{|S_N| \leq 2\sqrt{N\log N}}e^{-L_1^N t_N}1\!\!1_{-\epsilon\sqrt{N} \leq \alpha\sqrt{N} - \beta a S_N \leq +\epsilon\sqrt{N}}]
\end{aligned}
\tag{5.2}
$$

Note that, from Theorem 2.7, on the set where $|S_N| \leq 2\sqrt{N \log N}$ and $\alpha\sqrt{N} - \beta a S_N \leq -\epsilon\sqrt{N}$, we have $|\exp(-L_1^N t_N) - 1| \leq N^C \exp(-\epsilon\sqrt{N}) \to 0$. Therefore the first term in (5.2) is close to $Q[|S_N| \leq 2\sqrt{N \log N}; \alpha\sqrt{N} - \beta a S_N \leq -\epsilon\sqrt{N}]$. On the set where $|S_N| \leq 2\sqrt{N \log N}$ and $\alpha\sqrt{N} - \beta a S_N \geq \epsilon\sqrt{N}$ we have $L_1^N t_N \geq N^{-C}\exp(\epsilon\sqrt{N}) \to +\infty$.

Therefore the second term in (5.2) converges to 0. The third term in (5.2) is bounded by $Q[-\epsilon\sqrt{N} \le \alpha\sqrt{N} - \beta a S_N \le +\epsilon\sqrt{N}]$. Thus so far we have proved that

$$\begin{aligned}&\limsup |\mathbb{P}_{m^N}[\tau_N \ge t_N] - Q[|S_N| \le 2\sqrt{N\log N}; \alpha\sqrt{N} - \beta a S_N \le -\epsilon\sqrt{N}]|\\ \le &\limsup Q[-\epsilon\sqrt{N} \le \alpha\sqrt{N} - \beta a S_N \le +\epsilon\sqrt{N}]\end{aligned}$$

Apply the central limit theorem to S_N to deduce that

$$\limsup |\mathbb{P}_{m^N}[\tau_N \ge t_N] - Q[\beta a\mathcal{N} \ge \alpha + \epsilon]| \le Q[\alpha - \epsilon \le \beta a\mathcal{N} \le \alpha + \epsilon] \tag{5.3}$$

Since (5.3) holds for any $\epsilon > 0$, we also have

$$\mathbb{P}_{m^N}[\tau_N \ge t_N] \to Q[\beta a\mathcal{N} \ge \alpha]$$

■

Proof of (2.27): : let $0 < \alpha' < \Delta\mathcal{F}$ and define $s = \exp(\beta\alpha' N)$ and

$$S = s\exp(-K\sqrt{N}\log N) - K\log N$$

where K is the constant in Proposition 2.6. Then S tends to $+\infty$ and, from (2.38), we have

$$\Lambda_1^N s \le N^C e^{(\alpha' - \Delta\mathcal{F})\beta N + C\sqrt{N}\log N} \to 0$$

on the set $|S_N| \le 2\sqrt{N\log N}$. ϕ being bounded by 1, we deduce from Proposition 2.6 that, on the set $|S_N| \le 2\sqrt{N\log N}$, we have, for $N \ge N_0$,

$$|\psi_1^N(m^N)\mathcal{G}_N(\phi\psi_1^N)| \le C \tag{5.4}$$

Proceeeding as in the proof of (2.26) and using (5.4), it is easy to see that

$$\begin{aligned}&\limsup |\mathbb{E}_{m^N}[\phi(m_N(t_N))]\\ &- Q[\mathbb{I}_{|S_N| \le 2\sqrt{N\log N}}\left(\mathcal{G}_N(\phi) + \psi_1^N(m^N)\mathcal{G}_N(\phi\psi_1^N)e^{-\Lambda_1^N t_N}\right)]| = 0\end{aligned} \tag{5.5}$$

Using Theorem 2.7 and (5.4), we therefore have

$$\begin{aligned}&\limsup |\mathbb{E}_{m^N}[\phi(m_N(t_N))]\\ &- Q[\mathbb{I}_{|S_N| \le 2\sqrt{N\log N}}\left(\mathcal{G}_N(\phi) + \psi_1^N(m^N)\mathcal{G}_N(\phi\psi_1^N)\mathbb{I}_{\beta a|S_N| + \alpha\sqrt{N} \le -\epsilon\sqrt{N}}\right)]|\\ \le &C\limsup Q[-\epsilon\sqrt{N} \le \beta a|S_N| + \alpha\sqrt{N} \le \epsilon\sqrt{N}]\end{aligned} \tag{5.6}$$

¿From [8] formula (5.25), we know that, for almost all realization of h,

$$\psi_1^N(m^N)\mathcal{G}_N(\phi\psi_1^N) + \mathcal{G}_N(\phi) - \phi(m_1) \to 0$$

Using the bounded convergence Lemma (which is justified by (5.4)), we deduce from (5.6) that

$$\begin{aligned} &\limsup \left| I\!E_{m^N}[\phi(m_N(t_N))] \right. \\ &\left. - Q[\mathcal{G}_N(\phi) + (\phi(m_1) - \mathcal{G}_N(\phi))\mathbb{I}_{\beta a|S_N| + \alpha\sqrt{N} \le -\epsilon\sqrt{N}}] \right| \\ \le & C \limsup Q[-\epsilon\sqrt{N} \le \beta a|S_N| + \alpha\sqrt{N} \le \epsilon\sqrt{N}] \end{aligned} \tag{5.7}$$

Let A_N be the set $\beta a|S_N| + \alpha\sqrt{N} \le -\epsilon\sqrt{N}$. ¿From Proposition 2.1, we have

$$Q[\mathcal{G}_N(\phi)\mathbb{I}_{A_N}] - \phi(m_1)Q[\alpha_N\mathbb{I}_{A_N}] - \phi(m_2)Q[(1-\alpha_N)\mathbb{I}_{A_N}] \to 0$$

By symmetry

$$Q[\alpha_N\mathbb{I}_{A_N}] = Q[(1-\alpha_N)\mathbb{I}_{A_N}] = \frac{1}{2}Q[A_N]$$

Therefore

$$Q[\mathcal{G}_N(\phi)\mathbb{I}_{\beta a|S_N| + \alpha\sqrt{N} \le -\epsilon\sqrt{N}}] - \frac{1}{2}(\phi(m_1) + \phi(m_2))Q[\beta a|S_N| + \alpha\sqrt{N} \le -\epsilon\sqrt{N}] \to 0$$

Use now the central limit theorem for S_N to conclude that

$$\begin{aligned} &\limsup \left| I\!E_{m^N}[\phi(m_N(t_N))] \right. \\ &\left. - \left(\frac{1}{2}(\phi(m_1) + \phi(m_2)) + \frac{1}{2}(\phi(m_1) - \phi(m_2))Q[\beta a|\mathcal{N}| \le -\alpha - \epsilon]\right) \right| \\ &\le Q[-\epsilon - \alpha \le \beta a|\mathcal{N}| \le -\alpha + \epsilon] \end{aligned} \tag{5.8}$$

And since (5.8) is true for all $\epsilon > 0$, we have

$$I\!E_{m^N}[\phi(m_N(t_N))] \to \frac{1}{2}(\phi(m_1) + \phi(m_2)) + \frac{1}{2}(\phi(m_1) - \phi(m_2))Q[\beta a|\mathcal{N}| \le -\alpha]$$

■

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