UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

BI-LIPSCHITZ $\mathcal{G}$-TRIVIALITY AND NEWTON POLYHEDRA,
$\mathcal{G} = \mathcal{R},\ \mathcal{C},\ \mathcal{K},\ \mathcal{R}_V,\ \mathcal{C}_V,\ \mathcal{K}_V$

J. C. F. COSTA
M. J. SAIA
C. H. SOARES JÚNIOR

Nº 322

NOTAS DO ICMC

SÉRIE MATEMÁTICA

São Carlos – SP
Nov./2009

BI-LIPSCHITZ $\mathcal{G}$-TRIVIALITY AND NEWTON POLYHEDRA, $\mathcal{G}=\mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$, $\mathcal{R}_V$, $\mathcal{C}_V$, $\mathcal{K}_V$

J. C. F. COSTA M. J. SAIA C. H. SOARES JÚNIOR

Keywords: bi-Lipschitz determinacy, Newton filtration, controlled vector fields.

A.M.S. Mathematics Subject classification: 58C27

Resumo

Neste artigo são mostradas estimativas para a bi-Lipschitz $\mathcal{G}$-trivialidade, $\mathcal{G}$ é um dos subgrupos de Mather $\mathcal{C}$ ou $\mathcal{K}$, em familias de germes de aplicações satisfazendo uma condiç ão de Lojasiewicz em dois casos, para a classe dos germes quase homogêneos e tambem no contexto dos germes que são Newton não degenerados. Tambem consideramos a trivialidade bi-Lipschitz em familias de aplicações definidas em variedades analíticas, neste caso são mostradas estimativas para a bi-Lipschitz $\mathcal{G}_V$-trivialidade, com $\mathcal{G}=\mathcal{R},\mathcal{C}$ ou $\mathcal{K}$ no caso quase homogêneo, ou em outras palavras, quando o germede aplicação e a variedade são quase homogenoes de mesmo tipo. Estes resultados completam os resultados decritos em [13], [8], [15], [9] and [11]. Na últimaseção são apresentadas tabelas com todas as estimativas para a C^ℓ-$\mathcal{G}$ e bi-Lipschitz $\mathcal{G}$ trivialidade com $\mathcal{G}=\mathcal{R},\mathcal{C}$ or $\mathcal{K}$ e $\ell=0,1,\ldots,\infty$, nos dois casos, o quase homogeneo e o caso Newton não degenerado.

Abstract

Let $\mathcal{G}$ be one of Mather's groups $\mathcal{C}$ or $\mathcal{K}$. In this work we provide estimates for the bi-Lipschitz $\mathcal{G}$-triviality of a family of map germs satisfying a Lojasiewicz condition in two cases: for the class of weighted homogeneous map germs and then we go to the more general setup, in the Newton Polyhedron context, or the class of non-degenerate map germs with respect to some Newton polyhedra. We also consider the bi-Lipschitz triviality for families of map germs defined on analytic varieties, and in this case we give estimates for the bi-Lipschitz $\mathcal{G}_V$-triviality where $\mathcal{G}=\mathcal{R},\mathcal{C}$ or $\mathcal{K}$ in the weighted homogenous case, or in other words, when the map germ and the analytic variety are weighted homogeneous. These results complete the results contained in [13], [8], [15], [9] and [11]. In the last section, we present a table with all estimates for the C^ℓ-$\mathcal{G}$ and bi-Lipschitz $\mathcal{G}$ triviality, with $\mathcal{G}=\mathcal{R},\mathcal{C}$ or $\mathcal{K}$ and $\ell=0,1,\ldots,\infty$, for both cases: the weighted homogeneous and the Newton non-degenerate case.

This project was partially supported by CAPES-Projeto PROCAD. The first named author is partially supported by FAPESP-Grant 2007/01274-6. The second named author is partially supported by CNPq - Grant 300880/2003-6. The third named author is partially supported by CNPq - Grant 475147/2007-3 and by FUNCAP - Grant 9941/06.

1. Introduction

A basic problem in Singularity Theory is the local classification of mappings module diffeomorphisms. However, this problem presents very rigidity. Then seems natural to investigate classification of mappings by isomorphisms weaker than diffeomorphisms. In this paper we are interested in the study of mappings up to bi-Lipschitz maps.

A mapping $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* if there exists a constant $c > 0$ such that

$$\| \phi(x) - \phi(y) \| \leq c \| x - y \|, \quad \forall x, y \in U.$$

When $n = p$ and ϕ has a Lipschitz inverse, we say that ϕ is *bi-Lipschitz.*

Equivalence relations defined in terms of bi-Lipschitz maps are important tools in the study of equisingularity of mappings and sets from the metric point of view. Metric classification refers to classification up to bi-Lipschitz maps.

Note that the notion of bi-Lipschitz equivalence is stronger than the topological equivalence and weaker than any other C^ℓ-equivalence, with $\ell \geq 1$.

We can define Lipschitz versions of the equivalence relations involving the classical Mather's groups $\mathcal{R}, \mathcal{C}$ and $\mathcal{K}$. In fact, two map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are bi-Lipschitz $\mathcal{R}$-equivalent if there exists a germ of bi-Lipschitz homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $g = f \circ h^{-1}$. Two map germs f and g are bi-Lipschitz $\mathcal{K}$-equivalent if there exists a pair of bi-Lipschitz homeomorphisms (h, H), with $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $H = (H_1, H_2) : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ satisfying the conditions: $H_1 = h$, $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and $H \circ (Id, f) = (Id, g) \circ h$, where Id denotes the germ of the identity in $\mathbb{R}^n$. When $h = Id$ we call the bi-Lipschitz $\mathcal{K}$-equivalence by bi-Lipschitz $\mathcal{C}$-equivalence. In this way, the bi-Lipschitz $\mathcal{K}$-equivalence can be seen as the semi direct product between the bi-Lipschitz $\mathcal{R}$-equivalence and the bi-Lipschitz $\mathcal{C}$-equivalence.

In this article we are interested in the bi-Lipschitz-$\mathcal{G}$-triviality, with $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$, or in other words, the bi-Lipschitz $\mathcal{G}$-equivalence along of a family of map germs $F_t(x)$ with $F_0(x) = f$ satisfying a Lojasiewicz condition. The bi-Lipschitz $\mathcal{R}$-triviality of families of germs was studied recently by Fernandes and Ruas in [8] for the weighted homogenous case and by Fernandes and Soares Jr. in [9] for the Newton non-degenerate case. However, with respect to bi-Lipschitz triviality with respect to the groups $\mathcal{C}$ and $\mathcal{K}$ there are no results in the literature.

The main tool to obtain these triviality conditions is an application of a Thom-Levine type theorem with the construction of controlled vector fields in the presence of a suitable Lojasiewicz condition. This method is often used to obtain several triviality conditions and it is based on the fixing of an appropriate filtration in the space of germs, which depends of the initial germ f, to prove the triviality. If we consider a deformation $F_t(x)$ of f, the general form of the result is given by a filtration condition involving $\frac{\partial F_t}{\partial t}$ and the deformation F. The triviality is proven by solving localized equations of type $-\rho \frac{\partial F}{\partial t} = \xi(f)$ where ρ is a control function which depends of the chosen filtration and ξ is a smooth germ of vector field in the $\mathcal{G}$-tangent space of the germ. If the deformation F satisfies the filtration condition, we obtain the appropriate condition for the vector field $\rho^{-1}\xi$.

For instance in the C^ℓ-$\mathcal{G}$-actions, where $\mathcal{G}$ is one of the Mather's groups we see, among others, the works of Damon [7] and Kuo [10] for the characterization of topological classification and for the C^0-$\mathcal{G}$-triviality of map germs. More recently Abderrahmane in [1] studies the C^0-$\mathcal{R}$-triviality for function germs which satisfy a non-degeneracy condition which depends of a convenient Newton filtration. In the case of the C^ℓ-triviality for $0 < \ell < \infty$, we know the work of Bromberg and Lopes de Medrano [3] which gives estimates for the C^ℓ-$\mathcal{R}$-triviality in families of real functions germs of class at least $C^{\ell+1}$ in the weighted homogeneous case. Estimates for the C^ℓ-$\mathcal{G}$-triviality, where $0 \leq \ell < \infty$ and $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ appear in [13] for weighted homogeneous map germs and in [15] for map germs which satisfy a Newton non-degeneracy condition.

By now there is a gap in these results, since there is no study of the bi-Lipschitz case for the groups $\mathcal{C}$ and $\mathcal{K}$, which is the main purpose of this article. Here we complete this gap giving estimates for the bi-Lipschitz-$\mathcal{G}$-triviality, for $\mathcal{G} = \mathcal{K}$ or $\mathcal{C}$ in both cases, first for the class of weighted homogeneous map germs and then we go to the more general setup, the Newton Polyhedron context, or in other words, the class of non-degenerate map germs with respect to some Newton polyhedra.

Concerning the $\mathcal{G}_V$-triviality for families of map germs defined on analytic varieties V, we see estimates for the C^ℓ-$\mathcal{G}_V$-triviality with $\ell > 1$ and $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ in [11], but again the bi-Lipschitz case is not treated.

We also consider this case in this article and give estimates for the bi-Lipschitz-$\mathcal{G}_V$-triviality where $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ in the weighted homogenous case, that is, when the map germ and the analytic variety are weighted homogeneous.

Moreover, in the last section of this paper we compare our results with all other known results which appear as a consequence of the use of the method of construction of controlled vector fields. We show a table with all known results. As expected we conclude that the estimates depend only of the control functions obtained in each case and of the type of equivalence which we are looking.

2. Bi-Lipschitz $\mathcal{G}$-triviality, $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$

We denote by $\mathcal{E}_n$ the local ring of real analytic function germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ and its maximal ideal by $\mathcal{M}_n$, $\mathcal{E}_{n,p}$ denotes the $\mathcal{E}_n$-module of analytic map germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}^p$.

From now on we shall consider map germs in $\mathcal{E}_{n,p}$ with the additional property that $f(0) = 0$, these germs are denoted $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$

For any map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ in $\mathcal{E}_{n,p}$ we call $I_{\mathcal{R}}f$ the ideal of $\mathcal{E}_n$ generated by the $p \times p$ minors of the jacobian matrix of f; by $I_{\mathcal{C}}f$ the ideal generated by the coordinate functions of f, and by $I_{\mathcal{K}}f$ the ideal $I_{\mathcal{R}}f + I_{\mathcal{C}}f$.

Let $N_{\mathcal{C}}\, f(x) = |f(x)|^2$, $N_{\mathcal{R}}\, f(x) = |\mathrm{d}f(x)|^2 = \sum_j M_j^2$, where M_j are the generators of $I_{\mathcal{R}}f$ and $N_{\mathcal{K}}\, f = N_{\mathcal{R}}\, f + N_{\mathcal{C}}\, f$. For $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ we say that $N_{\mathcal{G}}\, f$ satisfies a Lojasiewicz condition if there exist constants $c > 0$ and $\alpha > 0$ with $N_{\mathcal{G}}\, f(x) \geq c|x|^\alpha$.

2.1. Weighted Homogeneous Case.

Definition 2.1. *A map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ in $\mathcal{E}_{n,p}$ is* weighted homogeneous of type $(r_1, \dots, r_n; d_1, \dots, d_p)$ *with r_i, $d_j \in \mathbb{Q}^+$, if for all $\lambda \in \mathbb{R} - \{0\}$:*

$$f(\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n) = (\lambda^{d_1} f_1(x), \dots, \lambda^{d_p} f_p(x)).$$

We remember that any weighted homogenous analytic map germ f in $\mathcal{E}_{n,p}$ is always a polynomial.

In this subsection we shall fix an n-tuple of weights $\overline{r} = (r_1, \dots, r_n)$ in such a way that we can define an appropriate filtration in the ring $\mathcal{E}_n$ which is suitable for the weighted homogeneous case.

Definition 2.2. *Given $\overline{r} = (r_1, \dots, r_n)$, for any monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, define $\mathrm{fil}_{\overline{r}}(x^\alpha) = \sum_{i=1}^n \alpha_i r_i$. We define the $\overline{r}$-filtration in $\mathcal{E}_n$ via the function defined for any f in $\mathcal{E}_n$ by*

$$\mathrm{fil}_{\overline{r}}(f) = \inf_\alpha \left\{ \mathrm{fil}_{\overline{r}}(x^\alpha) \mid \left(\frac{\partial^\alpha f}{\partial x^\alpha} \right)(0) \neq 0 \right\}.$$

This definition can be extended to $\mathcal{E}_{n+s}$, the ring of s-parameter families of germs in n-variables, by defining $\mathrm{fil}_{\overline{r}}(x^\alpha t^\beta) = \mathrm{fil}_{\overline{r}}(x^\alpha)$.

For any map germ $f = (f_1, \dots, f_p)$ in $\mathcal{E}_{n,p}$, we call $\mathrm{fil}_{\overline{r}}(f) = (d_1, \dots, d_p)$, where $d_i = \mathrm{fil}_{\overline{r}}(f_i)$, for each $i = 1, \dots, p$. As we are working with a fixed sets of weights $\overline{r} = (r_1, \dots, r_n)$, from now on we shall omit the subindex $\mathrm{fil}_{\overline{r}}$ and call $\mathrm{fil}_{\overline{r}}(f)$ =$\mathrm{fil}(f)$.

Analogously to the previous definitions for the functions $N_{\mathcal{G}}\, f$, for any polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ we define the weighted homogeneous versions of them. Call $N_{\mathcal{R}}^\star f := \sum_I M_I^{2k_I}$, where M_I denotes a $p \times p$ minor of the jacobian matrix of f with $I = \{i_1, \dots, i_p\}$ and $1 \leq i_1 \leq \dots \leq i_p \leq n$, $k_I = \dfrac{k}{s_I}$ with $s_I = \mathrm{fil}(M_I)$ and $k = lcm\,\{s_I\}$; and $N_{\mathcal{C}}^\star f := \sum_{i=1}^p (f_i)^{2\beta_i}$, where $\beta_i = \frac{lcm\,\{\mathrm{fil}\,(f_j), j=1,\dots,p\}}{\mathrm{fil}\,(f_i)}$.

Let $a, b \in \mathbb{N}$ be prime numbers with $gcd(a, b) = 1$ such that $\mathrm{fil}\,(N_{\mathcal{R}}^\star f)^a = \mathrm{fil}\,(N_{\mathcal{C}}^\star f)^b$.

We call $N_{\mathcal{K}}^\star f := (N_{\mathcal{R}}^\star f)^a + (N_{\mathcal{C}}^\star f)^b$.

In analogous way, for any family of polynomial map germs $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $t \in [0, 1]$, we call $N_{\mathcal{R}}^\star f_t := \sum_I M_{t_I}^{2k_I}$, $N_{\mathcal{C}}^\star f_t := \sum_{i=1}^p (f_{t_i})^{2\beta_i}$ and for the a, b, k_I and β_i given before we call

$$N_{\mathcal{K}}^\star f_t := (N_{\mathcal{R}}^\star f_t)^a + (N_{\mathcal{C}}^\star f_t)^b.$$

Then we have the following theorem:

Theorem 2.3. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a weighted homogeneous polynomial map germ of type $(r_1, \dots, r_n; d_1, \dots, d_p)$ with $r_1 \leq \dots \leq r_n$, $d_1 \leq \dots \leq d_p$, satisfying a Lojasiewcz condition $N_{\mathcal{K}}^\star f_t(x) \geq c|x|^\alpha$, for constants c and α. Then, small deformations of f defined by $f_t = f + t\Theta$, $\Theta = (\Theta_1, \dots, \Theta_p)$, with $\mathrm{fil}\,(\Theta_i) \geq d_p + r_n - r_1$ are bi-Lipschitz $\mathcal{K}$-trivial.*

Proof. Let $B = lcm\,\{\mathrm{fil}\,(f_j),\ j = 1, \dots, p\}$ and $k = lcm\,\{s_I\}$, then we get

$$\mathrm{fil}\,(N_{\mathcal{K}}^\star f_t) = \mathrm{fil}\,(N_{\mathcal{R}}^\star f_t)^a = \mathrm{fil}\,(N_{\mathcal{C}}^\star f_t)^b = 2ka = 2Bb \text{ and}$$

$$N_{\mathcal{K}}^{\star} f_t \frac{\partial f_t}{\partial t} = ((N_{\mathcal{R}}^{\star} f_t)^a + (N_{\mathcal{C}}^{\star} f_t)^b) \frac{\partial f_t}{\partial t} =$$

$$= (N_{\mathcal{R}}^{\star} f_t)^a \frac{\partial f_t}{\partial t} + (N_{\mathcal{C}}^{\star} f_t)^b \frac{\partial f_t}{\partial t} =$$
$$= (N_{\mathcal{R}}^{\star} f_t)^{a-1} N_{\mathcal{R}}^{\star} f_t \frac{\partial f_t}{\partial t} + (N_{\mathcal{C}}^{\star} f_t)^{b-1} N_{\mathcal{C}}^{\star} f_t \frac{\partial f_t}{\partial t} =$$
$$= (N_{\mathcal{R}}^{\star} f_t)^{a-1} \, \mathrm{d}f_t(W_{\mathcal{R}}) + (N_{\mathcal{C}}^{\star} f_t)^{b-1} \textstyle\sum_{i=1}^{p} L_i f_{ti},$$

here $W_{\mathcal{R}} = \sum_I M_I^{2k_I-1} W_I$, $W_I = \sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i}$ and $\begin{cases} w_i = 0, & i \notin I \\ w_{im} = \sum_{j=1}^{p} N_{ji_m} (\frac{\partial f_t}{\partial t})_j, & i_m \in I, \end{cases}$

where N_{ji_m} is the $(p-1) \times (p-1)$ minor cofactor of $\dfrac{\partial f_j}{\partial x_{i_m}}$ in the differential $\mathrm{d}f$ of f and $L_i = f_{ti}^{2\beta_i - 1} (\frac{\partial f_t}{\partial t})$, with $\beta_i = \frac{lcm\, fil\,(f_j)}{fil\,(f_i)} = \frac{B}{fil\,(f_i)}$, for $j = 1, \ldots, p$ and $i = 1, \ldots, p$.

Then, $\quad N_{\mathcal{K}}^{\star} f_t \frac{\partial f_t}{\partial t} = \mathrm{d}f_t((N_{\mathcal{R}}^{\star} f_t)^{a-1} W_{\mathcal{R}}) + \sum_{i=1}^{p} (N_{\mathcal{C}}^{\star} f_t)^{b-1} f_{ti}^{2\beta_i - 1} \left(\frac{\partial f_t}{\partial t}\right) f_{ti}.$

Dividing the above expression by $N_{\mathcal{K}}^{\star} f_t$ we obtain:

$$\frac{\partial f_t}{\partial t} = \mathrm{d}f_t \left(\frac{(N_{\mathcal{R}}^{\star} f_t)^{a-1} W_{\mathcal{R}}}{N_{\mathcal{K}}^{\star} f_t} \right) + \sum_{i=1}^{p} \frac{(N_{\mathcal{C}}^{\star} f_t)^{b-1} f_{ti}^{2\beta_i - 1} (\frac{\partial f_t}{\partial t}) f_{ti}}{N_{\mathcal{K}}^{\star} f_t}.$$

Now, first we define the vector field with respect to the group $\mathcal{R}$: $\xi = \dfrac{(N_{\mathcal{R}}^{\star} f_t)^{a-1} W_{\mathcal{R}}}{N_{\mathcal{K}}^{\star} f_t}$, call $\eta_j = \sum_{i=1}^{p} \frac{(N_{\mathcal{C}}^{\star} f_t)^{b-1} f_{ti}^{2\beta_i - 1} \Theta_j y_i}{N_{\mathcal{K}}^{\star} f_t}$ with $\eta_{ji} = \frac{(N_{\mathcal{C}}^{\star} f_t)^{b-1} f_{ti}^{2\beta_i - 1} \Theta_j y_i}{N_{\mathcal{K}}^{\star} f_t}$ and define the vector field with respect to the group $\mathcal{C}$: $\eta(x, y, t) = (\eta_1, \ldots, \eta_p)(x, y, t)$.

Now we show that the vector field ξ is Lipschitz. In fact observe that

$$\mathrm{fil}\,((N_{\mathcal{R}}^{\star} f_t)^{a-1} W_{\mathcal{R}}) = \mathrm{fil}\,(N_{\mathcal{R}}^{\star} f_t)^{a-1} + \mathrm{fil}\,(W_{\mathcal{R}})$$

$$\geq (a-1)\, \mathrm{fil}\,((N_{\mathcal{R}}^{\star} f_t)) + \min\{\mathrm{fil}\,(M_I^{2k_I - 1}) + \mathrm{fil}\,(W_I)\}$$

$$\geq (a-1)2k + \min\{(2k_I - 1)\mathrm{fil}\,(M_I) + \mathrm{fil}\,(N_{ji_m}) + \mathrm{fil}\,(\tfrac{\partial f_t}{\partial t})_j\}$$

$$\geq 2ka - 2k + \min\{(2\tfrac{k}{\mathrm{fil}\,(M_I)} - 1)\, \mathrm{fil}\,(M_I) + \mathrm{fil}\,(N_{ji_m}) + \mathrm{fil}\,(\Theta_j)\}$$

$$\geq 2ka - 2k + \min\{2k - \mathrm{fil}\,(M_I) + [\mathrm{fil}\,(M_I) - \mathrm{fil}\,(\tfrac{\partial f_j}{\partial x_{i_m}})] + \mathrm{fil}\,(\Theta_j)\}$$

$$\geq 2ka - 2k + \min\{2k - \mathrm{fil}\,(\tfrac{\partial f_j}{\partial x_{i_m}}) + \mathrm{fil}\,(\Theta_j)\}$$

$$\geq 2ka - 2k + \min\{2k - [\mathrm{fil}\,(f_j) - r_j] + \mathrm{fil}\,(\Theta_j)\}$$

$$\geq 2ka - 2k + \min\{2k - d_j + r_j + \mathrm{fil}\,(\Theta_j)\}$$

$$\geq 2ka - d_p + r_1 + \mathrm{fil}\,(\Theta_j)$$

$$\geq 2ka + r_n.$$

Moreover, $\mathrm{fil}\,(N^\star_{\mathcal{K}}\, f_t) = \mathrm{fil}\,((N^\star_{\mathcal{R}}\, f_t)^a) = 2ka$.

Then, as the filtration of the numerator of ξ is greater than or equal to $\mathrm{fil}\,(N^\star_{\mathcal{K}}\, f_t) + r_n$, it follows by the Lemma 3.2. of [8] that the vector field ξ is Lipschitz.

On the other side, for the vector field η we have:

$$\mathrm{fil}\,((N^\star_{\mathcal{C}}\, f_t)^{b-1} f_{ti}^{2\beta_i-1}(\tfrac{\partial f_t}{\partial t})) \geq (b-1)2B + (2\beta_i - 1)\,\mathrm{fil}\,(f_{ti}) + \mathrm{fil}\,(\Theta_j)$$

$$\geq 2Bb - 2B + 2\tfrac{B}{\mathrm{fil}\,(f_{ti})}\mathrm{fil}\,(f_{ti}) - \mathrm{fil}\,(f_{ti}) + \mathrm{fil}\,(\Theta_j)$$

$$\geq 2Bb - d_p + (d_p + r_n - r_1) = 2Bb + r_n - r_1,$$

and $\mathrm{fil}\,(N^\star_{\mathcal{K}}\, f_t) = \mathrm{fil}\,((N^\star_{\mathcal{C}}\, f_t)^b) = 2Bb$. Therefore we can not apply the Lemma 3.2. of [8] and we do not know if the vector field η is Lipschitz.

However, we can modify η in order to obtain a Lipschitz vector field $\tilde{\eta}$. For this we use the bump function, as done in the Lemma 4 of [13], which we recover here.

Call "weighted homogeneous standard control", the function:

$$\rho_d(x) = x_1^{2w_1} + \cdots + x_n^{2w_n}$$

in such a way that $\rho_d(x)$ is weighted homogeneous of type $(r_1, \ldots, r_n; 2d)$.

We remark that as $d_i \geq d_1$ for all $i = 1, \ldots, p$, we have that there exists a constant c such that $|f_{ti}(x)|^2 \leq c\rho_{d_1}(x)$ for all $i = 1, \ldots, p$.

Lemma 2.4. *([13]) Let c be such constant, V and U be neighborhoods of the region $|y| < c\rho_{d_1}^{\frac{1}{2}}(x)$ in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0,0,t\}$,*

$$V = \{(x,y,t) \mid |y| \leq c_1 \rho_{d_1}^{\frac{1}{2}}(x), \text{ with } c_1 > c\}$$

and U is chosen in such a way that $U \subset \overline{U} \subset V$. Then, there exists a conic bump function $p : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ such that $p_{|\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0,0,t\}}$ is smooth and

$$\begin{array}{ll} p(x,y,t) = 1, & (x,y,t) \in \overline{U} \\ p(x,y,t) = 0, & (x,y,t) \in V^c \\ 0 \leq p(x,y,t) \leq 1, & (x,y,t) \in V - \overline{U} \\ p(0,0,t) = 0, & \forall t \end{array}$$

Using the bump function p of Lemma 2.4, we define the vector field

$$\tilde{\eta}(x,y,t) = p(x,y,t)\eta(x,y,t).$$

As $\tilde{\eta}$ coincides with η in a conic neighborhood of the graph of f_t, the equation

$$\frac{\partial f_t}{\partial t} = \mathrm{d}f_t(\xi) + (p\eta) \circ f_t$$

also holds.

Moreover, we can redefine η such that $\tilde{\eta}$ is zero outside of conic neighborhood of the graph of f_t. With the bump function we can write

$$\tilde{\eta}_{ji} = \frac{(N^\star_{\mathcal{C}}\, f_t)^{b-1} f_{ti}^{2\beta_j - 1}\Theta_i\, p\, y_i}{N^\star_{\mathcal{K}}\, f_t}. \tag{1}$$

Then, when we compose $\eta \circ f_t$ we obtain that the filtration of the numerator of each $\tilde{\eta}_{ij}(f_{t_j})$ is greater than or equal to $fil\ (N^{\star}_{\mathcal{K}} f_t) + r_n$.

Now we can apply the Lemma 3.2 of [8] to conclude that $\tilde{\eta}_{ji}$ is Lipschitz and, hence, $\tilde{\eta}$ is a Lipschitz vector field in a conic neighbourhood of the graph of f_t. As $\tilde{\eta} = p\eta$ is zero out of conic neighbourhood of the graph of f_t, it follows that $\tilde{\eta}$ is a Lipschitz vector field, well defined in all its domain.

By integrating the vector field ξ and $\tilde{\eta}$, we obtain the bi-Lipschitz $\mathcal{K}$-triviality required and the result follows. □

We also have a version of the Theorem 2.3 with respect to the $\mathcal{C}$ group.

Theorem 2.5. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewcz condition $N^{\star}_{\mathcal{C}} f_t(x) \geq c|x|^{\alpha}$, for constants c and α. Then, small deformations of f defined by $f_t = f + t\Theta$ with $\Theta = (\Theta_1, \ldots, \Theta_p)$ and fil $(\Theta_i) \geq d_p + r_n - r_1$ are bi-Lipschitz $\mathcal{C}$-trivial.*

Proof. The proof of this result is analogous to the proof of the Theorem 2.3. We shall show here the main differences, which appear in the Thom-Levine type equation.

As we are considering the group $\mathcal{C}$, we write the following equation:

$$N^{\star}_{\mathcal{C}} f_t \frac{\partial f_t}{\partial t} = \sum_{i=1}^{p} f_{ti}^{2\beta_i - 1} \left(\frac{\partial f_t}{\partial t} \right) f_{ti}.$$

Dividing the above expression by $N^{\star}_{\mathcal{C}} f_t$ we obtain: $\dfrac{\partial f_t}{\partial t} = \displaystyle\sum_{i=1}^{p} \frac{f_{ti}^{2\beta_i - 1}(\frac{\partial f_t}{\partial t}) f_{ti}}{N^{\star}_{\mathcal{C}} f_t}$.

Define the vector field $\eta(x, y, t) = (\eta_1, \ldots, \eta_p)(x, y, t)$ with each $\eta_j = \displaystyle\sum_{i=1}^{p} \frac{f_{ti}^{2\beta_i - 1} \Theta_j y_i}{N^{\star}_{\mathcal{C}} f_t}$.

Now we use the the bump function again to construct the vector field $\tilde{\eta}(x, y, t) = p(x, y, t)\eta(x, y, t)$ and following the same steps as done in the proof of Theorem 2.3 the result follows. □

2.2. Newton Non Degenerate Case.

In this subsection we consider a Newton filtration for $\mathcal{E}_n$ which is determined from a Newton polyhedron and provide estimates for the bi-Lipschitz-$\mathcal{G}$-triviality, $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$ of analytic map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ satisfying some non-degeneracy condition with respect to this filtration.

To construct a Newton polyhedron we fix an $n \times m$ matrix $A = (a_i^j)$, with $i = 1, ..., n$, $j = 1, ..., m$, $a^j = (a_1^j, \ldots, a_n^j) \in \mathbb{Q}^n_+$ and $m \geq n$, such that the first n columns of A are $(0, ..., 0, a_j^j, 0, ..., 0)$ with $a_j^j > 0$, for all $j = 1, ..., n$.

Definition 2.6. *The Newton polyhedron $\Gamma_+(A)$ is the convex hull in $\mathbb{R}^n_+$ of Supp$(A) + \mathbb{R}^n_+$, where Supp$(A) = \{a^j,\ j = 1, ..., m\}$. The Newton diagram of A, denoted $\Gamma(A)$, is the union of the compact faces of $\Gamma_+(A)$.*

Now we define the control functions associated to a Newton polyhedron $\Gamma_+(A)$.

For any vector a^j of the matrix A and $k \in \mathbb{R}_+$, we denote $ka^j = (ka_1^j, ka_2^j, \ldots, ka_n^j)$. We fix the smallest integer p such that pa_i^j is integer for all i, j and for any non-negative rational number d we define the control function $\rho^d : \mathbb{R}^n \to \mathbb{R}$ as:

$$\rho^d(x) = \left(\sum_{j=1}^{m} x_1^{2pa_1^j} x_2^{2pa_2^j} \cdots x_n^{2pa_n^j} \right)^{\frac{d}{2p}}. \tag{2}$$

We remark that the function $\rho^{2p}(x) = \sum_{j=1}^{m} x_1^{2pa_1^j} x_2^{2pa_2^j} \cdots x_n^{2pa_n^j}$ is a polynomial.

We call ρ^d *control* of $\Gamma_+(dA)$, where dA denotes the matrix $dA = (da_i^j)$ and denote by $\Gamma_+(\rho^d)$, the Newton polyhedron of the matrix $dA = (da_i^j)$, i.e, $\Gamma_+(\rho^d) = \Gamma_+(dA)$.

Example 2.7. For $n = 2$, such an A is written as $A = \begin{pmatrix} a_1^1 & 0 & a_1^3 & \cdots & a_1^m \\ 0 & a_2^2 & a_2^3 & \cdots & a_2^m \end{pmatrix}$.

If we fix $A = \begin{pmatrix} \frac{1}{b} & 0 & \frac{b-1}{(b+1)b} \\ 0 & \frac{1}{b+1} & \frac{1}{(b+1)b} \end{pmatrix}$ with b a positive integer, we obtain $p = b(b+1)$ and the control function ρ of $\Gamma_+(A)$ is given as $\rho(x, y) = \left(x^{2b+2} + y^{2b} + x^{2b-2}y^2\right)^{\frac{1}{2b(b+1)}}$.

The corresponding Newton polyhedron $\Gamma_+(\rho^{2b(b+1)})$ has two faces with vertices $\{(2b+2, 0), (2b-2, 1), (0, 2b)\}$.

We remark that if the matrix A is a $n \times n$ diagonal matrix, we recover the weighted homogeneous case. We remember also that in this case the corresponding Newton polyhedron has only one face and the control function is the same given in the weighted homogeneous subsection.

The main condition to guarantee the bi-Lipshitz triviality in this case is given in terms of a non-degeneracy condition, called A-isolated, which we describe here.

For any germ g with Taylor series $g(x) = \sum a_\alpha x^\alpha$, if Δ is a subset of a Newton polyhedron $\Gamma_+(A)$, call g_Δ the germ $g_\Delta(x) = \sum_{\alpha \in \Delta} a_\alpha x^\alpha$.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ written as $f(x) = \sum a_\gamma x^\gamma$, where x^γ denotes the monomial $x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$. For any fixed germ f, call d the biggest rational integer such that $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_+(\rho^d)$ for all $a_\gamma \neq 0$.

Definition 2.8 ([1], p. 525). *The origin is an A-isolated point of a germ f if for each compact face Δ of $\Gamma(\rho^d)$, the equation $f_\Delta(x) = 0$ does not have solution in $(\mathbb{R} - \{0\})^n$.*

For a Newton polyhedron $\Gamma_+(A)$ and for each $w = (w_1, \ldots, w_n) \in \mathbb{R}_+^n$ define:

Definition 2.9. (a) $\ell(w) = \min\{\langle w, k\rangle : k \in \Gamma_+(A)\}$, $\langle w, k\rangle = \sum_{i=1}^n w_i k_i$.

(b) $\Delta(w) = \{k \in \Gamma_+(A) : \langle w, k\rangle = \ell(w)\}$.

(c) *Two vectors $a, a' \in (\mathbb{R}_+^*)^n$ are equivalent if $\Delta(a) = \Delta(a')$.*

Now we fix a Newton polyhedron $\Gamma_+(A)$ and for each $(n-1)$-dimensional compact face Δ_k of $\Gamma(A)$ we denote by $v^k = (v_1^k, ..., v_n^k)$, the vector in $\mathbb{Z}_+^n - \{0\}$ with minimum length which is associated to Δ_k i.e, $\Delta_k = \Delta(v^k)$.

Definition 2.10. *For an analytic real germ $f(x) = \sum a_\gamma x^\gamma$ call*

$$\mathrm{fil}(f) := \inf\{\mathrm{fil}(\gamma) \ / \ a_\gamma \neq 0\}, \quad \text{where} \quad \mathrm{fil}(\gamma) = \min_k \left\{ \frac{M}{\ell(v^k)} \langle \gamma, v^k \rangle \right\},$$

$$M = lcm\{\ell(v^k)\},\ R = \max_j \max_i \left\{ \frac{M}{\ell(v^k)} v_i^k \right\} \text{ and } r = \min_j \min_i \left\{ \frac{M}{\ell(v^k)} v_i^k \right\}.$$

Definition 2.11. *For a fixed matrix A, a germ f is called A-homogeneous of degree d if $f(x) = \sum_{\nu \in \Gamma(\rho^d)} c_\nu x^\nu$.*

Definition 2.12. *An analytic map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$; $f = (f_1, \ldots, f_p)$ is a A-homogenous of degree $d = (d_1, \ldots, d_p)$ if each f_i is A-homogenous of degree d_i.*

Now in an analogous way than the weighted homogeneous case, for any polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ we define, for some fixed Newton filtration, the appropriate control functions $N_{\mathcal{R}}^\star f := \sum_I M_I^{2k_I}$, $N_{\mathcal{C}}^\star f := \sum_{i=1}^p (f_i)^{2\beta_i}$ and

$$N_{\mathcal{K}}^\star f := (N_{\mathcal{R}}^\star f)^a + (N_{\mathcal{C}}^\star f)^b,$$

where $k_I = \frac{k}{s_I}$ with $s_I = \mathrm{fil}\,(M_I)$, $k = lcm\,\{s_I\}$ and $a, b \in \mathbb{N}$ be prime numbers with $gcd(a,b) = 1$ such that $\mathrm{fil}\,(N_{\mathcal{R}}^\star f)^a = \mathrm{fil}\,(N_{\mathcal{C}}^\star f)^b$.

Then for any family $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $t \in [0,1]$, of polynomial map germs with $f_o(x) = f(x)$, we also define $N_{\mathcal{R}}^\star f_t := \sum_I M_{t_I}^{2k_I}$, $N_{\mathcal{C}}^\star f_t := \sum_{i=1}^p (f_{t_i})^{2\beta_i}$, $N_{\mathcal{K}}^\star f_t = (N_{\mathcal{R}}^\star f_t)^a + (N_{\mathcal{C}}^\star f_t)^b$, such that a, b, k_I, β_i are given as before.

Next we show the bi-Lipschitz-$\mathcal{G}$-triviality in the cases: $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$, of map germs with respect to some Newton polyhedron.

Theorem 2.13. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ. Suppose that $N_{\mathcal{C}}^\star f$ is A-isolated for some matrix A. If $f_t = f + t\Theta$ is a deformation of f, $\Theta = (\Theta_1, \ldots, \Theta_p)$, with $fil(\Theta_i) \geq fil(f_p) + R - r$, then f_t is bi-Lipschitz $\mathcal{C}$-trivial.*

Theorem 2.14. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ. Suppose that $N_{\mathcal{K}}^\star f$ is A-isolated for some matrix A. If $f_t = f + t\Theta$ is a deformation of f, $\Theta = (\Theta_1, \ldots, \Theta_p)$, with $fil(\Theta_i) \geq fil(f_p) + R - r$, then f_t is bi-Lipschitz $\mathcal{K}$-trivial.*

The numbers R and r cited in the Theorems 2.13 and 2.14 are as in Definition 2.10.

The proofs of these theorems are already equal than the proofs done in the weighted homogeneous case, the main difference is that here we are working with the Newton filtration and the controls are that relative to this filtration. Then we can omit them here.

Remark 2.15. In [15] there are shown estimates for the C^ℓ-$\mathcal{G}$-triviality, with $\mathcal{G} = \mathcal{K}$ or $\mathcal{C}$ and $0 \leq \ell < \infty$ in the A-isolated case. However these estimates can be improved using the bump function, as done before. In the last section of this article we shall include these new estimates for all C^ℓ-$\mathcal{G}$-trivialities, with $\mathcal{G} = \mathcal{K}$ or $\mathcal{C}$ and $0 \leq \ell < \infty$, in a general table with all known estimates.

3. Examples

Example 3.1. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $f(x, y) = (xy, x^{2b+2} - y^{2b} + x^{2b-2}y^2)$, with $b > 2$ and fix the matrix $A = \begin{pmatrix} \frac{1}{b} & 0 & \frac{b-1}{(b+1)b} \\ 0 & \frac{1}{b+1} & \frac{1}{(b+1)b} \end{pmatrix}$.

Since $lcm\{\ell(v_1), \ell(v_2)\} = 2b(b+1)$, $\text{fil}(xy) = 2b+2$, $\text{fil}(y^{2b}) = 2b(b+1)$, $\text{fil}(x^{2b-2}y^2) = 2b(b+1)$ and $\text{fil}(x^{2b+2}) = 2b(b+1)$, then we obtain $R = 2b$ and $r = b$.

The 2×2 minor of df, $M(x, y) = -2((b+1)x^{2b+2} + by^{2b} + bx^{2b-2}y^2)$ is A-homogenous of degree $2b(b+1)$, which is the same degree as the second coordinate function of f.
To get the control function $N^\star_{\mathcal{K}} f(x, y)$, we have $N^\star_{\mathcal{R}} f(x, y) = N_{\mathcal{R}} f(x, y) = |M(x, y)|^2$ and $N^\star_{\mathcal{C}} f(x, y) = xy^{2b(b+1)} + (x^{2b+2} - y^{2b} + x^{2b-2}y^2)^{2b+2}$, therefore we obtain

$$N^\star_{\mathcal{K}} f(x, y) = |M(x, y)|^{2b+2} + xy^{2b(b+1)} + (x^{2b+2} - y^{2b} + x^{2b-2}y^2)^{2b+2}$$

which is A-isolated.

From the Theorem 2.14 we see that a family $f + t\Theta$ with $\Theta = (\Theta_1, \Theta_2)$ is bi-Lipschitz-$\mathcal{K}$-trivial if $\text{fil}(\Theta_i) \geq 2b(b+1) + 2b - b$.

It is interesting to compare this bi-Lipschitz $\mathcal{K}$ estimate with the C^1-$\mathcal{K}$ and C^0-$\mathcal{K}$ estimates. In fact, by Theorem 3.16 of [15] this family f_t is C^1-$\mathcal{K}$-trivial if $fil(\Theta_i) \geq 2b(b+1) + 2b + 1$. An analogous computation shows that this family is C^0-$\mathcal{K}$-trivial if $\text{fil}(\Theta_i) \geq 2b(b+1) + 1$.

Example 3.2. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $f(x, y) = (xy + x^2y^2, x^{2(c+1)} + xy - y^{2c})$ with $c \geq 3$.

Fix $A = \begin{pmatrix} 2(c+1) & 0 & 1 \\ 0 & 2c & 1 \end{pmatrix}$, then $lcm\{\ell(v^1), \ell(v^2)\} = 2c(c+1)$, $R = 2c^2 + c$ and $r = c$. We have also $\varphi(a, b) = \min\{(c+1)\langle(a, b), (2c-1, 1)\rangle, c\langle(a, b), (1, 2c+1)\rangle\}$ and $\text{fil}(f_1, f_2) = (2c(c+1), 2c(c+1))$.

Since $\text{fil}(f_1) = \text{fil}(f_2)$, we have $N^\star_{\mathcal{C}} f = N_{\mathcal{C}} f$ is A-isolated, then we apply the Theorem 2.13 to obtain that

$$f_t(x, y) = (xy + x^2y^2 + t\Theta_1(x, y), x^{2(c+1)} + xy - y^{2c} + t\Theta_2(x, y))$$

is bi-Lipschitz-$\mathcal{C}$-trivial if $fil(\Theta_i)(x, y) \geq 4c^2 + 2c$.

Example 3.3. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $f(x, y, z) = (3x^6 + 2y^6 + xz^4, x^6 + y^6 + yz^3)$.

Then $df = \begin{bmatrix} 18x^5 + z^4 & 12y^5 & 4xz^3 \\ 6x^5 & 6y^5 + z^3 & 3yz^2 \end{bmatrix}$ and its minors are $M_{12} = 36x^5y^5 + 18x^5z^3 + 6y^5z^4 + z^7$, $M_{13} = 54x^5yz^2 + 3yz^6 - 24x^6z^3$ and $M_{23} = 36y^6z^2 - 24xy^5z^3 - 4xz^6$.

Here $N^\star_{\mathcal{C}} f$ is not A-isolated for any matrix A, hence the Theorem 2.13 does not apply to this example. However we can obtain the estimate for the group $\mathcal{K}$ because

$N^{\star}_{\mathcal{K}}f$ is A-isolated for the matrix $A = \begin{pmatrix} 6 & 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \\ 0 & 0 & 7 & 4 & 3 \end{pmatrix}$. Then we obtain $R = 288$, $r = 84$ and $\mathrm{fil}(f_1) = \mathrm{fil}(f_2) = 504$.

Therefore we conclude that a family

$$f_t(x, y, z) = (3x^6 + 2y^6 + xz^4 + t\Theta_1(x, y, z), x^6 + y^6 + yz^3 + ty^5z^3t\Theta_2(x, y, z))$$

is bi-Lipschitz-$\mathcal{K}$-trivial if $fil(\Theta_i) \geq 504+288-84 = 708$. For instance, if we consider the monomial y^4z^3, we get the bi-Lipschitz-$\mathcal{K}$-triviality of the family

$$f_t(x, y, z) = (3x^6 + 2y^6 + xz^4, x^6 + y^6 + yz^3 + ty^4z^3).$$

4. Bi-Lipschitz $\mathcal{G}_V$-triviality, $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

In this section we show estimates for the bi-Lipschitz $\mathcal{G}_V$-triviality, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ of map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ defined on a germ of real analytic variety $(V, 0)$. The main difference here is that we fix the groups $\mathcal{G}_V$, which are the respective sub-groups of $\mathcal{G}$ preserving the variety V as described below.

We remark that in this case there is a different approach for each group of interest, or in other words, to work with the group $\mathcal{R}_V$ we need to consider a real analytic germ of variety $(V, 0)$ in $(\mathbb{R}^n, 0)$ defined by $V = \{x \in \mathbb{R}^n : g_1(x) = \cdots = g_s(x) = 0\}$ where each $g_i(x)$ is a germ of real analytic function for $i = 1, \dots, s$.

However, when the interest of study is the group $\mathcal{C}_V$, we need to consider the variety $(V, 0)$ as a subset of $(\mathbb{R}^p, 0)$, that is, $V = \{y \in \mathbb{R}^p : g_1(y) = \cdots = g_s(y) = 0\}$ where each $g_i(y)$ is a germ of real analytic function in $\mathcal{E}_p$ for all $i = 1, \dots, s$.

Moreover, the group $\mathcal{K}_V$ can be seen as the semi-direct product of the groups $\mathcal{R}$ and $\mathcal{C}_V$ and it is not related to the group $\mathcal{R}_V$.

4.1. Bi-Lipschitz $\mathcal{R}_V$-triviality.

Here we consider the real analytic variety $(V, 0)$ in the source, or $(V, 0) \subset (\mathbb{R}^n, 0)$, defined by $V = \{x : g_1(x) = \cdots = g_s(x) = 0\}$ where each $g_i(x)$ is a real analytic function germ, for $i = 1, \dots, s$.

We denote by θ_n the set of germs of tangent vector fields in $(\mathbb{R}^n, 0)$ and call the submodule of germs of tangent vector fields to V by $\theta_V = \{\psi \in \theta_n : \psi(I(V)) \subset I(V)\}$, where $I(V)$ is the ideal in $\mathcal{E}_n$ consisting of germs of real analytic functions vanishing on V.

We are interested in that case that V is defined by weighted homogeneous polynomials g_i of type $(r_1, \dots, r_n; \alpha_i)$ for some α_i, since in this case we see in [5] or [14] that we can always choose weighted homogeneous generators for θ_V. In this case we also need to consider weighted homogeneous function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of type $(r_1, \dots, r_n; d)$ which are consistent with V. In fact, we say that a function germ f is *consistent* with a weighted homogeneous variety V if f is weighted homogeneous of the same type than V, or in other words, if V is defined by weighted homogeneous polynomials g_i of type $(r_1, \dots, r_n; \alpha_i)$ for some α_i, then f is also weighted homogenous of type $(r_1, \dots, r_n; d)$ for some d.

We denote by θ_V^0 the tangent space of $\mathcal{R}_V(f) = \mathcal{M}_n\{\gamma_j\}_{1\leq j\leq p} + I(V)\mathcal{E}_n\left\{\frac{\partial}{\partial x_i} = \theta_V^0\right\}$.

In the usual way we extend the filtration to θ_V defining fil $(\frac{\partial}{\partial x_i}) = -r_i$ for $i = 1,\ldots,n$ so that given $\gamma = \sum_{j=1}^n \gamma_j \frac{\partial}{\partial x_i} \in \theta_V$ then $\text{fil}(\gamma) = \inf_j\{\text{fil}(\gamma_j) - r_j\}$

Let V be a weighted homogeneous analytic subvariety of $(\mathbb{R}^n, 0)$ and $\{\gamma_1,\ldots,\gamma_m\}$ be a system of weighted homogeneous generators of type $(r_1,\ldots,r_n; S_1,\ldots,S_m)$ for θ_V^0, with $r_1 \leq \cdots \leq r_n$, $r_i \in \mathbb{Z}$ and each $\gamma_j = \sum_{j=1}^n \gamma_{ji}\frac{\partial}{\partial x_i}$.

We denote $N_{\mathcal{R}_V} f = (df(\gamma_j))^2 = \sum_{j=1}^m \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\gamma_{ji}\right)^2$.

Then we have the following:

Theorem 4.1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a weighted homogeneous function germ of type $(r_1,\ldots,r_n; d)$ which is consistent with V. Suppose that $N_{\mathcal{R}_V} f$ satisfies a Lojasiewicz condition $N_{\mathcal{R}_V} f(x) \geq c|x|^\alpha$ for some constants c and α. Then deformations $f_t(x) = f(x) + t\Theta(x)$ with fil$(\Theta) \geq d + r_n - r_1$ are bi-Lipschitz $\mathcal{R}_V$-trivial for all $t \in [0,1]$.*

Proof. The proof of this Theorem is analogous to the proof of the Theorem 3.1. given by H. Liu and D. Zhang in [11], where it is given estimates for the C^ℓ-$\mathcal{R}_V$-determinacy, with $\ell \geq 2$, for weighted homogeneous function germs defined on weighted homogeneous analytic varieties. This proof is done with the construction of controlled vector fields in a similar way than the proof of other results which use controlled vector fields. The difference is that their estimates are given in terms of the C^ℓ-class of differentiability of such vector fields, while our estimates are given to obtain the Lipschitz condition for the vector field given in that proof. □

We remark here that the estimate for the C^0-$\mathcal{R}_V$-triviality was first showed by Damon in [6], see also [14]. We do not know by now any estimate for the C^1-$\mathcal{R}_V$-triviality, however it is also possible to show that the estimate given in Theorem 4.1 coincides with the estimate for the C^1-$\mathcal{R}_V$-triviality. This is the unique situation, the $\mathcal{R}_V$-triviality, where we find equal estimates for the C^1-triviality and the bi-Lipschitz-triviality. Our method does not allow us to improve this estimate or to show if there are germs which are bi-Lipschitz $\mathcal{R}_V$-trivial and are not C^1-$\mathcal{R}_V$-trivial. In the last section of this article we shall give a table with all estimates that we know which are given in terms of controlled vector fields.

4.2. Bi-Lipschitz $\mathcal{G}_V$-triviality, $\mathcal{G} = \mathcal{K}$ or $\mathcal{C}$.

To work with the $\mathcal{K}_V$ and $\mathcal{C}_V$ groups we need to consider an analytic subvariety $(V, 0)$ of $(\mathbb{R}^p, 0)$. The group $\mathcal{K}_V$ is the subgroup of $\mathcal{K}$ consisting of elements $(h, H) \in \mathcal{K}$ such that $H(\mathbb{R}^n \times V) = \mathbb{R}^n \times V$ and the group $\mathcal{C}_V$ is defined by the pairs $(h, H) \in \mathcal{K}_V$, where h is the identity mapping of $\mathbb{R}^n$.

In order to state the theorems for these cases we fix V be a germ at 0 of an analytic subvariety of $(\mathbb{R}^p, 0)$ defined by a weighted homogeneous polynomial g of type $(w_1,\ldots w_p; L)$ and $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a weighted homogeneous map germ of type $(r_1,\ldots,r_n; w_1,\ldots w_p)$ with $r_1 \leq \cdots \leq r_n$ and $w_1 \leq \cdots \leq w_p$.

Here we follow [11] to define $N_{\mathcal{C}_V} f = \sum_{i=1}^{p} \left(\sum_{i=1}^{p} \frac{\partial f}{\partial x_i} \gamma_{ji} \right)^2$ and $N_{\mathcal{K}_V} f$ is defined as the semi direct product $N_{\mathcal{K}_V} f = N_{\mathcal{R}} f + N_{\mathcal{C}_V} f$.

Now we give the estimates for the bi-Lipshitz $\mathcal{K}_V$-triviality in the next theorem.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; w_1, \ldots, w_p)$ with $r_1 \leq \cdots \leq r_n$ and $w_1 \leq \cdots \leq w_p$. Suppose that θ_V is a free $\mathcal{E}_p$ module and $\psi_1, \ldots, \psi_p$ is a basis for θ_V with any ψ_i is weighted homogeneous germ of type $(w_1, \ldots, w_p; d_i)$ with $d_1 \leq \cdots \leq d_p$.

Theorem 4.2. *Suppose that $N_{\mathcal{K}_V} f$ satisfies a Lojasiewicz condition $N_{\mathcal{K}_V} f(x) \geq c|x|^\alpha$ for some constants c and α. Then deformations $f_t(x) = f(x) + t\Theta(x)$, $\Theta = (\Theta_1, \ldots, \Theta_p)$, with fil$(\Theta_i) \geq d_i + w_p + r_n - r_1$ are bi-Lipshitz $\mathcal{K}_V$-trivial for all $t \in [0, 1]$.*

Proof. The proof of this Theorem is analogous to the proof of the Theorem 4.4. given by H. Liu and D. Zhang in [11], where there are given estimates for the C^ℓ-$\mathcal{K}_V$-determinacy, with $\ell \geq 2$, of weighted homogeneous map germs on weighted homogeneous analytic varieties, the difference here is that our estimates are given to obtain the Lipschitz condition for the vector field given in that proof, while they are interested only in these C^ℓ-class of differentiability with $\ell \geq 2$, of such vector fields. Also we observe that we are applying the bump function to obtain this estimate. □

It is interesting to remark here that this estimate depends of the weighted degrees of the generators $\psi_1, \ldots, \psi_p$ of θ_V, while in the estimate given for the $\mathcal{R}_V$-triviality this does not occurs.

We also have a version of this result for the $\mathcal{C}_V$-triviality. For this we consider the same notation given before the Theorem 4.2

Theorem 4.3. *Suppose that $N_{\mathcal{C}_V} f$ satisfies a Lojasiewicz condition $N_{\mathcal{C}_V} f(x) \geq c|x|^\alpha$ for some constants c and α. Then deformations $f_t(x) = f(x) + t\Theta(x)$, $\Theta = (\Theta_1, \ldots, \Theta_p)$, with fil$(\Theta_i) \geq d_i + w_p + r_n - r_1$ are bi-Lipschitz $\mathcal{C}_V$-trivial for all $t \in [0, 1]$.*

Proof. The proof of Theorem 4.3 is analogous to the proof of Theorem 4.2, the difference is that we consider only the vector field associate to the group $\mathcal{C}_V$, as done in the proof of the Theorem 2.5. Here we are also using the bump function to get this estimate. □

5. Tables of estimates

In the tables below we show all estimates that we know to compute the C^ℓ-$\mathcal{G}$-triviality with $0 \leq \ell < \infty$, the bi-Lipschitz triviality and also the C^ℓ-$\mathcal{G}_V$-triviality for real map germs and $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

5.1. The weighted homogenous case.

Group $\mathcal{R}$	Estimate	Reference
C^ℓ-$\mathcal{R}$-trivial, $\ell \geq 2$	$\text{fil}(\Theta_i) \geq d_i - r_1 + \ell r_n + 1$	[13]
C^1-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i - r_1 + r_n + 1$	[13]
bi-Lipschitz-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i - r_1 + r_n$	[8]
C^0-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i$	[13]

Here f is weighted homogenous of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq r_2 \leq \cdots \leq r_n$.

Groups $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$	**Estimate**	**Reference**
C^ℓ-$\mathcal{G}$-trivial, $\ell \geq 2$	$\text{fil}(\Theta_i) \geq d_p + \ell r_n$	[13]
C^1-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p + r_n$	[13]
bi-Lipschitz-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p - r_1 + r_n$	$--$
C^0-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p$	[13]

Here f is weighted homogenous of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq r_2 \leq \cdots \leq r_n$ and $d_1 \leq d_2 \leq \cdots \leq d_p$.

5.2. The Newton non-degenerate case.

Group $\mathcal{R}$	**Estimate**	**Reference**
C^ℓ-$\mathcal{R}$-trivial, $\ell \geq 2$	$\text{fil}(\Theta_i) \geq d_i - r + \ell R + 1$	[15]
C^1-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i - r + R + 1$	[15]
bi-Lipschitz-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i - r + R$	[9]
C^0-$\mathcal{R}$-trivial	$\text{fil}(\Theta_i) \geq d_i$	[15]

Groups $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$	**Estimate**	**Reference**
C^ℓ-$\mathcal{G}$-trivial, $\ell \geq 2$	$\text{fil}(\Theta_i) \geq d_p + \ell R$	$--$
C^1-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p + R$	$--$
bi-Lipschitz-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p - r + R$	$--$
C^0-$\mathcal{G}$-trivial	$\text{fil}(\Theta_i) \geq d_p$	$--$

In this table $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is A-homogenous of degree $d = (d_1, \ldots, d_p)$ with $d_1 \leq d_2 \leq \cdots \leq d_p$, $R = \max_j \max_i \left\{\frac{M}{\ell(v^k)} v_i^k\right\}$, $r = \min_j \min_i \left\{\frac{M}{\ell(v^k)} v_i^k\right\}$ and $M = lcm\{\ell(v^k)\}$ for a fixed Newton polyhedron $\Gamma_+(A)$.

We remember here that in [15] there are also shown estimates for the C^ℓ-$\mathcal{G}$-triviality with $\mathcal{G} = \mathcal{C}$ or $\mathcal{K}$ and $0 \leq \ell < \infty$, however these estimates were done without the use of the bump function, hence they differ from these in this table by adding one in the filtration of the deformations given above.

5.3. The $\mathcal{G}_V$-triviality.

Group $\mathcal{R}_V$	**Estimate**	**Reference**
C^ℓ-$\mathcal{R}_V$-trivial, $\ell \geq 2$	$\text{fil}(\Theta) \geq d - r_1 + \ell r_n$	[11]
C^1-$\mathcal{R}_V$-trivial	$\text{fil}(\Theta) \geq d - r_1 + r_n$	$--$
bi-Lipschitz-$\mathcal{R}_V$-trivial	$\text{fil}(\Theta) \geq d - r_1 + r_n$	$--$
C^0-$\mathcal{R}_V$-trivial	$\text{fil}(\Theta) \geq d$	[6]

Here f is a weighted homogeneous function germ of type $(r_1, \ldots, r_n; d)$ and V in $\mathbb{R}^n$ is defined by weighted homogeneous polynomials g_i of type $(r_1, \ldots, r_n; D_i)$ for some D_i with $r_1 \leq \cdots \leq r_n$.

Groups $\mathcal{G}_V = \mathcal{C}_V$ or $\mathcal{K}_V$	Estimate	Reference
C^ℓ-$\mathcal{G}_V$-trivial, $\ell \geq 2$	$\mathrm{fil}(\Theta_i) \geq d_i + w_p + \ell r_n + 1$	[11]
C^1-$\mathcal{G}_V$-trivial	$\mathrm{fil}(\Theta_i) \geq d_i + w_p + r_n + 1$	$--$
bi-Lipschitz-$\mathcal{G}_V$-trivial	$\mathrm{fil}(\Theta_i) \geq d_i + w_p + r_n - r_1$	$--$
C^0-$\mathcal{G}_V$-trivial	$\mathrm{fil}(\Theta_i) \geq d_i + w_p$	$--$

Here V is a germ at 0 of an analytic subvariety of $(\mathbb{R}^p, 0)$ defined by a weighted homogeneous polynomial g of type $(w_1, \ldots, w_p; L)$ and $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a weighted homogeneous map germ of type $(r_1, \ldots, r_n; w_1, \ldots, w_p)$ with $r_1 \leq \cdots \leq r_n$ and $w_1 \leq \cdots \leq w_p$. In this case, also we suppose that θ_V is a free $\mathcal{E}_p$ module and $\psi_1, \ldots \psi_p$ is a basis for θ_V formed by weighted homogeneous germs of type $(w_1, \ldots, w_p; d_1, \ldots, d_p)$ with $d_1 \leq d_2 \leq \cdots \leq d_p$.

References

[1] O. M. Abderrahmane, *Polyèdre de Newton et Trivialité en Famille*, J. Math. Soc. Japan, 54, 3 (2002) 513–550.

[2] L. Birbrair, J.C.F. Costa, A. Fernandes, M.A.S. Ruas, *$\mathcal{K}$-bi-Lipschitz equivalence of real function-germs*, Proc. A.M.S. 135 (2007) 1089–1095.

[3] S. Bromberg, L. Medrano, *C^r-sufficiency of quasihomogeneous functions.* Real and Complex Singularities, Pitman Res. Notes in Math. Series 333 (1995) 179–189.

[4] M. A. Buchner & W. Kucharz, *Topological triviality of family of zero-sets.* Proc. Amer. Math. Soc., 102, 3 (1988) 699–705.

[5] J. Damon, *On the freeness of equisingular deformations of plane curve singularities*, Topology and its applications, 118 (2002) 31–43.

[6] J. Damon, *Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Nonlinearity, 5 (1992) 373–412.

[7] J. Damon, *Topological invariants of μ-constant deformations of complete intersection singularities*, Quart. J. Math. Oxford, 2, 40 (1989) 139–159.

[8] A. Fernandes, M.A.S. Ruas, *Bi-Lipschitz determinacy of quasi-homogeneous germs*, Glasgow Math. J. 46, 1 (2004) 77–82.

[9] A. Fernandes, C.H. Soares Jr, *On the bi-Lipschitz triviality of families of real maps*, Real and Complex Singularities, Contemp. Math. 354, A.M.S., Providence, RI (2004) 95–103.

[10] T. C. Kuo, *On C^0-sufficience of jets of potencial functions.* Topology, 8 (1969) 167–171.

[11] H. Liu, D. Zhang, *C^ℓ-$\mathcal{G}_V$-determinacy of weighted homogeneous function germs on weighted homogeneous analytic varieties*, Hokkaido Maht. J. 37 (2008) 309–329.

[12] M.A.S. Ruas, *On the degree of C^ℓ-determinacy*, Math. Scand. 59 (1986) 59–70.

[13] M.A.S. Ruas, M.J. Saia, *C^ℓ-determinacy of weighted homogeneous germs*, Hokkaido Math. J. 26 (1997) 89–99.

[14] M.A.S. Ruas, J. N. Tomazella, *Topological triviality of families of functions on analytic varieties*, Nagoya Math. J. 175 (2004) 39–50.

[15] M.J. Saia, C.H. Soares Jr, *C^l-$\mathcal{G}$-triviality of map germs and Newton polyhedra, $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$*, Hokkaido Math. J., 27, 2 (2008) 331–348.

Departamento de Matemática, IBILCE-UNESP, São José do Rio Preto-SP, Brazil
E-mail address: jcosta@ibilce.unesp.br

Departamento de Matemática, ICMC-USP, 13560-970 São Carlos, SP, Brazil
E-mail address: mjsaia@icmc.usp.br

Departamento de Matemática, Universidade Regional do Cariri, Crato, Ceará, Brazil
E-mail address: humberto@urca.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

321/09	FENILLE, M. C. – On acyclic and simply connected open manifolds.
320/09	FENILLE, M. C. – Closed injective systems and its fundamental limit spaces .
319/09	BRASSELET, J. P.; GRULH JR., N. G. ; RUAS, M. A. S. – The Euler obstruction and the Chern obstruction.
318/09	BIANCONI, R.; FEDERSON, M. - A fredholm-type theorem for linear integral equations of stieltjes type.
317/09	CARBINNATO, M. C.; RYBAKOWSKY, K. P. - Conley index and parabolic problems with localized large diffusion and nonlinear boundary conditions.
316/09	CARBINNATO, M. C.; RYBAKOWSKY, K. P. - Conley index and homology index braids in singular perturbation problems without uniqueness of solutions.
315/09	HARTMANN, L.; SPREAFICO, M. - Analitic Torsion for Manifolds with Totally Geodesic Boundary.
314/09	CALLEJAS-BEDREGAL, R.; JORGE PERÉZ, V. H. - Mixed Multiplicities of Arbitrary Modules.
313/09	BRONZI, M.; TAHZIBI, A. - Homoclinic tangency and variation of entropy.
312/09	THAZIBI, A. - Carlos Teobaldo Gutiérrez Vidalon 1944-2008.